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New extensions of Hermite-Hadamard and Fejér type inequalities using fuzzy fractional integral operators through different fuzzy convexities

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Abstract

It is a familiar fact to develop inequalities using the popular method by adopting fractional operators, and such study of methods is the main core of modern research in recent year. Fuzzy interval valued (FIV) mappings not only used to generalize of different convex mappings but also developed fractional operators. In this paper, we investigate fuzzy fractional inequalities for different fuzzy convexities by successfully implementing generalized fuzzy fractional operators (G-FFO). We discuss the extension of Hermite–Hadamard, trapezoid-type inequalities on the basis of fuzzy convexities and fuzzy fractional operators. Moreover, we establish the Fejér and midpoint type fuzzy inequalities for (η_1, η_2)-convex fuzzy function.

Keywords: Hermite-Hadamard type fuzzy inequality, Fejér type fuzzy inequality, generalized fuzzy fractional operators, convex fuzzy interval valued function, (η_1, η_2)-convex fuzzy interval valued function.

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1. Introduction

Convex functions are of significant importance in the advancement of fractional calculus, influencing multiple domains within mathematics. The concept of convexity has been widely studied because of its applicability in various areas, particularly in optimization theory. The connection between control optimization and inequality theory is strongly influenced by the properties of convex functions, which have led researchers to formulate and investigate numerous well-established inequalities. Convexity inequalities have numerous applications in the study of different models from real-world applications [18, 33]. Meanwhile, the growing interest in fractional analysis has led to an increasing demand for fractional operators in different mathematical domains. To meet this need, researchers have worked on developing fractional operators by incorporating non-singular special functions as kernels, leading to modified

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versions of inequalities [2, 3, 6, 22]. Generalized fractional operators have emerged as a powerful tool for refining fractional inequalities related to various convexities and pre-invexities, which play a crucial role in mathematical analysis. The concept of inequalities and convexity plays an important role in the development of the minimization model and its analysis in different areas of mathematics. Modifications to inequalities depend on convexity and fractional operators. Due to this, has increased the demand of generalized fractional operators and the extension of convexity. There are many techniques to improve fractional operators. One of these techniques has to obtain the new family of generalized fractional operators by means of its kernel. The gradual advancement of novel technologies has led to the need for special functions and fractional operators. Over the past few decades, an enormous amount of effort has gone into creating fractional operators with generalized special functions as their basis. These operators have a wide range of applications in the field of fractional inequalities within operator theory. These kinds of fractional operator provided ways to solve various issues that the scientific community was confronting. Many authors discuss generalized fractional operators by using multi-index special functions as its kernel and modifies well-known inequalities [1, 17].

Fuzzy interval theory has numerous important applications, particularly in the handling of problems involving ambiguity, vagueness, and imprecise information. It plays a crucial role in decision-making processes for individual and group collaborations. This concept led to the development of interval analysis [21] by Moore Ramon in 1966. The field has attracted many researchers because of its effectiveness in decision-making evaluations. Interval analysis studies have proven valuable in global optimization and constraint-solving algorithms, making them increasingly popular among experts over the past few decades. It has provided accurate and reliable results, reduced errors, and improved precision. Motivated by these advances, many researchers have focused on inequalities to achieve more refined and meaningful outcomes.

Fractional integrals deal with the Hermite-Hadamard-Fejér inequality for (η_1, η_2)-convex functions. Various midpoint and trapezoid-type Hermite-Hadamard inequality-related inequalities originate when the examined function's absolute derivative is (η_1, η_2)-convex functions. Furthermore, in the case where a positive (η_1, η_2) function with convex properties is increasing, an improvement is offered for the standard Hermite-Hadamard inequality based on fractional integrals [29].

$$\Phi(\frac{a+b}{2})\int_a^b \Psi(y)dy \leqslant \int_a^b \Phi(y)\Psi(y)dy \leqslant \frac{\Phi(a)+\Phi(b)}{2}\int_a^b \Psi(y)dy.$$

The Hermite-Hadamard dual inequality was the first basic discovery for convex mappings formed on a real-numbered range, having a natural mathematical explanation and restricted value possibilities for specific variations. Many researchers have worked to generalize the fundamentals of the Hermite-Hadamard inequality for different convexities [8, 12, 19, 31], including derivable findings for particular means. Hermite-Hadamard-type inequalities have also been provided for various conceptions of convexity, which have a significant contribution in pure analysis [5, 11, 15, 25, 31, 32]. It highlighted the qualities of several functions, functionals, and sequences that can help modify the H-H result.

Many authors defined and studied different periods generalities of the function of Mittag-Leffler type in their research papers across the duration of the last period due to contribution in physics, engineering, and practical sciences [10, 17, 30]. These writers included Wiman, Prabhakar, Shukla and Prajapati, Salim and Faraj, and others. In order as examples of the subsistence and distinctiveness of the solution to the associated integral equation of the initial kind, Prabhakar analayzed many properties of the extended Mittag-Leffler function as well as the generalized integral measured in the kernel. In addition, Kilbas et al. dedicated their own to delving more into, as well as the integral transform operator defined in [13]. They provided formulas for its Riemann-Liouville generalized integral and variance operatives, as well as integral representation, differentiation, and integration features. See Srivastava and Tomovski's study for further results and interpretations. Convexity and its generalized version play a crucial role in optimization within the fuzzy domain, leading to fuzzy variational inequalities that characterize the optimality conditions of convex functions. The integration of variational inequality theory and fuzzy set theory has resulted in powerful techniques for solving mathematical problems related to minimization theory and interval analysis.

Motivated by above results of the paper, we will investigate fuzzy fractional inequalities for different fuzzy convexities by successfully implementing generalized fuzzy fractional operators (G-FFO). We will discuss the extension of Hermite-Hadamard, trapezoid-type inequalities on the basis of fuzzy convexities and fuzzy fractional operators. Moreover, we will establish the Fejér and midpoint type fuzzy inequalities for (η_1, η_2)-convex fuzzy function. Finally, some conclusions and future research will be given for interested readers.

2. Preliminaries

In this section, we discuss some basic definitions which help us to understand our main results.

Definition 2.1 ([26, 28])**.** $\Phi : I \to \mathbb{R}$ is defined for the convex function, where $u \in [1, 0], \forall x, y \in I$ as follows:

$$\Phi((1-u)y + ux) \leqslant (1-u)\Phi(y) + u\Phi(x).$$

Definition 2.2 ([9])**.** The Hermite-Hadamard (H-H) inequality defined as follows:

$$y\Big(\frac{\lambda+\eta}{2}\Big) \leqslant \frac{1}{\eta-\lambda}\int_{\lambda}^{\eta} y(\eta)d\eta \leqslant \frac{y(\eta)+y(\lambda)}{2},$$

was proved in [26], where $\eta, \lambda \in K \subseteq \mathbb{R}$ and $\eta < \lambda$.

Definition 2.3 ([4])**.** Let $I \subset \mathbb{R}$ be an invex set with respect to $\eta_1 : I \times I \to \mathbb{R}$, consider $\varphi : I \to \mathbb{R}$ and $\eta_2 : \varphi(I) \times \varphi(I) \to \mathbb{R}$. The function φ is said to be (η_1, η_2)-convex, if

$$\varphi(x + \lambda\eta_1(y, x)) \leqslant \varphi(x) + \lambda\eta_2(\varphi(y), \varphi(x)),$$

is valid for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 2.4 ([14])**.** Letting $\ell \in [0, 1]$, the FIV-mapping $y : \Pi \subseteq \mathbb{R} \to \Pi_{bc}$ is evaluated by $y_\ell(x) = \big[y_*(x, \ell), y^*(x, \ell)\big], \forall x \in \Pi$ in which the left- and right-sided mappings $y^*(x, \ell), y_*(x, \ell) : \Pi \to \mathbb{R}$, are named upper and lower functions of y.

Letting for $\ell \in [0, 1]$, both $y_*(x, \ell)$ and $y^*(x, \ell)$ be continuous at $x \in \Pi$, then the FIV-function $y : \Pi \subseteq \mathbb{R} \to \Pi_{bc}$ is continuous at $x \in \Pi$.

Definition 2.5 ([16])**.** Assume that Π_{bc} is the aforementioned class of intervals and $H \in \Pi_{bc}$. Then

$$\Lambda = [\Lambda_*, \Lambda^*] = \{\alpha \in \mathbb{R} \,|\, \Lambda_* \preccurlyeq \alpha \preccurlyeq \Lambda^*\} \ \ (\Lambda_*, \Lambda^* \in \mathbb{R}).$$

If $\Lambda_* = \Lambda^*$, then we say that Λ is degenerate.

In this study, we treat all fuzzy intervals as non-degenerate intervals. If $\Lambda_* \geqslant 0$, then $[\Lambda_*, \Lambda^*]$, which also called a positive interval, which all of them can be specified by the symbol Π^+_{bc} so that

$$\Pi^+_{bc} = \Big\{[\Lambda_*, \Lambda^*] : [\Lambda_*, \Lambda^*] \in \Pi_{bc} \ \text{ and } \ \Lambda_* \geqslant 0\Big\}.$$

Let $\sigma \in \mathbb{R}$. Then, we define the scalar multiplication as

$$\sigma \cdot \Lambda = \begin{cases} [\sigma\Lambda_*, \sigma\Lambda^*], & \text{if } \sigma \geqslant 0, \\ [\sigma\Lambda^*, \sigma\Lambda_*], & \text{if } \sigma \preccurlyeq 0. \end{cases}$$

Moreover, the inclusion " $\subseteq$ " means that $\Lambda_1 \subseteq \Lambda_2$ if and only if $[\Lambda_{1*}, \Lambda^*_1] \subseteq [\Lambda_{2*}, \Lambda^*_2]$ iff $\Lambda_{1*} \preccurlyeq \Lambda_{2*}, \Lambda^*_1 \preccurlyeq \Lambda^*_2$ [16]. By considering $F \subseteq \mathbb{R}$ as the fuzzy interval, let $\hbar_1, \hbar_2 \in F$. The partial order relation $\preccurlyeq$ is given by

$$\hbar_1 \preccurlyeq \hbar_2 \quad \Leftrightarrow \quad [\hbar_1]^\ell \preccurlyeq_I [\hbar_2]^\ell, \ \ \forall \ell \in [0, 1],$$

where $[\hbar]^\ell$ is called the ℓ-level set of $\hbar$ defined by

$$[\hbar]^\ell = \{r \in \mathbb{R} : \ \ell \preccurlyeq \hbar(r)\}.$$

Proposition 2.6 ([7]). *For* $\hbar_1, \hbar_2 \in F$, $c \in \mathbb{R}$ *and* $\ell \in [0,1]$, *we have*

$$\left[\hbar_1 \widetilde{+} \hbar_2\right]^\ell = [\hbar_1]^\ell + [\hbar_2]^\ell, \quad \left[\hbar_1 \widetilde{\times} \hbar_2\right]^\ell = [\hbar_1]^\ell \times [\hbar_2]^\ell, \quad [c \cdot \hbar_1]^\ell = c[\hbar_1]^\ell, \quad \left[c \widetilde{+} \hbar_1\right]^\ell = c + [\hbar_1]^\ell.$$

Definition 2.7 ([20, 23]). The FIV-function $y : [u, v] \to K_{bc}$ is convex on $[u, v]$, if

$$y\Big(tu + (1-t)v\Big) \preccurlyeq ty(u) \widetilde{+} (1-t)y(v),$$

for all $t \in [0,1]$, and $y(v) \succcurlyeq 0$.

Definition 2.8. Let $I \subset \mathbb{R}$ be an invex set with respect to $\eta_1 : I \times I \to \mathbb{R}$, consider $\phi : I \to \mathbb{R}$ and $\eta_2 : \phi(I) \times \phi(I) \to \mathbb{R}$. The ϕ is said to be (η_1, η_2)-convex FIV-function, if

$$\phi(x + \lambda \eta_1(y, x)) \preccurlyeq \phi(x) \widetilde{+} \lambda \eta_2(\phi(y), \phi(x)),$$

is valid for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 2.9 ([27]). The Mittag-Leffler operator is defined as:

$$E_\alpha(w) = \sum_{n=0}^{\infty} \frac{w^n}{\Gamma(\alpha n + 1)},$$

where $w \in \mathbb{C}$ and $\alpha > 0$.

Definition 2.10 ([27]). The Prabhakar fractional fractional operators is defined for $a < x$, $\alpha, \beta, \rho, \lambda \in \mathbb{C}$ and $\Re(\alpha) > 0, \Re(\rho) > 0$, as follows:

$$\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda} f(x) = \int_a^x (x-t)^{\beta-1} E^\rho_{\alpha,\beta} \lambda (x-t)^\alpha f(t) dt \quad (\Re(\beta) > 0),$$

and operator for $a > x$

$$\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda} f(x) = \int_x^a (t-x)^{\beta-1} E^\rho_{\alpha,\beta} \lambda (t-x)^\alpha f(t) dt \quad (\Re(\beta) > 0),$$

for any real number $\alpha \geqslant 0$, where the function

$$E^\rho_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{(\rho)_n x^n}{\Gamma(\alpha n + \beta) n!}, \quad \Re(\alpha) > 0,$$

is an entire function.

