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On the boundedness of singular integral operators on grand variable Herz-Hardy spaces

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Abstract

This study investigates the boundedness of various commutators on grand variable Herz-Hardy spaces, including the Marcinkiewicz integral operator, the Calderón-Zygmund singular integral operator and the fractional integral operator. Firstly we define the Lebesgue spaces with variable exponent and some basic lemmas including Hölder's inequality for Lebesgue spaces. We use the definition of variable Herz spaces to give the definition of grand variable Herz spaces. Then we apply the atomic decomposition of grand variable Herz-Hardy spaces to obtain the boundedness of Marcinkiewicz integral operator, Calderón-Zygmund singular integral operator and fractional integral operator on grand variable Herz-Hardy spaces.

Keywords: Fractional integral operator, Calderón-Zygmund singular integral operator, mathematical operators, grand variable Herz spaces, grand Herz-Hardy spaces.

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1. Introduction

Much effort has been spent investigating the theory of function spaces with variable exponent since the work of Kováčik and Rákosník [7]. The Herz-Hardy spaces were presented in [1] as natural extensions of Hardy spaces and Herz spaces [3]. The Herz-Hardy spaces have been extended to include important results like the boundedness of singular integral operators and sublinear operators [9, 18]. The idea of Herz-type Hardy spaces with variable exponent are introduced in [17] and established the characterizations of these spaces in terms of atomic and molecular decompositions. Finally by using these decompositions, the authors obtained the boundedness of some operators on the Herz-type Hardy spaces with variable exponent. In [4], authors proved the boundedness of singular integral operators and their commutators from products of variable exponent Herz spaces to variable exponent Herz spaces. For the boundedness of the rough fractional Hausdorff operator on variable exponent Herz-type spaces see [5].

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The idea of grand variable Herz-Morrey spaces was introduced in [12]. For the boundedness of higher order commutators on Hardy operators, Marcinkiewicz integral operators on variable order and other integral operators on grand variable Herz-Morrey spaces see [15]. The grand variable Herz spaces are the generalized version of variable Herz spaces, for the boundedness of higher order commutators of fractional integrals of variable order, commutators of rough Hardy operators, commutators on variable Marcinkiewicz fractional integral operators on grand variable Herz spaces see [14]. For more results on variable exponents function spaces see [13]. The topic of this manuscript has numerous applications in the study of partial differential equation, e.g., please see [8].

In this paper, our aim is to obtain the boundedness of some singular integral operators on grand variable Herz-Hardy spaces with some proper assumptions by using its atomic decomposition.

To make the presentation of the paper understandable, we divided the paper into different sections. In the first section we include some basic definitions, some important lemmas on Lebesgue spaces, definitions of grand variable Herz spaces and grand variable Herz-Hardy spaces, and atomic decomposition of grand varaible Herz-Hardy spaces. In Section 2, we prove that Calderón-Zygmund operator maps continuously from grand variable Herz-Hardy spaces to grand variable Herz spaces. Next we prove that fractional integral operator maps continuously from grand variable Herz-Hardy spaces to grand variable Herz spaces. Lastly we prove that the Marcinkiewicz integral operator is bounded on grand variable Herz-Hardy spaces.

Let $\mathbb{R}^n$ denote the n-dimensional real Euclidean space. We utilize C as a positive constant which may vary from line to line. For every $t \in \mathbb{Z}$, we denote $B_t = B(0, 2^t) = \{x \in \mathbb{R}^n : |x| < 2^t\}$. Let $\mathcal{A}$ be a measurable subset of $\mathbb{R}^n$ and $q(\cdot)$ be a real-valued measurable function on $\mathcal{A}$ with values in $[1, \infty)$. Let $q_- := \operatorname{ess}\inf_{y\in\mathcal{A}} q(y) > 1$ and $q_+ := \operatorname{ess}\sup_{y\in\mathcal{A}} q(y) < \infty$. We suppose that

$$1 \leqslant q_-(\mathcal{A}) \leqslant q(y) \leqslant q_+(\mathcal{A}) < \infty. \tag{1.1}$$

Let B denote the ball such that $B(z, r) := \{y \in \mathcal{A} : |z - y| < r\}$. The set $\mathcal{P}(\mathcal{A})$ consists of all $q(\cdot)$ satisfying $q_- > 1$ and $q_+ < \infty$. Now the variable Lebesgue space $L^{q(\cdot)}(\mathcal{A})$ is given as

$$L^{q(\cdot)}(\mathcal{A}) = \left\{ g \text{ is measurable: } \int_{\mathcal{A}} \left(\frac{|g(z)|}{\lambda}\right)^{q(z)} dz < \infty, \text{ where } \lambda \text{ is a constant} \right\}.$$

Lebesgue space is equipped with the norm

$$\|g\|_{L^{q(\cdot)}(\mathcal{A})} = \inf\left\{\lambda > 0 : \int_{\mathcal{A}} \left(\frac{|g(z)|}{\lambda}\right)^{q(z)} dz \leqslant 1\right\}.$$

The Hardy-Littlewood maximal operator M for $g \in L^1_{\text{loc}}(\mathcal{A})$ is defined as

$$Mg(z) := \sup_{0<r} \frac{1}{r^n} \int\limits_{B(z,r)} |g(z)| dz \quad (z \in \mathcal{A}).$$

The notion of $\mathcal{B}(\mathcal{A})$ denotes the collection of $q(\cdot) \in \mathcal{P}(\mathcal{A})$ such that M is bounded on $L^{q(\cdot)}(\mathcal{A})$. Let $x, y \in \mathcal{A}$ with $|x - y| \leqslant \frac{1}{2}$ and $q : \mathcal{A} \mapsto (0, \infty)$. Now log-Hölder's continuity condition (or Dini-Lipschitz condition) is given as

$$|q(x) - q(y)| \leqslant \frac{C}{-\ln|x - y|}, \tag{1.2}$$

where C is called log-Hölder continuity constant. Let log-Dini-Lipschitz constant (or decay constant) $C_\infty > 0$, $q(\cdot)$ satisfies the decay condition if $\lim_{|x|\to\infty} q(x) = q_\infty := q(\infty)$ such that

$$|q(x) - q_\infty| \leqslant \frac{C_\infty}{\ln(e + |x|)}. \tag{1.3}$$

Let $C_0 > 0$, $q(\cdot)$ satisfy the log-Hölder continuity condition at 0 for $|h| \leqslant \frac{1}{2}$, such that

$$|q(x) - q(0)| \leqslant \frac{C_0}{|\ln|x||}. \tag{1.4}$$

$\mathcal{P}^{\log} = \mathcal{P}^{\log}(\mathcal{A})$ consists of all functions $q(\cdot) \in \mathcal{P}(\mathcal{A})$ satisfying (1.1) and (1.2). Let $\mathcal{P}_\infty(\mathcal{A})$ (resp. $\mathcal{P}_{0,\infty}(\mathcal{A})$) is the subset of $\mathcal{P}(\mathcal{A})$ consisting of functions which satisfy condition (1.3) (resp. both conditions (1.3) and (1.4)). The set of positive integers is denoted by $\mathbb{Z}_+$. For each $t_0 \in \mathbb{Z}, \mathbf{1}_{A_{t_0}} = \mathbf{1}_{t_0}$; for $t_0 \in \mathbb{Z}_+$, $\mathbf{1}_{B_0} = \tilde{\mathbf{1}}_0$; and so on. Let $\mathbf{1}_{A_{t_0}}$ denote the characteristic function of A_{t_0}, where $A_{t_0} = B_{t_0} \setminus B_{t_0-1}$. By $a \leqslant b$ we denote $a \leqslant Cb$.

