



On pantograph ψ -Caputo integro-differential problems via upper and lower solutions



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Abstract

In this study, we explore the existence of positive solutions for a class of fractional differential pantograph equations that incorporate the ψ -Caputo fractional derivative under specified initial conditions. By reformulating the problem as an equivalent Riemann-Liouville integral equation, we rigorously derive our principal results using the upper and lower solutions method. Lastly, we present an example to demonstrate the validity of our results.

Keywords: Fractional differential pantograph equations, ψ -fractional integral and derivative, Arzelà-Ascoli theorem, upper and lower solutions.

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1. Introduction

In nowadays research, fractional differential equations are extensively employed across various domains, including physics, fluid dynamics, viscoelasticity, biological systems, control theory, and chemistry, among others, see [9, 13, 15, 17]. For a rigorous exposition of fundamental results in fractional calculus and the theoretical framework of fractional differential equations, we refer the reader to the comprehensive monographs of several scholars [14, 20, 21].

Several methodologies for defining fractional integrals and derivatives have been proposed in the literature. In 2017, Ricardo Almeida expanded the field by introducing the ψ -Caputo operator, thereby adding to the existing repertoire, which includes the Caputo, Caputo-Hadamard, Caputo-Erdélyi-Kober, and Caputo-Katugampola operators, see [2, 12, 16].

The pantograph equation is a versatile differential equation applied across diverse fields, including electrodynamics, astrophysics, and cellular growth modeling. This broad applicability has led to a surge of recent studies on the fractional order pantograph equation by various researchers, see [8, 10, 19].

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The following differential equation illustrates the classic form of the pantograph equation [6]:

$$\begin{cases} \chi'(\varepsilon) = \lambda\chi(\varepsilon) + \vartheta\chi(\gamma\varepsilon), \quad \varepsilon \geq 0, \\ \chi(0) = \chi_0, \end{cases}$$

under the condition $\lambda, \vartheta \in \mathbb{C}$ and $0 < \gamma < 1$. Additionally, Balachandran et al. [6] established the existence and uniqueness of solutions for a fractional pantograph equation characterized by the Caputo fractional derivative

$$\begin{cases} {}^C\mathbb{D}_{0+}^{\rho}\chi(\varepsilon) = \mathfrak{N}(\varepsilon, \chi(\varepsilon), \chi(\gamma\varepsilon)), \quad \varepsilon \in [0, T], \quad 0 < \gamma \leq 1, \\ \chi(0) = \chi_0, \end{cases}$$

where ${}^C\mathbb{D}_{0+}^{\rho}$ and ${}^C\mathbb{D}_{0+}^{\rho}$ denote the Caputo fractional derivative of order $\rho \in (0, 1)$ and $\mathfrak{N} : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a given continuous function. Many researchers have focused their attention on studying the existence and uniqueness of solutions for initial and boundary value problems involving fractional differential equations, see [4] and the references therein. Moreover, the technique involving upper and lower solutions has proven effective in investigating the existence of solutions for both differential equations and inclusions, as demonstrated by its successful application in previous studies, see [1, 7] and related literature.

In [5], Baitiche et al. investigated the Monotone iterative technique for a sequential ψ -Caputo fractional differential equations with nonlinear boundary conditions as given in the following form:

$$\begin{cases} \left({}^C\mathbb{D}_{a+}^{\rho+1, \psi} + \lambda {}^C\mathbb{D}_{a+}^{\rho, \psi} \right) \chi(\varepsilon) = \mathfrak{N}(\varepsilon, \chi(\varepsilon)), \quad \varepsilon \in [a, b], \quad 0 < \rho \leq 1, \\ g(\chi(a), \chi(b)) = 0, \quad \chi'(a) = 0, \end{cases}$$

where ${}^C\mathbb{D}_{0+}^{\rho, \psi}$ denote the ψ -Caputo fractional derivatives of orders ρ , $\mathfrak{N} : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$, $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are both continuous functions and a real parameter λ is a positive.

Building on the work above, in this paper we study the existence of solutions to the following nonlinear fractional differential pantograph equation with an initial value condition, using the method of lower and upper solutions, which involves the fractional ψ -Caputo derivative :

$${}^C\mathbb{D}_{0+}^{\rho, \psi}\chi(\varepsilon) = \mathfrak{N}(\varepsilon, \chi(\varepsilon), \chi(\gamma\varepsilon), \mathbb{I}_{0+}^{\rho, \psi}\chi(\gamma\varepsilon)), \quad \varepsilon \in \Omega := [0, \kappa], \quad \chi(0) = \sum_{i=1}^n \mu_i \mathbb{I}_{0+}^{\rho, \psi}\chi(\eta_i), \quad (1.1)$$

where ${}^C\mathbb{D}_{0+}^{\rho, \psi}$ is the ψ -Caputo fractional derivative of order $\rho \in (0, 1)$, $\mathbb{I}_{0+}^{\rho, \psi}$ is the ψ -Riemann-Liouville fractional integral of order ρ , $\rho \in (0, 1)$, $0 < \eta_i < \kappa$, $0 < \gamma < 1$, $n \in \mathbb{N}^*$ and μ_i is fixed real, $\eta_i \in \Omega$, $\mathfrak{N} \in C(\Omega \times \mathbb{R}^3, \mathbb{R})$.