Definition 2.11 ([24]). The left- and right-sided Prabhakar fuzzy fractional integral operators, associated with the left and right endpoint functions, are defined as follows:

$$\left[\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda} f(x)\right]^l = \int_a^x (x-t)^{\beta-1} E^\rho_{\alpha,\beta} \lambda (x-t)^\alpha f(l,t) dt = \int_a^x (x-t)^{\beta-1} E^\rho_{\alpha,\beta} \lambda (x-t)^\alpha [f^*(l,t), f_*(l,t)] dt,$$

where

$$\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda} f^*(x,l) = \int_a^x (x-t)^{\beta-1} E^\rho_{\alpha,\beta} \lambda (x-t)^\alpha f^*(l,t) dt \quad (x > a),$$
$$\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda} f_*(x,l) = \int_a^x (x-t)^{\beta-1} E^\rho_{\alpha,\beta} \lambda (x-t)^\alpha f_*(l,t) dt \quad (x > a),$$

and

$$\begin{aligned}\left[\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda} f(x)\right]^l = \left[\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda} f(x)\right]^l &= \int_x^a (t-x)^{\beta-1} E^\rho_{\alpha,\beta} \lambda (t-x)^\alpha f(l,t) dt \\ &= \int_x^a (t-x)^{\beta-1} E^\rho_{\alpha,\beta} \lambda (t-x)^\alpha [f^*(l,t), f_*(l,t)] dt.\end{aligned}$$

Definition 2.12 ([30]). Generalized form of Mittag-Leffler function is defined as:

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(w) = \sum_{n=0}^{\infty} \frac{w^n(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\delta)pn},$$

where $\alpha,\beta,\gamma,\delta \in \mathbb{C}$, and $\Re(\gamma) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\delta) > 0$; $q, p > 0$ and $\Re\alpha + p \geqslant q$.

Definition 2.13 ([17]). The left and right sided generalization fractional operator having multi-index Mittag-Leffler function its kernel, is defined for $\alpha,\beta,\gamma,\delta \in \mathbb{C}$; and $\Re(\gamma) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\delta) > 0$; $p, q > 0$ and $\Re\alpha + p \geqslant q$ as follows:

$$(\xi^{\gamma,\sigma,q}_{\alpha,\beta,p,\omega;a^+}\varphi)(x) = \int_a^x (-t+x)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(-t+x)^{\alpha})\varphi(t)dt,$$
$$(\xi^{\gamma,\sigma,q}_{\alpha,\beta,p,\omega;b^-}\varphi)(x) = \int_x^b (t-x)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(t-x)^{\alpha})\varphi(t)dt.$$

Definition 2.14. The left- and right-sided generalized fuzzy fractional operators, corresponding to the left and right endpoint functions, are defined as follows:

$$\begin{aligned}\left[\mathfrak{E}^{\gamma,\sigma,q}_{\alpha,\beta,p}f(x)\right]^l &= \int_a^x (x-t)^{\beta-1}E^{\gamma,\sigma,q}_{\alpha,\beta,p}\lambda(x-t)^{\alpha}f(l,t)dt\\ &= \int_a^x (x-t)^{\beta-1}E^{\gamma,\sigma,q}_{\alpha,\beta,p}\lambda(x-t)^{\alpha}[f^*(l,t), f_*(l,t)]dt,\end{aligned}$$

where

$$\mathfrak{E}^{\gamma,\sigma,q}_{\alpha,\beta,p,\lambda}f^*(x,l) = \int_a^x (x-t)^{\beta-1}E^{\gamma,\sigma,q}_{\alpha,\beta,p}\lambda(x-t)^{\alpha}f^*(l,t)dt \quad (x > a),$$
$$\mathfrak{E}^{\gamma,\sigma,q}_{\alpha,\beta,p,\lambda}f_*(x,l) = \int_a^x (x-t)^{\beta-1}E^{\gamma,\sigma,q}_{\alpha,\beta,p}\lambda(x-t)^{\alpha}f_*(l,t)dt \quad (x > a),$$

and

$$\begin{aligned}\left[\mathfrak{E}^{\gamma,\sigma,q}_{\alpha,\beta,p,\lambda}f(x)\right]^l &= \int_x^a (t-x)^{\beta-1}E^{\gamma,\sigma,q}_{\alpha,\beta,p}\lambda(t-x)^{\alpha}f(l,t)dt\\ &= \int_x^a (t-x)^{\beta-1}E^{\gamma,\sigma,q}_{\alpha,\beta,p}\lambda(t-x)^{\alpha}[f^*(l,t), f_*(l,t)]dt.\end{aligned}$$

3. Behaviour of generalized fuzzy Hermite-Hadamard-type fractional integral inequality

In this section, we obtain the refinement of the Hermite-Hadamard-type inequality by implementation of the generalized fuzzy fractional operators (GFFO) having Mittag-Leffler function as its kernel.

Theorem 3.1. *Let $\Lambda : [0,1] \to \mathbb{R}$ be a convex FIV- function, with $\psi > \alpha > 0$ and $\Lambda \in L_1[\alpha,\psi]$. If Λ is a non-decreasing function on $[\alpha,\psi]$, then the following inequality holds:*

$$\Lambda\Big(\frac{\alpha+\psi}{2}\Big)(\xi^{\gamma,\rho,q}_{\sigma,\rho,p,w;\psi^-})(\alpha) \preccurlyeq \frac{1}{2}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,w;\alpha^+}\Big)(\psi)\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,w;\psi^-}\Big)(\alpha)\Big] \preccurlyeq \frac{\Lambda(\alpha)\widetilde{+}\Lambda(\psi)}{2}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,w;1^+}\Big)(\psi).$$

Proof. Considering the convex FIV- function Λ on the interval $[\alpha,\psi]$, we have

$$\Lambda\Big(\frac{\alpha+\psi}{2}\Big) \preccurlyeq \frac{\Lambda(\alpha)\widetilde{+}\Lambda(\psi)}{2}.$$

Putting the value of $x = (1-t)\psi + t\alpha,\ y = t\psi + (1-t)\alpha$, we get

$$2\Lambda\Big(\frac{\alpha+\psi}{2}\Big) \preccurlyeq \Lambda(t\alpha+(1-t)\psi)\widetilde{+}\Lambda((1-t)\alpha+t\psi).$$

For each $l \in [0,1]$ and for the lower FIV-function y_*, we can write

$$2\Lambda_*\Big(\frac{\alpha+\psi}{2}, l\Big) \preccurlyeq \Lambda_*(t\alpha+(1-t)\psi, l)\widetilde{+}\Lambda_*((1-t)\alpha+t\psi, l).$$

Multiplying by $(-t+1)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p})(\varkappa(-t+1)^{\sigma})$, and integrating the resulting inequality from $[0,1]$, we have

$$\begin{aligned}
&2\Lambda_*\Big(\frac{\alpha+\psi}{2}, l\Big)\int_0^1 (1-t)^{\zeta-1}(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p})(\varkappa(1-t)^{\sigma})dt\\
&\quad\preccurlyeq \int_0^1 (1-t)^{\zeta-1}(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p})(\varkappa(1-t)^{\sigma})\Lambda_*(t\alpha+(1-t)\psi, l)dt\\
&\qquad+\int_0^1 (1-t)^{\zeta-1}(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p})(\varkappa(1-t)^{\sigma})\Lambda_*((1-t)\alpha+t\psi, l)dt,\\
&2\Lambda_*\Big(\frac{\alpha+\psi}{2}, l\Big)\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\int_0^1(1-t)^{\sigma n+\zeta-1}dt\\
&\quad\preccurlyeq \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\times\Big[\int_0^1(1-t)^{\sigma n+\zeta-1}\Lambda_*(t\alpha+(1-t)\psi, l)dt\\
&\qquad+\int_0^1(1-t)^{\sigma n+\zeta-1}\Lambda_*((1-t)\alpha+t\psi, l)dt\Big].
\end{aligned}\tag{3.1}$$

By substituting $(1-t)\psi + t\alpha = z$, and its differential in equation (3.1), we have

$$\begin{aligned}
2\Lambda_*\Big(\frac{\alpha+\psi}{2}, l\Big)\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\psi^-}1\Big)(\alpha) \preccurlyeq \Big[&\int_\alpha^\psi\Big(\frac{z-\alpha}{\psi-\alpha}\Big)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}\Big(\varkappa\Big(\frac{z-\alpha}{\psi-\alpha}\Big)^{\sigma}\Big)\Lambda_*(z,l)dz\\
&+\int_\alpha^\psi\Big(\frac{\psi-z}{\psi-\alpha}\Big)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}\Big(\varkappa\Big(\frac{\psi-z}{\psi-\alpha}\Big)^{\sigma}\Big)\Lambda_*(z,l)dz\Big]
\end{aligned}\tag{3.2}$$

or

$$\Lambda_*\Big(\frac{\alpha+\psi}{2}, l\Big)\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\psi^-}1\Big)(\alpha) \preccurlyeq \frac{1}{2}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\Lambda_*\Big)(\psi,l)+\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\psi^-}\Lambda_l\Big)(\alpha,l)\Big].$$

Similarly, continuing the calculation for upper FIV-function Λ^*, we obtain

$$\Lambda^*\Big(\frac{\alpha+\psi}{2}, l\Big)\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\psi^-}1\Big)(\alpha) \preccurlyeq \frac{1}{2}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\Lambda^*\Big)(\psi,l)+\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\psi^-}\Lambda^*\Big)(\alpha,l)\Big].\tag{3.3}$$

Combining the inequalities (3.2) and (3.3), we have

$$\begin{aligned}
&\Big[\Lambda_*\Big(\frac{\alpha+\psi}{2}, l\Big),\Lambda^*\Big(\frac{\alpha+\psi}{2}, l\Big)\Big]\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\psi^-}1\Big)(\alpha)\\
&\preccurlyeq \frac{1}{2}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\Lambda_*\Big)(\psi,l)+\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\psi^-}\Lambda_*\Big)(\alpha,l),\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\Lambda^*\Big)(\psi,l)+\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\psi^-}\Lambda^*\Big)(\alpha,l)\Big].
\end{aligned}$$

Hence,

$$\Lambda\Big(\frac{\alpha+\psi}{2}\Big)\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\psi^-}1\Big)(\alpha) \preccurlyeq \frac{1}{2}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\Lambda\Big)(\psi)\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\psi^-}\Lambda\Big)(\alpha)\Big].\tag{3.4}$$

To solve the second part of the inequality, once again considering the FIV convex function of Λ, we have

$$\Lambda((1-t)\psi+t\alpha)\preccurlyeq t\Lambda(\alpha)\widetilde{+}(1-t)\Lambda(\psi). \tag{3.5}$$

Again, rewrite the above inequality as

$$\Lambda(t\psi+(1-t)\alpha)\preccurlyeq(1-t)\Lambda(\alpha)\widetilde{+}t\Lambda(\psi). \tag{3.6}$$

By adding (3.5) and (3.6), we get

$$\Lambda\big[(1-t)\psi+t\alpha\big]+\Lambda\big[t\psi+(1-t)\alpha\big]\preccurlyeq(\Lambda(\alpha)\widetilde{+}\Lambda(\psi)).$$

For each $l\in[0,1]$ and for the lower FIV-function Λ_*, we obtain

$$\Lambda_*\big[(1-t)\psi+t\alpha,l\big]+\Lambda_*\big[t\psi+(1-t)\alpha,l\big]\preccurlyeq(\Lambda_*(\alpha,l)\widetilde{+}\Lambda_*(\psi,l)).$$

Multiplying by $(-t+1)^{\zeta-1}(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p})(\varkappa(-t+1)^{\sigma})$ and then integrating over the interval $[0,1]$, we have

$$\begin{aligned}&\int_0^1(1-t)^{\zeta-1}(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p})(\varkappa(1-t)^{\sigma})\Lambda_*(t\alpha+(1-t)\psi,l)dt\\&\quad+\int_0^1(1-t)^{\zeta-1}(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p})(\varkappa(1-t)^{\sigma})\Lambda_*((1-t)\alpha+t\psi,l)dt\\&\preccurlyeq\Big(\Lambda_*(\alpha,l)+\Lambda_*(\psi,l)\Big)\int_0^1(1-t)^{\zeta-1}(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p})(\varkappa(1-t)^{\sigma})dt.\end{aligned}$$

After making a suitable substitutions and simplification, we obtain

$$\frac{\Lambda_*(\alpha,l)+\Lambda_*(\psi,l)}{2}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}1\Big)(\psi)\geqslant\frac{1}{2}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\Lambda_*\Big)(\psi,l)+\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\psi^-}\Lambda_*\Big)(\alpha,l)\Big]. \tag{3.7}$$

Similarly, continuing our computation for the upper FIV-function Λ^*, we obtain

$$\frac{\Lambda^*(\alpha,l)+\Lambda^*(\psi,l)}{2}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}1\Big)(\psi)\geqslant\frac{1}{2}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\Lambda^*\Big)(\psi,l)+\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\psi^-}\Lambda^*\Big)(\alpha,l)\Big]. \tag{3.8}$$

By combining the equations (3.7) and (3.8), we have

$$\begin{aligned}&\frac{1}{2}\Big[\Lambda_*(\alpha,l)+\Lambda_*(\psi,l),\Lambda^*(\alpha,l)+\Lambda^*(\psi,l)\Big]\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}1\Big)(\psi)\\&\quad\geqslant\frac{1}{2}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\Lambda_*\Big)(\psi,l)+\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\psi^-}\Lambda_*\Big)(\alpha,l),\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\Lambda^*\Big)(\psi,l)+\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\psi^-}\Lambda^*\Big)(\alpha,l)\Big].\end{aligned}$$

Hence,

$$\frac{\Lambda(\alpha)\widetilde{+}\Lambda(\psi)}{2}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}1\Big)(\psi)\succcurlyeq\frac{1}{2}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\Lambda\Big)(\psi)\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\psi^-}\Lambda\Big)(\alpha)\Big]. \tag{3.9}$$

By combining the equations (3.4) and (3.9), we have the required inequality

$$\Lambda\Big(\frac{\alpha\widetilde{+}\psi}{2}\Big)\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\psi^-}1\Big)(\alpha)\preccurlyeq\frac{1}{2}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\psi^-}\Lambda\Big)(\alpha)\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\Lambda\Big)(\alpha)\Big]\preccurlyeq\frac{\Lambda(\alpha)\widetilde{+}\Lambda(\psi)}{2}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}1\Big)(\psi),$$

which completes the proof. □

4. Analyzing the refinements of Hermite-Hadamard-type fuzzy inequality

In this section, we discuss the refinements of Hermite-Hadamard-type fuzzy inequality by GFFO.