Lemma 1.1 ([7]). *Let* f *belong to the Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^n)$ *and* g *belong to* $L^{p'(\cdot)}(\mathbb{R}^n)$, *where* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. *Then, the product function* fg *is integrable over* $\mathbb{R}^n$, *and the following inequality holds:*

$$\int_{\mathbb{R}^n} \left|f(x)g(x)\right| dx \leqslant r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where r_p *is defined as* $r_p = 1 + \frac{1}{p^-} - \frac{1}{p^+}$.

Lemma 1.2 ([2]). *Let* $\tilde{q}(\cdot)$ *be a variable exponent defined as* $\frac{1}{p(z)} - \frac{1}{q} = \frac{1}{\tilde{q}(z)}$, *where* $(z \in \mathbb{R}^n)$. *Then for all measurable functions* f *and* g *we have* $\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C \|g\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}$.

Lemma 1.3 ([6]). *Let* $p(\cdot)$ *be a function within the class* $\mathcal{B}(\mathbb{R}^n)$. *For any ball* B *in* $\mathbb{R}^n$, *there exists* $C \geqslant 0$ *such that* $\frac{1}{|B|}\|\mathbf{1}_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C$.

Lemma 1.4 ([6]). *Assuming that* $p(\cdot)$ *is a function in the class* $\mathcal{B}(\mathbb{R}^n)$, *then there exists a positive constant* C *such that, for every ball* B *in* $\mathbb{R}^n$ *and every measurable subset* S *within* B, *the following inequalities hold:*

$$\frac{\|\mathbf{1}_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\mathbf{1}_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leqslant C\frac{|B|}{|S|}, \quad \frac{\|\mathbf{1}_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\mathbf{1}_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leqslant \left(\frac{|S|}{|B|}\right)^{\delta_1}, \quad \frac{\|\mathbf{1}_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\mathbf{1}_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \leqslant \left(\frac{|S|}{|B|}\right)^{\delta_2},$$

where $0 < \delta_1, \delta_2 < 1$.

Assume that $0 < \eta \leqslant 1$, the Lipschitz space Lip_η is given as

$$\mathrm{Lip}_\eta = \left\{ f : \|f\|_{\mathrm{Lip}_\eta} = \sup_{z,y\in\mathbb{R}^n; z\neq y} \frac{|f(z)-f(y)|}{|z-y|^\eta} < \infty \right\}.$$

Now we define the grand variable Herz spaces (GVH spaces).

Definition 1.5. To define homogeneous GVH spaces, let $p \in \mathcal{P}(\mathbb{R}^n)$, $q_1 \in (1, \infty)$, $\theta > 0$, and $\beta \in \mathbb{R}$. The norm of GVH spaces are given by:

$$\dot{K}^{\beta,q_1),\theta}_{p(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{\dot{K}^{\beta,q_1),\theta}_{p(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|g\|_{\dot{K}^{\beta,q_1),\theta}_{p(\cdot)}(\mathbb{R}^n)} = \sup_{\epsilon>0} \left(\epsilon^\theta \sum_{t_0=-\infty}^{\infty} 2^{k_0\beta q_1(1+\epsilon)} \|g\mathbf{1}_{t_0}\|^{q_1(1+\epsilon)}_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{\frac{1}{q_1(1+\epsilon)}}.$$

The non-homogeneous GVH spaces are defined similarly.

The grand maximal function $G_N g$ of g is given as $G_N g(y) := \sup_{\varphi \in \mathcal{A}_N} \left|\varphi^*_\nabla(g)(y)\right|$, $y \in \mathbb{R}^n$, where $n + 1 < N$ and $\mathcal{A}_N := \left\{ \varphi \in \mathcal{S} : \sup_{|\alpha|, |\eta| \leqslant N, \forall y \in \mathbb{R}^n} |y^\alpha D^\eta \varphi(y)| \leqslant 1 \right\}$. Here $\mathcal{S}$ is the Schwartz space of all rapidly decreasing infinitely differentiable functions and $\mathcal{S}'$ is its the dual space. The nontangential maximal operator φ^*_∇ is given as $\varphi^*_\nabla(g)y := \sup_{t>|x-y|} |\varphi_t * g(x)|$, with $\varphi_t(y) = t^{-n}\varphi\left(\frac{x}{t}\right)$.

Definition 1.6. Let $\beta \in \mathbb{R}$, $1 < q_1 < \infty$, and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous grand variable Herz-Hardy space $H\dot{K}^{\beta, q_1), \theta}_{p(\cdot)}$ is given as

$$H\dot{K}^{\beta, q_1), \theta}_{p(\cdot)} := \left\{ g \in \mathcal{S}' : \|g\|_{H\dot{K}^{\beta, q_1), \theta}_{p(\cdot)}} := \|G_N g\|_{\dot{K}^{\beta, q_1), \theta}_{p(\cdot)}} < \infty \right\}.$$

For $u \in \mathbb{R}$, $[u]$ denotes the largest integer equal to or less than u.

Definition 1.7. Assume that $\beta \in \mathbb{R}$ is log-Hölder continuous at infinity and origin, and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Let $s \geqslant [\beta - n\delta_2]$ such that $n\delta_2 < \beta < \infty$, where δ_2 is given in Lemma 1.4.

(i) A function a is referred to as a central $(\beta, p(\cdot))$ atom, if it satisfies
  (1) $\operatorname{supp} a \subseteq B(0, r)$;
  (2) $\|a\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant |B(0, r)|^{-\beta/n}$;
  (3) $\int_{\mathbb{R}^n} a(x) x^\eta dx = 0, |\eta| \leqslant s$.

(ii) A function a on $\mathbb{R}^n$ is referred to as a central $(\beta, p(\cdot))$-atom of restricted type, if conditions (2), (3), and (4) are satisfied and
  (4) $\operatorname{supp} a \subseteq B(0, r)$, $r \geqslant 1$.

