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Compactness and Boundedness of composition operator on weighted Lorentz spaces with variable exponents

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Abstract

In this paper, we give an overview of the weighted Lorentz spaces with variable exponents and also characterize the boundedness and compactness of the composition operator on these spaces.

Keywords: Composition operator, weighted Lorentz spaces, mathematical operators, boundedness and compactness of the composition operator, distribution function, non-increasing rearrangement.

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1. Introduction

Let $(\mathfrak{U}, \mathcal{A}, m)$ be the σ-finite measure space and g be a complex-valued measurable function. Letting $\Lambda \geqslant 0$, then the distribution function $\mathcal{D}_g(\Lambda)$ is given as

$$\mathcal{D}_g(\Lambda) = m(\{x \in \mathfrak{U} : \Lambda < |g(x)|\}).$$

Let g^* denote the non-increasing rearrangement of g defined as

$$g^*(\tau) = \inf\{\Lambda > 0 : \tau \geqslant \mathcal{D}_g(\Lambda)\},$$

where $\inf \emptyset = \infty$. Let $w : (0, \infty) \to (0, \infty)$ be a locally integrable function and define as

$$\int_0^\infty \omega(\tau) d\tau < \infty.$$

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Let Ω be a measurable subset of $\mathbb{R}^n$ and let $q\colon \Omega \to [1,\infty)$ be a measurable function. We suppose that

$$1 \leqslant \nu_-(\Omega) \leqslant q(x) \leqslant \nu_+(\Omega) < \infty,$$

where $\nu_- := \operatorname{ess}\inf_{x\in\Omega} q(x)$, $\quad \nu_+ := \operatorname{ess}\sup_{x\in\Omega} q(x)$. Lebesgue space with variable exponent $L^{u(\cdot)}(\Omega)$ is the class of functions of Ω such that

$$L^{u(\cdot)}(\Omega) = \left\{ f : \int_\Omega \left(\frac{|f(y)|}{\gamma} \right)^{p(y)} dy < \infty, \text{ where } \gamma \text{ is a constant} \right\}.$$

Norm in $L^{u(\cdot)}(\Omega)$ is defined as follows

$$\|f\|_{L^{u(\cdot)}(\Omega)} = \inf \left\{ \gamma > 0 : \int_\Omega \left(\frac{|f(y)|}{\gamma} \right)^{p(y)} dy \leqslant 1 \right\}.$$

Lorentz spaces were introduced in [20, 21] as a generalization of classical Lebesgue spaces and have become a standard tool in mathematical analysis, cf. [5, 7, 10]. It is noticed that Kempka and Vybíral [18] introduced the variable Lorentz spaces and showed that these spaces arise through real interpolation between variable Lebesgue spaces and $L^\infty(\mathbb{R}^n)$ when $\nu(\cdot) = q$ is constant. Similarly to the classical case, the variable Lorentz space becomes the variable Lebesgue space if $u(\cdot) = \nu(\cdot)$. We also mention that Ephremidze et al. [13] introduced another kind of variable exponent Lorentz space via non-increasing rearrangement function.

For boundedness of the Hilbert transform and the Hilbert maximal operator on weighted classical Lorentz spaces see [3], and for Calderón-Zygmund operators and commutators see [9]. For characterization of the Hardy-Littlewood maximal operator see [11]. Agora et al. [4] characterized the weak-type boundedness of the Hilbert transform H on weighted Lorentz spaces. This topic has numerous applications in the study of partial differential equations; e.g., please see [19]. In [6] the authors proved the boundedness of important operators such as the Bochner-Riesz operator, the rough operators, or the sparse operators among many others. For the weak-type boundedness of the Hardy-Littlewood maximal operator M on weighted Lorentz spaces see [2]. Guliyev et al. [15] obtained the boundedness of the generalized B-potential integral operators in the Lorentz spaces. Lorentz-Shimogaki and Boyd theorems for weighted Lorentz spaces were obtained in [1]. As a consequence, they gave the complete characterization of the strong boundedness of the Hilbert transform on these spaces. They also obtained the complete solution of the weak-type boundedness of the Hardy-Littlewood operator on these spaces. Castillo et al. [12] gave an overview of weighted Lorentz spaces and characterize the boundedness and compactness of the composition operator in these spaces. For more results on function spaces, see [16, 17, 22–31]. Inspired by the concept, in this paper we will find the boundedness and compactness of composition operator in weighted Lorentz spaces with variable exponents.

The Lorentz space denoted by $L(u(\cdot), \nu(\cdot))$ is the generalization of the Lebesgue space $L^{u(\cdot)}$. Let $\mathcal{F}(\mathfrak{U}, \mathcal{A})$ be the set of all $\mathcal{A}$-measurable functions on $\mathfrak{U}$. Letting $g \in \mathcal{F}(\mathfrak{U}, \mathcal{A})$, and $u(\cdot)$ and $\nu(\cdot)$ be measurable functions with values in $[1,\infty]$, then we can write

$$\|g\|_{(u(\cdot),\nu(\cdot))} = \left(\int_0^\infty \left(\tau^{1/u(\cdot)} g^*(\tau) \right)^{\nu(\cdot)} \frac{d\tau}{\tau} \right)^{1/\nu(\cdot)}.$$

Letting $0 < u(\tau) < \infty$, then the Lorentz space $\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)$ is given as

$$\|g\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} = \left(\int_0^\infty (g^*(\tau))^{u(\cdot)} \omega(\tau) d\tau \right)^{1/u(\cdot)} < \infty$$

for $\mathfrak{U} = (0, 1)$. Lorentz introduced Lorentz spaces [20]. In this paper we study some basic properties of weighted Lorentz spaces with variable exponents which are generalizations of classical Lorentz spaces. We discuss some properties of the aforementioned spaces such as lattice property, weak Fatou property, and inclusion relations. We also study boundedness, compactness, and closed range of composition operators on weighted Lorentz spaces with variable exponents.

2. Weighted Lorentz spaces with variable exponents

Let $(\mathfrak{U}, \mathcal{A}, \mathfrak{m})$ be a general measure space, except if we otherwise mentioned. Let w be the weight, and define as

$$W(r) = \int_0^r \omega(\tau)d\tau < \infty$$

with $0 \leqslant r < \infty$. Let $dm(\tau) = w(\tau)d\tau$, and we define $\mathcal{D}_g^w(\Lambda)$ and $g_w^*(\tau)$ is depending on the weight w. Let $(\mathfrak{U}, \mathcal{A}, \mathfrak{m}) = (\mathbb{R}^n, \mathcal{B}, \omega(\tau)d\tau)$ for simplicity we define $L_{(u(\cdot),v(\cdot))}(\omega)$ for $L_{(u(\cdot),v(\cdot))}(\mathfrak{U})$.