Lemma 4.1. *Letting* $\Lambda : \mathrm{I} \to \mathbb{R}$ *be a differentiable FIV-function on* $\mathrm{I} = [\alpha, \psi] \subseteq \mathbb{R}$ *in which* $\Lambda^{'} \in L_1[\alpha, \psi]$, *then*

$$\frac{\Lambda(\alpha)\widetilde{+}\Lambda(\psi)}{2}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma}) - \frac{1}{2(\psi-\alpha)}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\alpha^{+}}\Lambda\Big)(\psi)\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\psi^{-}}\Lambda\Big)(\alpha)\Big] = \frac{\psi-\alpha}{2}\mathrm{I},$$

where

$$\mathrm{I} = \int_0^1 (1-t)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1-t)^{\sigma})\Lambda'(\alpha t + (1-t)\psi)dt + \int_0^1 -t^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(t)^{\sigma})\Lambda'(\alpha t + (1-t)\psi)dt.$$

Proof. Consider the integral

$$\mathrm{I} = \int_0^1 (1-t)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1-t)^{\sigma})\Lambda'(\alpha t + (1-t)\psi)dt + \int_0^1 -t^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(t)^{\sigma})\Lambda'(\alpha t + (1-t)\psi)dt.$$

For every $l \in [0, 1]$, we have

$$\mathrm{I} = \int_0^1 (1-t)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1-t)^{\sigma})\Lambda'_{*}(\alpha t + (1-t)\psi, l)dt + \int_0^1 -t^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(t)^{\sigma})\Lambda'_{*}(\alpha t + (1-t)\psi, l)dt.$$

Let $\mathrm{I} = \mathrm{I}_1 + \mathrm{I}_2$.
Firstly, let us consider the integral I_1,

$$\begin{aligned}
\mathrm{I}_1 &= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\int_0^1(1-t)^{\sigma n+\zeta-1}\Lambda'_{*}(\alpha t+(1-t)\psi, l)dt \\
&= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n(1-t)^{\sigma n+\zeta-1}\frac{\Lambda_{*}(\alpha t+(1-t)\psi, l)}{\alpha-\psi}|_0^1 + \frac{\sigma n+\zeta-1}{\alpha-\psi},
\end{aligned}$$

$$\begin{aligned}
&\int_0^1(1-t)^{\sigma n+\zeta-1}\Lambda'_{*}(\alpha t+(1-t)\psi, l)dt \\
&= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big[\frac{\Lambda_{*}(\psi, l)}{\psi-\alpha} - \frac{\sigma n+\zeta-1}{(\alpha-\psi)^2}\int_{\alpha}^{\psi}(\frac{x-\alpha}{\psi-\alpha})^{\zeta+\sigma n-2}\Lambda'_{*}(x)dx\Big] \\
&= \frac{\Lambda_{*}(\psi)}{\psi-\alpha}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma}) - \frac{\sigma n+\zeta-1}{(\psi-\alpha)^2}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\alpha^{+}}\Lambda_{*}\Big)(\psi, l).
\end{aligned}$$

Similarly, we get

$$\mathrm{I}_2 = \frac{\Lambda_{*}(\alpha, l)}{\psi-\alpha}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma}) - \frac{\sigma n+\zeta-1}{(\psi-\alpha)^2}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\psi^{-}}\Lambda_{*}\Big)(\alpha, l).$$

By adding both integrals I_1 and I_2, we have

$$\frac{\Lambda_{*}(\alpha, l)+\Lambda_{*}(\psi, l)}{\psi-\alpha}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma}) - \frac{1}{(\psi-\alpha)^2}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\alpha^{+}}\Lambda_{*}\Big)(\psi, l) + \Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,\varkappa;\psi^{-}}\Lambda_{*}\Big)(\alpha, l)\Big] = \mathrm{I}.$$

Multiplying by $\frac{\psi-\alpha}{2}$, we get the required results

$$\frac{\Lambda_{*}(\alpha, l)+\Lambda_{*}(\psi, l)}{2}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\mu}, p) - \frac{1}{2(\psi-\alpha)}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,\varkappa;\alpha^{+}}\Lambda_{*}\Big)(\psi, l) + \Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,\varkappa;\psi^{-}}\Lambda_{*}\Big)(\alpha, l)\Big] = \frac{\psi-\alpha}{2}\mathrm{I}.$$

Now, for the upper FIV-function Λ^*, we obtain

$$\frac{\Lambda^*(\alpha,l)+\Lambda^*(\psi,l)}{2}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\mu},p)-\frac{1}{2(\psi-\alpha)}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,\varkappa;\alpha^+}\Lambda^*\Big)(\psi,l)+\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,\varkappa;\psi^-}\Lambda^*\Big)(\alpha,l)\Big]=\frac{\psi-\alpha}{2}\mathrm{I}.$$

Combining the equations for lower and upper FIV-functions, we have

$$\frac{\Lambda(\alpha)\widetilde{+}\Lambda(\psi)}{2}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\mu},p)-\frac{1}{2(\psi-\alpha)}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,\varkappa;\alpha^+}\Lambda\Big)(\psi)\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,\varkappa;\psi^-}\Lambda\Big)(\alpha)\Big]=\frac{\psi-\alpha}{2}\mathrm{I},$$

which completes the proof. □

Theorem 4.2. *Let $\Lambda:\mathrm{I}=[\alpha,\psi]\to\mathbb{R},\ \mathrm{I}\subset\mathbb{R}$ be a differentiable function on (α,ψ). Moreover supposing that $|\Lambda'|$ is a convex FIV-function on I, then the following inequality holds:*

$$\begin{aligned}&\Big|\frac{\Lambda(\alpha)\widetilde{+}\Lambda(\psi)}{2}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma})-\frac{1}{2(\psi-\alpha)}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\alpha^+}\Lambda\Big)(\psi)\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\psi^-}\Lambda\Big)(\alpha)\Big]\Big|\\&\preccurlyeq\frac{\psi-\alpha}{2}\Big|\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p}(\varkappa(1)^{\sigma})-\frac{1}{2^{\psi'}}\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p}(\varkappa(\frac{1}{2})^{\sigma})\Big|[|\Lambda'(\alpha)\widetilde{+}\Lambda'(\psi)|].\end{aligned}$$

Proof. From Lemma 4.1 and using the properties of modulus, we have

$$\begin{aligned}&\Big|\frac{\Lambda(\alpha)\widetilde{+}\Lambda(\psi)}{2}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma})-\frac{1}{2(\psi-\alpha)}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\alpha^+}\Lambda\Big)(\psi)\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p,\varkappa;\psi^-}\Lambda\Big)(\alpha)\Big]\Big|\\&=\Big|\frac{\psi-\alpha}{2}\mathrm{I}\Big|\\&\preccurlyeq\frac{\psi-\alpha}{2}\sum_{n=0}^{\infty}\Big|\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big|\int_0^1|(1-t)^{\sigma n+\zeta-1}-t^{\sigma n+\zeta-1}||\Lambda'(\alpha t+(1-t)\psi)|dt\\&\preccurlyeq\frac{\psi-\alpha}{2}\sum_{n=0}^{\infty}\Big|\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big|\int_0^1|(1-t)^{\sigma n+\zeta-1}-t^{\sigma n+\zeta-1}|[|\Lambda'(\alpha)|+(1-t)|\Lambda'(\psi)|]dt\\&=\frac{\psi-\alpha}{2}\sum_{n=0}^{\infty}\Big|\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big|\int_0^{\frac{1}{2}}((1-t)^{\sigma n+\zeta-1}-t^{\sigma n+\zeta-1})[|\Lambda'(\alpha)|+(1-t)|\Lambda'(\psi)|]dt\\&\quad+\int_{\frac{1}{2}}^1((t)^{\sigma n+\zeta-1}-(1-t)^{\sigma n+\zeta-1})[|\Lambda'(\alpha)|+(1-t)|\Lambda'(\psi)|]dt.\end{aligned}$$

For every $l\in[0,1]$, we get

$$\begin{aligned}&\Big|\frac{\Lambda(\alpha)\widetilde{+}\Lambda(\psi)}{2}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma})-\frac{1}{2(\psi-\alpha)}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\alpha^+}\Lambda\Big)(\psi)\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\psi^-}\Lambda\Big)(\alpha)\Big]\Big|\\&\preccurlyeq\frac{\psi-\alpha}{2}\sum_{n=0}^{\infty}\Big|\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big|\int_0^{\frac{1}{2}}((1-t)^{\sigma n+\zeta-1}-t^{\sigma n+\zeta-1})[|\Lambda_*'(\alpha,l)|+(1-t)|\Lambda_*'(\psi,l)|]dt\\&\quad+\int_{\frac{1}{2}}^1((t)^{\sigma n+\zeta-1}-(1-t)^{\sigma n+\zeta-1})[|\Lambda_*'(\alpha,l)|+(1-t)|\Lambda_*'(\psi,l)|]dt.\end{aligned}$$

After integrating by parts, we obtain

$$\begin{aligned}&\Big|\frac{\Lambda(\alpha)\widetilde{+}\Lambda(\psi)}{2}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma})-\frac{1}{2(\psi-\alpha)}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\alpha^+}\Lambda\Big)(\psi)\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\psi^-}\Lambda\Big)(\alpha)\Big]\Big|\\&\preccurlyeq\frac{\psi-\alpha}{2}\Big|\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p}(\varkappa(1)^{\sigma})-\frac{1}{2^{\psi'}}\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p}(\varkappa(\frac{1}{2})^{\sigma})\Big|[|\Lambda_*'(\alpha,l)\widetilde{+}\Lambda_*'(\psi,l)|].\end{aligned}$$

Similarly, for upper FIV-functions, we have

$$\Big|\frac{\Lambda(\alpha)\widetilde{+}\Lambda(\psi)}{2}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma})-\frac{1}{2(\psi-\alpha)}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\alpha^{+}}\Lambda\Big)(\psi)\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\psi^{-}}\Lambda\Big)(\alpha)\Big]\Big|$$
$$\preccurlyeq\frac{\psi-\alpha}{2}\Big|\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p}(\varkappa(1)^{\sigma})-\frac{1}{2^{\psi'}}\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p}(\varkappa(\frac{1}{2})^{\sigma})\Big|[|\Lambda^{*'}(\alpha,l)\widetilde{+}\Lambda^{*'}(\psi,l)|].$$

Combining the inequalities for lower and upper FIV-functions, we get

$$\Big|\frac{\Lambda(\alpha)\widetilde{+}\Lambda(\psi)}{2}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma})-\frac{1}{2(\psi-\alpha)}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\alpha^{+}}\Lambda\Big)(\psi)\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\psi^{-}}\Lambda\Big)(\alpha)\Big]\Big|$$
$$\preccurlyeq\frac{\psi-\alpha}{2}\Big|\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p}(\varkappa(1)^{\sigma})-\frac{1}{2^{\psi'}}\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p}(\varkappa(\frac{1}{2})^{\sigma})\Big|[|\Lambda'(\alpha)\widetilde{+}\Lambda'(\psi)|],$$

which completes the proof. □

5. Behavior of Fejér-type fuzzy inequalities for (η_1,η_2)-convex FIV-function

Here, we discuss the Fejér type inequalities for (η_1,η_2)-convex FIV-function by using the GFFO.