Theorem 1.8 ([10]). *Assume that $\beta \in \mathbb{R}$ is log-Hölder continuous at infinity and origin such that $n\delta_2 < \beta < \infty$, $\theta > 0$, $1 < q_1 < \infty$, and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Let $s \geqslant [\beta - n\delta_2]$, where δ_2 is given in Lemma 1.4. Then $g \in H\dot{K}^{\beta, q_1), \theta}_{p(\cdot)}$ iff $g = \sum_{k_0=-\infty}^{\infty} \lambda_{k_0} a_{k_0}$ in the sense of $\mathcal{S}'$, where a_{k_0} is a central $(\beta, p(\cdot))$-atom with support contained in B_{k_0} and $\sup_{\epsilon>0} \epsilon^\theta \sum_{k_0=-\infty}^{\infty} \left|\lambda_{k_0}\right|^{q_1(1+\epsilon)} < \infty$. Moreover,*

$$\|g\|_{H\dot{K}^{\beta, q_1), \theta}_{p(\cdot)}} \approx \inf \left(\sup_{\epsilon>0} \left(\epsilon^\theta \sum_{k_0=-\infty}^{\infty} \left|\lambda_{k_0}\right|^{q_1(1+\epsilon)} \right)^{1/(q_1(1+\epsilon))} \right).$$

2. Main results

Now we will prove that commutators of the Calderón-Zygmund operator maps continuously from grand variable Herz-Hardy spaces into grand variable Herz spaces. If $b \in \mathrm{Lip}_\eta$ and $\omega \in C^2\left(S^{n-1}\right)$ is homogeneous of degree zero and has mean value zero on the unit sphere, then the Calderón-Zygmund singular integral operator T is given as

$$Tg(z) = \text{ p.v.} \int_{\mathbb{R}^n} \frac{\omega(z-y)}{|z-y|^n} g(y) dy.$$

The commutator $[b, T]$ is given as $[b, T]g(z) = b(z)Tg(z) - T(bg)(z)$. Letting $0 < \eta < n$, then the fractional integral operator I_η satisfies the inequality

$$|[b, T]g(x)| \leqslant \|b\|_{\mathrm{Lip}_\eta} I_\eta(|g|)(x). \tag{2.1}$$

Theorem 2.1. *Assume that $b \in \mathrm{Lip}_\eta$, $0 < \eta \leqslant 1$ and $\theta > 0$. If $p_1(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ with $\frac{n}{\eta} > p_1^+$, $\frac{1}{p_1(z)} = \frac{\eta}{n} + \frac{1}{p_2(z)}$, $1 < q_1 \leqslant q_2 < \infty$, and $n\delta_2 \leqslant \beta < n\delta_2 + \eta$, then $[b, T]$ maps $H\dot{K}^{\beta, q_1), \theta}_{p_1(\cdot)}$ continuously into $\dot{K}^{\beta, q_2), \theta}_{p_2(\cdot)}$.*

Proof. Let $g \in H\dot{K}^{\beta,q_1),\theta}_{p_1(\cdot)}$ and $b \in \mathrm{Lip}_\eta$. By Theorem 1.8, we get $g = \sum_{t_0=-\infty}^{\infty} \lambda_{t_0} a_{t_0}$, where $\|g\|_{H\dot{K}^{\beta,q_1),\theta}_{p_1(\cdot)}} \approx \inf\left(\sup_{\epsilon>0}\left(\epsilon^\theta \sum_{k_0=-\infty}^{\infty} |\lambda_{k_0}|^{q_1(1+\epsilon)}\right)^{1/(q_1(1+\epsilon))}\right)$. If a_{t_0} is a dyadic central $(\beta, p_1(\cdot))$ atom with support B_{t_0} and $q_1 \leqslant q_2$, we get

$$
\begin{aligned}
\|[b,T](g)\|^{q_1(1+\epsilon)}_{\dot{K}^{\beta,q_2),\theta}_{p_2(\cdot)}} &= \sup_{\epsilon>0} \epsilon^\theta \left\{ \sum_{k_0=-\infty}^{\infty} 2^{k_0\beta q_2(1+\epsilon)} \left\|[b,T](g)\mathbf{1}_{k_0}\right\|^{q_2(1+\epsilon)}_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right\}^{\frac{q_1(1+\epsilon)}{q_2(1+\epsilon)}} \\
&\leqslant \sup_{\epsilon>0} \epsilon^\theta \sum_{k_0=-\infty}^{\infty} 2^{k_0\beta q_1(1+\epsilon)} \left\|[b,T]\,(g)\mathbf{1}_{k_0}\right\|^{q_1(1+\epsilon)}_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
&\leqslant \sup_{\epsilon>0} \epsilon^\theta \sum_{k_0=-\infty}^{\infty} 2^{k_0\beta q_1(1+\epsilon)} \left(\sum_{t_0=-\infty}^{k_0-2} |\lambda_{t_0}| \left\|[b,T]\,(a_{t_0})\,\mathbf{1}_{k_0}\right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1(1+\epsilon)} \\
&\quad + C \sup_{\epsilon>0} \epsilon^\theta \sum_{k_0=-\infty}^{\infty} 2^{k_0\beta q_1(1+\epsilon)} \left(\sum_{t_0=k_0-1}^{\infty} |\lambda_{t_0}| \left\|[b,T]\,(a_{t_0})\,\mathbf{1}_{k_0}\right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1(1+\epsilon)} \\
&=: E_1 + E_2.
\end{aligned}
$$

We now calculate E_1. For every $z \in a_{k_0}$ and every $k_0 \in \mathbb{Z}$, $t_0 \leqslant k_0 - 2$, based on the the vanishing property of a_{t_0} and Hölder's inequality, we get

$$
\begin{aligned}
&|[b,T]\,(a_{t_0})\,(z)| \\
&\quad\leqslant \left|(b(z)-b(0))\int_{B_{t_0}} \left(\frac{\omega(z-y)}{|z-y|^n} - \frac{\omega(z)}{|z|^n}\right) a_{t_0}(y)dy\right| + \left|\int_{B_{t_0}} \frac{\omega(z-y)}{|z-y|^n}(b(y)-b(0))a_{t_0}(y)dy\right| \\
&\quad\leqslant \|b\|_{\mathrm{Lip}_\eta}\left(|z|^\eta \int_{B_{t_0}} \frac{|y|}{|z|^{n+1}}\,|a_{t_0}(y)|\,dy + \int_{B_{t_0}} \frac{|y|^\eta}{|z-y|^n}\,|a_{t_0}(y)|\,dy\right) \\
&\quad\leqslant \|b\|_{\mathrm{Lip}_\eta}\left(|z|^{\eta-n-1}2^{t_0}\,\|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} + |z|^{-n}2^{t_0\eta}\,\|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right) \\
&\quad\leqslant 2^{t_0\eta-k_0 n}\|b\|_{\mathrm{Lip}_\eta}\,\|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Since

$$
I_\eta\left(\mathbf{1}_{B_{k_0}}\right)(z) \geqslant \int_{B_{k_0}} \frac{dy}{|z-y|^{n-\eta}} \mathbf{1}_{B_{k_0}}(z) \geqslant C2^{k_0\eta}\mathbf{1}_{B_{k_0}}(z).
$$

