J. Math. Computer Sci., 40 (2026), 415–443

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Some new variants of fractional Hermite-Hadamard and Pachpatte-type integral inequalities involving Raina's and Mittag-Leffer functions with applications

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Abstract

Integral inequalities combined with convexity in the frame of fractional calculus is an interesting research topic. Mathematical inequalities and convex functions have become vital to the growth of many pure and applied fields of science. In this article, we demonstrate a few generalized fractional integral inequalities involving Raina's function that represent the Mittag-Leffler function. This article provides to an intriguing connection between convex functions, special functions, and fractional calculus. First, we present and investigate the concept of a generalized convex involving Raina's function in a polynomial context and discuss its algebraic properties. We establish the new mathematical approach of Hermite-Hadamard inequality and Pachpatte-type inequality involving the newly introduced definition via Caputo-Fabrizio fractional integral operator. Furthermore, to improve our results, we establish a new fractional lemma and utilizing this, provides some new fractional perspectives of the Hermite-Hadamard-type inequality with the aid of generalized m-convex function involving Raina's function. Applications of some of our presented results to special means are given as well. The study's conclusions provide fresh and noteworthy improvements over previous research, offering special perspectives and contributions to the area.

Keywords: Convex function, Raina function, Generalized convex involving Raina's function, improved power-mean integral inequality, Hölder-İscan inequality, Caputo-Fabrizio operator.

2020 MSC: 26A51, 26A33, 26D07, 26D10, 26D15.

Abbreviations

H-H	Hermite-Hadamard
C-FFIO	Caputo-Fibrizio fractional integral operator
w.r.t.	with respect to
$PG_m CRF$	n-polynomial generalized m-convex involving Raina's function

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doi: 10.22436/jmcs.040.03.08

Received: 2025-05-26 *Revised:* 2025-06-09 *Accepted:* 2025-06-26

1. Introduction

Convexity has spurred a fascinating and useful branch of research over the past several decades, spanning a wide range of problems in pure and applied science. The study of convex functions has blossomed in many different ways with many different tools and calculations. Convexity is a powerful framework for creating mathematical tools to solve complex problems. The close relationship between convexity and inequalities has already attracted the interest of specialists in the domain and has given rise to many new extensions of the classical Hadamard inequality, which have been widely studied and abundant in the literature. Geared with the concept of a convex function, formed relatively in the theory of inequalities by being introduced in fractional analysis, it has important brains and astonishing implementations in various areas of applied sciences. For the literature, see references [4–6, 10, 16, 17, 20, 22].

Fractional calculus is a branch of mathematics that investigates the theory and applications of fractional integration over complicated domains. It is the extension of traditional calculus to deal with integration and differentiation of orders of any noninteger value. Because of its enormous practical potential, fractional calculus has attracted a lot of attention from mathematicians in recent years. Worth mentioning is that J. Liouville is considered to be one of the earliest in this field, with a strict definition of fractional derivatives and a series of nine articles on fractional calculus between 1832 and 1837, with a last contribution in 1855. Between 1802 and 1829, N.H. Abel is said to have initiated the first useful application of fractional calculus. Fractional operators are essential in the construction of fractional calculus, which has attracted much interest over the past decade. The most important reason for investigating fractional integral inequalities lies in the incessant extensions of classical inequalities, i.e., Ostrowski, Simpson, and Hadamard inequalities, via different kinds of fractional operators. Researchers have been making regular inputs to fractional analysis, bringing new ideas and structures that have stretched the limits of applied mathematics. The C-FFIO has wide range of applications due to its non-singular, exponentially decaying kernel, which efficiently models interprets with seamless memory effects. It is employed to solve fractional differential equations that arise in anomalous diffusion, heat and mass transfer, and viscoelasticity. By including memory-dependent behavior, it improves the modeling of biological systems and disease dynamics in epidemiology and biomedical engineering. It is also used in fields like finance, control systems, and environmental science where non-local effects and long-term dependencies are crucial. In real-world modeling, its non-singularity also improves physical interpretability and numerical stability. Because of its wide-ranging implications and applications, C-FFIO has emerged as an interesting research operator for mathematicians and scientists. The fractional integral inequality theory, specifically, has wide-ranging applications and significance in applied mathematics.

Furthermore, integral inequalities on the Raina function have been addressed for a long time. In order to familiarize the readers with the current trend of developing research in a variety of fields, including Mittag-Leffler and Raina-type functions, an almost comprehensive overview of references pertaining to these functions is presented. Analytic inequalities of this type, as well as particularly related strategies, have applications for various fields where symmetry plays an important role. Due to their potential to be prolonged, some variations have been established by a number of mathematicians; see references [30, 31].

Building upon this theme and motivated by the growing body of research in this area, this work is organized as follows. To begin with, in Section 2, we incorporate some well-known remarks, corollaries, theorems, and definitions because all of these will be needed in future sections. In Section 3, we present the new idea, that is, the n-polynomial generalized m-convex involving Raina's function. Subsequently, we incorporate in this section its algebraic features. In Section 4, we establish the new approach of H-H inequality through a newly presented concept via C-FFIO. In Section 5, we discuss a new lemma and by using this new presented lemma along with power mean and Hölder inequality, we establish some new fractional estimations of the H-H inequality through a new presented idea. In Section 6, on the basis of a newly presented lemma with the help of an improved version of the power mean and Hölder inequality, we achieve some new fractional forms of the H-H inequality through a new established idea. In Section 7, we study a new mathematical fractional approach of Pachpatte-type inequality through a

newly established definition. Subsequently, in Section 8, we present some applications in the context of established results. Finally, in Section 9, we provide a conclusion.

2. Preliminaries

In order to make readers' comprehension and participation easier, this section will concentrate on a limited number of definitions, comments, and theorems that are pivotal to our discussion. The main purpose of this section is to establish the foundation by presenting and investigating key concepts and definitions that will be essential to our analysis in later sections. We start by considering the concept of convex functions and the classical H-H inequality in the context of classical calculus, setting the stage for our subsequent discussion of fractional calculus.

Definition 2.1 ([18]). A real-valued function Π is said to be convex, if

$$\Pi\left(\mathfrak{e}\mathfrak{x}_1+(1-\mathfrak{e})\,\mathfrak{x}_2\right)\leqslant \mathfrak{e}\Pi\left(\mathfrak{x}_1\right)+(1-\mathfrak{e})\,\Pi\left(\mathfrak{x}_2\right),$$

holds for all $\mathfrak{x}_1,\mathfrak{x}_2\in I$ and $\mathfrak{e}\in[0,1]$.

The most famous inequality involving convex functions is the H-H inequality [12] stated as follows.