Definition 2.1. Let $0 < u(\tau) \leqslant \infty$ and $\|\cdot\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} : \mathcal{F}(\mathfrak{U}, \mathcal{A}) \to [0, \infty]$ be defined as

$$\|g\|_{\Lambda_{\mathfrak{U}}^{p}(\omega)} = \left(\int_0^\infty (g^*(\tau))^{u(\tau)}\, \omega(\tau)d\tau\right)^{1/u(\tau)}.$$

Lorentz space $\Lambda^{u(\cdot)}(\omega) = \Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)$ is given as $\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega) = \left\{g \in \mathcal{F}(\mathfrak{U}, \mathcal{A}) : \|g\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} < \infty\right\}$. It is not difficult to note that $\|g\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} = \|g^*\|_{L^{u(\cdot)}(\omega)}$. Also observe that

$$\Lambda_{\mathfrak{U}}^{u(\cdot),v(\cdot)}(\omega) = \left\{g \in \mathcal{F}(\mathfrak{U}, \mathcal{A}) : \|g\|_{\Lambda_{\mathfrak{U}}^{(u(\cdot),v(\cdot))}} = \|g^*\|_{L_{(u(\cdot),v(\cdot))}(\omega)} < \infty\right\}.$$

Remark 2.2. If $0 < u(\cdot), v(\cdot) < \infty, \Lambda_{\mathfrak{U}}^{v(\cdot)}\left(\tau^{\frac{v(\cdot)}{u(\cdot)-1}}\right) = L_{(u(\cdot),v(\cdot))}\left(\tau^{\frac{v(\cdot)}{u(\cdot)-1}}\right)$ in this case $W(\tau) = \frac{u(\cdot)}{v(\cdot)}\tau^{\frac{v(\cdot)}{u(\cdot)}}, t \geqslant 0$.

Proposition 2.3. *Let $(\mathfrak{U}, \mathcal{A}, \mathfrak{m})$ be a σ-finite measure space. For $0 < u(\cdot) < \infty$, then*

$$\|g\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} \leqslant \left(\int_0^\infty u_+\tau^{u_+-1} W\left(\mathcal{D}_g(\tau)\right) d\tau\right)^{1/u_-}.$$

Proof.

$$\begin{aligned}
\|g\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} &= \left(\int_0^\infty (g^*(\tau))^{u(\tau)}\, \omega(\tau)d\tau\right)^{1/u(\tau)} \\
&\leqslant \left(\int_0^\infty \left(\int_0^{g^*(\tau)} u_+\Lambda^{u_+-1}d\Lambda\right) \omega(\tau)d\tau\right)^{1/u_-} \\
&= \left(\int_0^\infty \left(\int_0^\infty u_+\Lambda^{u_+-1}\mathbf{1}_{(0,g^*(\tau))}(\Lambda)d\Lambda\right) \omega(\tau)d\tau\right)^{1/u_-} \\
&= \left(\int_0^\infty \left(\int_0^\infty u_+\Lambda^{u_+-1}\mathbf{1}_{\{\Lambda>0:g^*(\tau)>\Lambda\}}(\Lambda)d\Lambda\right) \omega(\tau)d\tau\right)^{1/u_-} \\
&= \left(\int_0^\infty \left(\int_0^\infty u_+\Lambda^{u_+-1}\mathbf{1}_{\{t>0:\mathcal{D}_g(\Lambda)>t\}}(\tau)d\Lambda\right) \omega(\tau)d\tau\right)^{1/u_-}.
\end{aligned}$$

Applying the Fubini's theorem, we obtain

$$\begin{aligned}
&\left(\int_0^\infty \left(\int_0^\infty u_+\Lambda^{u_+-1}\mathbf{1}_{\{t>0:\mathcal{D}_g(\Lambda)>t\}}(\tau)d\Lambda\right) \omega(\tau)d\tau\right)^{1/u_-} \\
&= \left(\int_0^\infty u_+\Lambda^{u_+-1}\left(\int_0^\infty \mathbf{1}_{\{t>0:\mathcal{D}_g(\Lambda)>t\}}(\tau)\omega(\tau)d\tau\right) d\Lambda\right)^{1/u_-} \\
&= \left(\int_0^\infty u_+\Lambda^{u_+-1}\left(\int_0^{\mathcal{D}_g(\Lambda)} \omega(\tau)d\tau\right) d\Lambda\right)^{1/u_-} = \left(\int_0^\infty u_+\Lambda^{u_+-1}W\left(\mathcal{D}_g(\Lambda)\right) d\Lambda\right)^{1/u_-},
\end{aligned}$$

which completes the proof. □

Proposition 2.4. *Let* $(\mathfrak{U}, \mathcal{A}, m)$ *be a measure space. For* $0 < u(\cdot), v(\cdot) < \infty$*, and* $g \in \mathcal{F}(\mathfrak{U}, \mathcal{A})$*, then*

(i) $\|g\|_{\Lambda_{X}^{u(\cdot),v(\cdot)}(\omega)} \leqslant \left(\int_0^\infty u_+ \tau^{v_+ - 1} \left(W\left(\mathcal{D}_g(\tau)\right)\right)^{v_+/u_-} d\tau\right)^{1/v_-}$.

(ii) $\|g\|_{\Lambda_{\mathfrak{U}}^{u(\cdot),\infty}(\omega)} \leqslant \sup_{t>0} t \left(W\left(\mathcal{D}_g(\tau)\right)\right)^{1/u_-} = \sup_{t>0} g^*(\tau)(W(\tau))^{1/u_-}$.

Proof. Since $\mathcal{D}_g(\tau) = \mathcal{D}_{g^*}(\tau)$, we get

$$\begin{aligned} W\left(\mathcal{D}_g(\tau)\right) = W\left(\mathcal{D}_{g^*}(\tau)\right) &= \int_0^{\mathcal{D}_{g^*}(\tau)} \omega(s) ds \\ &= \int_0^\infty \mathcal{X}_{(0, \mathcal{D}_{g^*}(\tau))}(s) \omega(s) ds \\ &= \int_0^\infty (\mathcal{X})^*_{\{g^*(s)>t\}}(s)\omega(s) ds = \int_{\{g^*(s)>t\}} \omega(s) ds = \mathcal{D}^w_{g^*}(\tau). \end{aligned}$$