The rest of this paper is structured as follows. Section 2 presents the fundamental definitions and preliminary results necessary for the subsequent analysis. The main results, focusing on the existence of solutions to the ψ -Caputo fractional problem (1.1), are also discussed in this section. Finally, in the last section, we provide an example to illustrate and interpret our findings.

2. Main results

In this section, we introduce notations, definitions and preliminary facts which are used throughout this paper. Let $\Omega = [0, \kappa]$. By $C(\Omega, \mathbb{R})$, we denote the Banach space of all continuous functions from $\Omega \rightarrow \mathbb{R}$ with the norm

$$\|\chi\|_{\infty} = \sup\{|\chi(\varepsilon)| : \varepsilon \in \Omega\}.$$

By $C^n(\Omega, \mathbb{R})$, we denote the space of functions that are n -times continuously differentiable on Ω . Let $\psi \in C^1(J, \mathbb{R}^+)$ be an increasing differentiable function such that $\psi'(\varepsilon) \neq 0$, for all $\varepsilon \in J$.

Definition 2.1 ([2, 3]). For $\rho > 0$, the ψ -Riemann-Liouville fractional integral of order ρ for an integrable function $\chi : \Omega \mapsto \mathbb{R}$ is given by

$$\mathbb{I}_{0+}^{\rho, \psi} \chi(\varepsilon) = \frac{1}{\Gamma(\rho)} \int_0^\varepsilon \psi'(s) (\psi(\varepsilon) - \psi(s))^{\rho-1} \chi(s) ds,$$

where Γ is the classical Euler Gamma function.

Definition 2.2 ([2, 3]). Let $n-1 < \rho < n$, $\chi : \Omega \mapsto \mathbb{R}$ be an integrable function and ψ is defined as in Definition 2.1. The ψ -Riemann-Liouville fractional derivative of order ρ of a function χ is defined by

$$\mathbb{D}_{a+}^{\rho, \psi} \chi(\varepsilon) = \left[\frac{1}{\psi'(\varepsilon)} \frac{d}{d\varepsilon} \right]^n \mathbb{I}_{0+}^{n-\rho, \psi} \chi(\varepsilon) = \frac{1}{\Gamma(n-\rho)} \left[\frac{1}{\psi'(\varepsilon)} \frac{d}{d\varepsilon} \right]^n \int_0^\varepsilon \psi'(s) (\psi(\varepsilon) - \psi(s))^{n-\rho-1} \chi(s) ds,$$

where $n = [\rho] + 1$ and $[\rho]$ denotes the integer part of the real number ρ .

Definition 2.3 ([2, 3]). Let $n-1 < \rho < n$, $\chi \in C^n(\Omega, \mathbb{R})$ and ψ be defined as in Definition 2.1. The ψ -Caputo fractional derivative of a function χ of order ρ is determined as

$${}^C\mathbb{D}_{0+}^{\rho, \psi} \chi(\varepsilon) = \mathbb{I}_{0+}^{n-\rho, \psi} \chi_{\psi}^{[n]}(\varepsilon),$$

where $n = [\rho] + 1$ for $\rho \notin \mathbb{N}$, $n = \rho$ for $\rho \in \mathbb{N}$ and $\chi_{\psi}^{[n]}(\varepsilon) = \left[\frac{1}{\psi'(\varepsilon)} \frac{d}{d\varepsilon} \right]^n \chi(\varepsilon)$. Further, if $\chi \in C^n(\Omega)$ and $\rho \notin \mathbb{N}$,

$${}^C\mathbb{D}_{0+}^{\rho, \psi} \chi(\varepsilon) = \mathbb{I}_{0+}^{n-\rho, \psi} \left[\frac{1}{\psi'(\varepsilon)} \frac{d}{d\varepsilon} \right]^n \chi(\varepsilon) = \frac{1}{\Gamma(n-\rho)} \int_0^\varepsilon \psi'(s) (\psi(\varepsilon) - \psi(s))^{n-\rho-1} \chi_{\psi}^{[n]}(s) ds.$$

Thus, if $\rho \in \mathbb{N}$, one has ${}^C\mathbb{D}_{0+}^{\rho, \psi} \chi(\varepsilon) = \chi_{\psi}^{[n]}(\varepsilon)$.