Theorem 5.1. *Let $\Lambda : I \to \mathbb{R}$ be an (η_1,η_2)-convex FIV-function in which $\eta_2 \in \Lambda(I)\times\Lambda(I)$ and for each value of $\psi,\alpha \in I$, $\eta_1(\psi,\alpha)$ with $\Lambda \in L_1[\alpha,\alpha+\eta_1(\psi,\alpha)]$, and the function $\phi : [\alpha,\alpha+\eta_1(\psi,\alpha)] \to \mathbb{R}^{+}$ is integrable, and invariant to $\alpha+\frac{1}{2}\eta_1(\psi,\alpha)$, i.e., $\phi(2\alpha+\eta_1(\psi,\alpha)-x)=\phi(x)$, where $I\subseteq\mathbb{R}$ is to be considered an invex set relative to η_1, such that*

$$\eta_1(t_2\eta_1(\psi,\alpha)+\psi,t_1\eta_1(\psi,\alpha)+\psi)=(-t_1+t_2)\eta_1(\psi,\alpha), \tag{5.1}$$

for all $t_1,t_2\in[0,1]$. Then, the following Fejér type inequality holds:

$$\Lambda\Big(\frac{2\alpha\widetilde{+}\eta_1(\alpha,\psi)}{2}\Big)\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^{-}}\phi\Big)(\alpha)\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^{+}}\phi\Big)(\alpha+\eta_1(\psi,\alpha))\Big]$$
$$-\frac{1}{2}\int_{\alpha}^{\alpha+\eta_1(\psi,\alpha)}\Big[(\alpha+\eta_1(\psi,\alpha)-x)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(\alpha+\eta_1(\psi,\alpha)-x)^{\sigma})$$
$$+(x-\alpha)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(x-\alpha)^{\sigma})\Big]\times\eta_2(\Lambda(x),\Lambda(2\alpha+\eta_1(\psi,\alpha)-x))\phi(x)dx$$
$$\preccurlyeq\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^{-}}\Lambda\phi(\alpha)\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^{+}}\Lambda\phi\Big)(\alpha+\eta_1(\psi,\alpha)).$$

Proof. Considering the (η_1,η_2)-convex function and using the equation (5.1), we have

$$\Lambda\Big(\frac{2\alpha+\eta_1(\psi,\alpha)}{2}\Big)=\Lambda\Big(\frac{2\alpha+(1+t)\eta_1(\psi,\alpha)}{2}-\frac{t\eta_1(\psi,\alpha)}{2}\Big).$$

For every $\ell\in[0,1]$, using the (η_1,η_2)-convex fuzzy interval valued function and the equation (5.1), we get

$$\begin{aligned}\Lambda_*\Big(\frac{2\alpha+\eta_1(\psi,\alpha)}{2},\ell\Big)&=\Lambda_*\Big(\frac{2\alpha+(1+t)\eta_1(\psi,\alpha)}{2}-\frac{t\eta_1(\psi,\alpha)}{2},\ell\Big)\\&=\Lambda_*\Bigg(\frac{2\alpha+(1+t)\eta_1(\psi,\alpha)}{2}+\frac{1}{2}\eta_1\Big(\alpha+\frac{(1-t)}{2}\eta_1(\psi,\alpha),\alpha+\frac{(1-t)}{2}\eta_1(\psi,\alpha)\Big),\ell\Bigg),\\\Lambda_*\Big(\frac{2\alpha+\eta_1(\psi,\alpha)}{2},\ell\Big)&\\\preccurlyeq\Lambda_*\Big(\frac{2\alpha+(1+t)\eta_1(\psi,\alpha)}{2},\ell\Big)&+\frac{1}{2}\eta_2\Big(\Lambda_*(\frac{2\alpha+(1-t)\eta_1(\psi,\alpha)}{2},\ell),\Lambda_*(\frac{2\alpha+(1+t)\eta_1(\psi,\alpha)}{2},\ell)\Big).\end{aligned}\tag{5.2}$$

Now by using the same technique as above, we obtain

$$\begin{aligned}\Lambda_*\Big(\frac{2\alpha+\eta_1(\psi,\alpha)}{2},\ell\Big) \preccurlyeq \Lambda_*\Big(\frac{2\alpha+(1-t)\eta_1(\psi,\alpha)}{2},\ell\Big) \\ +\frac{1}{2}\eta_2\Big(\Lambda_*(\frac{2\alpha+(1+t)\eta_1(\psi,\alpha)}{2},\ell),\Lambda_*(\frac{2\alpha+(1-t)\eta_1(\psi,\alpha)}{2},\ell)\Big).\end{aligned} \tag{5.3}$$

Considering the generalized fractional integral operators defined in (2.14), we have

$$\begin{aligned}I_1 &= \Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\alpha,\psi))^-}\Lambda_*\Phi\Big)(\alpha,\ell)\\ &= \int_\alpha^{\alpha+\eta_1(\alpha,\psi)}[(x-\alpha)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(x-\alpha)^\sigma)\Lambda_*(x,\ell)\Phi(x)dx\\ &= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\int_\alpha^{\alpha+\eta_1(\alpha,\psi)}(x-\alpha)^{\sigma n+\zeta-1}\Lambda_*(x,\ell)\Phi(x)dx.\\ I_1 &= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big[\int_\alpha^{\alpha+\frac{1}{2}\eta_1(\alpha,\psi)}(x-\alpha)^{\sigma n+\zeta-1}\Lambda_*(x,\ell)\Phi(x)dx\\ &\quad+\int_{\alpha+\frac{1}{2}\eta_1(\alpha,\psi)}^{\alpha+\eta_1(\alpha,\psi)}(x-\alpha)^{\sigma n+\zeta-1}\Lambda_*(x,\ell)\Phi(x)dx\Big].\end{aligned}$$

By substituting $x=\frac{2\alpha+(1-t)\eta_1(\psi,\alpha)}{2}$ and $x=\frac{2\alpha+(1+t)\eta_1(\psi,\alpha)}{2}$ in first and second integral, respectively, we get

$$\begin{aligned}I_1 &= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n(\frac{\eta_1(\psi,\alpha)}{2})^{\sigma n+\zeta}\\ &\quad\times\Big[\int_0^1(1-t)^{\sigma n+\zeta-1}\Lambda_*\Big(\frac{2\alpha+(1-t)\eta_1(\psi,\alpha)}{2},\ell\Big)\Phi\Big(\frac{2\alpha+(1-t)\eta_1(\psi,\alpha)}{2}\Big)dt\\ &\quad+\int_0^1(1+t)^{\sigma n+\zeta-1}\Lambda_*\Big(\frac{2\alpha+(1+t)\eta_1(\psi,\alpha)}{2},\ell\Big)\Phi\Big(\frac{2\alpha+(1-t)\eta_1(\psi,\alpha)}{2}\Big)dx\Big].\end{aligned}$$

By using the inequalities (5.2) and (5.3), we obtain

$$\begin{aligned}I_1 &\geqslant \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big(\frac{\eta_1(\psi,\alpha)}{2}\Big)^{\sigma n+\zeta}\Big[\int_0^1(1-t)^{\sigma n+\zeta-1}\Lambda_*\Big(\frac{2\alpha+\eta_1(\psi,\alpha)}{2},\ell\Big)\\ &\quad\times\Phi\Big(\frac{2\alpha+(1-t)\eta_1(\psi,\alpha)}{2}\Big)dt-\frac{1}{2}\int_0^1(1-t)^{\sigma n+\zeta-1}\eta_2\\ &\quad\times\Big(\Lambda_*(\frac{2\alpha+(1+t)\eta_1(\psi,\alpha)}{2},\ell),\Lambda_*(\frac{2\alpha+(1-t)\eta_1(\psi,\alpha)}{2},\ell)\Big)\Phi\Big(\frac{2\alpha+(1-t)\eta_1(\psi,\alpha)}{2}\Big)dt\\ &\quad+\int_0^1(1+t)^{\sigma n+\zeta-1}\Lambda_*\Big(\frac{2\alpha+\eta_1(\psi,\alpha)}{2},\ell),\Phi(\frac{2\alpha+(1+t)\eta_1(\psi,\alpha)}{2}\Big)dt-\frac{1}{2}\int_0^1(1+t)^{\sigma n+\zeta-1}\eta_2\\ &\quad\times\Big(\Lambda_*(\frac{2\alpha+(1-t)\eta_1(\psi,\alpha)}{2},\ell),\Lambda_*(\frac{2\alpha+(1+t)\eta_1(\psi,\alpha)}{2},\ell)\Big)\Phi\Big(\frac{2\alpha+(1-t)\eta_1(\psi,\alpha)}{2}\Big)dt\Big],\\ I_1 &\geqslant \Lambda_*\Big(\frac{2\alpha+\eta_1(\psi,\alpha)}{2},\ell\Big)\Big(\frac{\eta_1(\psi,\alpha)}{2}\Big)^{\sigma n+\zeta}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\\ &\quad\times\Big[\int_0^1(1-t)^{\sigma n+\zeta-1}\Phi\Big(\frac{2\alpha+(1-t)\eta_1(\psi,\alpha)}{2}\Big)dt-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\end{aligned}$$

$$\times \Big(\frac{\eta_1(\psi,\alpha)}{2}\Big)^{\sigma n+\zeta-1}\int_0^1 (1-t)^{\sigma n+\zeta-1}\eta_2\Big(\Lambda_*(\frac{2\alpha+(1+t)\eta_1(\psi,\alpha)}{2},\ell),\Lambda_*(\frac{2\alpha+(1-t)\eta_1(\psi,\alpha)}{2},\ell)\Big)$$
$$\times \Phi\Big(\frac{2\alpha+(1-t)\eta_1(\psi,\alpha)}{2}\Big)dt+\Lambda_*\Big(\frac{2\alpha+\eta_1(\psi,\alpha)}{2},\ell\Big)\Big(\frac{\eta_1(\psi,\alpha)}{2}\Big)^{\sigma n+\zeta}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n$$
$$\times \int_0^1 (1+t)^{\sigma n+\zeta-1}\Phi\Big(\frac{2\alpha+(1+t)\eta_1(\psi,\alpha)}{2}\Big)dt-\frac{1}{2}\Big(\frac{\eta_1(\psi,\alpha)}{2}\Big)^{\sigma n+\zeta}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n$$
$$\times \int_0^1 (1+t)^{\sigma n+\zeta-1}\eta_2\Big(\Lambda_*(\frac{2\alpha+(1-t)\eta_1(\psi,\alpha)}{2},\ell\Big),\Lambda_*\Big(\frac{2\alpha+(1+t)\eta_1(\psi,\alpha)}{2},\ell)\Big)$$
$$\times \Phi\Big(\frac{2\alpha+(1-t)\eta_1(\psi,\alpha)}{2}\Big)dt\Big].$$

Again by simplifying and using the above substitution as well, we have

$$\begin{aligned}I_1 \geqslant{}& \Lambda_*\Big(\frac{2\alpha+\eta_1(\psi,\alpha)}{2},\ell\Big)\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\\ &\times\int_\alpha^{\alpha+\eta_1(\psi,\alpha)}(x-\alpha)^{\sigma n+\zeta-1}\Phi(x)dx-\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\\ &\times\int_\alpha^{\alpha+\eta_1(\psi,\alpha)}(\alpha+\eta_1(\psi,\alpha)-x)^{\sigma n+\zeta-1}\eta_2(\Lambda_*(x,\ell),\Lambda_*(2\alpha+\eta_1(\psi,\alpha)-x,\ell))\Phi(x)dx.\end{aligned}$$

By solving this, we get

$$\begin{aligned}I_1 \geqslant{}& \Lambda_*\Big(\frac{2\alpha+\eta_1(\psi,\alpha)}{2},\ell\Big)\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Phi\Big)(\alpha)-\frac{1}{2}\int_\alpha^{\alpha+\eta_1(\psi,\alpha)}(\alpha+\eta_1(\psi,\alpha)-x)^{\zeta-1}\\ &\times\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(\alpha+\eta_1(\psi,\alpha)-x)^{\sigma})\times\eta_2(\Lambda_*(x,\ell),\Lambda_*(2\alpha+\eta_1(\psi,\alpha)-x,\ell))\Phi(x)dx.\end{aligned}$$

Now, consider

$$\begin{aligned}I_2 &= \Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda_*\Phi\Big)((\alpha+\eta_1(\psi,\alpha),p),\ell)\\ &=\int_\alpha^{\alpha+\eta_1(\psi,\alpha)}(\alpha+\eta_1(\psi,\alpha)-x)^{\zeta-1}\times\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(\alpha+\eta_1(\psi,\alpha)-x)^{\sigma}))\Lambda_*(x,\ell)\Phi(x)dx.\end{aligned}$$

Similarly, we have

$$\begin{aligned}I_2 \geqslant{}& \Lambda_*\Big(\frac{2\alpha+\eta_1(\psi,\alpha)}{2},\ell\Big)\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Phi\Big)(\alpha+\eta_1(\psi,\alpha))-\frac{1}{2}\int_\alpha^{\alpha+\eta_1(\psi,\alpha)}(x-\alpha)^{\zeta-1}\\ &\times\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(x-\alpha)^{\sigma}))\times\eta_2(\Lambda_*(x,\ell),\Lambda_*(2\alpha+\eta_1(\psi,\alpha)-x,\ell))\Phi(x)dx.\end{aligned}$$

By applying the generalized fractional integral operators I_1 and I_2, and adding, we get

$$\begin{aligned}&\Lambda_*\Big(\frac{2\alpha+\eta_1(\alpha,\psi)}{2},\ell\Big)\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\phi\Big)(\alpha)+\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\phi\Big)(\alpha+\eta_1(\psi,\alpha))\Big]\\ &\quad-\frac{1}{2}\int_\alpha^{\alpha+\eta_1(\psi,\alpha)}\Big[(\alpha+\eta_1(\psi,\alpha)-x)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(\alpha+\eta_1(\psi,\alpha)-x)^{\sigma})\\ &\quad+(x-\alpha)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(x-\alpha)^{\sigma})\Big]\eta_2(\Lambda_*(x,\ell),\Lambda_*(2\alpha+\eta_1(\psi,\alpha)-x,\ell)).\phi(x)dx\\ &\preccurlyeq\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda_*\phi\Big)(\alpha,\ell)+\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\Lambda_*\phi\Big)(\alpha+\eta_1(\psi,\alpha),\ell).\end{aligned}$$

Now, for upper FIV-function Λ^*, we obtain

$$\begin{aligned}&\Lambda^*\Big(\frac{2\alpha+\eta_1(\alpha,\psi)}{2},\ell\Big)\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\phi\Big)(\alpha)+\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\phi\Big)(\alpha+\eta_1(\psi,\alpha))\Big]\\&\quad-\frac{1}{2}\int_\alpha^{\alpha+\eta_1(\psi,\alpha)}\Big[(\alpha+\eta_1(\psi,\alpha)-x)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(\alpha+\eta_1(\psi,\alpha)-x)^\sigma)\\&\quad+(x-\alpha)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(x-\alpha)^\sigma)\Big]\eta_2(\Lambda^*(x,\ell),\Lambda^*(2\alpha+\eta_1(\psi,\alpha)-x,\ell)).\phi(x)dx\\&\quad\preccurlyeq\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda^*\phi\Big)(\alpha,\ell)+\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\Lambda^*\phi\Big)(\alpha+\eta_1(\psi,\alpha),\ell).\end{aligned}$$