Also, noting that $\mathcal{D}^w_{g^*}(\tau) = (g^*)^*_w(\tau)$, and using the relation $\frac{1}{u_+} + \frac{1}{v_-} = 1$, thus we obtain

$$\begin{aligned} \|g\|_{\Lambda_x^{u(\cdot),v(\cdot)}(\omega)} = \|g^*\|_{L_{(u(\cdot),v(\cdot))}(\omega)} &= \left(\int_0^\infty \left(\tau^{1/u(\tau)} (g^*)^*_w(\tau)\right)^{v(\tau)} \frac{d\tau}{\tau}\right)^{1/v(\tau)} \\ &\leqslant \left(\int_0^\infty \left(\tau^{1/u_-} (g^*)^*_w(\tau)\right)^{v_+} \frac{d\tau}{\tau}\right)^{1/v_-} \\ &\leqslant \left(\int_0^\infty \tau^{\frac{v_+}{u_- - 1}} \int_0^{(g^*)^*_w(\tau)} v_+ s^{v_+ - 1} ds d\tau\right)^{1/v_-} \\ &\leqslant \left(\int_0^\infty u_+ \tau^{v_+ - 1} \left(\mathcal{D}^w_{g^*}(\tau)\right)^{\frac{v_+}{u_-}} d\tau\right)^{1/v_-} \\ &\leqslant \left(\int_0^\infty u_+ \tau^{v_+ - 1} \left[W\left(\mathcal{D}_g(\tau)\right)\right]^{\frac{v_+}{u_-}} d\tau\right)^{1/v_-}. \end{aligned}$$

(ii) It is not difficult to note that

$$\|g\|_{\Lambda_{\mathfrak{R}}^{u(\cdot),v(\cdot)}(\omega)} = \|g^*\|_{L_{(u(\cdot),\infty)}(\omega)} \leqslant \sup_{t>0} t \left(\mathcal{D}^w_{g^*}(\tau)\right)^{1/u_-} = \sup_{t>0} t \left(W\left(\mathcal{D}_g(\tau)\right)\right)^{1/u_-} = \sup_{t>0} g^*(\tau)(W(\tau))^{1/u_-},$$

which ends the proof. □

Remark 2.5.

(a) By comparing Propositions 2.3 and 2.4 (i) we note that $v(\tau) < \infty$, $\|g\|_{\Lambda_{\mathcal{X}}^{u(\cdot),v(\cdot)}(\omega)} = \|g\|_{\Lambda_{\mathfrak{U}}^{v(\cdot)}(w_0)}$, where $w_0(\tau) = (W(\tau))^{\frac{v_+}{u_- - 1}} \omega(\tau)$, with $0 < t < m(\mathfrak{U})$. Hence every Lorentz space reduces to $\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)$ and its weak version $\Lambda_{\mathfrak{U}}^{u(\cdot),\infty}(\omega)$.

(b) From Proposition 2.4 (ii) we deduce that $\Lambda_{\mathfrak{U}}^{u(\cdot),\infty}(\omega) = \Lambda_{\mathfrak{U}}^{v(\cdot),\infty}\left(\frac{v_+}{u_-} w_0\right)$, for $0 < u(\cdot), v(\cdot) < \infty$.

(c) Observe that (a) makes sense because $g^*(\tau) = 0$ if $t \geqslant m(\mathfrak{U})$.

For the space $L_{(u(\cdot),\infty)}(\mathfrak{U})$ is the quasi-norm $\|g\|_{(u(\cdot),\infty)}$ for $v(\tau) < u(\tau)$, equivalent to the functional

$$\sup_{E \subset \mathfrak{U}} \|g \mathcal{X}_E\|_{v(\cdot)} (m(E))^{\frac{1}{u_-} - \frac{1}{v_-}}.$$

Proposition 2.6 (Chebyschev's type inequality)**.** *Letting* $g \in \Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)$*, then*

$$W\left(\mathcal{D}_g(\tau)\right) \leqslant \frac{\|g\|^{u(\tau)}_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)}}{\tau^{u(\tau)}}.$$

Proof. Letting $E = \{x \in \mathfrak{U} : |g(x)| > t\ |\}$, then observe that $t\mathcal{X}_E < |g(x)|$, for this we have

$$\tau^{u(\cdot)}\mathcal{X}_{(0,m(E))}(\tau) < (g^*(\tau))^{u(\tau)}.$$

Thus

$$\tau^{u(\tau)}\int_0^{m(E)} \omega(\tau)d\tau \leqslant \int_0^{\infty} (g^*(\tau))^{u(\tau)}\,\omega(\tau)d\tau.$$

Finally

$$W(\mathcal{D}_g(\tau)) \leqslant \frac{\|g\|^{u(\tau)}_{\Lambda^{u(\cdot)}_{\mathfrak{U}}(\omega)}}{\tau^{u(\tau)}}.$$

□

Proposition 2.7. *Let $0 < v(\tau) < u(\tau) < \infty$, and $g \in \mathcal{F}(\mathfrak{U}, \mathcal{A})$. Then*

$$\|g\|_{\Lambda^{u(\cdot),\infty}_{\mathfrak{U}}(\omega)} \leqslant \sup_{E\subset\mathfrak{U}} \|g\mathcal{X}_E\|_{\Lambda^{v(\cdot)}_{\mathfrak{U}}(\omega)} [W(m(E))]^{\frac{1}{u(\tau)}-\frac{1}{v(\tau)}} \leqslant \left(\frac{v_+}{u_- - v_-}\right)^{1/v(\tau)} \|g\|_{\Lambda^{u(\cdot),\infty}_{\mathfrak{U}}(\omega)}.$$

Proof. To show the first inequality, and set E of $\mathfrak{U}$ given by $E = \{x \in \mathfrak{U} : |g(x)| > t\}$, and also let

$$S = \sup_{E\subset\mathfrak{U}} \|g\mathcal{X}_E\|_{\Lambda^{v(\cdot)}_{\mathfrak{U}}(\omega)} (W(m(E)))^{\frac{1}{u(\tau)}-\frac{1}{v(\tau)}}.$$

Then

$$\begin{aligned}
S &\geqslant \|g\mathcal{X}_E\|_{\Lambda^{v(\cdot)}_{\mathcal{X}}(\omega)} (W(m(E)))^{\frac{1}{u(\tau)}-\frac{1}{v(\tau)}}\\
&= \left(\int_0^{\infty} \left[(g\mathcal{X}_E)^*(s)\right]^{v(\cdot)} \omega(s)ds\right)^{1/v(\tau)} (W(m(E)))^{\frac{1}{u(\tau)}-\frac{1}{v(\tau)}}\\
&\geqslant \left(\tau^{v(\cdot)}\int_0^{m(E)} \omega(s)ds\right)^{1/v(\tau)} (W(m(E)))^{\frac{1}{u(\tau)}-\frac{1}{v(\tau)}}\\
&= t(W(m(E)))^{\frac{1}{v(\tau)}}(W(m(E)))^{\frac{1}{u(\tau)}-\frac{1}{v(\tau)}} = t(W(m(E)))^{\frac{1}{u(\tau)}} = t\,(W(\mathcal{D}_g(\tau)))^{\frac{1}{u(\tau)}}.
\end{aligned}$$