The following lemma provides a list of the basic properties of the ψ -fractional operators.

Lemma 2.4 ([2, 3]). Let $\rho > 0$. The following holds.

1. If $\chi \in C(\Omega, \mathbb{R})$, then ${}^C\mathbb{D}_{0+}^{\rho, \psi} \mathbb{I}_{0+}^{\rho, \psi} \chi(\varepsilon) = \chi(\varepsilon)$.
2. If $\chi \in C(\Omega, \mathbb{R})$ and $\rho \in (0, 1)$, then $\mathbb{I}_{0+}^{\rho, \psi} {}^C\mathbb{D}_{0+}^{\rho, \psi} \chi(\varepsilon) = \chi(\varepsilon) - \chi(0)$.

Lemma 2.5 ([2, 11, 18]). Let $\rho > 0$ and $\chi : \Omega \mapsto \mathbb{R}$. Then we have

- $\mathbb{I}_{0+}^{\rho, \psi} [\psi(\varepsilon) - \psi(0)]^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\rho + \sigma)} [\psi(\varepsilon) - \psi(0)]^{\rho + \sigma - 1}$, $\sigma > 0$;
- ${}^C\mathbb{D}_{0+}^{\rho, \psi} [\psi(\varepsilon) - \psi(0)]^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma - \rho)} [\psi(\varepsilon) - \psi(0)]^{\sigma - \rho - 1}$, $\sigma > n \in \mathbb{N}$;
- ${}^C\mathbb{D}_{0+}^{\rho, \psi} [\psi(\varepsilon) - \psi(0)]^k = 0$, $\forall k \in \{0, 1, \dots, n-1\}$, $n \in \mathbb{N}$;
- $\mathbb{I}_{0+}^{\rho, \psi} \mathbb{I}_{0+}^{\sigma, \psi} \chi(\varepsilon) = \mathbb{I}_{0+}^{\rho + \sigma, \psi} \chi(\varepsilon)$, $\sigma > 0$;
- ${}^C\mathbb{D}_{0+}^{\rho, \psi} \beta = 0$, for any constant β .

3. The existence of solutions

Let $v : \Omega \rightarrow \mathbb{R}$ be a continuous function. We consider the following linear fractional differential equation:

$${}^C\mathbb{D}_{0+}^{\rho, \psi} \chi(\varepsilon) = v(\varepsilon), \quad \varepsilon \in \Omega = [0, \kappa], \quad (3.1)$$

where $\rho \in (0, 1]$, with the initial condition

$$\chi(0) = \sum_{i=1}^n \mu_i \mathbb{I}_{0+}^{\rho, \psi} \chi(\eta_i), \quad (3.2)$$

where $\rho \in (0, 1]$, $\eta_i \in \Omega$, $n \in \mathbb{N}^*$ and μ_i are fixed reals. The following theorem shows that the problem (3.1)-(3.2) has a unique solution given by:

$$\chi(\varepsilon) = \int_0^\varepsilon \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \nu(s) ds + \sum_{i=1}^n \int_0^{\eta_i} \mu_i \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \chi(s) ds. \quad (3.3)$$

Theorem 3.1. *The function χ satisfies equations (3.1) and (3.2) if and only if it satisfies (3.3).*

Proof. Assume χ satisfies the equations (3.1). We prove that χ is a solution to the equation (3.3). Applying the fractional integral $\mathbb{I}_{0+}^{\rho, \psi}$ to both sides of (3.1) and using Lemma 2.4, we have $\chi(\varepsilon) - \zeta_0 = \mathbb{I}_{0+}^{\rho, \psi} \nu(\varepsilon)$, which implies that

$$\chi(\varepsilon) = \int_0^\varepsilon \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \nu(s) ds + \zeta_0, \quad (3.4)$$

such as $\zeta_0 \in \mathbb{R}$. Now, condition (3.2) gives

$$\zeta_0 := \sum_{i=1}^n \int_0^{\eta_i} \mu_i \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \chi(s) ds. \quad (3.5)$$

Substituting (3.5) into (3.4), we obtain (3.3). Conversely, assume χ satisfies the equation (3.3). Applying operator ${}^C\mathbb{D}_{0+}^{\rho, \psi}$ on both sides of (3.3) for all $\varepsilon \in \Omega$, then, from Lemmas 2.4 and 2.5, we obtain ${}^C\mathbb{D}_{0+}^{\rho, \psi} \chi(\varepsilon) = \nu(\varepsilon)$. Taking the limit t tends towards of 0 in (3.3) we obtain $\chi(0) = \zeta_0$. This shows that the initial conditions (3.2) are satisfied. $\square$

Definition 3.2. A function $\chi \in C(\Omega, \mathbb{R})$ that satisfies the equations (3.1) and (3.2) on Ω is considered a solution to the fractional problem (3.1)-(3.2).