Combining the equations of upper and lower FIV-functions, we have

$$\begin{aligned}&\Lambda\Big(\frac{2\alpha\widetilde{+}\eta_1(\alpha,\psi)}{2}\Big)\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\phi\Big)(\alpha)\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\phi\Big)(\alpha+\eta_1(\psi,\alpha))\Big]\\&\quad-\frac{1}{2}\int_\alpha^{\alpha+\eta_1(\psi,\alpha)}\Big[(\alpha+\eta_1(\psi,\alpha)-x)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(\alpha+\eta_1(\psi,\alpha)-x)^\sigma)\\&\quad+(x-\alpha)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(x-\alpha)^\sigma)\Big]\times\eta_2(\Lambda(x),\Lambda(2\alpha+\eta_1(\psi,\alpha)-x)).\phi(x)dx\\&\quad\preccurlyeq\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda\phi\Big)(\alpha)\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\Lambda\phi\Big)(\alpha+\eta_1(\psi,\alpha)),\end{aligned}$$

which completes the proof. □

Lemma 5.2. *Let* I *be an invex subset of* $\mathbb{R}$ *related to* $\eta_1 : I\times I\to\mathbb{R}$. *Consider* $\alpha,\psi\in I$ *satisfying* $0<\eta_1(\alpha,\psi)$, *and let* Φ :$[\alpha,\alpha+\eta_1(\psi,\alpha)]\to\mathbb{R}$ *be integrable and symmetric about* $\frac{1}{2}\eta_1(\psi,\alpha)+\alpha$. *Then for the integral defined in* (2.14), *we have*

$$\begin{aligned}&\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda\Big)(\alpha+\eta_1(\psi,\alpha))\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda\Big)(\alpha)\\&\quad=\frac{1}{2}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda\Big)(\alpha+\eta_1(\psi,\alpha))\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda\Big)(\alpha)\Big].\end{aligned}$$

Proof. Let us consider,

$$\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda\Big)(\alpha+\eta_1(\psi,\alpha))=\int_\alpha^{\alpha+\eta_1(\psi,\alpha)}(\alpha+\eta_1(\psi,\alpha)-x)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(\alpha+\eta_1(\psi,\alpha)-x)^\sigma)\Lambda(x)dx.$$

For every $\ell\in[0,1]$, we have

$$\begin{aligned}&\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda_*\Big)(\alpha+\eta_1(\psi,\alpha),\ell)\\&\quad=\int_\alpha^{\alpha+\eta_1(\psi,\alpha)}(\alpha+\eta_1(\psi,\alpha)-x)^{\zeta-1}\times\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(\alpha+\eta_1(\psi,\alpha)-x)^\sigma)\Lambda_*(x,\ell)dx.\end{aligned}\tag{5.4}$$

Substituting $x=2\alpha+\eta_1(\psi,\alpha)-t$ in equation (5.4), we get

$$\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda_*\Big)(\alpha+\eta_1(\psi,\alpha),\ell)=\int_\alpha^{\alpha+\eta_1(\psi,\alpha)}(t-\alpha)^{\zeta-1}\times\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(t-\alpha)^\sigma)\Lambda_*(2\alpha+\eta_1(\psi,\alpha)-t,\ell)dt.$$

Hence,

$$\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda_*\Big)(\alpha+\eta_1(\psi,\alpha),\ell)=\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda_*\Big)(\alpha,\ell).\tag{5.5}$$

Adding $\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda_*\Big)(\alpha,\ell)$ in (5.5) on both sides, we obtain

$$\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda_*\Big)(\alpha+\eta_1(\psi,\alpha),\ell)+\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda_*\Big)(\alpha,\ell)$$

$$= \frac{1}{2}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda_*\Big)(\alpha+\eta_1(\psi,\alpha),\ell) + \Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda_*\Big)(\alpha,\ell)\Big].$$

Now for upper FIV-function, we have

$$\begin{aligned}&\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda^*\Big)(\alpha+\eta_1(\psi,\alpha),\ell) + \Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda^*\Big)(\alpha,\ell)\\ &\quad= \frac{1}{2}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda^*\Big)(\alpha+\eta_1(\psi,\alpha),\ell) + \Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda_*\Big)(\alpha,\ell)\Big].\end{aligned}$$

Combining upper and lower FIV-function, we get

$$\begin{aligned}&\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda\Big)(\alpha+\eta_1(\psi,\alpha))\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda\Big)(\alpha)\\ &\quad= \frac{1}{2}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda\Big)(\alpha+\eta_1(\psi,\alpha))\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda\Big)(\alpha)\Big],\end{aligned}$$

which completes the proof. □

Theorem 5.3. *Let $\Lambda : I \to \mathbb{R}$ be an (η_1,η_2)-convex FIV-function, η_2 be an integrable bifunction on $\Lambda(I)\times\Lambda(I)$ and for each $\psi,\alpha \in I$, $\eta_1(\psi,\alpha)$ with $\Lambda \in L_1\ [\alpha,\alpha+\eta_1(\psi,\alpha)]$ and the function $\phi : [\alpha,\alpha+\eta_1(\psi,\alpha)] \to \mathbb{R}^+$ be integrable and symmetric with $\alpha+\frac{1}{2}\eta_1(\psi,\alpha)$ such that $\phi(2\alpha+\eta_1(\psi,\alpha)-x) = \phi(x)$, where $I \subseteq\mathbb{R}$ is an invex convex set with respect to η_1. Then, the following generalized fractional integrals defined in (2.14), holds:*

$$\begin{aligned}&\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda\Big)(\alpha)\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda\Big)(\alpha+\eta_1(\psi,\alpha))\\ &\quad\preccurlyeq\left(\frac{(2\Lambda(\alpha)\widetilde{+}\eta_2(\Lambda(\psi)\Lambda(\alpha))}{2}\right)\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda\Big)(\alpha+\eta_1(\psi,\alpha))\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda\Big)(\alpha,)\Big].\end{aligned}$$

Proof. Consider the integral

$$\begin{aligned}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda\Phi\Big)(\alpha) &= \int_\alpha^{\alpha+\eta_1(\psi,\alpha)}(x-\alpha)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(x-\alpha)^\sigma)\Lambda(x)\Phi(x)dx\\ &= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\int_\alpha^{\alpha+\eta_1(\psi,\alpha)}(x-\alpha)^{\sigma n+\zeta-1}\Lambda(x)\Phi(x)dx.\end{aligned}$$

For every $\ell \in [0,1]$, solving for lower FIV-function, we have

$$\begin{aligned}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda_*\Phi\Big)(\alpha,\ell) &= \int_\alpha^{\alpha+\eta_1(\psi,\alpha)}(x-\alpha)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(x-\alpha)^\sigma)\Lambda_*(x,\ell)\Phi(x)dx\\ &= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\int_\alpha^{\alpha+\eta_1(\psi,\alpha)}(x-\alpha)^{\sigma n+\zeta-1}\Lambda_*(x,\ell)\Phi(x)dx.\end{aligned}$$

By substituting $x = t\eta_1(\psi,\alpha)+\alpha$, we get

$$\begin{aligned}&\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda_*\Phi\Big)(\alpha,\ell)\\ &\quad= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n(\eta_1(\psi,\alpha))^{\sigma n+\zeta}\int_0^1(t)^{\sigma n+\zeta-1}\Lambda_*(\alpha+t\eta_1(\psi,\alpha))\Phi(\alpha+t\eta_1(\psi,\alpha))dt.\end{aligned}$$

By using (η_1,η_2)-convexity of Λ_*, we obtain

$$\preccurlyeq \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n(\eta_1(\psi,\alpha))^{\sigma n+\zeta}\int_0^1(t)^{\sigma n+\zeta-1}(\Lambda_*(\alpha,\ell)+t\eta_2\Lambda_*(\psi,\ell),\Lambda_{*(\alpha))})\Phi(\alpha+t\eta_1(\psi,\alpha))dt$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}} (\varkappa)^n (\eta_1(\psi,\alpha))^{\sigma n+\zeta} \Big[\Lambda_*(\alpha,\ell) \int_0^1 (t)^{\sigma n+\zeta-1} \Phi(\alpha+t\eta_1(\psi,\alpha))dt \\
&\quad + \eta_2(\Lambda_*(\psi,\ell),\Lambda_*(\alpha,\ell)) \int_0^1 (t)^{\sigma n+\zeta} \Phi(\alpha+t\eta_1(\psi,\alpha))dt\Big].
\end{aligned} \tag{5.6}
$$

Now, consider the GFFO, we have

$$
\begin{aligned}
&\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\Lambda_*\Phi\right)((\alpha+\eta_1(\psi,\alpha),\ell)) \\
&\quad = \int_{\alpha}^{\alpha+\eta_1(\psi,\alpha)} (\alpha+\eta_1(\psi,\alpha)-x)^{\zeta-1} \xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(\alpha+\eta_1(\psi,\alpha)-x)^{\sigma})\Lambda_*(x,\ell)\Phi(x)dx \\
&\quad = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}} (\varkappa)^n \int_{\alpha}^{\alpha+\eta_1(\psi,\alpha)} (\alpha+\eta_1(\psi,\alpha)-x)^{\sigma n+\zeta-1}\Lambda_*(x,\ell)\Phi(x)dx.
\end{aligned}
$$

By substituting $x = \alpha+(1-t)\eta_1(\psi,\alpha)$, we get

$$
\begin{aligned}
&\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\Lambda_*\Phi\right)(\alpha+\eta_1(\psi,\alpha),\ell) \\
&= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}} (\varkappa)^n \Big(\eta_1(\psi,\alpha)\Big)^{\sigma n+\zeta} \int_0^1 (t)^{\sigma n+\zeta-1}\Lambda_*(\alpha+(1-t)\eta_1(\psi,\alpha),\ell)\Phi(\alpha+(1-t)\eta_1(\psi,\alpha))dt.
\end{aligned}
$$

Using (η_1,η_2)-convexity of Λ_*, we obtain

$$
\begin{aligned}
&\preccurlyeq \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}} (\varkappa)^n (\eta_1(\psi,\alpha))^{\sigma n+\zeta} \\
&\quad \times \int_0^1 (t)^{\sigma n+\zeta-1} (\Lambda(\alpha)+(1-t)\eta_2\Lambda_*(\psi,\ell),\Lambda_*(\alpha,\ell)))\Phi(\alpha+t\eta_1(\psi,\alpha))dt \\
&= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}} (\varkappa)^n (\eta_1(\psi,\alpha))^{\sigma n+\zeta} \Big[\Lambda_*(\alpha,\ell) \int_0^1 (t)^{\sigma n+\zeta-1}\Phi(\alpha+(1-t)\eta_1(\psi,\alpha))dt \\
&\quad + \eta_2(\Lambda_*(\psi,\ell),\Lambda_*(\alpha,\ell)) \int_0^1 (t)^{\sigma n+\zeta-1}(1-t)\Phi(\alpha+(1-t)\eta_1(\psi,\alpha))dt\Big].
\end{aligned} \tag{5.7}
$$

By adding (5.6) and (5.7), we have

$$
\begin{aligned}
&\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\Lambda_*\Phi\right)(\alpha+\eta_1(\psi,\alpha),\ell) + \left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda_*\Phi\right)(\alpha,\ell) \\
&\quad \preccurlyeq \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}} (\varkappa)^n (\eta_1(\psi,\alpha))^{\sigma n+\zeta} \Big[2\Lambda_*(\alpha,\ell) \int_0^1 t^{\sigma n+\zeta-1}\Phi(\alpha+t\eta_1(\psi,\alpha))\, dt \\
&\qquad + \eta_2(\Lambda_*(\psi,\ell),\Lambda_*(\alpha,\ell)) \int_0^1 t^{\sigma n+\zeta}\Phi(\alpha+t\eta_1(\psi,\alpha))\, dt\Big] \\
&\quad = \Big(2\Lambda_*(\alpha,\ell)+\eta_2(\Lambda_*(\psi,\ell),\Lambda_*(\alpha,\ell))\Big) \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}} (\varkappa)^n (\eta_1(\psi,\alpha))^{\sigma n+\zeta} \\
&\qquad \times \int_0^1 t^{\sigma n+\zeta}\Phi(\alpha+t\eta_1(\psi,\alpha))\, dt \\
&\quad = \left(\frac{2\Lambda_*(\alpha,\ell)+\eta_2(\Lambda_*(\psi,\ell),\Lambda_*(\alpha,\ell))}{2}\right) \left(2\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda_*\Phi\right)(\alpha,\ell).
\end{aligned}
$$

By using Lemma 5.2, we get

$$\begin{aligned}&\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda_*\right)(\alpha,\ell)+\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda_*\right)(\alpha+\eta_1(\psi,\alpha),\ell)\\&\quad\preccurlyeq\left(\frac{(2\Lambda_*(\alpha,\ell)+\eta_2(\Lambda_*(\psi,\ell)\Lambda_*(\alpha,\ell))}{2}\right)\\&\quad\times\left[\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda_*\right)(\alpha+\eta_1(\psi,\alpha),\ell)+\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda_*\right)(\alpha,\ell)\right].\end{aligned}\tag{5.8}$$

Similarly, for upper FIV-function, we obtain

$$\begin{aligned}&\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda^*\right)(\alpha,\ell)+\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda^*\right)(\alpha+\eta_1(\psi,\alpha),\ell)\\&\quad\preccurlyeq\left(\frac{(2\Lambda^*(\alpha,\ell)+\eta_2(\Lambda^*(\psi,\ell)\Lambda^*(\alpha,\ell))}{2}\right)\\&\quad\times\left[\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda^*\right)(\alpha+\eta_1(\psi,\alpha),\ell)+\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda^*\right)(\alpha,\ell)\right].\end{aligned}\tag{5.9}$$