Theorem 2.2. *If $\Pi:[\mathfrak{x}_1,\mathfrak{x}_2]\to\mathbb{R}$ is a convex function, then*

$$\Pi\left(\frac{\mathfrak{x}_1+\mathfrak{x}_2}{2}\right)\leqslant\frac{1}{\mathfrak{x}_2-\mathfrak{x}_1}\int_{\mathfrak{x}_1}^{\mathfrak{x}_2}\Pi(x)dx\leqslant\frac{\Pi(\mathfrak{x}_1)+\Pi(\mathfrak{x}_2)}{2}. \tag{2.1}$$

The above inequality (2.1) becomes in the revert if Π is concave over the interval $[\mathfrak{x}_1,\mathfrak{x}_2]$ and the sharp constant $\frac{1}{2}$ cannot be swapped by a comparable constant. Since then, different articles with new ideas, different results and hypotheses, and progress have been suggested in the field. Due to application and perception in the domain of mathematical analysis, this type of inequality has attracted a lot of attention. Raina [21] proposed a family of functions formally stated by

$$\mathcal{R}^{\rho}_{\epsilon,\theta}(z)=\ \mathcal{R}^{\rho(0),\,\rho(1),\ldots}_{\epsilon,\theta}(z)=\sum_{k=0}^{+\infty}\frac{\rho(\mathfrak{v})}{\Gamma(\epsilon k+\theta)}z^k, \tag{2.2}$$

where $\rho=(\rho(0),\ldots,\rho(\mathfrak{v}),\ldots)$ and $\epsilon,\theta>0,|z|<R$. Equation (2.2) is the extension of classical Mittag-Leffler function.

If $\epsilon=1,\theta=0$, and $\rho(\mathfrak{v})=\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}$ for $k=0,1,2,\ldots$, where α,β, and γ are parameters which can take arbitrary real or complex values (provided that $\gamma\neq0,-1,-2,\ldots$), and the symbol α_k denotes the quantity

$$(\alpha)_k=\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}=\alpha(\alpha+1)\cdots(\alpha+k-1),\quad k=0,1,2,\ldots,$$

and restricts its domain to $|z|\leqslant1$ (with $z\in\mathbb{C}$), then we have the classical hypergeometric function, that is

$$\mathcal{R}(\alpha,\beta;\gamma;z)=\sum_{k=0}^{+\infty}\frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k}z^k.$$

Moreover, if $\rho=(1,1,\ldots)$ with $\epsilon=\alpha$, $(Re(\alpha)>0),\theta=1$, then

$$\mathfrak{E}_\alpha(z)=\sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(1+\alpha k)}. \tag{2.3}$$

Equation (2.3) is referred to as a classical Mittag-Leffler function and was initially examined by Magnus Gustaf (Gösta) Mittag-Leffler in 1903 and Anders Wiman in 1905. Since then, the Mittag-Leffler function

has been generalized and investigated in many elective methods and contexts. The applications of the Mittag-Leffler function have been extended across diverse disciplines, encompassing mathematics, natural sciences, physics, engineering, and statistics.

The Mittag-Leffler function appears usually in the study of fractional calculus and especially in the studies of fractional conjecture of the kinetic equation, super diffusive transport, random walks, Lévy flights, and in the studies of complicated structures. It has become remarkably popular in many recent research papers, primarily due to their potential applications in reaction-diffusion and other applied problems; see the references [2, 8, 9, 14, 24, 25]. Cortez presented the generalized convex set and the convex function pertaining to Raina's function in [30, 31].

Definition 2.3 ([30]). Let $\rho = (\rho(0), \dots, \rho(\upsilon), \dots)$ and $\epsilon, \theta > 0$. A set $X \neq \emptyset$ is said to be generalized convex, if $\mathfrak{x}_1 + \mathfrak{e}\, \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{x}_1) \in X$, for all $\mathfrak{x}_1, \mathfrak{x}_2 \in X$ and $\mathfrak{e} \in [0,1]$.

Definition 2.4 ([30]). Let ρ represent a bounded sequence then $\rho = (\rho(0), \dots, \rho(\upsilon), \dots)$ and $\epsilon, \theta > 0$. If real-valued Π holds the following inequality

$$\Pi\Big(\mathfrak{x}_1 + \mathfrak{e}\, \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{x}_1)\Big) \leqslant \mathfrak{e}\Pi(\mathfrak{x}_2) + (1-\mathfrak{e})\Pi(\mathfrak{x}_1),$$

for all $\mathfrak{x}_1, \mathfrak{x}_2 \in X$, where $\mathfrak{x}_1 < \mathfrak{x}_2$ and $\mathfrak{e} \in [0,1]$, then Π is said to be generalized convex function.

Remark 2.5. If $\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{x}_1) = \mathfrak{x}_2 - \mathfrak{x}_1 > 0$, then we achieve Definition 2.1.

The following Condition-A first time explored by Ahmad et al. [3].

Condition A. Let X be generalized convex subset w.r.t. $\mathcal{R}^{\rho}_{\epsilon,\theta}(\cdot)$. For any $\mathfrak{x}_1, \mathfrak{x}_2 \in X$ and $\mathfrak{e} \in [0,1]$,

$$\mathcal{R}^{\rho}_{\epsilon,\theta}\Big(\mathfrak{x}_1 - (\mathfrak{x}_1 + \mathfrak{e}\, \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{x}_1))\Big) = -\mathfrak{e}\, \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{x}_1),$$

$$\mathcal{R}^{\rho}_{\epsilon,\theta}\Big(\mathfrak{x}_2 - \Big(\mathfrak{x}_1 + \mathfrak{e}\, \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{x}_1)\Big)\Big) = (1-\mathfrak{e})\, \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{x}_1).$$

Note that, for every $\mathfrak{x}_1$, $\mathfrak{x}_2 \in X$ and for all $\mathfrak{e}_1$, $\mathfrak{e}_2 \in [0,1]$ from Condition-A, we have

$$\mathcal{R}^{\rho}_{\epsilon,\theta}\Big(\mathfrak{x}_1 + \mathfrak{e}_2\, \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{x}_1) - (\mathfrak{x}_1 + \mathfrak{e}_1\, \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{x}_1))\Big) = (\mathfrak{e}_2 - \mathfrak{e}_1)\, \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{x}_1).$$

Definition 2.6 ([29]). Two functions Π_1 and Π_2 are said to be similarly ordered if

$$(\Pi_1(\mathfrak{x}_1) - \Pi_2(\mathfrak{x}_2))(\Pi_2(\mathfrak{x}_1) - \Pi_2(\mathfrak{x}_2)) \geqslant 0, \quad \forall \mathfrak{x}_1, \mathfrak{x}_2 \in \mathbb{R}.$$

Hölder's inequality and power mean inequality both play vital roles in functional analysis and are important techniques for the consideration of how functions behave in various function spaces. Both inequalities provide a fundamental format for the rigorous investigation of integrable functions, allowing a deeper analysis of functional relationships, operator behavior, and intrinsic depth of functional spaces.