If $t > 0$, we have $\|g\|_{\Lambda^{u(\cdot),\infty}_{\mathfrak{U}}(\omega)} \leqslant s$. If $g \in \mathcal{F}(\mathfrak{U}, \mathcal{A})$, $E \subset \mathfrak{U}$, letting $a = \|g\|_{\Lambda^{u(\cdot),\infty}_{\mathcal{X}}(\omega)} (W(m(E)))^{-\frac{1}{u(\tau)}}$, then

$$\begin{aligned}
\|g\mathcal{X}_E\|_{\Lambda^{v(\cdot)}_{\mathcal{X}}(\omega)} &= \int_0^{\infty} v(\tau)\tau^{v(\tau)-1} W\left(\mathcal{D}_{g\mathcal{X}_E}(\tau)\right) d\tau\\
&= \int_0^{a} v(\tau)\tau^{v(\tau)-1} W\left(\mathcal{D}_{g\mathcal{X}_E}(\tau)\right) d\tau + \int_a^{\infty} v(\tau)\tau^{v(\tau)-1} W\left(\mathcal{D}_{g\mathcal{X}_E}(\tau)\right) d\tau.
\end{aligned}$$

Note that $\mathcal{D}_{g\mathcal{X}_E}(\tau) = m(\{x \in \mathfrak{U} : |g\mathcal{X}_E(x)| > t\}) \leqslant m(E)$ and since $g\mathcal{X}_E \leqslant f$, then $\mathcal{D}_{g\mathcal{X}_E}(\tau) \leqslant \mathcal{D}_g(\tau)$, thus

$$\begin{aligned}
\|g\mathcal{X}_E\|^{v(\tau)}_{\Lambda^{v(\cdot)}_{\mathfrak{U}}(\omega)} &\leqslant \omega(m(E))\int_0^{a} v(\tau)\tau^{v(\tau)-1}d\tau + \int_a^{\infty} v(\tau)\tau^{v(\tau)-1} W(\mathcal{D}_g(\tau))\,d\tau\\
&\leqslant \omega(m(E))a^{v(\tau)} + \int_a^{\infty} v(\tau)\tau^{v(\tau)-1}\,\|g\|^{u(\tau)}_{\Lambda^{u(\cdot),\infty}_{\mathfrak{U}}(\omega)} \frac{d\tau}{\tau^{u(\tau)}}\\
&\leqslant \omega(m(E))a^{v(\tau)} + \frac{v_+}{u_- - v_-}\|g\|^{u(\tau)}_{\Lambda^{u(\cdot),\infty}_{\mathfrak{U}}(\omega)} a^{v(\tau)-u(\tau)}\\
&\leqslant \frac{v_+}{u_- - v_-}\|g\|^{v(\tau)}_{\Lambda^{u(\cdot),\infty}_{\mathfrak{U}}(\omega)} (W(m(E)))^{\frac{u(\tau)-v(\tau)}{u(\tau)}}.
\end{aligned}$$

Hence

$$\|g\mathcal{X}_E\|_{\Lambda^{v(\cdot)}_{\mathfrak{U}}(\omega)} (W(m(E)))^{\frac{1}{u(\tau)}-\frac{1}{v(\tau)}} \leqslant \left(\frac{v_+}{u_- - v_-}\right)^{\frac{1}{v(\tau)}} \|g\|_{\Lambda^{u(\cdot),\infty}_{\mathfrak{U}}(\omega)}.$$

□

Proposition 2.8. *Letting* $0 < u(\tau) < \infty$, *and* $f, g, g_k, k \geqslant 1$ *functions belonging to* $\mathcal{F}(\mathfrak{U}, \mathcal{A})$, *then we get*

(i) $|f| \leqslant |g|$, *implies* $\|f\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} \leqslant \|g\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)}$.

(ii) $\|\alpha g\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} = |\alpha| \, \|g\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)}$.

(iii) $0 \leqslant g_k \leqslant g_{k+1} \to g$, *a.e., then* $\lim_{k\to\infty} \|g_k\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} = \|g\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)}$.

(iv) $\|\liminf |g_k|\|_{\Lambda_{\mathfrak{X}}^{u(\cdot)}(\omega)} \leqslant \liminf \|g_k\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)}$.

(v) $\Lambda_{\mathfrak{U}}^{\nu(\cdot)}(\omega) \subset \Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)$ *for* $0 < u(\cdot) < \nu(\cdot) < \infty, \omega(m(\mathfrak{U})) < \infty$.

(vi) $\mathbf{1}_E \in \Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)$ *if* $m(E) < \infty$.

Proof. Properties (i) and (ii) follow from (1) and Definition 2.1. To prove (iii) just observe that $0 \leqslant g_k \leqslant g_{k+1} \to g$, implies that $\lim_{k\to\infty} \mathcal{D}_{g_k}(\Lambda) = \mathcal{D}_g(\Lambda)$. Next, letting $F_k(\tau) = \mathcal{D}_{g_k}(\tau)$, then $g_k^*(\tau) = m(\{\Lambda > 0 : \mathcal{D}_{g_k}(\Lambda) > t\}) = \mathcal{D}_{F_k}(\tau)$, since $\mathcal{D}_{g_k}(\tau) \leqslant \mathcal{D}_{g_{k+1}}(\tau)$ we have $F_k(\tau) \leqslant F_{k+1}(\tau)$. Hence $E_{F_1}(\tau) \subseteq E_{F_2}(\tau) \subseteq \cdots$, then

$$E_F(\tau) = \bigcup_{k=1}^{\infty} E_{F_k}(\tau), \text{ where } E_{F_k}(\tau) = \{F_k > t\}.$$

Therefore $\lim_{k\to\infty} \mathcal{D}_{F_k}(\tau) = \mathcal{D}_g(\tau)$. Hence $\lim_{k\to\infty} f_k^*(\tau) = g^*(\tau)$. Applying the monotone convergence theorem we get $\lim_{k\to\infty} \|g_k\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} = \|g\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)}$.

(iv) The distribution function of $\liminf |g_n|$ satisfies

$$\begin{aligned}
\mathrm{limi}^{D}_{\mathrm{inf}|g_n|}(\Lambda) &= m(\{x : \liminf |g_n(x)| > \Lambda\}) \\
&= m(\liminf \{x : |g_n(x)| > \Lambda\}) \\
&= m\left(\bigcup_{n=1} \bigcap_{k=n} \{x : |g_k(x)| > \Lambda\} \right) \\
&= \liminf m(\{x : |g_n(x)| > \Lambda\}) \\
&= \liminf \mathcal{D}_{g_n}(\Lambda) \leqslant \liminf \mathcal{D}_{|g_n|}(\Lambda),
\end{aligned}$$

thus, we have $D_{\liminf |g_n|}(\Lambda) \leqslant \liminf \mathcal{D}_{|g_n|}(\Lambda)$. From this we obtained

$$\begin{aligned}
\inf\left\{\Lambda > 0 : D_{\liminf|g_n|}(\Lambda) \leqslant t\right\} &\leqslant \inf\left\{\Lambda > 0 : \liminf \underset{|g_n|}{D}(\Lambda) \leqslant t\right\} \\
&\leqslant \liminf\left(\inf\left\{\Lambda > 0 : \underset{|g_n|}{D}(\Lambda) \leqslant t\right\}\right) \ (\liminf |g_n|)^*(\tau) \leqslant \liminf\left(|g_n|^*(\tau)\right).
\end{aligned}$$

The estimate (iv) follows immediately from the latter inequality and Fatou's Lemma. Remaining properties are not difficult to obtain. □

Proposition 2.9. *Assume that* W *be a positive function on* $(0, \infty)$. *Let* $\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)$ *be a variable Lorentz space and* $(g_n)_n$ *is a sequence of measurable functions on* $\mathfrak{U}$.