Combining the equations (5.8) and (5.9), we have

$$\begin{aligned}&\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda\right)(\alpha)\widetilde{+}\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda\right)(\alpha+\eta_1(\psi,\alpha))\\&\quad\preccurlyeq\left(\frac{(2\Lambda(\alpha)\widetilde{+}\eta_2(\Lambda(\psi)\Lambda(\alpha))}{2}\right)\left[\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda\right)(\alpha+\eta_1(\psi,\alpha))\widetilde{+}\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda\right)(\alpha,)\right],\end{aligned}$$

which completes the proof. □

Corollary 5.4. *Replacing* $\eta_1(\alpha,\psi)=-\psi+\alpha$ *in Theorems 5.1 and 5.3, we obtain the following Fejér type inequalities:*

$$\begin{aligned}&\Lambda\left(\frac{\alpha+\psi}{2}\right)\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\Phi\right)(\psi)\widetilde{+}\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\psi)^-}\Phi\right)(\alpha)-\frac{1}{2}\int_\alpha^\psi\left[(\psi-x)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(\psi-x)^\sigma)\right.\\&\quad\left.+(\psi-x)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(\psi-x)^\sigma)\right]\eta_2(\Lambda(x),\Lambda(\alpha+\psi-x))\Phi(x)dx\\&\quad\preccurlyeq\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\psi)^-}\Lambda\Phi\right)(\alpha)\widetilde{+}\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda\Phi\right)(\psi)\\&\quad\preccurlyeq\left(\frac{(2\Lambda(\alpha)\widetilde{+}\eta_2(\Lambda(\psi),\Lambda(\alpha))}{2}\right)\left[\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda\right)(\psi)\widetilde{+}\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\psi)^-}\Lambda\right)(\alpha)\right].\end{aligned}$$

Corollary 5.5. *Replacing* $\eta_2(\alpha,\psi)=-\psi+\alpha$ *in Theorems 5.1 and 5.3, we get the following Fejér type inequalities:*

$$\begin{aligned}&\Lambda\left(\frac{2\alpha\widetilde{+}\eta_1(\psi,\alpha)}{2}\right)\left[\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;\alpha^+}\Phi\right)(\alpha+\eta_1(\psi,\alpha))\widetilde{+}\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Phi\right)(\alpha)\right]\\&\quad\preccurlyeq\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda\Phi\right)(\alpha)\widetilde{+}\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda\Phi\right)(\alpha+\eta_1(\psi,\alpha))\\&\quad\preccurlyeq\left(\frac{(\Lambda(\alpha)\widetilde{+}\Lambda(\psi))}{2}\right)\left[\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda\right)(\alpha+\eta_1(\psi,\alpha))\widetilde{+}\left(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda\right)(\alpha)\right].\end{aligned}$$

6. Trapezoid and mid-point type fuzzy inequalities

The mid-point and trapezoidal type inequalities related to Hermite-Hadamard inequalities are examined for (η_1,η_2)-convex FIV-function. Moreover, we extract some corollaries showing well-known inequalities in the literature.

Lemma 6.1. *Let* $\Lambda : \mathrm{I} \to \mathbb{R}$, $\mathrm{I} \subseteq \mathbb{R}, \Lambda' \in L_1[\alpha, \alpha+\eta_1(\psi,\alpha)]$ *be a differentiable FIV-function, and* I *be an open invex set with respect to* $\eta_1 : \mathrm{I} \times \mathrm{I} \to \mathbb{R}$ *with* $\eta_1(\psi,\alpha) > 0$ *for* $\psi, \alpha \in \mathrm{I}$. *Then, for generalized fractional integrals, we have*

$$\frac{\eta_1(\psi,\alpha)}{2}\sum_{k=1}^{4} \mathrm{I}_k = \Lambda\Big(\frac{2\alpha\widetilde{+}\eta_1(\psi,\alpha)}{2}\Big)\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^\sigma) - \frac{1}{2\eta_1(\psi,\alpha)} \times \Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda\Big)(\alpha+\eta_1(\psi,\alpha))\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda\Big)(\alpha)\Big],$$

where

$$\begin{aligned}
\mathrm{I}_1 &:= \int_0^{\frac12} (t)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(t)^\sigma)\Lambda'(\alpha+t\eta_1(\psi,\alpha))dt,\\
\mathrm{I}_2 &:= \int_0^{\frac12} -(t)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(t)^\sigma)\Lambda'(\alpha+(1-t)\eta_1(\psi,\alpha))dt,\\
\mathrm{I}_3 &:= \int_{\frac12}^1 (t)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(t)^\sigma)\Lambda'(\alpha+t\eta_1(\psi,\alpha))dt - \int_{\frac12}^1 \xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^\sigma)\Lambda'(\alpha+t\eta_1(\psi,\alpha))dt,\\
\mathrm{I}_4 &:= \int_{\frac12}^1 \xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^\mu))\Lambda'(\alpha+(1-t)\eta_1(\psi,\alpha))dt - \int_{\frac12}^1 (t)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(t)^\sigma)\Lambda'(\alpha+(1-t)\eta_1(\psi,\alpha))dt.
\end{aligned}$$

Proof. Consider the GFFO, we have

$$\begin{aligned}
\mathrm{I}_1 &= \int_0^{\frac12} (t)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(t)^\sigma)\Lambda'(\alpha+t\eta_1(\psi,\alpha))dt,\\
\mathrm{I}_1 &= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\int_0^{\frac12}(t)^{\sigma n+\zeta-1}\Lambda'(\alpha+t\eta_1(\psi,\alpha))dt.
\end{aligned}$$

For every $\ell \in [0,1]$, we have

$$\begin{aligned}
\mathrm{I}_1 &= \int_0^{\frac12} (t)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(t)^\sigma)\Lambda_*{}'(\alpha+t\eta_1(\psi,\alpha),\ell)dt,\\
\mathrm{I}_1 &= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\int_0^{\frac12}(t)^{\sigma n+\zeta-1}\Lambda_*{}'(\alpha+t\eta_1(\psi,\alpha),\ell)dt.
\end{aligned}$$

After simplification, we have

$$\begin{aligned}
\mathrm{I}_1 &= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big[(t)^{\sigma n+\zeta-1}\frac{\Lambda_*(\alpha+t\eta_1(\psi,\alpha),\ell)}{\eta_1(\psi,\alpha)}\Big|_0^{\frac12}\\
&\quad - \frac{\sigma n+\zeta-1}{\eta_1(\psi,\alpha)}\int_0^{\frac12}(t)^{\zeta+\sigma n-2}\Lambda_*(\alpha+t\eta_1(\psi,\alpha),\ell)dt\Big]\\
&= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big[\frac{(2)^{-(\sigma n+\zeta-1)}}{\eta_1(\psi,\alpha)}\Lambda_*\frac{(2\alpha+\eta_1(\psi,\alpha),\ell)}{2}\\
&\quad - \frac{\sigma n+\zeta-1}{\eta_1(\psi,\alpha)}\int_0^{\frac12}(t)^{\zeta+\sigma n-2}\Lambda_*(\alpha+t\eta_1(\psi,\alpha),\ell)dt\Big].
\end{aligned}$$

Applying the same manner for resolving I_2, we have

$$\mathrm{I}_2 = \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big[\frac{(2)^{-(\zeta+\sigma n-1)}}{\eta_1(\psi,\alpha)}\Lambda_*\frac{(2\alpha+\eta_1(\psi,\alpha),\ell)}{2}$$

$$-\frac{\sigma n+\zeta-1}{\eta_1(\psi,\alpha)}\int_0^{\frac{1}{2}}(t)^{\zeta+\sigma n-2}\Lambda_*(\alpha+(1-t)\eta_1(\psi,\alpha),\ell)dt\Big].$$

Now, for I_3, we get

$$I_3=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\int_{\frac{1}{2}}^1(t^{\zeta+\sigma n}-2){\Lambda_*}'(\alpha+t\eta_1(\psi,\alpha),\ell)dt.$$

Integrating by parts, we obtain

$$\begin{aligned}I_3=&\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big[(t^{\zeta+\sigma n}-2)\frac{\Lambda_*(\alpha+t\eta_1(\psi,\alpha),\ell)}{\eta_1(\psi,\alpha)}\Big|_{\frac{1}{2}}^1\\&-\frac{\sigma n+\zeta-1}{\eta_1(\psi,\alpha)}\int_{\frac{1}{2}}^1(t)^{\zeta+\sigma n-2}\Lambda_*(\alpha+t\eta_1(\psi,\alpha),\ell)dt\Big]\\=&\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big[\frac{1-(2)^{-(\sigma n+\zeta-1)}}{\eta_1(\psi,\alpha)}\Lambda_*\frac{(2\alpha+\eta_1(\psi,\alpha),\ell)}{2}\\&-\frac{\sigma n+\zeta-1}{\eta_1(\psi,\alpha)}\int_{\frac{1}{2}}^1(t)^{\zeta+\sigma n-2}\Lambda_*(\alpha+(1-t)\eta_1(\psi,\alpha),\ell)dt\Big].\end{aligned}$$

Similarly,

$$\begin{aligned}I_4=&\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big[\frac{1-(2)^{-(\sigma n)+\zeta-1}}{\eta_1(\psi,\alpha)}\Lambda_*\frac{(2\alpha+\eta_1(\psi,\alpha),\ell)}{2}\\&-\frac{\sigma n+\zeta-1}{\eta_1(\psi,\alpha)}\int_{\frac{1}{2}}^1(t)^{\zeta+\sigma n-2}\Lambda_*(\alpha+(1-t)\eta_1(\psi,\alpha),\ell)dt\Big].\end{aligned}$$

By adding I_1, I_2, I_3, and I_4, we have

$$\begin{aligned}\sum_{k=1}^{4}I_K=&\frac{2}{\eta_1(\psi,\alpha)}\Lambda_*\Big(\frac{2\alpha+\eta_1(\psi,\alpha),\ell}{2}\Big)\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n-\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\frac{\sigma n+\zeta-1}{\eta_1(\psi,\alpha)}\\&\times\Big[\Big[\int_0^1(t)^{\zeta+\sigma n-2}\Lambda_*(\alpha+(1-t)\eta_1(\psi,\alpha),\ell)dt+\int_0^1(t)^{\zeta+\sigma n-2}\Lambda_*(\alpha+(1-t)\eta_1(\psi,\alpha),\ell)dt\Big].\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\eta_1(\psi,\alpha)}{2}\sum_{k=1}^{4}I_k=&\Lambda_*\Big(\frac{2\alpha+\eta_1(\psi,\alpha),\ell}{2}\Big)\xi_{\sigma,\zeta,p}^{\gamma,\rho,q}(\varkappa(1)^{\sigma})-\frac{1}{2\eta_1(\psi,\alpha)}\\&\times\Big[\Big(\xi_{\sigma,\zeta,p,\varkappa;(\alpha)^+}^{\gamma,\rho,q}\Lambda_*\Big)(\alpha+\eta_1(\psi,\alpha),\ell)+\Big(\xi_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}^{\gamma,\rho,q}\Lambda_*\Big)(\alpha,\ell)\Big].\end{aligned}\tag{6.1}$$

For upper FIV-function, we obtain

$$\begin{aligned}\frac{\eta_1(\psi,\alpha)}{2}\sum_{k=1}^{4}I_K=&\Lambda^*\Big(\frac{2\alpha+\eta_1(\psi,\alpha),\ell}{2}\Big)\xi_{\sigma,\zeta,p}^{\gamma,\rho,q}(\varkappa(1)^{\sigma})-\frac{1}{2\eta_1(\psi,\alpha)}\\&\times\Big[\Big(\xi_{\sigma,\zeta,p,\varkappa;(\alpha)^+}^{\gamma,\rho,q}\Lambda^*\Big)(\alpha+\eta_1(\psi,\alpha),\ell)+\Big(\xi_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}^{\gamma,\rho,q}\Lambda^*\Big)(\alpha,\ell)\Big].\end{aligned}\tag{6.2}$$

Combining the equations (6.1) and (6.2), we have

$$\frac{\eta_1(\psi,\alpha)}{2}\sum_{k=1}^{4}I_K=\Lambda\Big(\frac{2\alpha\tilde{+}\eta_1(\psi,\alpha)}{2}\Big)\xi_{\sigma,\zeta,p}^{\gamma,\rho,q}(\varkappa(1)^{\sigma})-\frac{1}{2\eta_1(\psi,\alpha)}$$

$$\times\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha)^+}\Lambda\Big)(\alpha+\eta_1(\psi,\alpha))\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda\Big)(\alpha)\Big],$$

which completes the proof. □

Theorem 6.2. *Let $\Lambda : I \to \mathbb{R}$, $I \in \mathbb{R}$, $|\Lambda'| \in L_1[\alpha, \alpha+\eta_1(\psi,\alpha)]$, be a differentiable (η_1,η_2)-convex FIV-function, where I is assumed to be an open invex set related to $\eta_1 : I\times I \to \mathbb{R}$, where $\eta_1(\psi,\alpha) > 0$ for $\alpha,\psi \in I$. Then, for the generalized fractional integrals, we have*

$$\begin{aligned}&\Big|\Lambda(\frac{2\alpha\widetilde{+}\eta_1(\psi,\alpha)}{2})\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^\sigma)-\frac{1}{2\eta_1(\psi,\alpha)}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha)^+}\Lambda\Big)(\alpha+\eta_1(\psi,\alpha))\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda\Big)(\alpha)\Big]\Big|\\
&\preccurlyeq\sum_{n=0}^{\infty}\Big|\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big|\frac{\eta_1(\alpha,\psi)}{2^{\zeta+\sigma n}(\zeta+\sigma n)}[|\Lambda'(\alpha)|\widetilde{+}|\Lambda'(\psi)|+\frac{1}{2}\eta_2(|\Lambda'(\alpha)|,|\Lambda'(\psi)|)\widetilde{+}\frac{1}{2}\eta_2(|\Lambda'(\psi)|,|\Lambda'(\alpha)|)]\\
&\quad\times\frac{\eta_1(\psi,\alpha)}{2^{\psi+1}}\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p,\varkappa}(\varkappa(\frac{1}{2})^\sigma)[|\Lambda'(\alpha)|\widetilde{+}|\Lambda'(\psi)|\widetilde{+}\frac{1}{2}\eta_2(|\Lambda'(\alpha)|,|\Lambda'(\psi)|)\widetilde{+}\frac{1}{2}\eta_2(|\Lambda'(\psi)|,|\Lambda'(\alpha)|)].\end{aligned}$$