Theorem 2.7 ([13]). *If Π_1 and Π_2 are real functions defined on $[\mathfrak{x}_1, \mathfrak{x}_2]$ and $|\Pi_1|^p, |\Pi_2|^q$ are integrable functions on $[\mathfrak{x}_1, \mathfrak{x}_2]$, then for $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\int_{\mathfrak{x}_1}^{\mathfrak{x}_2} |\Pi_1(x)\Pi_2(x)|dx \leqslant \frac{1}{\mathfrak{x}_2 - \mathfrak{x}_1}\Bigg\{\left(\int_{\mathfrak{x}_1}^{\mathfrak{x}_2} (\mathfrak{x}_2 - x)|\Pi_1(x)|^p dx\right)^{\frac{1}{p}} \left(\int_{\mathfrak{x}_1}^{\mathfrak{x}_2} (\mathfrak{x}_2 - x)|\Pi_2(x)|^q dx\right)^{\frac{1}{q}}$$
$$+ \left(\int_{\mathfrak{x}_1}^{\mathfrak{x}_2} (x - \mathfrak{x}_1)|\Pi_1(x)|^p dx\right)^{\frac{1}{p}} \left(\int_{\mathfrak{x}_1}^{\mathfrak{x}_2} (x - \mathfrak{x}_1)|\Pi_2(x)|^q dx\right)^{\frac{1}{q}}\Bigg\}.$$

Theorem 2.8 ([15]). *If Π_1 and Π_2 are real functions defined on $[\mathfrak{x}_1,\mathfrak{x}_2]$ and $|\Pi_1|,|\Pi_2|^q$ are integrable functions on $[\mathfrak{x}_1,\mathfrak{x}_2]$, then for $q \geqslant 1$, we have*

$$\int_{\mathfrak{x}_1}^{b}|\Pi_1(x)\Pi_2(x)|dx \leqslant \frac{1}{\mathfrak{x}_2-\mathfrak{x}_1}\left\{\left(\int_{\mathfrak{x}_1}^{\mathfrak{x}_2}(\mathfrak{x}_2-x)|\Pi_1(x)|dx\right)^{1-\frac{1}{q}}\left(\int_{\mathfrak{x}_1}^{\mathfrak{x}_2}(\mathfrak{x}_2-x)|\Pi_1(x)||\Pi_2(x)|^q dx\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\mathfrak{x}_1}^{\mathfrak{x}_2}(x-\mathfrak{x}_1)|\Pi_1(x)|dx\right)^{1-\frac{1}{q}}\left(\int_{\mathfrak{x}_1}^{\mathfrak{x}_2}(x-\mathfrak{x}_1)|\Pi_1(x)||\Pi_2(x)|^q dx\right)^{\frac{1}{q}}\right\}.$$

We discuss several key definitions from the concept of fractional calculus that will be used in the following results.

Definition 2.9 ([26]). Suppose $p \in [1,\infty)$ and $(\mathfrak{x}_1,\mathfrak{x}_2) \subseteq \mathbb{R}$, then the Sobolev space $H^p(\mathfrak{x}_1,\mathfrak{x}_2)$ is defined by

$$H^p(\mathfrak{x}_1,\mathfrak{x}_2) = \{\Pi \in L^2(\mathfrak{x}_1,\mathfrak{x}_2) : \Delta^u\Pi \in L^2(\mathfrak{x}_1,\mathfrak{x}_2), \forall\ |u| \leqslant p\}.$$

Definition 2.10 ([11]). Let $\Pi \in H^1(\mathfrak{x}_1,\mathfrak{x}_2)$, $\sigma \in [0,1]$ and $\mathfrak{x}_1 < \mathfrak{x}_2$. The C-FFIO in the left mode is given as

$$\left({}^{\mathcal{CF}}_{\mathfrak{x}_1}I^{\sigma}\Pi\right)(\mathfrak{e}) = \frac{(1-\sigma)}{\mathcal{B}(\sigma)}\Pi(\mathfrak{e}) + \frac{\sigma}{\mathcal{B}(\sigma)}\int_{\mathfrak{x}_1}^{\mathfrak{e}}\Pi(x)dx.$$

Similarly, the right C-FFIO is given as

$$\left({}^{\mathcal{CF}}I^{\sigma}_{\mathfrak{x}_2}\Pi\right)(\mathfrak{e}) = \frac{(1-\sigma)}{\mathcal{B}(\sigma)}\Pi(\mathfrak{e}) + \frac{\sigma}{\mathcal{B}(\sigma)}\int_{\mathfrak{e}}^{\mathfrak{x}_2}\Pi(x)dx,$$

where $\mathcal{B}(\sigma) > 0$ is a normalization function satisfying $\mathcal{B}(0) = \mathcal{B}(1) = 1$.

Generalizations and refinements of inequalities pertaining to fractional operators can exist in the literature (see for example, [1, 19, 23, 27, 28]).

In this paper, we modify the equality employing C-FFIO, which is introduced by Dragomir [7].

Theorem 2.11 ([11]). *Let $\Pi : \mathbb{I} \to \mathbb{R}$ be a convex function on $\mathbb{I}$. If $\mathfrak{x}_1,\mathfrak{x}_2 \in \mathbb{I}$ with $\mathfrak{x}_1 < \mathfrak{x}_2$ and $\Pi \in \mathcal{L}[\mathfrak{x}_1,\mathfrak{x}_2]$, then*

$$\Pi\left(\frac{\mathfrak{x}_1+\mathfrak{x}_2}{2}\right) \leqslant \frac{\mathcal{B}(\sigma)}{\sigma(\mathfrak{x}_2-\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}I^{\sigma}_{\mathfrak{x}_1}\Pi\right)(\mathfrak{v}) + \left({}^{\mathcal{CF}}I^{\sigma}_{\mathfrak{x}_2}\Pi\right)(\mathfrak{v}) - \frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi(\mathfrak{v})\right] \leqslant \frac{\Pi(\mathfrak{x}_1)+\Pi(\mathfrak{x}_2)}{2}, \tag{2.4}$$

where $\sigma \in [0,1]$ and $k \in [\mathfrak{x}_1,\mathfrak{x}_2]$.

3. n-polynomial generalized m-convex involving Raina's function and its properties

Owing to the theory of convexity's wide applications in optimization problems and applied sciences, over the last few decades it has developed remarkably. Most of the world's problems are nonconvex, despite the fact that convexity has produced a wide range of outcomes. Therefore, it is always worthwhile to investigate nonconvex functions, which are roughly similar to convex functions. Convex functions have been widely praised by many famous mathematicians in the twentieth century, such as Jensen, Hermite, Holder, and Stolz. In the 20th century, an unprecedented volume of research was conducted, producing major results in the areas of geometric functional analysis, convex analysis, and nonlinear programming.

Here, we shall introduce and explore the new definition, i.e., PG_mCRF, an interesting and useful concept for convex functions and examine some of its algebraic properties.