(i) *Letting* $\lim_{m,n} \|g_m - g_n\|_{\Lambda_x^{u(\cdot)}(\omega)} = 0$, *then* $(g_n)_n$ *be a Cauchy sequence and* $g \in \mathcal{F}(\mathfrak{U}, \mathcal{A})$ *such that* $\lim_n \|g_n - g\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} = 0$.

(ii) *If* $g \in \mathcal{F}(\mathfrak{U}, \mathcal{A})$ *and* $\lim_n \|g_n - g\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} = 0$, *then* $(g_n)_n$ *converges to* g *and there exists a partial* $(g_{n_k})_k$ *convergent to* g *a.e..*

Proof. For $u(\cdot) = \infty$, it is not difficult to note that $\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega) = L_\infty$. If $u(\cdot) < \infty$, then by Chebyschev's type inequality we have

$$W(\mathcal{D}_g(\tau)) \leqslant \frac{\|g\|^{u(\tau)}_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)}}{\tau^{u(\tau)}}, \quad t > 0,$$

by using (i) we get $W(\mathcal{D}_{g_m - g_n}(\tau)) \overrightarrow{m,n}\, 0$ for every $t > 0$, which (since $W > 0$), implies $\mathcal{D}_{g_m-g_n}(\tau)\overrightarrow{m,n}0$, $t > 0$, so $(g_n)_n$ is a Cauchy sequence. Applying the Proposition 2.7 (iv) we get

$$\|f - g_n\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} \leqslant \liminf_k \|g_{n_k} - g_n\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)},$$

hence $\lim_n \|f - g_n\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} = 0$. □

Theorem 2.10. *Letting* $0 < u(\tau) < \infty$, *the space* $\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)$ *is quasi-normed iff*

$$0 < W(m(A \cup B)) \leqslant C(W(m(A)) + W(m(B)))$$

such that $A, B \subset \mathfrak{U}$ *and* $m(A \cup B) > 0$.

Proof.

Sufficiency: The hypothesis, implies that $W(m(A)) > 0$ if $m(A) > 0$. If $\|g\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} = 0$ by using the Proposition 2.6, we have $W(\mathcal{D}_g(\tau)) = 0, t > 0$, and hence $\mathcal{D}_g(\tau) = 0$ for every $t > 0$, that is $f = 0$ almost everywhere. If $0 \leqslant f, g \in \Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)$ and $t > 0$, then $\{f + g > t\} \subset \left\{f > \frac{\tau}{2}\right\} \cup \left\{g > \frac{\tau}{2}\right\}$ and then we get

$$0 < W(\mathcal{D}_{f+g}(\tau)) \leqslant C\left(W\left(\mathcal{D}_f\left(\frac{\tau}{2}\right)\right) + W\left(\mathcal{D}_g\left(\frac{\tau}{2}\right)\right)\right).$$

Thus by Proposition 2.3 we have

$$\|f + g\|_{\Lambda_{\mathfrak{U}}^{p}(\omega)} \leqslant C_{u(\cdot)}\left(\|f\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} + \|g\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)}\right).$$

Note that $\mathcal{X}_{A\cup B} \leqslant \mathcal{X}_A + \mathcal{X}_B$, so we get

$$\begin{aligned} 0 < [W(m(A \cup B))]^{1/u(\cdot)} &= \|\mathcal{X}_{A\cup B}\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} \\ &\leqslant \|\mathcal{X}_A + \mathcal{X}_B\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} \\ &\leqslant C\left(\|\mathcal{X}_A\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} + \|\mathcal{X}_B\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)}\right) \\ &= C_{u(\cdot)}\left([W(m(A))]^{1/u(\cdot)} + [W(m(B))]^{1/u(\cdot)}\right), \end{aligned}$$

which is equivalent to the condition of the statement. □

3. Boundedness of composition operator

Let I be measurable non-singular transformation and C_I denotes the composition operator from $\mathcal{F}(X, \mathcal{A}, m)$ (linear space of all equivalence classes of $\mathcal{A}$-measurable functions on X) into itself given as

$$C_I(g)(x) = g(I(x)),\ x \in X,\ g \in \mathcal{F}(X, \mathcal{A}, m).$$

Transformation C_I from $\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)$ into the space of all complex-valued measurable functions on X is given as

$$(C_I g)(x) = \begin{cases} g(I(x)), & \text{if } x \in Y, \\ 0, & \text{otherwise.} \end{cases}$$

where Y is a measurable subset of X.

Theorem 3.1. *Let* $T : \mathfrak{U} \to \mathfrak{U}$ *be a non-singular measurable transformation. Then* $C_I(f \mapsto f \circ T)$ *induced by* I *is bounded iff there exists a constant* $M > 0$ *such that*

$$\int_0^{m\left(I^{-1}(A)\right)} \omega(\tau)d\tau \leqslant M^{u(\cdot)} \int_0^{m(A)} \omega(\tau)d\tau$$

for all $A \in \mathcal{A}$*. Moreover*

$$\|C_I(f)\| = \sup_{0<W(m(A))<\infty} \left(\frac{W\left(m\left(I^{-1}(A)\right)\right)}{W(m(A))} \right)^{1/u(\tau)}.$$

Proof.