Proof. By using the properties of modulus and Lemma 6.1, we have

$$\begin{aligned}&\Big|\Lambda\Big(\frac{2\alpha\widetilde{+}\eta_1(\psi,\alpha)}{2}\Big)\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^\sigma)-\frac{1}{2\eta_1(\psi,\alpha)}\\
&\quad\times\Big[(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha)^+}\Lambda\Big)(\alpha+\eta_1(\psi,\alpha))\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda\Big)(\alpha)\Big]\Big|\preccurlyeq\frac{\eta_1(\psi,\alpha)}{2}\sum_{k=1}^{4}I_k.\end{aligned}$$

To solve $|I_k|$, $k = 1,2,3,4$, using (η_1,η_2)-convexity of $|\Lambda_*'|$ for every $\ell \in [0,1]$, we get

$$\begin{aligned}|I_1|&\preccurlyeq\int_0^{\frac12}t^{\zeta-1}|\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(t)^\sigma)||\Lambda_*{}'(\alpha+t\eta_1(\psi,\alpha),\ell)|dt\\
&\preccurlyeq\int_0^{\frac12}t^{\zeta-1}|\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(t)^\sigma)||\Lambda_*{}'(\alpha),\ell|dt+\int_0^{\frac12}(t)^{\zeta}|\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(t)^\sigma))|\eta_2(|\Lambda_*'(\psi,\ell)|+|\Lambda_*'(\alpha,\ell)|)dt\\
&\preccurlyeq\sum_{n=0}^{\infty}\Big|\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big|\Big[|\Lambda_*{}'(\alpha,\ell)|\frac{1}{2^{\zeta+\sigma n}(\zeta+\sigma n)}+\eta_2(|\Lambda_*'(\psi,\ell)|,|\Lambda_*'(\alpha,\ell)|)\frac{1}{2^{\zeta+\sigma n}(\zeta+\sigma n+1)}\Big].\end{aligned}$$

Continuing the same manner for I_2, we obtain

$$|I_2|\preccurlyeq\sum_{n=0}^{\infty}\Big|\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big|\Big[|\Lambda_*{}'(\psi,\ell)|\frac{1}{2^{\zeta+\sigma n}(\zeta+\sigma n)}+\eta_2(|\Lambda_*'(\alpha,\ell)|,|\Lambda_*'(\psi,\ell)|)\frac{1}{2^{\zeta+\sigma n}(\zeta+\sigma n+1)}\Big].$$

Let us calculate $|I_k|$ for $k = 3,4$. We apply the fact that for every $j \in (0,1]$ and $\alpha_1,\alpha_1 \in [0,1]$, we have $|\alpha_1^j,\alpha_2^j| \preccurlyeq |\alpha_1,\alpha_2|^j$. Then,

$$\begin{aligned}|I_3|\preccurlyeq\sum_{n=0}^{\infty}&\Big|\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big|\Big[|\Lambda_*{}'(\psi,\ell)|\frac{1}{2^{\zeta+\sigma n}(\zeta+\sigma n)}\\
&+\eta_2(|\Lambda_*'(\psi,\ell)|,|\Lambda_*'(\alpha,\ell)|)\frac{\zeta+\sigma n+2}{2^{\zeta+\sigma n+1}(\zeta+\sigma n)(\zeta+\sigma n+1)}\Big]\end{aligned}$$

and

$$\begin{aligned}|I_4|\preccurlyeq\sum_{n=0}^{\infty}&\Big|\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big|\Big[|\Lambda_*{}'(\psi,\ell)|\frac{1}{2^{\zeta+\sigma n}(\zeta+\sigma n)}\\
&+\eta_2(|\Lambda_*'(\alpha,\ell)|,|\Lambda_*'(\psi,\ell)|)\frac{\zeta+\sigma n+2}{2^{\zeta+\sigma n+1}(\zeta+\sigma n)(\zeta+\sigma n+1)}\Big].\end{aligned}$$

Hence,

$$\begin{aligned}
&\Big|\Lambda_*(\frac{2\alpha+\eta_1(\psi,\alpha)}{2},\ell)\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma})-\frac{1}{2\eta_1(\psi,\alpha)}\\
&\quad\times\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha)^+}\Lambda_*\Big)(\alpha+\eta_1(\psi,\alpha),\ell)+\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda_*\Big)(\alpha,\ell)\Big]\Big|\\
&\preccurlyeq\sum_{n=0}^{\infty}\Big|\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big|\frac{\eta_1(\alpha,\psi)}{2^{\zeta+\sigma n}(\zeta+\sigma n)}[|\Lambda_*'(\alpha,\ell)|+|\Lambda_*'(\psi,\ell)|\\
&\quad+\frac{1}{2}\eta_2(|\Lambda_*'(\alpha,\ell)|,|\Lambda_*'(\psi,\ell)|)+\frac{1}{2}\eta_2(|\Lambda_*'(\psi,\ell)|,|\Lambda_*'(\alpha,\ell)|)]\frac{\eta_1(\psi,\alpha)}{2^{\psi+1}}\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p,\varkappa}(\varkappa(\frac{1}{2})^{\sigma})\\
&\quad\times[|{\Lambda_*}'(\alpha,\ell)|+|{\Lambda_*}'(\psi,\ell)|+\frac{1}{2}\eta_2(|{\Lambda_*}'(\alpha,\ell)|,|{\Lambda_*}'(\psi,\ell)|)+\frac{1}{2}\eta_2(|{\Lambda_*}'(\psi,\ell)|,|\Lambda_*'(\alpha,\ell)|)].
\end{aligned}\tag{6.3}$$

Similarly for upper FIV-function, we have

$$\begin{aligned}
&\Big|\Lambda^*(\frac{2\alpha+\eta_1(\psi,\alpha)}{2},\ell)\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma})-\frac{1}{2\eta_1(\psi,\alpha)}\\
&\quad\times\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha)^+}\Lambda^*\Big)(\alpha+\eta_1(\psi,\alpha),\ell)+\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda^*\Big)(\alpha,\ell)\Big]\Big|\\
&\preccurlyeq\sum_{n=0}^{\infty}\Big|\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big|\frac{\eta_1(\alpha,\psi)}{2^{\zeta+\sigma n}(\zeta+\sigma n)}[|{\Lambda^*}'(\alpha,\ell)|+|{\Lambda^*}'(\psi,\ell)|\\
&\quad+\frac{1}{2}\eta_2(|{\Lambda^*}'(\alpha,\ell)|,|{\Lambda^*}'(\psi,\ell)|)+\frac{1}{2}\eta_2(|{\Lambda^*}'(\psi,\ell)|,|\Lambda_*'(\alpha,\ell)|)]\frac{\eta_1(\psi,\alpha)}{2^{\psi+1}}\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p,\varkappa}(\varkappa(\frac{1}{2})^{\sigma})\\
&\quad\times[|{\Lambda^*}'(\alpha,\ell)|+|{\Lambda^*}'(\psi,\ell)|+\frac{1}{2}\eta_2(|{\Lambda_*}'(\alpha,\ell)|,|{\Lambda^*}'(\psi,\ell)|)+\frac{1}{2}\eta_2(|{\Lambda^*}'(\psi,\ell)|,|{\Lambda^*}'(\alpha,\ell)|)].
\end{aligned}\tag{6.4}$$

Combining the equations (6.3) and (6.4), we have

$$\begin{aligned}
&\Big|\Lambda(\frac{2\alpha\tilde{+}\eta_1(\psi,\alpha)}{2})\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma})-\frac{1}{2\eta_1(\psi,\alpha)}\\
&\quad\times\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha)^+}\Lambda\Big)(\alpha+\eta_1(\psi,\alpha))\tilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda\Big)(\alpha)\Big]\Big|\\
&\preccurlyeq\sum_{n=0}^{\infty}\Big|\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big|\frac{\eta_1(\alpha,\psi)}{2^{\zeta+\sigma n}(\zeta+\sigma n)}[|\Lambda'(\alpha)|\tilde{+}|\Lambda'(\psi)|\\
&\quad+\frac{1}{2}\eta_2(|\Lambda'(\alpha)|,|\Lambda'(\psi)|)\tilde{+}\frac{1}{2}\eta_2(|\Lambda'(\psi)|,|\Lambda'(\alpha)|)]\frac{\eta_1(\psi,\alpha)}{2^{\psi+1}}\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p,\varkappa}(\varkappa(\frac{1}{2})^{\sigma})\\
&\quad\times[|\Lambda'(\alpha)|\tilde{+}|\Lambda'(\psi)|\tilde{+}\frac{1}{2}\eta_2(|\Lambda'(\alpha)|,|\Lambda'(\psi)|)\tilde{+}\frac{1}{2}\eta_2(|\Lambda'(\psi)|,|\Lambda'(\alpha)|)],
\end{aligned}$$

which completes the proof. □

Corollary 6.3. *In Theorem 6.2, if we replace* $\eta_1(\alpha,\psi)$ = -ψ+α *for all* α, $\psi \in I$, *we have*

$$\begin{aligned}
&\Big|\Lambda(\frac{\alpha\tilde{+}\psi}{2})\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma})-\frac{1}{2(\psi,\alpha)}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha)^+}\Lambda\Big)(\alpha+\eta_1(\psi,\alpha))\tilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda)(\alpha)\Big]\Big|\\
&\preccurlyeq\frac{\eta_1(\psi,\alpha)}{2^{\psi+1}}|\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p,\varkappa}(\varkappa(\frac{1}{2})^{\sigma}))|[|\Lambda'(\alpha)|\tilde{+}|\Lambda'(\psi)|+\frac{1}{2}\eta_2(|\Lambda'(\alpha)|,|\Lambda'(\psi)|)\tilde{+}\frac{1}{2}\eta_2(|\Lambda'(\psi)|,|\Lambda'(\alpha)|)].
\end{aligned}$$

Corollary 6.4. *In Theorem 6.2, if we take* $\eta_1(\alpha,\psi)=-\psi+\alpha$ *and* $\eta_2(x,y)=x-y$ *for each value of* $x,y\in\Lambda(I)$, *and* $\alpha,\psi\in I$, *we get*

$$\begin{aligned}
&\Big|\Lambda\Big(\frac{\alpha\tilde{+}\psi}{2}\Big)\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma}))-\frac{1}{2(\psi,\alpha)}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha)^+}\Lambda\Big)(\psi)\tilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\psi)^-}\Lambda\Big)(\alpha)\Big]\Big|\\
&\preccurlyeq\frac{\eta_1(\psi,\alpha)}{2^{\psi+1}}|\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p,\varkappa}(\varkappa(\frac{1}{2})^{\sigma}))|[|\Lambda'(\alpha)|\tilde{+}|\Lambda'(\psi)|].
\end{aligned}$$

Lemma 6.5. *Let a function $\Lambda : I \to \mathbb{R}$ in which $I \in \mathbb{R}, \Lambda' \in L_1[\alpha, \alpha\eta_1(\psi, \alpha)]$ be a differentiable FIV-function, where I is assumed to be an open convex set with respect to $\eta_1 : I \times I \to \mathbb{R}$ with $\eta_1(\psi, \alpha) > 0$ for $\psi, \alpha \in I$. Then, for the generalized fractional integral, we have*

$$\begin{aligned}&\Big(\frac{\Lambda(\alpha)\widetilde{+}\Lambda(\alpha+\eta_1(\psi,\alpha))}{2}\Big)\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^\sigma)) - \frac{1}{2\eta_1(\psi,\alpha)}\\ &\quad\times\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha)^+}\Lambda\Big)(\alpha+\eta_1(\psi,\alpha))\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda\Big)(\alpha)\Big] = \frac{\eta_1(\psi,\alpha)}{2}I,\end{aligned}$$

where

$$I := \int_0^1 t^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(t)^\sigma))\Lambda'(\alpha+t\eta_1(\psi,\alpha))dt + \int_0^1 -(1-t)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(t)^\sigma))\Lambda'(\alpha+(1-t)\eta_1(\psi,\alpha))dt.$$

Proof. Considering the fractional integral, we have

$$I = \int_0^1 t^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(t)^\sigma))\Lambda'(\alpha+t\eta_1(\psi,\alpha))dt + \int_0^1 -(1-t)^{\zeta-1}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(t)^\sigma))\Lambda'(\alpha+(1-t)\eta_1(\psi,\alpha))dt.$$