Now, we focus our attention on the existence of solutions for problems (1.1), and before proceeding, we introduce definitions for the lower and upper solutions of the problem (1.1).

Lemma 3.3. *Let $\rho, \rho \in (0, 1]$, $\eta_i \in \Omega$ and $\mathfrak{N} : \Omega \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuous. If $\chi \in C(\Omega, \mathbb{R})$, then χ is a solution to the problem (1.1) if and only if χ is the fixed point of the operator $\mathcal{A} : C(\Omega, \mathbb{R}) \rightarrow C(\Omega, \mathbb{R})$ defined by*

$$\begin{aligned} \mathcal{A}\chi(\varepsilon) := & \int_0^\varepsilon \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \mathfrak{N}(s, \chi(s), \chi(\gamma s), \mathbb{I}_{0+}^{\rho, \psi} \chi(\gamma s)) ds \\ & + \sum_{i=1}^n \int_0^{\eta_i} \mu_i \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \chi(s) ds. \end{aligned} \quad (3.6)$$

Since the function $\mathfrak{N} \in C(\Omega \times \mathbb{R}^3, \mathbb{R})$, it is clear that $\mathcal{A}\chi \in C(\Omega, \mathbb{R})$.

Definition 3.4. Let $\underline{\chi} \in C(\Omega, \mathbb{R})$ and $\bar{\chi} \in C(\Omega, \mathbb{R})$. A pair of functions $(\underline{\chi}, \bar{\chi})$ is called an upper solution (u-solution) and lower solution (l-solution) of problem (1.1), respectively, if

$$\begin{aligned} \underline{\chi}(\varepsilon) \leq & \int_0^\varepsilon \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \mathfrak{N}(s, \underline{\chi}(s), \underline{\chi}(\gamma s), \mathbb{I}_{0+}^{\rho, \psi} \underline{\chi}(\gamma s)) ds \\ & + \sum_{i=1}^n \int_0^{\eta_i} \mu_i \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \underline{\chi}(s) ds \end{aligned}$$

and

$$\begin{aligned}\bar{\chi}(\varepsilon) \leq & \int_0^\varepsilon \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \mathfrak{K}(s, \bar{\chi}(s), \bar{\chi}(\gamma s), \mathbb{I}_{0+}^{\rho, \psi} \bar{\chi}(\gamma s)) ds \\ & + \sum_{i=1}^n \int_0^{\eta_i} \mu_i \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \bar{\chi}(s) ds.\end{aligned}$$

Let the upper-lower solution (u-sol, l-sol) to the fractional integral equation (3.6) of constant order be $(\underline{\chi}, \bar{\chi})$. In the following, we refer to the set of acceptable solutions for the fractional integral equation (3.6), which is governed by a pair of upper and lower solutions $(\underline{\chi}, \bar{\chi})$ as follows:

$$\nabla_{(\underline{\chi}, \bar{\chi})} := \{\chi \in C(\Omega, \mathbb{R}) : \underline{\chi}(\varepsilon) \leq \chi(\varepsilon) \leq \bar{\chi}(\varepsilon), \varepsilon \in \Omega\}.$$

Theorem 3.5. Let $\mathfrak{K} \in C(\Omega \times \mathbb{R}^3, \mathbb{R})$. Assume that $(\underline{\chi}, \bar{\chi}) \in C(\Omega, \mathbb{R}) \times C(\Omega, \mathbb{R})$ are two solutions of the fractional integral equation (3.6), upper and lower, with $\underline{\chi}(\varepsilon) \leq \bar{\chi}(\varepsilon)$ for $\varepsilon \in \Omega$. If $(\varepsilon, \lambda_1, \lambda_2, \lambda_3) \rightarrow \mathfrak{K}(\varepsilon, \lambda_1, \lambda_2, \lambda_3)$ is nondecreasing, that is

$$\mathfrak{K}(\varepsilon, \lambda_1, \lambda_2, \lambda_3) - \mathfrak{K}(\varepsilon, \lambda'_1, \lambda'_2, \lambda'_3) \leq 0, \text{ for } \lambda_1 \leq \lambda'_1, \lambda_2 \leq \lambda'_2, \text{ and } \lambda_3 \leq \lambda'_3,$$

then, there exist minimum and maximum solutions $\chi_M(\varepsilon), \chi_L(\varepsilon) \in \nabla_{(\underline{\chi}, \bar{\chi})}$, for each $\chi \in \nabla_{(\underline{\chi}, \bar{\chi})}$ such that

$$\chi_L(\varepsilon) \leq \chi(\varepsilon) \leq \chi_M(\varepsilon) \text{ for all } \varepsilon \in \Omega.$$