According to Lemmas 1.3 and 1.4, we get

$$
\begin{aligned}
\left\|[b,T]\,(a_{t_0})\,\mathbf{1}_{k_0}\right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\leqslant 2^{t_0\eta-k_0 n}\|b\|_{\mathrm{Lip}_\eta}\,\|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \left\|\mathbf{1}_{B_{k_0}}\right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
&\leqslant 2^{t_0\eta-k_0(n+\eta)}\|b\|_{\mathrm{Lip}_\eta}\,\|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \left\|I_\eta(\mathbf{1}_{B_{k_0}})\right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
&\leqslant 2^{t_0\eta-k_0(n+\eta)}\|b\|_{\mathrm{Lip}_\eta}\,\|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \left\|\mathbf{1}_{B_{k_0}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&\leqslant 2^{(t_0-k_0)\eta}\|b\|_{\mathrm{Lip}_\eta}\,\|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \frac{\left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{\left\|\mathbf{1}_{B_{k_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}} \\
&\leqslant 2^{-t_0\beta+(t_0-k_0)(\eta+n\delta_2)}\|b\|_{\mathrm{Lip}_\eta}.
\end{aligned}
$$

As a result, we get

$$\begin{aligned}E_1 &\leqslant \|b\|_{\mathrm{Lip}_\eta}^{q_1(1+\epsilon)} \sup_{\epsilon>0} \epsilon^\theta \sum_{k_0=-\infty}^{\infty} 2^{k_0\beta q_1(1+\epsilon)} \left(\sum_{t_0=-\infty}^{k_0-2} 2^{-t_0\beta+(t_0-k_0)(\eta+n\delta_2)} |\lambda_{t_0}| \right)^{q_1(1+\epsilon)} \\ &= C\|b\|_{\mathrm{Lip}_\eta}^{q_1(1+\epsilon)} \sup_{\epsilon>0} \epsilon^\theta \sum_{k_0=-\infty}^{\infty} \left(\sum_{t_0=-\infty}^{k_0-2} 2^{(\eta+n\delta_2-\beta)(t_0-k_0)} |\lambda_{t_0}| \right)^{q_1(1+\epsilon)}.\end{aligned}$$

Take $\frac{1}{q_1(1+\epsilon)} + \frac{1}{(q_1(1+\epsilon))'} = 1$ for $1 < q_1 < \infty$. Due to Hölder's inequality, since $\eta + n\delta_2 - \beta > 0$, we obtain

$$\begin{aligned}E_1 &\leqslant \|b\|_{\mathrm{Lip}_\eta}^{q_1(1+\epsilon)} \sup_{\epsilon>0} \epsilon^\theta \sum_{k_0=-\infty}^{\infty} \left(\sum_{t_0=-\infty}^{k_0-2} |\lambda_{t_0}|^{q_1(1+\epsilon)} 2^{(t_0-k_0)(\eta+n\delta_2-\beta)q_1(1+\epsilon)/2} \right) \\ &\quad \times \left(\sum_{t_0=-\infty}^{k_0-2} 2^{(t_0-k_0)(\eta+n\delta_2-\beta)(q_1(1+\epsilon)'/2} \right)^{q_1(1+\epsilon)/(q_1(1+\epsilon))'} \\ &\leqslant \|b\|_{\mathrm{Lip}_\eta}^{q_1(1+\epsilon)} \sup_{\epsilon>0} \epsilon^\theta \sum_{k_0=-\infty}^{\infty} \left(\sum_{t_0=-\infty}^{k_0-2} |\lambda_{t_0}|^{q_1(1+\epsilon)} 2^{(t_0-k_0)(\eta+n\delta_2-\beta)q_1(1+\epsilon)/2} \right) \\ &= C\|b\|_{\mathrm{Lip}_\eta}^{q_1(1+\epsilon)} \sup_{\epsilon>0} \epsilon^\theta \sum_{t_0=-\infty}^{\infty} |\lambda_{t_0}|^{q_1(1+\epsilon)} \left(\sum_{k_0=t_0+2}^{\infty} 2^{(t_0-k_0)(\eta+n\delta_2-\beta)q_1(1+\epsilon)/2} \right) \\ &\leqslant \|b\|_{\mathrm{Lip}_\eta}^{q_1(1+\epsilon)} \sup_{\epsilon>0} \epsilon^\theta \sum_{t_0=-\infty}^{\infty} |\lambda_{t_0}|^{q_1(1+\epsilon)}.\end{aligned}$$

Now, we will find the estimate of E_2. We have obtained from boundedness of I_η from $L^{q_1(\cdot)}$ to $L^{q_2(\cdot)}$ by utilizing the fact (2.1),

$$\begin{aligned}E_2 &\leqslant \|b\|_{\mathrm{Lip}_\eta}^{q_1(1+\epsilon)} \sup_{\epsilon>0} \epsilon^\theta \sum_{k_0=-\infty}^{\infty} 2^{k_0\beta q_1(1+\epsilon)} \left(\sum_{t_0=k_0-1}^{\infty} |\lambda_{t_0}| \, \|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1(1+\epsilon)} \\ &\leqslant \|b\|_{\mathrm{Lip}_\eta}^{q_1(1+\epsilon)} \sup_{\epsilon>0} \epsilon^\theta \sum_{k_0=-\infty}^{\infty} \left(\sum_{t_0=k_0-1}^{\infty} |\lambda_{t_0}| 2^{(k_0-t_0)\beta} \right)^{q_1(1+\epsilon)},\end{aligned}$$

for $1 < q_1 < \infty$. An application of Hölder's inequality gives us

$$\begin{aligned}E_2 &\leqslant \|b\|_{\mathrm{Lip}_\eta}^{q_1(1+\epsilon)} \sup_{\epsilon>0} \epsilon^\theta \sum_{k_0=-\infty}^{\infty} \left(\sum_{t_0=k_0-1}^{\infty} |\lambda_{t_0}|^{q_1(1+\epsilon)} 2^{(k_0-t_0)\beta q_1(1+\epsilon)/2} \right) \\ &\quad \times \left(\sum_{t_0=k_0-1}^{\infty} 2^{(k_0-t_0)\beta(q_1(1+\epsilon))'/2} \right)^{q_1(1+\epsilon)/(q_1(1+\epsilon))'} \\ &\leqslant \|b\|_{\mathrm{Lip}_\eta}^{q_1(1+\epsilon)} \sup_{\epsilon>0} \epsilon^\theta \sum_{t_0=-\infty}^{\infty} |\lambda_{t_0}|^{q_1(1+\epsilon)}.\end{aligned}$$