$(\Leftarrow)$ Let $A = \{x \in \mathfrak{U} : |g(x)| > \Lambda\}$, observe that

$$\begin{aligned}
\int_0^{\mathcal{D}_{C_I(f)}(\Lambda)} \omega(\tau)d\tau &= \int_0^{m(\{x\in\mathfrak{U}:|g(I(x))|>\Lambda\})} \omega(\tau)d\tau \\
&= \int_0^{m\left(I^{-1}\{x\in\mathfrak{U}:|g(x)|>\Lambda\}\right)} \omega(\tau)d\tau \\
&\leqslant M^{u(\cdot)} \int_0^{m(\{x\in\mathfrak{U}:|g(x)|>\Lambda\})} \omega(\tau)d\tau \leqslant M^{u(\cdot)} \int_0^{\mathcal{D}_g(\Lambda)} \omega(\tau)d\tau.
\end{aligned}$$

Thus

$$W\left(\mathcal{D}_{C_I(f)}(\Lambda)\right) \leqslant M^{u(\cdot)} W\left(\mathcal{D}_g(\Lambda)\right).$$

The above inequality together with Proposition 2.3 gives us

$$\begin{aligned}
\|C_I(f)\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} &\leqslant \left(\int_0^{\infty} u_+ \Lambda^{u_+-1} W\left(\mathcal{D}_{C_I(f)}(\Lambda)\right) d\Lambda \right)^{1/u_-} \\
&\leqslant M \left(\int_0^{\infty} u_+ \Lambda^{u_+-1} W\left(\mathcal{D}_g(\Lambda)\right) d\Lambda \right)^{1/u_-} \leqslant M \|g\|_{\Lambda_{\mathfrak{U}}^{u_+}(\omega)}
\end{aligned}$$

and thus, C_I is bounded, also note that $\|C_I(f)\| \leqslant M$.

$(\Rightarrow)$ Suppose that C_I is bounded on $\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)$. That is, there exists $M > 0$ such that

$$\|C_I(f)\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} \leqslant M \|g\|_{\Lambda_{\mathfrak{U}}^{u_+}(\omega)}.$$

Next, choose $f = \mathcal{X}_E$ for each $E \in \mathcal{A}$. Then

$$\int_0^{m\left(I^{-1}(A)\right)} \omega(\tau)d\tau = \left\|\mathcal{X}_{I^{-1}(E)}\right\|^{u(\tau)}_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} = \|C_I(f)\|^{u(\tau)}_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} \leqslant M^{u(\cdot)} \|g\|^{u_+}_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} = M^{u(\cdot)} \int_0^{m(E)} \omega(\tau)d\tau.$$

Further, it is not hard to see that:

$$\|C_I(f)\| = \sup_{0<W(m(A))<\infty} \left(\frac{W\left(m\left(I^{-1}(A)\right)\right)}{W(m(A))} \right)^{1/u(\tau)}. \qquad \square$$

4. Compactness on Lorentz spaces with variable exponents

In this section, we discuss the compactness of composition operator on Lorentz spaces.

Definition 4.1. Suppose m is a (non-negative countably additive) measure on the σ-algebra $\mathcal{A}$. A set $E \in \mathcal{A}$ will be called an atom for m if

(i) $m(E) > 0$; and
(ii) given $g \in \mathcal{A}$, either $m(E \cap F)$ or $m(E \backslash F)$ is 0.

We shall say that m is purely atomic or simple atomic if every measurable set of positive measurable contain an atom.

Theorem 4.2. *Assume that* $W \in \Delta_2(\mathfrak{U})$. *Let* $\mathfrak{U} = \bigcup_{n=1}^{\infty} B_n$ *be a purely atomic measure space and let* $T : \mathfrak{U} \to \mathfrak{U}$ *be a non-singular measurable transformation such that the sequence*

$$b_n = \frac{W\left(m\left(I^{-1}\left(B_n\right)\right)\right)}{W\left(m\left(B_n\right)\right)} \to 0$$

as $n \to \infty$. *Then* C_I *is a compact composition operator on the Lorentz space* $\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)$, *where* w *is a non-decreasing weight.*

Proof. Since $(\mathfrak{U}, \mathcal{A}, m)$ is purely atomic with atoms B_n and $b_n \to 0$ as $n \to \infty$, observe that f and $\sum f(B_n)\mathcal{X}_{B_n}$ are equal almost everywhere. For each $N \in \mathbb{N}$, define $C_I^{(N)}$ by

$$C_I^{(N)}(f) = \sum_{n \leqslant N} f\left(B_n\right) \mathcal{X}_{I^{-1}\left(B_n\right)}.$$

Then for each $\Lambda > 0$, we have

$$\begin{aligned}
W\left(\mathcal{D}_{\left(C_I - C_I^{(N)}\right)(f)}(\Lambda)\right) &= \int_0^{D_{\left(C_I - C_I^{(N)}\right)(f)}(\Lambda)} \omega(\tau)d\tau \\
&= \int_0^m \left(\left\{x \in \mathfrak{U} : \left|\sum_{n>N} f\left(B_n\right) \mathcal{X}_{I^{-1}\left(B_n\right)}(x)\right| > \Lambda\right\}\right) \omega(\tau)d\tau \\
&\leqslant \int_0^m \left(\left\{x \in \mathfrak{U} : \sum_{n>N} \left|f\left(B_n\right)\right| \mathcal{X}_{I^{-1}\left(B_n\right)}(x) > \Lambda\right\}\right) \omega(\tau)d\tau \\
&\leqslant \int_0^{\sum_{n>N} m\left(I^{-1}\left(\{B_n : |f(B_n)| > \Lambda\}\right)\right)} \omega(\tau)d\tau \leqslant C \sum_{n>N} \int_0^{m\left(I^{-1}\left(\{B_n : |f(B_n)| > \Lambda\}\right)\right)} \omega(\tau)d\tau.
\end{aligned}$$

From the above inequality and the fact that w is a non-decreasing weight, we have

$$\begin{aligned}
W\left(\mathcal{D}_{\left(C_I - C_I^{(N)}\right)(f)}(\Lambda)\right) &\leqslant C \sum_{\substack{n>N \\ |f(B_n)| > \Lambda}} \frac{W\left(m\left(I^{-1}\left(B_n\right)\right)\right)}{W\left(m\left(B_n\right)\right)} W\left(m\left(B_n\right)\right) \\
&\leqslant C\left(\sup_{n>N} b_n\right) \sum_{n>N} W\left(m\left(\{B_n : |f\left(B_n\right)| > \Lambda\}\right)\right) = C\mathcal{D}_N W\left(\mathcal{D}_g(\Lambda)\right),
\end{aligned}$$

where $\mathcal{D}_N = \sup_{n>N} b_n \in (0,1)$. Finally, by Proposition 2.3 we obtain

$$\left\|\left(C_I - C_I^{(N)}\right)(f)\right\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} \leqslant \left(\mathcal{D}_N\right)^{\frac{1}{u_-}} \|g\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} \to 0$$

since $\mathcal{D}_N \to 0$ as $N \to \infty$. Thus, C_I, being the limit of finite rank operators $C_I^{(N)}$, is compact. □

Theorem 4.3. *Let C_I be a composition operator on the Lorentz spaces with variable exponent $\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)$, where w is a non-increasing weight, and $\{B_n\}_{n\in\mathbb{N}}$ are atoms of $\mathfrak{U}$. If C_I is compact, then m is purely atomic and*

$$b_n = \frac{W\left(m\left(I^{-1}\left(B_n\right)\right)\right)}{W\left(m\left(B_n\right)\right)} \to 0, \quad \text{as} \quad n \to \infty.$$