Proof. We define the following two sequences $\{\Phi_n\}_{n \in \mathbb{N}}$ and $\{\Psi_n\}_{n \in \mathbb{N}}$:

$$\left\{ \begin{array}{l} \Phi_0 = \underline{\chi}, \\ \Phi_{n+1}(\varepsilon) = \int_0^\varepsilon \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \mathfrak{K}(s, \Phi_n(s), \Phi_n(\gamma s), \mathbb{I}_{0+}^{\rho, \psi} \Phi_n(\gamma s)) ds \\ \quad + \sum_{i=1}^n \int_0^{\eta_i} \mu_i \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \Phi_n(s) ds, \varepsilon \in \Omega, \text{ for all } n \in \mathbb{N}, \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \Psi_0 = \bar{\chi}, \\ \Psi_{n+1}(\varepsilon) = \int_0^\varepsilon \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \mathfrak{K}(s, \Psi_n(s), \Psi_n(\gamma s), \mathbb{I}_{0+}^{\rho, \psi} \Psi_n(\gamma s)) ds \\ \quad + \sum_{i=1}^n \int_0^{\eta_i} \mu_i \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \Psi_n(s) ds, \varepsilon \in \Omega, \text{ for all } n \in \mathbb{N}. \end{array} \right.$$

The proof is now divided into three parts.

Step 1: Sequences $\{\Phi_n\}_{n \in \mathbb{N}}$ and $\{\Psi_n\}_{n \in \mathbb{N}}$ satisfy the following relation:

$$\underline{\chi}(\varepsilon) = \Phi_0(\varepsilon) \leq \Phi_1(\varepsilon) \leq \cdots \leq \Phi_n(\varepsilon) \leq \cdots \leq \Psi_n(\varepsilon) \leq \cdots \leq \Psi_1(\varepsilon) \leq \Psi_0(\varepsilon) = \bar{\chi}(\varepsilon), \forall \varepsilon \in \Omega. \quad (3.7)$$

Our first task is to demonstrate that the sequence $\{\Phi_n\}_{n \in \mathbb{N}}$ is nondecreasing and $\Phi_n(\varepsilon) \leq \Psi_0(\varepsilon)$, $\varepsilon \in \Omega$, $\forall n \in \mathbb{N}$. Therefore, by a recurrence relation, we prove

$$\Phi_{n-1}(\varepsilon) - \Phi_n(\varepsilon) \leq 0, \forall \varepsilon \in \Omega. \quad (3.8)$$

By the definition of $\Phi_0(\varepsilon)$, we have $\Phi_0(\varepsilon) \leq \Phi_1(\varepsilon)$ for each $\varepsilon \in \Omega$. We suppose that (3.8) is true for n and we prove for $(n+1)$: $\Phi_{n-1}(\varepsilon) \leq \Phi_n(\varepsilon)$, $\forall \varepsilon \in \Omega$ and $n \in \mathbb{N}$ we have

$$\left\{ \begin{array}{l} \Phi_n(\varepsilon) = \int_0^\varepsilon \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \mathfrak{N}(s, \Phi_{n-1}(s), \Phi_{n-1}(\gamma s), \mathbb{I}_{0+}^{\rho, \psi} \Phi_{n-1}(\gamma s)) ds \\ \quad + \sum_{i=1}^n \int_0^{\eta_i} \mu_i \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \Phi_{n-1}(s) ds, \\ \Phi_{n+1}(\varepsilon) = \int_0^\varepsilon \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \mathfrak{N}(s, \Phi_n(s), \Phi_n(\gamma s), \mathbb{I}_{0+}^{\rho, \psi} \Phi_n(\gamma s)) ds \\ \quad + \sum_{i=1}^n \int_0^{\eta_i} \mu_i \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \Phi_n(s) ds. \end{array} \right.$$

Using that $(\lambda_1, \lambda_2, \lambda_3) \rightarrow \mathfrak{N}(\varepsilon, \lambda_1, \lambda_2, \lambda_3)$ is nondecreasing, we have

$$\mathfrak{N}(\varepsilon, \Phi_{n-1}(\varepsilon), \Phi_{n-1}(\gamma\varepsilon), \mathbb{I}_{0+}^{\rho, \psi} \Phi_{n-1}(\gamma\varepsilon)) \leq \mathfrak{N}(\varepsilon, \Phi_n(\varepsilon), \Phi_n(\gamma\varepsilon), \mathbb{I}_{0+}^{\rho, \psi} \Phi_n(\gamma\varepsilon)), \quad \forall \varepsilon \in \Omega.$$

So, $\Phi_n(\varepsilon) \leq \Phi_{n+1}(\varepsilon)$, $\forall \varepsilon \in \Omega$. As $\Phi_n(\varepsilon)$ is noncreasing, by the definition of $\Psi_0(\varepsilon)$, we have

$$\Phi_n(\varepsilon) \leq \Phi_{n+1}(\varepsilon) \leq \Psi_0(\varepsilon), \quad \forall \varepsilon \in \Omega.$$