Hence we obtained our desired results. □

Let $b \in \mathrm{Lip}_\eta$. The fractional integral operator with $0 < \sigma < n$ is denoted by I_σ. The fractional integral operator $[b, I_\sigma]$ and its commutator are given as

$$[b, I_\sigma]\, g(z) = \int_{\mathbb{R}^n} \frac{(b(z)-b(y))}{|z-y|^{n-\sigma}} g(y)dy.$$

In [16], the authors proved the boundedness of $[b, I_\sigma]$ from $L^{p_1(\cdot)}(\mathbb{R}^n)$ to $L^{p_2(\cdot)}(\mathbb{R}^n)$ for $\frac{1}{p_1(z)} - \frac{1}{p_2(z)} = \frac{(\sigma+\eta)}{n}$ and $p_1(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ with $p_1^+ < n/(\sigma+\eta)$. Next, we will prove that commutators of the fractional integral operator map continuously from grand variable Herz-Hardy spaces into grand variable Herz spaces.

Theorem 2.2. *Assume that* $b \in \mathrm{Lip}_\eta$ *with* $0 < \eta \leqslant 1$, $0 < \sigma < n - \eta$ *and* $\theta > 0$ *and* $p_1(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ *with* $p_1^+ < n/(\sigma+\eta)$. *Let* $1/p_1(z) - 1/p_2(z) = \frac{\sigma+\eta}{n}, 1 < q_1 \leqslant q_2 < \infty$, *and* $n\delta_2 \leqslant \beta < n\delta_2 + \eta$, *then* $[b, I_\sigma]$ *maps* $H\dot{K}^{\beta, q_1), \theta}_{p_1(\cdot)}$ *continuously into* $\dot{K}^{\beta, q_2), \theta}_{p_2(\cdot)}$.

Proof. Let $g \in H\dot{K}^{\beta, q_1), \theta}_{p_1(\cdot)}$ and $b \in \mathrm{Lip}_\eta$. By Theorem 1.8 we get $g = \sum_{t_0=-\infty}^{\infty} \lambda_{t_0} a_{t_0}$, $\|g\|_{H\dot{K}^{\beta, q_1), \theta}_{p_1(\cdot)}} \approx \inf\left(\sup_{\epsilon>0}\left(\epsilon^\theta \sum_{k_0=-\infty}^{\infty} |\lambda_{k_0}|^{q_1(1+\epsilon)}\right)^{1/(q_1(1+\epsilon))}\right)$. Letting a_{t_0} be a dyadic central $(\beta, p_1(\cdot))$ atom with support B_{t_0} and $q_1 \leqslant u q_2$, then we get

$$\begin{aligned}
\|[b, I_\sigma](g)\|^{q_1(1+\epsilon)}_{\dot{K}^{\beta, q_2), \theta}_{p_2(\cdot)}} &= \sup_{\epsilon>0} \epsilon^\theta \left\{ \sum_{k_0=-\infty}^{\infty} 2^{k_0 \beta q_2(1+\epsilon)} \left\| [b, I_\sigma](g) \mathbf{1}_{k_0} \right\|^{q_2(1+\epsilon)}_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right\}^{q_1(1+\epsilon)/q_2(1+\epsilon)} \\
&\leqslant \sup_{\epsilon>0} \epsilon^\theta \sum_{k_0=-\infty}^{\infty} 2^{k_0 \beta q_1(1+\epsilon)} \left\| [b, I_\sigma](g) \mathbf{1}_{k_0} \right\|^{q_1(1+\epsilon)}_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
&\leqslant \sup_{\epsilon>0} \epsilon^\theta \sum_{k_0=-\infty}^{\infty} 2^{k_0 \beta q_1(1+\epsilon)} \left(\sum_{t_0=-\infty}^{k_0-2} |\lambda_{t_0}| \left\| [b, I_\sigma](a_{t_0}) \mathbf{1}_{k_0} \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1(1+\epsilon)} \\
&\quad + C \sup_{\epsilon>0} \epsilon^\theta \sum_{k_0=-\infty}^{\infty} 2^{k_0 \beta q_1(1+\epsilon)} \left(\sum_{t_0=k_0-1}^{\infty} |\lambda_{t_0}| \left\| [b, I_\sigma](a_{t_0}) \mathbf{1}_{k_0} \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1(1+\epsilon)} \\
&=: F_1 + F_2.
\end{aligned}$$

First, we compute F_1. Let $z \in a_{k_0}$ and $k_0 \in \mathbb{Z}$, such that $t_0 \leqslant k_0 - 2$. Then by using Hölder's inequality and the vanishing moments of a_{t_0}, we get

$$\begin{aligned}
&|[b, I_\sigma](a_{t_0})(z)| \\
&\quad \leqslant \int_{B_{t_0}} |b(z)-b(y)| \frac{|a_{t_0}(y)|\,|y|^\eta}{|z-y|^{n-\sigma+\eta}} dy \\
&\quad \leqslant 2^{-k_0(n-\sigma+\eta)+t_0\eta} \int_{B_{t_0}} |b(z)-b(y)|\,|a_{t_0}(y)|\, dy \\
&\quad \leqslant 2^{-k_0(n-\sigma+\eta)+t_0\eta} \left(|b(z)-b(0)| \int_{B_{t_0}} |a_{t_0}(y)|\, dy + \int_{B_{t_0}} |a_{t_0}(y)|\,|b(y)-b(0)| dy \right) \\
&\quad \leqslant 2^{-k_0(n-\sigma+\eta)+t_0\eta} \|b\|_{\mathrm{Lip}_\eta} \left(|z|^\eta \int_{B_{t_0}} |a_{t_0}(y)|\, dy + \int_{B_{t_0}} |y|^\eta\, |a_{t_0}(y)|\, dy \right)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant 2^{-k_0(n-\sigma+\eta)+t_0\eta}\|b\|_{Lip_\eta}\left(|z|^\eta\|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}+2^{t_0\eta}\|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)\\
&\leqslant 2^{-k_0(n-\sigma)+t_0\eta}\|b\|_{Lip_\eta}\|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Lemmas 1.3 and 1.4 thus give us