Definition 3.1. Let $\rho = (\rho(0),\ldots,\rho(\mathfrak{v}),\ldots)$ and $\epsilon,\theta > 0$. A set $X \neq \emptyset$ is said to be generalized m-convex, if $m\mathfrak{x}_1 + \mathfrak{e}\,\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1) \in X$, for all $\mathfrak{x}_1,\mathfrak{x}_2 \in X$ and $\mathfrak{e},m \in [0,1]$.

Definition 3.2. A function Π defined on the generalized m-convex set X is said to be n-polynomial generalized m-convex involving Raina's function, i.e., PG_mCRF, if

$$\Pi(m\mathfrak{x}_1 + \mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1)) \leqslant \frac{m}{n}\sum_{\mathfrak{z}=1}^{n}(1-\mathfrak{e}^{\mathfrak{z}})\,\Pi(\mathfrak{x}_1) + \frac{1}{n}\sum_{\mathfrak{z}=1}^{n}(1-(1-\mathfrak{e})^{\mathfrak{z}})\Pi(\mathfrak{x}_2),$$

holds for every $\mathfrak{x}_1, \mathfrak{x}_2 \in X$, $m \in (0,1]$, $\mathfrak{e} \in [0,1]$ and $n \in \mathbb{N}$.

Remark 3.3.

1. If $n = 1$, then we obtained the new definition namely, generalized m-convex involving Raina's function, i.e., G_mCRF,
$$\Pi(m\mathfrak{x}_1 + \mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1)) \leqslant m(1-\mathfrak{e})\,\Pi(\mathfrak{x}_1) + \mathfrak{e}\Pi(\mathfrak{x}_2).$$
2. If $n = m = 1$, then Definition 3.2 reverts to the Definition 2.4.
3. If $n = m = 1$ and $\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1) = \mathfrak{x}_2 - m\mathfrak{x}_1$, then Definition 3.2 reverts to the idea of convex function, which was investigated by Niculescu [18].

Please note that every convex function is G_mCRF, but the converse does not hold in general.

Here, we are going to introduce the new condition, namely extended Condition-A, in the following way.

Extended Condition-A. Let X be generalized m-convex subset w.r.t. $\mathcal{R}^{\rho}_{\epsilon,\theta}(\cdot)$. For any $\mathfrak{x}_1, \mathfrak{x}_2 \in X$ and $\mathfrak{e} \in [0,1]$,

$$\mathcal{R}^{\rho}_{\epsilon,\theta}\Big(\mathfrak{x}_1 - (m\mathfrak{x}_1 + \mathfrak{e}\,\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1))\Big) = -\mathfrak{e}\,\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1),$$
$$\mathcal{R}^{\rho}_{\epsilon,\theta}\Big(\mathfrak{x}_2 - \Big(m\mathfrak{x}_1 + \mathfrak{e}\,\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1)\Big)\Big) = (1-\mathfrak{e})\,\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1).$$

Note that, for every $\mathfrak{x}_1$, $\mathfrak{x}_2 \in X$ and for all $\mathfrak{e}_1$, $\mathfrak{e}_2 \in [0,1]$ from extended Condition-A, we have

$$\mathcal{R}^{\rho}_{\epsilon,\theta}\Big(m\mathfrak{x}_1 + \mathfrak{e}_2\,\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1) - (m\mathfrak{x}_1 + \mathfrak{e}_1\,\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1))\Big) = (\mathfrak{e}_2 - \mathfrak{e}_1)\,\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1).$$

Lemma 3.4. *The following inequalities*

$$\mathfrak{e} \leqslant \frac{1}{n}\sum_{\mathfrak{z}=1}^{n}(1-(1-\mathfrak{e})^{\mathfrak{z}}) \quad \textit{and} \quad m(1-\mathfrak{e}) \leqslant \frac{m}{n}\sum_{\mathfrak{z}=1}^{n}(1-\mathfrak{e}^{\mathfrak{z}})$$

are hold, $\forall\ \mathfrak{e} \in [0,1]$, $\mathfrak{z} \in \mathbb{N}$, *and* $m \in (0,1]$.

Proof. The proof is obvious. □

Proposition 3.5. *Every non-negative* G_mCRF *is* PG_mCRF.

Proof. Using G_mCRF and Lemma 3.4, we have

$$\Pi(m\mathfrak{x}_1 + \mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1)) \leqslant m(1-\mathfrak{e})\Pi(\mathfrak{x}_1) + \mathfrak{e}\Pi(\mathfrak{x}_2) \leqslant \frac{m}{n}\sum_{\mathfrak{z}=1}^{n}(1-\mathfrak{e}^{\mathfrak{z}})\,\Pi(\mathfrak{x}_1) + \frac{1}{n}\sum_{\mathfrak{z}=1}^{n}(1-(1-\mathfrak{e})^{\mathfrak{z}})\,\Pi(\mathfrak{x}_2).$$

□

Proposition 3.6. *Every* PG_mCRF *is Generalized* (h,m)*-convex involving Raina's function with* $h(\mathfrak{e}) = \frac{1}{n}\sum_{\mathfrak{z}=1}^{n}(1-(1-\mathfrak{e})^{\mathfrak{z}})$.

Proof. Using $PG_m CRF$ and given condition, we have

$$\Pi(m\mathfrak{x}_1 + \epsilon\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1)) \leqslant \frac{m}{n}\sum_{\mathfrak{z}=1}^{n}(1-\epsilon^{\mathfrak{z}})\,\Pi(\mathfrak{x}_1) + \frac{1}{n}\sum_{\mathfrak{z}=1}^{n}(1-(1-\epsilon)^{\mathfrak{z}})\,\Pi(\mathfrak{x}_2) \leqslant mh(1-\epsilon)\Pi(\mathfrak{x}_1) + h(\epsilon)\Pi(\mathfrak{x}_2).$$

□

From the above proposition, it appears evident that the newly introduced idea is highly big compared to already published functions, like convex and generalized m-convex. This is the most attractive attribute of the planned new Definition 3.2.

Next, we employ a newly established concept to present some examples.

Example 3.7. Suppose $\Pi(\mathfrak{x}) = |\mathfrak{x}|$, $\forall \mathfrak{x} \geqslant 0$ is a convex function, $\Rightarrow \Pi(\mathfrak{x}) = |\mathfrak{x}|$ is a $G_m CRF$. By utilizing Proposition 3.5, it is an $PG_m CRF$.

Example 3.8. Suppose $\Pi(\mathfrak{x}) = e^{\mathfrak{x}}$, $\forall \mathfrak{x} \geqslant 0$ is a convex function, $\Rightarrow \Pi(\mathfrak{x}) = e^{\mathfrak{x}}$ is a $G_m CRF$. By utilizing Proposition 3.5, it is an $PG_m CRF$.

We are going to look at and develop a few examples of the recently presented concept.