Additionally, we will prove that $\Phi_n(\varepsilon) \leq \Psi_n(\varepsilon)$, $\forall \varepsilon \in \Omega$. For $n = 0$, it's evident that $\underline{\chi}(\varepsilon) = \Phi_0(\varepsilon) \leq \Psi_0(\varepsilon) = \bar{\chi}(\varepsilon)$ assuming inductively, now we make

$$\Phi_n(\varepsilon) \leq \Psi_n(\varepsilon), \quad \forall \varepsilon \in \Omega, \quad n \in \mathbb{N}.$$

Given that $\mathfrak{N}$ is monotonic with respect to the second, three and four variables, it's simple to conclude accordingly

$$\Phi_{n+1}(\varepsilon) \leq \Psi_{n+1}(\varepsilon), \quad \forall \varepsilon \in \Omega, \quad n \in \mathbb{N}.$$

Also, we have that the sequence $\{\Phi_n\}_{n \in \mathbb{N}}$ is nonincreasing.

Step 2: Sequences $\{\Phi_n\}_{n \in \mathbb{N}}$ and $\{\Psi_n\}_{n \in \mathbb{N}}$ are both relatively compact in $C(\Omega, \mathbb{R})$. Since $\mathfrak{N}$ is continuous and $(\underline{\chi}, \bar{\chi}) \in C(\Omega, \mathbb{R}) \times C(\Omega, \mathbb{R})$, from case 1, we conclude that $\{\Phi_n\}_{n \in \mathbb{N}}$ and $\{\Psi_n\}_{n \in \mathbb{N}}$ belong to $C(\Omega, \mathbb{R})$. It follows from (3.7) that $\{\Phi_n\}_{n \in \mathbb{N}}$ and $\{\Psi_n\}_{n \in \mathbb{N}}$ are uniformly bounded. On the other hand, for any $\varepsilon_1, \varepsilon_2 \in \Omega$, without losing generality, let $\varepsilon_1 \leq \varepsilon_2$, we obtain

$$\begin{aligned} |\Phi_{n+1}(\varepsilon_1) - \Phi_{n+1}(\varepsilon_2)| &= \left| \int_0^{\varepsilon_2} \frac{\psi'(s)(\psi(\varepsilon_2) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \mathfrak{N}(s, \Phi_n(s), \Phi_n(\gamma s), \mathbb{I}_{\Phi_1^+}^{\rho, \psi} \Phi_n(\gamma s)) ds \right. \\ &\quad \left. - \int_0^{\varepsilon_1} \frac{\psi'(s)(\psi(\varepsilon_1) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \mathfrak{N}(s, \Phi_n(s), \Phi_n(\gamma s), \mathbb{I}_{\Phi_1^+}^{\rho, \psi} \Phi_n(\gamma s)) ds \right| \\ &= \left| \int_0^{\varepsilon_1} \left[\frac{\psi'(s)(\psi(\varepsilon_2) - \psi(s))^{\rho-1}}{\Gamma(\rho)} - \frac{\psi'(s)(\psi(\varepsilon_1) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \right] \right. \\ &\quad \times \mathfrak{N}(s, \Phi_n(s), \Phi_n(\gamma s), \mathbb{I}_{\Phi_1^+}^{\rho, \psi} \Phi_n(\gamma s)) ds \\ &\quad \left. + \int_{\varepsilon_1}^{\varepsilon_2} \frac{\psi'(s)(\psi(\varepsilon_2) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \mathfrak{N}(s, \Phi_n(s), \Phi_n(\gamma s), \mathbb{I}_{\Phi_1^+}^{\rho, \psi} \Phi_n(\gamma s)) ds \right| \\ &\leq \frac{L}{\Gamma(\rho)} |2[\psi(\varepsilon_2) - \psi(\varepsilon_1)]^\rho - [\psi(\varepsilon_2) - \psi(0)]^\rho + [\psi(\varepsilon_1) - \psi(0)]^\rho|. \end{aligned}$$

which converges to zero as $|\varepsilon_2 - \varepsilon_1| \rightarrow 0$, with $L > 0$ is a constant independent of n, ε_1 and ε_2 . It implies that $\{\Phi_n\}_{n \in \mathbb{N}}$ is equicontinuous in $C(\Omega, \mathbb{R})$, so $\{\Phi_n\}_{n \in \mathbb{N}}$ is relatively compact in $C(\Omega, \mathbb{R})$ based on the Arzelà-Ascoli theorem. Similarly, we obtain $\{\Psi_n\}_{n \in \mathbb{N}}$ is also relatively compact in $C(\Omega, \mathbb{R})$.

Step 3: In $\nabla_{(\underline{\chi}, \bar{\chi})}$, there exist minimum and maximum solutions.