$$
\begin{aligned}
\left\|[b,I_\sigma](a_{t_0})\mathbf{1}_{k_0}\right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\leqslant 2^{-k_0(n-\sigma)+t_0\eta}\|b\|_{Lip_\eta}\|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\mathbf{1}_{B_{k_0}}\right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\\
&\leqslant 2^{-k_0n+(t_0-k_0)\eta}\|b\|_{Lip_\eta}\|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\left\|I_\eta(\mathbf{1}_{B_{k_0}})\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\leqslant 2^{-k_0n+(t_0-k_0)\eta}\|b\|_{Lip_\eta}\|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\left\|\mathbf{1}_{B_{k_0}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\leqslant 2^{(t_0-k_0)\eta}\|b\|_{Lip_\eta}\|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{\left\|\mathbf{1}_{B_{k_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}\\
&\leqslant 2^{-t_0\beta+(t_0-k_0)(\eta+n\delta_2)}\|b\|_{Lip_\eta}.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
F_1 &\leqslant \|b\|_{Lip_\eta}^{q_1(1+\epsilon)}\sup_{\epsilon>0}\epsilon^\theta\sum_{k_0=-\infty}^{\infty}2^{k_0\beta q_1(1+\epsilon)}\left(\sum_{t_0=-\infty}^{k_0-2}2^{(t_0-k_0)(\eta+n\delta_2-t_0\beta)}|\lambda_{t_0}|\right)^{q_1(1+\epsilon)}\\
&= C\|b\|_{Lip_\eta}^{q_1(1+\epsilon)}\sup_{\epsilon>0}\epsilon^\theta\sum_{k_0=-\infty}^{\infty}\left(\sum_{t_0=-\infty}^{k_0-2}2^{(\eta+n\delta_2-\beta)(t_0-k_0)}|\lambda_{t_0}|\right)^{q_1(1+\epsilon)}.
\end{aligned}
$$

Take $\frac{1}{q_1(1+\epsilon)}+\frac{1}{q_1(1+\epsilon)'}=1$, when $1<q_1<\infty$. Due to Hölder's inequality, since $\eta+n\delta_2-\beta>0$, we obtain

$$
\begin{aligned}
F_1 &\leqslant \|b\|_{Lip_\eta}^{q_1(1+\epsilon)}\sup_{\epsilon>0}\epsilon^\theta\sum_{k_0=-\infty}^{\infty}\left(\sum_{t_0=-\infty}^{k_0-2}|\lambda_{t_0}|^{q_1(1+\epsilon)}2^{(t_0-k_0)(\eta+n\delta_2-\beta)q_1(1+\epsilon)/2}\right)\\
&\quad\times\left(\sum_{t_0=-\infty}^{k_0-2}2^{(t_0-k_0)(\eta+n\delta_2-\beta)(q_1(1+\epsilon))'/2}\right)^{q_1(1+\epsilon)/(q_1(1+\epsilon))'}\\
&\leqslant \|b\|_{Lip_\eta}^{q_1(1+\epsilon)}\sup_{\epsilon>0}\epsilon^\theta\sum_{k_0=-\infty}^{\infty}\left(\sum_{t_0=-\infty}^{k_0-2}|\lambda_{t_0}|^{q_1(1+\epsilon)}2^{(t_0-k_0)(\eta+n\delta_2-\beta)q_1(1+\epsilon)/2}\right)\\
&= \|b\|_{Lip_\eta}^{q_1(1+\epsilon)}\sup_{\epsilon>0}\epsilon^\theta\sum_{t_0=-\infty}^{\infty}|\lambda_{t_0}|^{q_1(1+\epsilon)}\left(\sum_{k_0=t_0+2}^{\infty}2^{(t_0-k_0)(\eta+n\delta_2-\beta)q_1(1+\epsilon)/2}\right)\\
&\leqslant \|b\|_{Lip_\eta}^{q_1(1+\epsilon)}\sup_{\epsilon>0}\epsilon^\theta\sum_{t_0=-\infty}^{\infty}|\lambda_{t_0}|^{q_1(1+\epsilon)}.
\end{aligned}
$$

For the estimate of F_2, we apply the boundedness of $[b,I_\sigma]$ to get

$$
F_2 \leqslant \|b\|_{Lip_\eta}^{q_1(1+\epsilon)}\sup_{\epsilon>0}\epsilon^\theta\sum_{k_0=-\infty}^{\infty}2^{k_0\beta q_1(1+\epsilon)}\left(\sum_{t_0=k_0-1}^{\infty}|\lambda_{t_0}|\,\|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\epsilon)}
$$

$$\leqslant \|b\|_{\mathrm{Lip}_\eta}^{q_1(1+\epsilon)} \sup_{\epsilon>0} \epsilon^\theta \sum_{k_0=-\infty}^{\infty} \left(\sum_{t_0=k_0-1}^{\infty} |\lambda_{t_0}| 2^{(k_0-t_0)\beta} \right)^{q_1(1+\epsilon)}$$
$$\leqslant \|b\|_{\mathrm{Lip}_\eta}^{q_1(1+\epsilon)} \sup_{\epsilon>0} \epsilon^\theta \sum_{t_0=-\infty}^{\infty} |\lambda_{t_0}|^{q_1(1+\epsilon)}.$$

Hence we obtained our desired results. □

Let $n \geqslant 2$, $0 < \nu \leqslant 1$, S^{n-1} be the unit sphere in $\mathbb{R}^n$ with the normalized Lebesgue measure, and $\omega \in \mathrm{Lip}_\nu\left(S^{n-1}\right)$ be the homogeneous function of degree zero such that

$$\int_{S^{n-1}} \omega\left(z'\right) d\sigma\left(z'\right) = 0,$$

where $z' = z/|z|$. For the idea of Marcinkiewicz integral operator see [11] in 1958. It is associated with the Littlewood-Paley g-function given as

$$\mu(g)(z) = \left(\int_0^\infty |F_t(g)(z)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_t(g)(z) = \int_{|z-y|<t} \frac{\omega(z-y)}{|z-y|^{n-1}} g(y) dy.$$

Let $b \in \mathrm{Lip}_\eta$. The commutator of Marcinkiewicz integral operator is given as

$$[b,\mu](g)(z) = \left(\int_0^\infty \left| \int_{|z-y|<t} \frac{[b(z)-b(y)]\omega(z-y)}{|z-y|^{n-1}} g(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2}.$$

For the boundedness of $[b,\mu]$ on Lebesgue spaces see [16]. Finally, We will prove that commutators of the Marcinkiewicz integral operator maps continuously from grand variable Herz-Hardy spaces into grand variable Herz spaces.