Example 3.9. Suppose $\Pi : \mathbb{R}^+ \to \mathbb{R}^+$ is stated by

$$\Pi(\mathfrak{x}) = \begin{cases} \mathfrak{x}, & 0 \leqslant \mathfrak{x} \leqslant 1, \\ 1, & \mathfrak{x} > 1, \end{cases} \quad \text{and} \quad \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1) = \begin{cases} \mathfrak{x}_2 - m\mathfrak{x}_1 & \mathfrak{x}_2 \leqslant 0, \mathfrak{x}_1 \leqslant 0, \\ \mathfrak{x}_2 - m\mathfrak{x}_1, & 0 \leqslant \mathfrak{x}_2 \leqslant 1, \mathfrak{x}_1 \leqslant 1, \\ -2 - m\mathfrak{x}_1, & \mathfrak{x}_2 \leqslant 0, 0 \leqslant \mathfrak{x}_1 \leqslant 1, \\ 2 - m\mathfrak{x}_1, & 0 \leqslant \mathfrak{x}_2 \leqslant 1, \mathfrak{x}_1 \leqslant 0. \end{cases}$$

The non-negative function $\Pi(x)$ is $G_m CRF$, but not convex. Employing to Proposition 3.5, it is an $PG_m CRF$.

Example 3.10. Let $\Pi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$\Pi(\mathfrak{x}) = \begin{cases} \mathfrak{x} + 1, & 0 \leqslant \mathfrak{x} \leqslant 1, \\ 1, & \mathfrak{x} > 1, \end{cases} \quad \text{and} \quad \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1) = \begin{cases} \mathfrak{x}_2 + m\mathfrak{x}_1, & \mathfrak{x}_2 \leqslant \mathfrak{x}_1, \\ 2(\mathfrak{x}_2 + m\mathfrak{x}_1), & \mathfrak{x}_2 > \mathfrak{x}_1, \end{cases}$$

$\forall \mathfrak{x}_1, \mathfrak{x}_2 \in \mathbb{R}^+ = [0, +\infty)$. Implies that the function $\Pi(x)$ is $G_m CRF$, but it is not convex. Utilizing to Proposition 3.5, it is an $PG_m CRF$.

Theorem 3.11. *If Π_1, Π_2 are two $PG_m CRF$, then $(\Pi_1 + \Pi_2)$ is also a $PG_m CRF$.*

Proof. Since given that Π_1 and Π_2 be two $PG_m CRF$, then

$$\begin{aligned}
(\Pi_1 + \Pi_2)(m\mathfrak{x}_1 + \epsilon\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1)) &= \Pi_1(m\mathfrak{x}_1 + \epsilon\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1)) + \Pi_2(m\mathfrak{x}_1 + \epsilon\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1)) \\
&\leqslant \frac{m}{n}\sum_{\mathfrak{z}=1}^{n}(1-\epsilon^{\mathfrak{z}})\,\Pi_1(\mathfrak{x}_1) + \frac{1}{n}\sum_{\mathfrak{z}=1}^{n}(1-(1-\epsilon)^{\mathfrak{z}})\Pi_1(\mathfrak{x}_2) \\
&\quad + \frac{m}{n}\sum_{\mathfrak{z}=1}^{n}(1-\epsilon^{\mathfrak{z}})\,\Pi_2(\mathfrak{x}_1) + \frac{1}{n}\sum_{\mathfrak{z}=1}^{n}(1-(1-\epsilon)^{\mathfrak{z}})\Pi_2(\mathfrak{x}_2) \\
&= \frac{m}{n}\sum_{\mathfrak{z}=1}^{n}(1-\epsilon^{\mathfrak{z}})\,[\Pi_1(\mathfrak{x}_1) + \Pi_2(\mathfrak{x}_1)] + \frac{1}{n}\sum_{\mathfrak{z}=1}^{n}(1-(1-\epsilon)^{\mathfrak{z}})\,[\Pi_1(\mathfrak{x}_2) + \Pi_2(\mathfrak{x}_2)] \\
&= \frac{m}{n}\sum_{\mathfrak{z}=1}^{n}(1-\epsilon^{\mathfrak{z}})\,(\Pi_1 + \Pi_2)(\mathfrak{x}_1) + \frac{1}{n}\sum_{\mathfrak{z}=1}^{n}(1-(1-\epsilon)^{\mathfrak{z}})(\Pi_1 + \Pi_2)(\mathfrak{x}_2).
\end{aligned}$$

This is the required proof. □

Theorem 3.12. *If* Π *is* $PG_m CRF$, *then* $(c\Pi)$ *is also an* $PG_m CRF$.

Proof. Since Π is $PG_m CRF$, and c is any constant number, then

$$\begin{aligned}(c\Pi)\left(m\mathfrak{x}_1+\mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)\right) &\leqslant c\left[\frac{m}{n}\sum_{\mathfrak{z}=1}^{n}(1-\mathfrak{e}^{\mathfrak{z}})\,\Pi\left(\mathfrak{x}_1\right)+\frac{1}{n}\sum_{\mathfrak{z}=1}^{n}(1-(1-\mathfrak{e})^{\mathfrak{z}})\Pi\left(\mathfrak{x}_2\right)\right]\\ &=\frac{m}{n}\sum_{\mathfrak{z}=1}^{n}(1-\mathfrak{e}^{\mathfrak{z}})\,c\Pi\left(\mathfrak{x}_1\right)+\frac{1}{n}\sum_{\mathfrak{z}=1}^{n}(1-(1-\mathfrak{e})^{\mathfrak{z}})c\Pi\left(\mathfrak{x}_2\right)\\ &=\frac{m}{n}\sum_{\mathfrak{z}=1}^{n}(1-\mathfrak{e}^{\mathfrak{z}})\,(c\Pi)\left(\mathfrak{x}_1\right)+\frac{1}{n}\sum_{\mathfrak{z}=1}^{n}(1-(1-\mathfrak{e})^{\mathfrak{z}})\,(c\Pi)\left(\mathfrak{x}_2\right).\end{aligned}$$

This completes the proof. □

Theorem 3.13. *Composition of two* $PG_m CRF$ *is also an* $PG_m CRF$.