In Steps 1 and 2, it is demonstrated that the sequence $\{\Phi_n\}_{n \in \mathbb{N}}$ and $\{\Psi_n\}_{n \in \mathbb{N}}$ exhibit monotonicity and relatively compact in $C(\Omega, \mathbb{R})$. Apparently, there exist continuous functions Φ and Ψ with $\Phi_n(\varepsilon) \leq \Phi(\varepsilon) \leq \psi(\varepsilon) \leq \Psi_n(\varepsilon)$, for each $\varepsilon \in \Omega$ and for all $n \in \mathbb{N}$, such that $\{\Phi_n\}_{n \in \mathbb{N}}$ and $\{\Psi_n\}_{n \in \mathbb{N}}$ converge uniformly to Φ and Ψ in $C(\Omega, \mathbb{R})$, respectively. So, Φ and Ψ are two solutions of (1.1), i.e.,

$$\begin{aligned}\Phi(\varepsilon) &= \int_0^\varepsilon \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \mathfrak{N}(s, \Phi(s), \Phi(\gamma s), \mathbb{I}_{0+}^{\rho, \psi} \Phi(\gamma s)) ds \\ &\quad + \sum_{i=1}^n \int_0^{\eta_i} \mu_i \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \Phi(s) ds, \quad \forall \varepsilon \in \Omega, \\ \Psi(\varepsilon) &= \int_0^\varepsilon \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \mathfrak{N}(s, \Psi(s), \Psi(\gamma s), \mathbb{I}_{0+}^{\rho, \psi} \Psi(\gamma s)) ds \\ &\quad + \sum_{i=1}^n \int_0^{\eta_i} \mu_i \frac{\psi'(s)(\psi(\varepsilon) - \psi(s))^{\rho-1}}{\Gamma(\rho)} \Psi(s) ds, \quad \forall \varepsilon \in \Omega.\end{aligned}$$

Therefore, $\underline{\chi}(\varepsilon) \leq \Phi(\varepsilon) \leq \psi(\varepsilon) \leq \bar{\chi}(\varepsilon)$, $\forall \varepsilon \in \Omega$. Lastly, we will demonstrate that Φ and ψ represent the the minimal and maximal solutions in $\nabla_{(\underline{\chi}, \bar{\chi})}$, for each $\chi \in \nabla_{(\underline{\chi}, \bar{\chi})}$, then we obtain

$$\underline{\chi}(\varepsilon) \leq \chi(\varepsilon) \leq \bar{\chi}(\varepsilon), \quad \forall \varepsilon \in \Omega.$$

Recall that $\mathfrak{N}$ is nondecreasing with respect to the second, third and the fourth arguments, we induct

$$\underline{\chi}(\varepsilon) \leq \Phi_n(\varepsilon) \leq \chi(\varepsilon) \leq \Psi_n(\varepsilon) \leq \bar{\chi}(\varepsilon), \quad \forall \varepsilon \in \Omega.$$

Using limits as $n \rightarrow +\infty$ in the above inequality, it implies that

$$\underline{\chi}(\varepsilon) \leq \Phi(\varepsilon) \leq \chi(\varepsilon) \leq \Psi(\varepsilon) \leq \bar{\chi}(\varepsilon), \quad \forall \varepsilon \in \Omega \text{ and for } n \in \mathbb{N}.$$

Thus, $\mathbb{X}_{\text{Max}} = \Psi$ and $\mathbb{X}_{\text{Min}} = \Phi$ are the minimal and maximal solutions in $C(\Omega, \mathbb{R})$, respectively, which completes the proof of the theorem. $\square$

Theorem 3.6. Assume that the hypotheses of Theorem 3.5 are satisfied. Then nonlinear fractional differential pantograph equation (1.1) has at least one solution in $C(\Omega, \mathbb{R})$.

Proof. Based on the assumptions stated in Theorem 3.5, we deduce that $\nabla_{(\mathcal{A}\underline{\chi}, \overline{\mathcal{A}\chi})}$ holds. Consequently, the solution set for the fractional equations (3.3) is non-empty within the space $C(\Omega, \mathbb{R})$. Moreover, by applying Lemma 3.3, we confirm the existence of at least one solution for the nonlinear fractional differential pantograph equation (1.1) in $C(\Omega, \mathbb{R})$. This completes the proof of the theorem. $\square$