Theorem 2.3. *Let $\omega \in \mathrm{Lip}_\nu\left(S^{n-1}\right)$ with $0 < \nu \leqslant 1$ and $b \in \mathrm{Lip}_\eta, 0 < \eta \leqslant \nu/2$. If $p_1(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ with $p_1^+ < n/\sigma, 1/p_1(z) - 1/p_2(z) = \eta/n, 1 < q_1 \leqslant q_2 < \infty$, and $n\delta_2 \leqslant \beta < n\delta_2 + \eta$, then $[b,\mu]$ maps $H\dot{K}_{p_1(\cdot)}^{\beta,q_1),\theta}$ continuously into $\dot{K}_{p_2(\cdot)}^{\beta,q_2),\theta}$.*

Proof. Let $g \in H\dot{K}_{p_1(\cdot)}^{\beta,q_1),\theta}$ and $b \in \mathrm{Lip}_\eta$. By Theorem 1.8 we get $g = \sum_{t_0=-\infty}^{\infty} \lambda_{t_0} a_{t_0}$, where $\|g\|_{H\dot{K}_{p_1(\cdot)}^{\beta,q_1),\theta}} \approx \inf\left(\sup_{\epsilon>0} \left(\epsilon^\theta \sum_{k_0=-\infty}^{\infty} |\lambda_{k_0}|^{q_1(1+\epsilon)} \right)^{1/(q_1(1+\epsilon))} \right)$. Letting a_{t_0} be a dyadic central $(\beta, p_1(\cdot))$ atom with support B_{t_0} and $q_1 \leqslant q_2$, we get

$$\|[b,\mu](g)\|_{\dot{K}_{p_2(\cdot)}^{\beta,q_2),\theta}}^{q_1(1+\epsilon)} = \sup_{\epsilon>0} \epsilon^\theta \left\{ \sum_{k_0=-\infty}^{\infty} 2^{k_0\beta q_2(1+\epsilon)} \left\|[b,\mu](g)\mathbf{1}_{k_0}\right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_2(1+\epsilon)} \right\}^{q_1(1+\epsilon)/q_2(1+\epsilon)}$$
$$\leqslant \sup_{\epsilon>0} \epsilon^\theta \sum_{k_0=-\infty}^{\infty} 2^{k_0\beta q_1(1+\epsilon)} \left\|[b,\mu](g)\mathbf{1}_{k_0}\right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1(1+\epsilon)}$$
$$\leqslant \sup_{\epsilon>0} \epsilon^\theta \sum_{k_0=-\infty}^{\infty} 2^{k_0\beta q_1(1+\epsilon)} \left(\sum_{t_0=-\infty}^{k_0-2} |\lambda_{t_0}| \left\|[b,\mu](a_{t_0})\mathbf{1}_{k_0}\right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1(1+\epsilon)}$$

$$+\sup_{\epsilon>0}\epsilon^{\theta}\sum_{k_0=-\infty}^{\infty}2^{k_0\beta q_1(1+\epsilon)}\left(\sum_{t_0=k_0-1}^{\infty}|\lambda_{t_0}|\left\|[b,\mu]\left(a_{t_0}\right)\mathbf{1}_{k_0}\right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\epsilon)} =: G_1+G_2.$$

First, we compute G_1. Take note of that

$$|[b,\mu]\left(a_{t_0}\right)(z)|\leqslant\left(\int_0^{|x|+2^{t_0+1}}\left|\iint_{|z-y|\leqslant t}\frac{[b(z)-b(y)]\omega(z-y)}{|z-y|^{n-1}}a_{t_0}(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2}$$
$$+\left(\int_{|x|+2^{t_0+1}}^{\infty}\left|\iint_{|z-y|\leqslant t}\frac{[b(z)-b(y)]\omega(z-y)}{|z-y|^{n-1}}a_{t_0}(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2} =: I_1+I_2.$$

Let $|z-y|<t$ with $z\in a_{k_0}$ and $t<|z|+2^{t_0+1}$. Let $k_0-2\geqslant t_0$ such that $|z-y|\sim|z|\sim|z|+2^{t_0+1}$. Letting $\operatorname{Lip}_{\nu}\left(S^{n-1}\right)\subset L^{\infty}\left(S^{n-1}\right)$, we get

$$\begin{aligned}
I_1&\leqslant\int_{\mathbb{R}^n}\frac{|b(z)-b(y)|\,|a_{t_0}(y)|}{|z-y|^{n-1}}dy\left(\int_{|z-y|}^{|z|+2^{t_0+1}}\frac{dt}{t^3}\right)^{1/2}\\
&\leqslant\|b\|_{\operatorname{Lip}_\eta}\int_{\mathbb{R}^n}\frac{|z|^{\eta}\,|a_{t_0}(y)|}{|z|^{n-1}}\frac{|y|^{1/2}}{|z|^{3/2}}dy\\
&\leqslant\|b\|_{\operatorname{Lip}_\eta}|z|^{\eta-n-1/2}\int_{B_{t_0}}|a_{t_0}(y)|\,|y|^{1/2}dy\\
&=C\|b\|_{\operatorname{Lip}_\eta}2^{t_0\eta}2^{j(1/2-\eta)}|z|^{-n}|z|^{\eta-1/2}\left\|a_{t_0}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\\
&\leqslant\|b\|_{\operatorname{Lip}_\eta}2^{-kn+j\eta}\left\|a_{t_0}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Observe that $t\geqslant|z|+|y|\geqslant|\,z-y\,|$, since $t\geqslant|z|+2^{t_0+1}$ and $y\in B_{t_0}$. Based on Hölder's inequality, we have

$$\begin{aligned}
I_2&\leqslant\left(\int_{|z|+2^{t_0+1}}^{\infty}\left|\int_{\mathbb{R}^n}\frac{\omega(z-y)}{|z-y|^{n-1}}[b(z)-b(y)]a_{t_0}(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2}\\
&=\left|\int_{B_{t_0}}\frac{\omega(z-y)}{|z-y|^{n-1}}[b(z)-b(y)]a_{t_0}(y)dy\right|\left(\int_{|z|+2^{t_0+1}}^{\infty}\frac{dt}{t^3}\right)^{1/2}\\
&\leqslant\frac{1}{|z|+2^{t_0+1}}\left|\int_{B_{t_0}}\frac{[b(z)-b(y)]\omega(z-y)}{|z-y|^{n-1}}a_{t_0}(y)dy\right|\\
&\leqslant\frac{1}{|z|+2^{t_0+1}}\left|\int_{B_{t_0}}\frac{\omega(z-y)}{|z-y|^{n-1}}[b(y)-b(0)]a_{t_0}(y)dy\right|\\
&\quad+\frac{1}{|z|+2^{t_0+1}}\left|[b(z)-b(0)]\int_{B_{t_0}}\left(\frac{\omega(z-y)}{|z-y|^{n-1}}-\frac{\omega(z)}{|z|^{n-1}}\right)a_{t_0}(y)dy\right|\\
&\leqslant\|b\|_{\operatorname{Lip}_\eta}\left(|z|^{\eta}\int_{B_{t_0}}\frac{|y|^{\nu}\,|a_{t_0}(y)|}{|z|^{n-1+\nu}}dy+\int_{B_{t_0}}\frac{|y|^{\eta}\,|a_{t_0}(y)|}{|z-y|^{n-1}}dy\right)\frac{1}{|z|+2^{t_0+1}}\\
&\leqslant\|b\|_{\operatorname{Lip}_\eta}\left(|z|^{\eta}\int_{B_{t_0}}\frac{|y|^{\nu}\,|a_{t_0}(y)|}{|z|^{n+\nu}}dy+\int_{B_{t_0}}\frac{|y|^{\eta}\,|a_{t_0}(y)|}{|z-y|^{n}}dy\right)\\
&\leqslant\|b\|_{\operatorname{Lip}_\eta}\left(|z|^{-n}2^{t_0\eta}+|z|^{\eta-n-\nu}2^{j\nu}\right)\left\|a_{t_0}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\\
&\leqslant 2^{-k_0n+t_0\eta}\|b\|_{\operatorname{Lip}_\eta}\left\|a_{t_0}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Lemmas 1.3 and 1.4 thus give us