Proof.

$$\begin{aligned}(\Pi_2\circ\Pi_1)\left(m\mathfrak{x}_1+\mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)\right) &=\Pi_2(\Pi_1(m\mathfrak{x}_1+\mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)))\\ &\leqslant\Pi_2\left[\frac{m}{n}\sum_{\mathfrak{z}=1}^{n}(1-\mathfrak{e}^{\mathfrak{z}})\,\Pi_1\left(\mathfrak{x}_1\right)+\frac{1}{n}\sum_{\mathfrak{z}=1}^{n}(1-(1-\mathfrak{e})^{\mathfrak{z}})\Pi_1\left(\mathfrak{x}_2\right)\right]\\ &\leqslant\frac{m}{n}\sum_{\mathfrak{z}=1}^{n}(1-\mathfrak{e}^{\mathfrak{z}})\,\Pi_2(\Pi_1\left(\mathfrak{x}_1\right))+\frac{1}{n}\sum_{\mathfrak{z}=1}^{n}(1-(1-\mathfrak{e})^{\mathfrak{z}})\Pi_2(\Pi_1\left(\mathfrak{x}_2\right))\\ &=\frac{m}{n}\sum_{\mathfrak{z}=1}^{n}(1-\mathfrak{e}^{\mathfrak{z}})\,(\Pi_2\circ\Pi_1)\left(\mathfrak{x}_1\right)+\frac{1}{n}\sum_{\mathfrak{z}=1}^{n}(1-(1-\mathfrak{e})^{\mathfrak{z}})(\Pi_2\circ\Pi_1)\left(\mathfrak{x}_2\right).\end{aligned}$$

This is the required proof. □

Theorem 3.14. *Let* $0<\mathfrak{x}_1<\mathfrak{x}_2$, $\Pi_j:\mathbb{X}=[\mathfrak{x}_1,\mathfrak{x}_2]\to[0,+\infty)$ *be a family of* $PG_m CRF$ *and* $\Pi(u)=\sup_j\Pi_j(u)$. *Then,* Π *is an* $PG_m CRF$ *for* $m\in(0,1]$, $\mathfrak{e}\in[0,1]$, *and* $\mathcal{U}=\{\Pi\in[\mathfrak{x}_1,\mathfrak{x}_2]:\Pi(\Pi_{\mathfrak{z}})<\infty\}$ *is an interval.*

Proof. Let $\mathfrak{x}_1,\mathfrak{x}_2\in\mathcal{U}$, $m\in(0,1]$, and $\mathfrak{e}\in[0,1]$, then

$$\begin{aligned}\Pi(m\mathfrak{x}_1+\mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)) &=\sup_j\Pi_j(m\mathfrak{x}_1+\mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))\\ &\leqslant\frac{m}{n}\sum_{\mathfrak{z}=1}^{n}(1-\mathfrak{e}^{\mathfrak{z}})\sup_j\Pi_j\left(\mathfrak{x}_1\right)+\frac{1}{n}\sum_{\mathfrak{z}=1}^{n}(1-(1-\mathfrak{e})^{\mathfrak{z}})\sup_j\Pi_j\left(\mathfrak{x}_2\right)\\ &=\frac{m}{n}\sum_{\mathfrak{z}=1}^{n}(1-\mathfrak{e}^{\mathfrak{z}})\,\Pi\left(\mathfrak{x}_1\right)+\frac{1}{n}\sum_{\mathfrak{z}=1}^{n}(1-(1-\mathfrak{e})^{\mathfrak{z}})\Pi\left(\mathfrak{x}_2\right)<\infty.\end{aligned}$$

This is the required proof. □

Theorem 3.15. *If* $\Pi_{\mathfrak{z}}:\mathbb{R}^n\to\mathbb{R}$ *is an* $PG_m CRF$, *then* $\mathbb{M}=\{x\in\mathbb{R}:\Pi_{\mathfrak{z}}(x)\leqslant 0,\mathfrak{z}=1,2,3,\ldots,n\}$ *is a generalized* m*-convex set.*

Proof. Since $\Pi_{\mathfrak{z}}(x)$, $(\mathfrak{z}=1,2,3,\ldots,n)$ is $PG_m CRF$ for $m\in(0,1]$ and $\mathfrak{e}\in[0,1]$, then for all $\mathfrak{x}_1,\mathfrak{x}_2\in\mathbb{R}^n$,

$$\Pi_{\mathfrak{z}}(m\mathfrak{x}_1+\mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))\leqslant\frac{m}{n}\sum_{\mathfrak{z}=1}^{n}(1-\mathfrak{e}^{\mathfrak{z}})\,\Pi\left(\mathfrak{x}_1\right)+\frac{1}{n}\sum_{\mathfrak{z}=1}^{n}(1-(1-\mathfrak{e})^{\mathfrak{z}})\Pi\left(\mathfrak{x}_2\right)$$

holds. When $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathbb{M}$, we know $\Pi_\mathfrak{z}(\mathfrak{x}_1) \leqslant 0$ and $\Pi_\mathfrak{z}(\mathfrak{x}_2) \leqslant 0$, then the above inequality implies that

$$\Pi_\mathfrak{z}(\mathfrak{m}\mathfrak{x}_1 + \mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)) \leqslant 0, \quad \mathfrak{z} = 1, 2, 3, \ldots, \mathfrak{n}.$$

That is, $\mathfrak{m}\mathfrak{x}_1 + \mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1) \in \mathbb{M}$. $\Rightarrow \mathbb{M}$ is a generalized $\mathfrak{m}$-convex set. □

Theorem 3.16. *If Π is a $\mathrm{PG}_\mathfrak{m}\mathrm{CRF}$ on generalized $\mathfrak{m}$-convex set $\mathbb{A}$, then Π is also quasi $\mathrm{PG}_\mathfrak{m}\mathrm{CRF}$.*

Proof.

$$\begin{aligned}\Pi(\mathfrak{m}\mathfrak{x}_1 + \mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)) &\leqslant \frac{\mathfrak{m}}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}(1-\mathfrak{e}^\mathfrak{z})\,\Pi(\mathfrak{x}_1) + \frac{1}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}(1-(1-\mathfrak{e})^\mathfrak{z})\Pi(\mathfrak{x}_2)\\ &\leqslant \left[\frac{\mathfrak{m}}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}(1-\mathfrak{e}^\mathfrak{z}) + \frac{1}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}(1-(1-\mathfrak{e})^\mathfrak{z})\right]\Pi(\mathfrak{x}_2) \leqslant \Pi(\mathfrak{x}_2).\end{aligned}$$

In the same manner, if we let $\Pi(\mathfrak{x}_2) \leqslant \Pi(\mathfrak{x}_1)$ for all $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathbb{X}$, we can also get

$$\Pi(\mathfrak{m}\mathfrak{x}_1 + \mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)) \leqslant \Pi(\mathfrak{x}_1).$$

Consequently,

$$\Pi(\mathfrak{m}\mathfrak{x}_1 + \mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)) \leqslant \max\{\Pi(\mathfrak{x}_1), \Pi(\mathfrak{x}_2)\}.$$

This completes the proof. □

Theorem 3.17. *If $\Pi_\mathfrak{z} : \mathbb{A} \subseteq \mathbb{R}^\mathfrak{n} \to \mathbb{R}(\mathfrak{z} = 1, 2, \ldots, \mathfrak{n})$ is $\mathrm{PG}_\mathfrak{m}\mathrm{CRF}$, then the function $\Pi = \sum_{\mathfrak{z}=1}^{\mathfrak{n}} \mathfrak{x}_\mathfrak{z}\Pi_\mathfrak{z}$, $\mathfrak{x}_\mathfrak{z} \geqslant 0$, $\mathfrak{z} = 1, 2, 3, \ldots, \mathfrak{n}$ is also an $\mathrm{PG}_\mathfrak{m}\mathrm{CRF}$, where $\mathfrak{x}_\mathfrak{z}$ is constant.*