4. Illustrative example

Let us examine the following ψ -Caputo fractional differential equation

$$\begin{cases} {}^C \mathbb{D}_{1+}^{\frac{1}{3}, \psi} \chi(\varepsilon) = \mathfrak{N}(\varepsilon, \chi(\varepsilon), \chi(\frac{1}{4}\varepsilon), \mathbb{I}_{1+}^{\frac{1}{3}, \psi} \chi(\frac{1}{4}\varepsilon)), & \Omega = [0, 1], \\ \chi(0) = \mu_1 \mathbb{I}_{1+}^{\frac{1}{2}, \psi} \chi(\eta_1), & n = 1, \end{cases} \quad (4.1)$$

where $\rho = \frac{1}{3} \in (0, 1]$, $\rho = \frac{1}{2} \in (0, 1]$, $\gamma = \frac{1}{4} \in (0, 1)$, $\mu_1 = \frac{33}{100}$, $\eta_1 = \frac{5}{7}$, $\kappa = 1$, and $\mathfrak{N} : \Omega \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is given by

$$\mathfrak{N}(\varepsilon, \chi(\varepsilon), \chi(\frac{1}{4}\varepsilon), \mathbb{I}_{1+}^{\frac{1}{3}, \psi} \chi(\frac{1}{4}\varepsilon)) = 3\sqrt{\varepsilon} + \chi(\varepsilon)^2 + \chi(\gamma\varepsilon) + 2 + \frac{1}{3} \mathbb{I}_{1+}^{\frac{1}{3}, \psi} \chi(\frac{1}{4}\varepsilon),$$

for $\varepsilon \in \Omega$ and $\chi(\varepsilon) \in \mathbb{R}$, we take $\underline{\chi}(\varepsilon) = 0$ as the lower solution (l-sol) and $\bar{\chi}(\varepsilon) = 3\varepsilon + 2$ as the lower solution (u-sol) of problem (4.1), and we take $\underline{\chi}(\varepsilon) \leq \bar{\chi}(\varepsilon)$, $\forall \varepsilon \in \Omega$. So, Theorem 3.5 holds. We now proceed to investigate fourth unique possibilities for ψ :

$$\psi_1(\varepsilon) = \varepsilon \text{ (C-D)}, \quad \psi_2(\varepsilon) = \ln \varepsilon \text{ (H-D)}, \quad \psi_3(\varepsilon) = 3^\varepsilon \text{ (C-E-K-D)}, \quad \psi_4(\varepsilon) = 3\varepsilon^{1/3} \text{ (C-Ka)}.$$

Since all assumptions of Theorem 3.5 are readily satisfied, we proceed to construct the sequences $\{\varphi_n\}_{n \in \mathbb{N}}$ and $\{\psi_n\}_{n \in \mathbb{N}}$ as follows:

$$\begin{cases} \Phi_0(\varepsilon) = \underline{\chi}(\varepsilon), \\ \Phi_{n+1}(\varepsilon) = \mathbb{I}_{1+}^{\frac{1}{3}, \psi} \mathfrak{N}(\varepsilon, \Phi_n(\varepsilon), \Phi_n(\frac{1}{4}\varepsilon), \mathbb{I}_{1+}^{\frac{1}{3}, \psi} \Phi_n(\frac{1}{4}\varepsilon)), \quad \varepsilon \in \Omega, \text{ for all } n \in \mathbb{N}, \end{cases}$$

and

$$\begin{cases} \Psi_0(\varepsilon) = \bar{\chi}(\varepsilon), \\ \Psi_{n+1}(\varepsilon) = \mathbb{I}_{1+}^{\frac{1}{3}, \psi} \mathfrak{N}(\varepsilon, \Psi_n(\varepsilon), \Psi_n(\frac{1}{4}\varepsilon), \mathbb{I}_{1+}^{\frac{1}{3}, \psi} \Psi_n(\frac{1}{4}\varepsilon)), \quad \varepsilon \in \Omega, \text{ for all } n \in \mathbb{N}. \end{cases}$$

By the Theorem 3.5, it follows that $\Phi_n \rightarrow \Phi$ in $C(\Omega, \mathbb{R})$ and $\Psi_n \rightarrow \Psi$ in $C(\Omega, \mathbb{R})$ as $n \rightarrow +\infty$ in the meantime, we can get $\Phi(\varepsilon) = \Psi(\varepsilon) = 2\sqrt{\varepsilon}$ for $\varepsilon \in \Omega$. The approximation of sequences $\{\Phi_n\}_{n \in \mathbb{N}}$ and $\{\Psi_n\}_{n \in \mathbb{N}}$ is $2\sqrt{\varepsilon}$. Thus, all conditions of Theorem 3.5 are met, and the problem (4.1) admits extremal solutions on the interval $[0, 1]$.

5. Conclusion

This study investigates the existence of solutions for a nonlinear fractional differential pantograph equation using the ψ -Caputo derivative and integral boundary conditions. By transforming the problem into an equivalent integral equation, we employ the upper-lower solution technique to establish the relevant existence theorems. Additionally, we present an example to illustrate the applicability and effectiveness of our method. This innovative analytical framework not only enhances our understanding of such fractional models but also paves the way for future research in complex dynamic systems.

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