$$\begin{aligned}
\left\|[b,\mu]\left(a_{t_0}\right)\mathbf{1}_{k_0}\right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\leqslant 2^{-k_0 n+t_0\eta}\|b\|_{\mathrm{Lip}_\eta}\|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\left\|\mathbf{1}_{B_{k_0}}\right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\
&\leqslant 2^{-k_0 n+(t_0-k_0)\eta}\|b\|_{\mathrm{Lip}_\eta}\|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\left\|I_\eta(\mathbf{1}_{B_{k_0}})\right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\
&\leqslant 2^{-k_0 n+(t_0-k_0)\eta}\|b\|_{\mathrm{Lip}_\eta}\|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\left\|\mathbf{1}_{B_{k_0}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\leqslant 2^{(t_0-k_0)\eta}\|b\|_{\mathrm{Lip}_\eta}\|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\left\|\mathbf{1}_{B_{t_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{\left\|\mathbf{1}_{B_{k_0}}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}\\
&\leqslant 2^{-t_0\beta+(t_0-k_0)(\eta+n\delta_2)}\|b\|_{\mathrm{Lip}_\eta}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
G_1 &\leqslant \|b\|^{q_1(1+\epsilon)}_{\mathrm{Lip}_\eta}\sup_{\epsilon>0}\epsilon^\theta\sum_{k_0=-\infty}^{\infty}2^{k_0\beta q_1(1+\epsilon)}\left(\sum_{t_0=-\infty}^{k_0-2}2^{(t_0-k_0)(\eta+n\delta_2)-t_0\beta}|\lambda_{t_0}|\right)^{q_1(1+\epsilon)}\\
&= C\|b\|^{q_1(1+\epsilon)}_{\mathrm{Lip}_\eta}\sup_{\epsilon>0}\epsilon^\theta\sum_{k_0=-\infty}^{\infty}\left(\sum_{t_0=-\infty}^{k_0-2}2^{(t_0-k_0)(\eta+n\delta_2-\beta)}|\lambda_{t_0}|\right)^{q_1(1+\epsilon)}.
\end{aligned}$$

Due to Hölder's inequality, since $\eta+n\delta_2-\beta>0$, we obtain

$$\begin{aligned}
G_1 &\leqslant \|b\|^{q_1(1+\epsilon)}_{\mathrm{Lip}_\eta}\sup_{\epsilon>0}\epsilon^\theta\sum_{k_0=-\infty}^{\infty}\left(\sum_{t_0=-\infty}^{k_0-2}|\lambda_{t_0}|^{q_1(1+\epsilon)}2^{(t_0-k_0)(\eta+n\delta_2-\beta)q_1(1+\epsilon)/2}\right)\\
&\quad\times\left(\sum_{t_0=-\infty}^{k_0-2}2^{(t_0-k_0)(\eta+n\delta_2-\beta)(q_1(1+\epsilon))'/2}\right)^{q_1(1+\epsilon)/(q_1(1+\epsilon))'}\\
&\leqslant \|b\|^{q_1(1+\epsilon)}_{\mathrm{Lip}_\eta}\sup_{\epsilon>0}\epsilon^\theta\sum_{k_0=-\infty}^{\infty}\left(\sum_{t_0=-\infty}^{k_0-2}|\lambda_{t_0}|^{q_1(1+\epsilon)}2^{(t_0-k_0)(\eta+n\delta_2-\beta)q_1(1+\epsilon)/2}\right)\\
&= \|b\|^{q_1(1+\epsilon)}_{\mathrm{Lip}_\eta}\sup_{\epsilon>0}\epsilon^\theta\sum_{t_0=-\infty}^{\infty}|\lambda_{t_0}|^{q_1(1+\epsilon)}\left(\sum_{k_0=t_0+2}^{\infty}2^{(t_0-k_0)(\eta+n\delta_2-\beta)q_1(1+\epsilon)/2}\right)\\
&\leqslant \|b\|^{q_1(1+\epsilon)}_{\mathrm{Lip}_\eta}\sup_{\epsilon>0}\epsilon^\theta\sum_{t_0=-\infty}^{\infty}|\lambda_{t_0}|^{q_1(1+\epsilon)}.
\end{aligned}$$

For the estimate of G_2, we apply the boundedness of $[b,\mu]$ to get

$$\begin{aligned}
G_2 &\leqslant \|b\|^{q_1(1+\epsilon)}_{\mathrm{Lip}_\eta}\sup_{\epsilon>0}\epsilon^\theta\sum_{k_0=-\infty}^{\infty}2^{k_0\beta p_1}\left(\sum_{t_0=k_0-1}^{\infty}|\lambda_{t_0}|\,\|a_{t_0}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\epsilon)}\\
&\leqslant \|b\|^{q_1(1+\epsilon)}_{\mathrm{Lip}_\eta}\sup_{\epsilon>0}\epsilon^\theta\sum_{k_0=-\infty}^{\infty}\left(\sum_{t_0=k_0-1}^{\infty}|\lambda_{t_0}|2^{(k_0-t_0)\beta}\right)^{q_1(1+\epsilon)}\\
&\leqslant \|b\|^{q_1(1+\epsilon)}_{\mathrm{Lip}_\eta}\sup_{\epsilon>0}\epsilon^\theta\sum_{t_0=-\infty}^{\infty}|\lambda_{t_0}|^{q_1(1+\epsilon)}.
\end{aligned}$$

Thus by using the estimates of G_1 and G_2, we conclude that the commutator of Marcinkiewicz integral operator is bounded on grand variable Herz-Hardy spaces. Thus we obtained our desired results. □

3. Conclusion

In this paper, we proved the boundedness results for commutators of the Marcinkiewicz integral operator, the Calderón-Zygmund singular integral operator and the fractional integral operator on grand variable Herz-Hardy spaces.

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