Proof. The proof is obvious. □

Theorem 3.18. *Let $\Pi : \mathbb{R}_0 \to \mathbb{R}_0 = [0, \infty)$ be $\mathrm{PG}_\mathfrak{m}\mathrm{CRF}$ w.r.t $\mathcal{R}^{\rho}_{\epsilon,\theta} : \mathbb{R}_0 \times \mathbb{R}_0 \times (0, 1] \to \mathbb{R}_0$ for $\mathfrak{m} \in (0, 1]$ and $\mathfrak{e} \in [0, 1]$. Suppose that $\mathcal{R}^{\rho}_{\epsilon,\theta}$ is monotonic increasing and Π is monotonic decreasing for $\mathfrak{m}$ for fixed $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathbb{R}_0$ and $\mathfrak{m}_1 \leqslant \mathfrak{m}_2$ $(\mathfrak{m}_1, \mathfrak{m}_2 \in (0, 1])$. If Π is $\mathrm{PG}_{\mathfrak{m}_1}\mathrm{CRF}$ on $\mathbb{R}_0$ with respect to $\mathcal{R}^{\rho}_{\epsilon,\theta}$, then Π is $\mathrm{PG}_{\mathfrak{m}_2}\mathrm{CRF}$ on $\mathbb{R}_0$ w.r.t. $\mathcal{R}^{\rho}_{\epsilon,\theta}$.*

Proof. Given that Π is $\mathrm{PG}_{\mathfrak{m}_1}\mathrm{CRF}$, for all $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathbb{R}_0$,

$$\Pi(\mathfrak{m}_1\mathfrak{x}_1 + \mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)) \leqslant \frac{\mathfrak{m}_1}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}(1-\mathfrak{e}^\mathfrak{z})\,\Pi(\mathfrak{x}_1) + \frac{1}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}(1-(1-\mathfrak{e})^\mathfrak{z})\Pi(\mathfrak{x}_2).$$

Combining Π with $\mathcal{R}^{\rho}_{\epsilon,\theta}$ regarding $\mathfrak{m}$ for fixed $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathbb{R}_0$ and $\mathfrak{m}_1 \leqslant \mathfrak{m}_2$, it follows that

$$\begin{aligned}&\Pi(\mathfrak{m}_2\mathfrak{x}_1 + \mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}_2\mathfrak{x}_1)) \leqslant \Pi(\mathfrak{m}_1\mathfrak{x}_1 + \mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}_1\mathfrak{x}_1))\\ &\Rightarrow \Pi(\mathfrak{m}_2\mathfrak{x}_1 + \mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}_2\mathfrak{x}_1)) \leqslant \frac{\mathfrak{m}_1}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}(1-\mathfrak{e}^\mathfrak{z})\,\Pi(\mathfrak{x}_1) + \frac{1}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}(1-(1-\mathfrak{e})^\mathfrak{z})\Pi(\mathfrak{x}_2)\\ &\Rightarrow \Pi(\mathfrak{m}_2\mathfrak{x}_1 + \mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}_2\mathfrak{x}_1)) \leqslant \frac{\mathfrak{m}_2}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}(1-\mathfrak{e}^\mathfrak{z})\,\Pi(\mathfrak{x}_1) + \frac{1}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}(1-(1-\mathfrak{e})^\mathfrak{z})\Pi(\mathfrak{x}_2).\end{aligned}$$

This completes the proof. □

Theorem 3.19. *Product of two $\mathrm{PG}_\mathfrak{m}\mathrm{CRF}$ is also an $\mathrm{PG}_\mathfrak{m}\mathrm{CRF}$.*

Proof. For $m \in (0,1]$ and $\epsilon \in [0,1]$, then

$$\begin{aligned}
&\Pi_1(mx_1 + \epsilon\mathcal{R}^{\rho}_{\epsilon,\theta}(x_2 - mx_1))\Pi_2(mx_1 + \epsilon\mathcal{R}^{\rho}_{\epsilon,\theta}(x_2 - mx_1)) \\
&\leqslant \left[\frac{m}{n}\sum_{j=1}^{n}(1-\epsilon^j)\,\Pi_1(x_1) + \frac{1}{n}\sum_{j=1}^{n}(1-(1-\epsilon)^j)\Pi_1(x_2)\right]\left[\frac{m}{n}\sum_{j=1}^{n}(1-\epsilon^j)\,\Pi_2(x_1) + \frac{1}{n}\sum_{j=1}^{n}(1-(1-\epsilon)^j)\Pi_2(x_2)\right] \\
&\leqslant \frac{m^2}{n^2}\sum_{j=1}^{n}(1-\epsilon^j)^2\,\Pi_1(x_1)\Pi_2(x_1) + \frac{1}{n^2}\sum_{j=1}^{n}(1-(1-\epsilon)^j)^2\Pi_1(x_2)\Pi_2(x_2) \\
&\quad + \frac{m}{n^2}\sum_{j=1}^{n}(1-(1-\epsilon)^j)\,(1-\epsilon^j)\,[\Pi_1(x_1)\Pi_2(x_2) + \Pi_1(x_2)\Pi_2(x_1)] \\
&\leqslant \frac{m^2}{n^2}\sum_{j=1}^{n}(1-\epsilon^j)^2\,\Pi_1(x_1)\Pi_2(x_1) + \frac{1}{n^2}\sum_{j=1}^{n}(1-(1-\epsilon)^j)^2\Pi_1(x_2)\Pi_2(x_2) \\
&\quad + \frac{m}{n^2}\sum_{j=1}^{n}(1-(1-\epsilon)^j)\,(1-\epsilon^j)\,[\Pi_1(x_1)\Pi_2(x_2) + \Pi_1(x_2)\Pi_2(x_2)] \\
&= \left[\frac{m}{n}\sum_{j=1}^{n}(1-\epsilon^j)\,\Pi_1(x_1)\Pi_2(x_1) + \frac{1}{n}\sum_{j=1}^{n}(1-(1-\epsilon)^j)\Pi_1(x_2)\Pi_2(x_2)\right]\left(\frac{1}{n}\sum_{j=1}^{n}(1-(1-\epsilon)^j) + \frac{m}{n}\sum_{j=1}^{n}(1-\epsilon^j)\right) \\
&\leqslant \frac{m}{n}\sum_{j=1}^{n}(1-\epsilon^j)\,\Pi_1(x_1)\Pi_2(x_1) + \frac{1}{n}\sum_{j=1}^{n}(1-(1-\epsilon)^j)\,\Pi_1(x_2)\Pi_2(x_2).
\end{aligned}$$

□

4. New mathematical approach of H-H inequality via n-polynomial generalized m-convex involving Raina's function pertaining to Caputo-Fabrizio fractional integral operator

The primary goal of this portion is to present a new mathematical fractional approach of H-H inequality in the mode of a newly explored idea.