Proof. We should decompose $\mathfrak{U}$ into two parts, say $\mathfrak{U}_1$, be the non-atomic part and $\mathfrak{U}_2$ be the atomic part of $\mathfrak{U}$, we may note that $\mathfrak{U}_1 \cap \mathfrak{U}_2 = \emptyset$ and $\mathfrak{U} = \mathfrak{U}_1 \cup \mathfrak{U}_2$, where $m_1 = m \mid \mathfrak{U}_1$ is non-atomic and $m_2 = m \mid \mathfrak{U}_2$ is atomic such that $m = m_1 + m_2$. Since $m\left(I^{-1}\right) \ll m$, by using the Radon-Nikodym Theorem we get a function g on $\mathfrak{U}_1$ such that

$$m\left(I^{-1}(A)\right) = \int_A g(x)dm(x), \text{ for all } A \in \mathcal{A} \mid \mathfrak{U}_1.$$

Let $A = \{x \in \mathfrak{U}_1 : g(x) > 0\}$. To prove $m(A) = 0$, letting $m(A) > 0$, then for $0 < \varepsilon < 1$ we get $A_\varepsilon = \{x \in \mathfrak{U}_1 : g(x) \geqslant \varepsilon\}$ has positive measure. Next, we may observe that the subsequence of A_ε given by

$$B_n = \{x \in A_\varepsilon : g(x) > n\}, \text{ for } n = 1, 2, \ldots$$

satisfies $B_n \subseteq A_{n-1}$ and for some $n_0 \in \mathbb{N}$,

$$0 < m\left(B_n\right) = m_1\left(B_n\right) < \frac{1}{n}, \text{ for all } n > n_0.$$

The latter equality holds since $(\mathfrak{U}_1, \mathcal{A} \mid \mathfrak{U}_1, m_1)$ is non-atomic. Hence for each $n \in \mathbb{N}$, define

$$g_n(x) = \frac{\mathbf{1}_{B_n}(x)}{\|\mathbf{1}_{B_n}\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)}}, x \in \mathfrak{U}.$$

Then it is not hard to see that $g_n \to 0$ weakly, $\|g_n\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} = 1$ and for $n > n_0$, we have

$$\|C_I g_n\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} = \frac{W\left(m\left(I^{-1}\left(B_n\right)\right)\right)}{W\left(m\left(B_n\right)\right)} = \frac{W\left(\int_{B_n} g(x)\mathcal{D}_m\right)}{W\left(m\left(B_n\right)\right)} \geqslant \frac{W\left(\varepsilon m\left(B_n\right)\right)}{W\left(m\left(B_n\right)\right)} \geqslant \varepsilon,$$

which implies that $C_I g_n \nrightarrow 0$ strongly. This contradicts the compactness of C_I. Thus $m(A) = 0$. Therefore $g = 0$ a.e. on $\mathfrak{U}_1$. Then $m\left(I^{-1}\left(\mathfrak{U}_1\right)\right) = 0$, since $\mathfrak{U} = I^{-1}\left(\mathfrak{U}_1\right) \cup I^{-1}\left(\mathfrak{U}_2\right)$. We have $\mathfrak{U} = I^{-1}\left(\mathfrak{U}_2\right)$, which implies that $\mathfrak{U} = \mathfrak{U}_2$. Hence $m = m_2$, shows that m is an atomic measure. Next, we claim that $b_n \to 0$. Suppose the contrary. Then for $\varepsilon > 0$ we get $b_n \geqslant \varepsilon$ where $n \in \mathbb{N}$. Let $\mathfrak{U} = \cup_{n=1}^{\infty} B_n$ where each B_n is an atom. For each $n \in \mathbb{N}$, let $g_n = \frac{\mathbf{1}_{B_n}}{\|\mathbf{1}_I^{-1}(B_n)\|_{\Lambda_{\mathcal{X}}^{u(\cdot)}(\omega)}}$. Then for each n, we have

$$\|g_n\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} = \frac{\|\mathbf{1}_{B_n}\|_{\mathbf{1}_{\mathfrak{U}}^{u(\cdot)}(\omega)}}{\left\|\mathbf{1}_{I^{-1}(B_n)}\right\|_{\Lambda_{\mathcal{X}}^{u(\cdot)}(\omega)}} = \left(\frac{W\left(m\left(B_n\right)\right)}{W\left(m\left(I^{-1}\left(B_n\right)\right)\right)}\right)^{1/u(\tau)} \leqslant \frac{1}{\left(b_n\right)^{1/u_-}} \leqslant \frac{1}{\varepsilon^{1/u_-}}$$

and $\|C_I g_n\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} = 1$, then $\varepsilon^{1/u_-} \|g_n\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} \leqslant \|C_I g_n\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)}$. On the other hand, for $n \neq m$ we have $B_n \cap A_m = \varnothing$ and so

$$\|C_I g_n - C_I g_m\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} = \|C_I g_n + C_I g_m\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)}.$$

Thus

$$\begin{aligned} 1 &= \frac{1}{2} \|C_I g_n + C_I g_m\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} \\ &\leqslant \frac{1}{2}\left(\|C_I g_n - C_I g_m\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} + \|C_I g_n + C_I g_m\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)}\right) = \|C_I g_n - C_I g_m\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)}. \end{aligned}$$

Finally these estimates contradict the compactness of C_I. Hence $b_n \to 0$. □

Next, we give a condition under which C_I is 1-1, has closed range, and its inverse is bounded.

Theorem 4.4. *Let* $(X, \mathcal{A}, m)$ *be a complete* σ*-finite measure space and* $T : X \to X$ *be a non-singular measurable transformation. Then* C_I *is bounded below if and only if there exists* $\varepsilon > 0$ *such that*

$$\int_0^{m(I^{-1}(A))} \omega(\tau)d\tau \geqslant \varepsilon^{u_-} \int_0^{m(A)} \omega(\tau)d\tau$$

for all $A \in \mathcal{A}$.