So, $I = I_1 + I_2$. Firstly, considering I_1 for every $\ell \in [0,1]$ and solving for lower FIV-function, we get

$$I_1 = \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\int_0^{\frac{1}{2}}(t)^{\sigma n+\zeta-1}{\Lambda_*}'(\alpha+t\eta_1(\psi,\alpha),\ell)dt.$$

Taking integration by parts on both sides, we obtain

$$\begin{aligned}I_1 &= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big[(t)^{\sigma n+\zeta-1}\frac{\Lambda_*(\alpha+t\eta_1(\psi,\alpha),\ell)}{\eta_1(\psi,\alpha)}\Big|_0^1\\ &\quad - \frac{\sigma n+\zeta-1}{\eta_1(\psi,\alpha)}\int_0^1(t)^{\sigma n+\zeta-2}\Lambda_*(\alpha+t\eta_1(\psi,\alpha),\ell)dt\Big]\\ &= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big[\frac{\Lambda_*(\alpha+\eta_1(\psi,\alpha),\ell)}{\eta_1(\psi,\alpha)} - \frac{\sigma n+\zeta-1}{\eta_1(\psi,\alpha)}\int_0^1(t)^{\sigma n+\zeta-2}\Lambda_*(\alpha+t\eta_1(\psi,\alpha),\ell)dt\Big].\\ &= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big[\frac{\Lambda_*(\alpha+\eta_1(\psi,\alpha),\ell)}{\eta_1(\psi,\alpha)}\\ &\quad - \frac{\sigma n+\zeta-1}{\eta_1(\psi,\alpha)}\int_\alpha^{\alpha+\eta_1(\psi,\alpha)}\Big(\frac{x-\alpha}{\alpha+\eta_1(\psi,\alpha)-\alpha}\Big)^{\sigma n+\zeta-2}\Lambda_*(x,\ell)dx\Big],\\ I_1 &= \frac{\Lambda_*(\alpha+\eta_1(\psi,\alpha),\ell)}{\eta_1(\psi,\alpha)}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^\sigma)) - \frac{1}{(\eta_1(\psi,\alpha))^2}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda_*\Big)(\alpha,\ell).\end{aligned}$$

Similarly, we have

$$\begin{aligned}I_2 &= \frac{\Lambda_*(\alpha,\ell)}{\eta_1(\psi,\alpha)}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^\sigma)) - \frac{1}{(\eta_1(\psi,\alpha))^2}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha)^+}\Lambda_*\Big)(\eta_1(\psi,\alpha),\ell),\\ I &= \frac{\Lambda_*(\alpha,\ell)+\Lambda_*(\alpha+\eta_1(\psi,\alpha),\ell)}{2}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^\sigma)) - \frac{1}{(\eta_1(\psi,\alpha))^2}\\ &\quad\times\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha)^+}\Lambda_*\Big)(\alpha+\eta_1(\psi,\alpha),\ell) + \Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda_*\Big)(\alpha,\ell)\Big].\end{aligned}$$

Multiplying both sides by $\frac{\eta_1(\psi,\alpha)}{2}$, we get

$$\begin{aligned}&\Big(\frac{\Lambda_*(\alpha,\ell)+\Lambda_*(\alpha+\eta_1(\psi,\alpha),\ell)}{2}\Big)\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^\sigma)) - \frac{1}{2\eta_1(\psi,\alpha)}\\ &\quad\times\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha)^+}\Lambda_*\Big)(\alpha+\eta_1(\psi,\alpha),\ell) + \Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda_*\Big)(\alpha,\ell)\Big] = \frac{\eta_1(\psi,\alpha)}{2}I.\end{aligned} \tag{6.5}$$

Similarly, for upper FIV-function, we obtain

$$\begin{aligned}&\Big(\frac{\Lambda^*(\alpha,\ell)+\Lambda^*(\alpha+\eta_1(\psi,\alpha),\ell)}{2}\Big)\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma}))-\frac{1}{2\eta_1(\psi,\alpha)}\\&\times\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha)^+}\Lambda^*\Big)(\alpha+\eta_1(\psi,\alpha),\ell)+\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda^*\Big)(\alpha,\ell)\Big].\end{aligned}\tag{6.6}$$

Combining the equations (6.5) and (6.6), we have the desired result. □

Theorem 6.6. *Assuming a function $\Lambda : I \to \mathbb{R}$ in which $I \in \mathbb{R}, \Lambda' \in L_1[\alpha, \alpha+\eta_1(\psi,\alpha)]$ is a differentiable function, where I is assumed to be an open convex set with respect to $\eta_1 : I\times I \to \mathbb{R}$ in which $\eta_1(\psi,\alpha) > 0$ for $\alpha,\psi \in I$, and letting $|\Lambda'|$ be an (η_1,η_2)-convex FIV-function for interval I, then the following integral inequality holds:*

$$\begin{aligned}&\Big|\frac{\Lambda(\alpha)\widetilde{+}\Lambda(\alpha+\eta_1(\psi,\alpha))}{2}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma}))-\frac{1}{2\eta_1(\psi,\alpha)}\\&\quad\times\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha)^+}\Lambda\Big)(\alpha+\eta_1(\psi,\alpha))\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda\Big)(\alpha)\Big]\Big|\\&\preccurlyeq\frac{\eta_1(\psi,\alpha)}{2}|\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p}(\varkappa(1)^{\sigma})-\frac{1}{2^{\zeta-1}}\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p}(\varkappa(\frac{1}{2})^{\sigma})|[2|\Lambda'(\alpha)|\widetilde{+}\eta_2(|\Lambda'(\psi)|,|\Lambda'(\psi)|).\end{aligned}$$

Proof. Consider the following integral, we have

$$\begin{aligned}&\Big|\frac{\Lambda(\alpha)\widetilde{+}\Lambda(\alpha+\eta_1(\psi,\alpha))}{2}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma})-\frac{1}{2\eta_1(\psi-\alpha)}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\alpha^+}\Lambda\Big)(\alpha+\eta_2(\psi,\alpha))\\&\quad\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha+\eta_2(\psi,\alpha))^-}\Lambda\Big)(\alpha)\Big]\Big|=\Big|\frac{\eta_1(\psi,\alpha)}{2}I\Big|.\end{aligned}$$

For lower FIV-function, we get

$$\begin{aligned}&\preccurlyeq\frac{\eta_1(\psi-\alpha)}{2}\sum_{n=0}^{\infty}\Big|\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big|\int_0^1|t^{\zeta+\sigma n-1}-(1-t)^{\zeta+\sigma n-1}||\Lambda_*'(\alpha+t\eta_1(\alpha,\psi),\ell)|dt\\&\preccurlyeq\frac{\eta_1(\psi-\alpha)}{2}\sum_{n=0}^{\infty}\Big|\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big|\\&\quad\times\int_0^1|t^{\zeta+\sigma n-1}-(1-t)^{\zeta+\sigma n-1}|[|\Lambda_*'(\alpha,\ell)|+t\eta_2(|\Lambda_*'(\psi,\ell)|,|\Lambda_*'(\alpha,\ell)|)]dt\\&=\frac{\eta_1(\psi,\alpha)}{2}\sum_{n=0}^{\infty}\Big|\frac{(\gamma)_{qn}}{\Gamma(\sigma n+\zeta)(\rho)_{pn}}(\varkappa)^n\Big|\int_0^{\frac{1}{2}}((1-t)^{\psi'+\mu n}-t^{\psi'+\mu n})[|\Lambda_*'(\alpha,\ell)|+t\eta_2(|\Lambda_*'(\psi,\ell)|,|\Lambda_*'(\alpha,\ell)|)]dt\\&\quad+\int_{\frac{1}{2}}^1(t^{\zeta+\sigma n-1}-(1-t)^{\zeta+\sigma n-1})[|\Lambda_*'(\alpha,\ell)|+t\eta_1(|\Lambda_*'(\psi,\ell)|,|\Lambda_*'(\alpha,\ell)|)]dt.\end{aligned}$$

By taking integration by parts on both sides, we obtain

$$\begin{aligned}&\Big|\frac{\Lambda_*(\alpha,\ell)+\Lambda_*(\alpha+\eta_1(\psi,\alpha),\ell)}{2}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma}))-\frac{1}{2\eta_1(\psi,\alpha)}\\&\quad\times\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha)^+}\Lambda_*\Big)(\alpha+\eta_1(\psi,\alpha),\ell)+\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda_*\Big)(\alpha,\ell)\Big]\Big|\\&\preccurlyeq\frac{\eta_1(\psi,\alpha)}{2}|\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p}(\varkappa(1)^{\mu})-\frac{1}{2^{\zeta-1}}\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p}(\varkappa(\frac{1}{2})^{\sigma})|[2|\Lambda_*'(\alpha,\ell)|+\eta_2(|\Lambda_*'(\psi,\ell)|,|\Lambda_*'(\psi,\ell)|)].\end{aligned}\tag{6.7}$$

For upper FIV-function, we have

$$\begin{aligned}\Big|&\frac{\Lambda^*(\alpha,\ell)+\Lambda^*(\alpha+\eta_1(\psi,\alpha),\ell)}{2}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma}))-\frac{1}{2\eta_1(\psi,\alpha)}\\&\times\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha)^+}\Lambda^*\Big)(\alpha+\eta_1(\psi,\alpha),\ell)+\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda^*\Big)(\alpha,\ell)\Big]\Big|\\&\preccurlyeq\frac{\eta_1(\psi,\alpha)}{2}|\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p}(\varkappa(1)^{\mu})-\frac{1}{2^{\zeta-1}}\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p}(\varkappa(\frac{1}{2})^{\sigma})|[2|\Lambda^{*'}(\alpha,\ell)|+\eta_2(|\Lambda^{*'}(\psi,\ell)|,|\Lambda^{*'}(\psi,\ell)|)].\end{aligned}\tag{6.8}$$

Combining the inequalities (6.7) and (6.8), we obtain

$$\begin{aligned}\Big|&\frac{\Lambda(\alpha)\widetilde{+}\Lambda(\alpha+\eta_1(\psi,\alpha))}{2}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma}))-\frac{1}{2\eta_1(\psi,\alpha)}\\&\times\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha)^+}\Lambda\Big)(\alpha+\eta_1(\psi,\alpha))\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;(\alpha+\eta_1(\psi,\alpha))^-}\Lambda\Big)(\alpha)\Big]\Big|\\&\preccurlyeq\frac{\eta_1(\psi,\alpha)}{2}|\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p}(\varkappa(1)^{\mu})-\frac{1}{2^{\zeta-1}}\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p}(\varkappa(\frac{1}{2})^{\sigma})|[2|\Lambda'(\alpha)|\widetilde{+}\eta_2(|\Lambda'(\psi)|,|\Lambda'(\psi)|)],\end{aligned}$$

which completes the proof. □

Corollary 6.7. *If we replace* $\eta_1(\alpha,\psi)=-\psi+\alpha$ *in Theorem* 6.6, *we have*

$$\begin{aligned}\Big|&\frac{\Lambda(\alpha)\widetilde{+}\Lambda(\psi)}{2}\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p}(\varkappa(1)^{\sigma})-\frac{1}{2(\psi-\alpha)}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\alpha^+}\Lambda\Big)(\psi)\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\psi^-}\Lambda\Big)(\alpha)\Big]\Big|\\&\preccurlyeq\frac{\psi-\alpha}{2}|\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p}(\varkappa(1)^{\sigma})-\frac{1}{2^{\zeta-1}}\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p}(\varkappa(\frac{1}{2})^{\sigma})|[2|\Lambda'(\alpha)|\widetilde{+}\eta_2(|\Lambda'(\psi)|,|\Lambda'(\alpha)|)].\end{aligned}$$

Corollary 6.8. *If we replace* $\eta_1(\alpha,\psi)=-\psi+\alpha$ *and* $\eta_2(x,y)=x-y$ *for each* $x,y\in\Lambda(I)$, *and* $\psi,\alpha\in I$ *in Theorem* 6.6, *we get*

$$\begin{aligned}\Big|&\frac{\Lambda(\alpha)\widetilde{+}\Lambda(\psi)}{2}\xi^{\gamma,\rho,q}_{\sigma,\zeta,p}(\varkappa(1)^{\sigma})-\frac{1}{2(\psi-\alpha)}\Big[\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\alpha^+}\Lambda\Big)(\psi)\widetilde{+}\Big(\xi^{\gamma,\rho,q}_{\sigma-1,\zeta,p,\varkappa;\psi^-}\Lambda\Big)(\alpha)\Big]\Big|\\&\preccurlyeq\frac{\psi-\alpha}{2}|\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p}(\varkappa(1)^{\sigma})-\frac{1}{2^{\zeta-1}}\xi^{\gamma,\rho,q}_{\sigma+1,\zeta,p}(\varkappa(\frac{1}{2})^{\sigma})|[|\Lambda'(\alpha)|\widetilde{+}|\Lambda'(\psi)|].\end{aligned}$$

7. Conclusions

In this paper, we examined the behavior of some well-known inequalities by implementation of generalized fuzzy fractional operators (GFFO) having generalized Mittag-Leffler (GML) function as its kernel. Additionally, we modified the Hermite-Hadamard, trapezoid, Fejér, and mid-point type inequalities by GFFO, which have great contribution in the field of analysis. Moving forward, we aim to extend this work to generalized convex FIV functions, particularly in the domain of fuzzy interval-valued nonlinear programming. This research has potential contributions to the broader field of fuzzy optimization theory. These results are expected to be enough to encourage further studies in all these evolving areas.

Competing objectives

The authors declare that they have no conflicts of interest regarding the publication of this research.

Author's contributions

The authors contributed equally in this work.

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