Proof. Letting C_I be bounded below, then there exists $\varepsilon > 0$ such that

$$\|C_I(f)\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} \geqslant \varepsilon \|g\|_{\Lambda_{\mathfrak{U}}^{u_+}(\omega)}$$

for all $g \in \Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)$. Thus for $A \in \mathcal{A}$, then $m(A) < \infty$ and then

$$\int_0^{m(I^{-1}(A))} \omega(\tau)d\tau = \|C_I(f)\|^{u(\cdot)}_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} \geqslant \varepsilon^{-} \|\mathbf{1}_A\|^{u(\cdot)}_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} = \varepsilon^{u_-} \int_0^{m(A)} \omega(\tau)d\tau.$$

Conversely, if

$$\int_0^{m(I^{-1}(A))} \omega(\tau)d\tau \geqslant \varepsilon^{u_-} \int_0^{m(A)} \omega(\tau)d\tau, \quad \text{for all } A \in \mathcal{A},$$

let us consider $A = \{x \in X : |g(x)| > t\}$, then we have

$$W(\mathcal{D}_{C_I}(\tau)) \geqslant \varepsilon^{u_-} W(\mathcal{D}_g(\tau))$$

and thus

$$\|C_I(f)\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)} \geqslant \varepsilon \|g\|_{\Lambda_{\mathfrak{U}}^{u(\cdot)}(\omega)}.$$

□

Corollary 4.5. C_I *is 1-1, has closed range, and its inverse* $[C_I(f)]^{-1}(x) = [g(I(x))]^{-1} = I^{-1} \circ f^{-1}(x)$ *is bounded.*

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References

[1] E. Agora, J. Antezana, M. J. Carro, J. Soria, *Lorentz-Shimogaki and Boyd theorems for weighted Lorentz spaces*, J. Lond. Math. Soc. (2), **89** (2014), 321–336. 1

[2] E. Agora, J. Antezana, M. J. Carro, *Weak-type boundedness of the Hardy-Littlewood operator on weighted Lorentz spaces*, J. Fourier Anal. Appl., **22** (2016), 1431–1439. 1

[3] E. Agora, M. J. Carro, J. Soria, *Boundedness of the Hilbert transform on weighted Lorentz spaces*, J. Math. Anal. Appl., **395** (2012), 218–229. 1

[4] E. Agora, M. J. Carro, J. Soria, *Characterization of the weak-type boundedness of the Hilbert transform on weighted Lorentz spaces*, J. Fourier Anal. Appl., **19** (2013), 712–730. 1

[5] M. A. Ariño, B. Muckenhoupt, *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions*, Trans. Amer. Math. Soc., **320** (1990), 727–735. 1

[6] S. Baena-Miret, M. J. Carro, *Boundedness of sparse and rough operators on weighted Lorentz spaces*, J. Fourier Anal. Appl., **27** (2021), 22 pages. 1

[7] C. Bennett, R. Sharpley, *Interpolation of operators*, Academic Press, Boston, MA, (1988). 1

[8] F. E. Browder, W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl., **20** (1967), 197–228.

[9] M. J. Carro, H. Li, J. Soria, Q. Sun, *Calderón-Zygmund operators and commutators on weighted Lorentz spaces*, J. Geom. Anal., **31** (2021), 8979–8990. 1

[10] M. Carro, L. Pick, J. Soria, V. D. Stepanov, *On embeddings between classical Lorentz spaces*, Math. Inequal. Appl., **4** (2001), 397–428. 1

[11] M. J. Carro, J. A. Raposo, J. Soria, *Recent developments in the theory of Lorentz spaces and weighted inequalities*, Mem. Amer. Math. Soc., **187** (2007), 128 pages. 1

[12] R. E. Castillo, C. Suarez, E. Trousselot, *Composition operator on* $\Lambda_x^p(\omega)$ *spaces*, Quaest. Math., **40** (2017), 833–848. 1

[13] L. Ephremidze, V. Kokilashvili, S. Samko, *Fractional, maximal and singular operators in variable exponent Lorentz spaces*, Fract. Calc. Appl. Anal., **11** (2008), 407–420. 1

[14] J. García-Cuerva, J. L. Rubio de Francia, *Weighted norm inequaities and related topics*, North-Holland Publishing Co., Amsterdam, (1985).

[15] V. S. Guliyev, A. Serbetci, I. Ekincioglu, *On boundedness of the generalized* B*-potential integral operators in the Lorentz spaces*, Integral Transforms Spec. Funct., **18** (2007), 885–895. 1

[16] A. Hussain, G. Gao, *Multilinear singular integrals and commutators on Herz space with variable exponent*, ISRN Math. Anal., **2014** (2014), 10 pages. 1

[17] A. Hussain, I. Khan, A. Mohamed, *Variable Her-Morrey estimates for rough fractional Hausdorff operator*, J. Inequal. Appl., **2024** (2024), 14 pages. 1

[18] H. Kempka, J. Vybíral, *Lorentz spaces with variable exponents*, Math. Nachr., **287** (2014), 938–954. 1

[19] T. Li, D. Acosta-Soba, A. Columbu, G. Viglialoro, *Dissipative gradient nonlinearities prevent* δ*-formations in local and nonlocal attraction-repulsion chemotaxis models*, Stud. Appl. Math., **154** (2025), 19 pages. 1

[20] G. Lorentz, *Some new Functions Spaces*, Ann. Math., **51** (1950), 37–55. 1

[21] G. Lorentz, *On the theory of spaces* Λ, Pacific J. Math., **1** (1951), 411–429. 1

[22] B. Sultan, A. Hussain, M. Sultan, *Chracterization of generalized Campanato spaces with variable exponents via fractional integrals*, J. Pseudo-Differ. Oper. Appl., **16** (2025), 16 pages. 1

[23] B. Sultan, M. Sultan, *Boundedness of commutators of rough Hardy operators on grand variable Herz spaces*, Forum Math., **36** (2024), 717–733.

[24] B. Sultan, M. Sultan, *Boundedness of higher order commutators of Hardy operators on grand Herz-Morrey spaces*, Bull. Sci. Math., **190** (2024), 19 pages.

[25] B. Sultan, M. Sultan, *Sobolev-type theorem for commutators of Hardy operators in grand Herz spaces*, Ukrainian Math. J., **76** (2024), 1196–1213.

[26] M. Sultan, B. Sultan, *A note on the boundedness of Marcinkiewicz integral operator on continual Herz-Morrey spaces*, Filomat, **39** (2025), 2017–2027.

[27] M. Sultan, B. Sultan, A. Aloqaily, N. Mlaiki, *Boundedness of some operators on grand Herz spaces with variable exponent*, AIMS Math., **8** (2023), 12964–12985.

[28] M. Sultan, B. Sultan, A. Hussain, *Grand Herz-Morrey spaces with variable exponent*, Math. Notes., **114** (2023), 957–977.

[29] B. Sultan, M. Sultan, A. Hussain, *Boundedness of the Bochner-Riesz operators on the weighted Herz-Morrey type Hardy spaces*, Complex Anal. Oper. Theory, **19** (2025), 26 pages.

[30] B. Sultan, M. Sultan, I. Khan, *On Sobolev theorem for higher commutators of fractional integrals in grand variable Herz spaces*, Commun. Nonlinear Sci. Numer. Simul., **126** (2023), 11 pages.

[31] B. Sultan, M. Sultan, A. Khan, T. Abdeljawad, *Boundedness of commutators of variable Marcinkiewicz fractional integral operator in grand variable Herz spaces*, J. Inequal. Appl., **2024** (2024), 16 pages. 1