Theorem 4.1. *Suppose $\mathbb{A}^\circ$ be an open generalized m-convex subset. Let $x_1, x_2 \in \mathbb{A}^\circ$, $x_1 < x_2$ with $mx_1 \leqslant mx_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(x_2 - mx_1)$. Suppose $\Pi : [mx_1, mx_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(x_2 - mx_1)] \to \mathbb{R}$ is $\mathrm{PG}_m\mathrm{CRF}$ and satisfies extended condition-A. Then*

$$\begin{aligned}
&\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)\Pi\left(mx_1 + \frac{1}{2}\mathcal{R}^{\rho}_{\epsilon,\theta}(x_2 - mx_1)\right) \\
&\quad\leqslant \frac{1}{\mathcal{R}^{\rho}_{\epsilon,\theta}(x_2 - mx_1)}\int_{mx_1}^{mx_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(x_2-mx_1)}\Pi(x)\,dx \leqslant \frac{m\Pi(x_1) + \Pi(x_2)}{n}\sum_{j=1}^{n}\frac{j}{j+1}.
\end{aligned} \tag{4.1}$$

Proof. Using $\mathrm{PG}_m\mathrm{CRF}$, we have that

$$\Pi(mx_1 + \epsilon\mathcal{R}^{\rho}_{\epsilon,\theta}(x_2 - mx_1)) \leqslant \frac{m}{n}\sum_{j=1}^{n}[1-\epsilon^j]\Pi(x_1) + \frac{1}{n}\sum_{j=1}^{n}[1-(1-\epsilon)^j]\Pi(x_2),$$

$$\int_0^1 \Pi(mx_1 + \epsilon\mathcal{R}^{\rho}_{\epsilon,\theta}(x_2 - mx_1))\,d\epsilon \leqslant \frac{m\Pi(x_1)}{n}\sum_{j=1}^{n}\int_0^1[1-\epsilon^j]\,d\epsilon + \frac{\Pi(x_2)}{n}\sum_{j=1}^{n}\int_0^1[1-(1-\epsilon)^j]\,d\epsilon.$$

But

$$\int_0^1 \Pi(mx_1 + \epsilon\mathcal{R}^{\rho}_{\epsilon,\theta}(x_2 - mx_1))\,d\epsilon = \frac{1}{\mathcal{R}^{\rho}_{\epsilon,\theta}(x_2 - mx_1)}\int_{mx_1}^{mx_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(x_2-mx_1)}\Pi(x)\,dx,$$

so

$$\frac{1}{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\int_{m\mathfrak{x}_1}^{m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(x)dx\leqslant\frac{m\Pi(\mathfrak{x}_1)+\Pi(\mathfrak{x}_2)}{n}\sum_{\mathfrak{z}=1}^{n}\frac{\mathfrak{z}}{\mathfrak{z}+1}.$$

For left side, using $PG_m CRF$, extended condition A and integration over $[0,1]$, we get

$$\begin{aligned}
&\Pi\Big(m\mathfrak{x}_1+\frac{1}{2}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)\Big)\\
&=\Pi(m\mathfrak{x}_1+\epsilon\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))+\frac{1}{2}\mathcal{R}^{\rho}_{\epsilon,\theta}(m\mathfrak{x}_1+(1-\epsilon)\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1),m\mathfrak{x}_1+\epsilon\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))\\
&\leqslant\frac{1}{n}\sum_{\mathfrak{z}=1}^{n}\left[1-\left(\frac{1}{2}\right)^{\mathfrak{z}}\right]\left[\int_0^1\Pi(m\mathfrak{x}_1+\epsilon\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))d\epsilon+\int_0^1\Pi(m\mathfrak{x}_1+(1-\epsilon)\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))d\epsilon\right]\\
&\leqslant\frac{1}{n}\sum_{\mathfrak{z}=1}^{n}\left[1-\left(\frac{1}{2}\right)^{\mathfrak{z}}\right]\frac{2}{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\int_{m\mathfrak{x}_1}^{m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(x)dx\\
&\leqslant\left[\frac{n+2^{-n}-1}{n}\right]\frac{2}{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\int_{m\mathfrak{x}_1}^{m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(x)dx.
\end{aligned}$$

This is the required proof. □

Remark 4.2. Considering Theorem 4.1, we establish the following new mathematical approach of H-H inequality pertaining to the classical Mittag-Leffler function if we pick $\rho=(1,1,\ldots)$ with $\epsilon=\alpha$, $\theta=1$:

$$\begin{aligned}
&\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)\Pi\Big(m\mathfrak{x}_1+\frac{1}{2}\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)\Big)\\
&\quad\leqslant\frac{1}{\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\int_{m\mathfrak{x}_1}^{m\mathfrak{x}_1+\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(x)dx\leqslant\frac{m\Pi(\mathfrak{x}_1)+\Pi(\mathfrak{x}_2)}{n}\sum_{\mathfrak{z}=1}^{n}\frac{\mathfrak{z}}{\mathfrak{z}+1}.
\end{aligned}$$

Remark 4.3. Considering Theorem 4.1, we establish the following new mathematical approach of H-H inequality pertaining to the classical Mittag-Leffler function if we pick $\rho=(1,1,\ldots)$ with $\epsilon=\alpha$, $\theta=1$ and $n=1$:

$$\Pi\Big(m\mathfrak{x}_1+\frac{1}{2}\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)\Big)\leqslant\frac{1}{\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\int_{m\mathfrak{x}_1}^{m\mathfrak{x}_1+\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(x)dx\leqslant\frac{m\Pi(\mathfrak{x}_1)+\Pi(\mathfrak{x}_2)}{2}.$$

Corollary 4.4. *Choosing* $\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)=\mathfrak{x}_2-m\mathfrak{x}_1$ *in Theorem* 4.1, *then*

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)\Pi\left(\frac{m\mathfrak{x}_1+\mathfrak{x}_2}{2}\right)\leqslant\frac{1}{(\mathfrak{x}_2-m\mathfrak{x}_1)}\int_{m\mathfrak{x}_1}^{\mathfrak{x}_2}\Pi(x)dx\leqslant\frac{m\Pi(\mathfrak{x}_1)+\Pi(\mathfrak{x}_2)}{n}\sum_{\mathfrak{z}=1}^{n}\frac{\mathfrak{z}}{\mathfrak{z}+1}.$$

Corollary 4.5. *Choosing* $n=1$ *and* $\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)=\mathfrak{x}_2-m\mathfrak{x}_1$ *in Theorem* 4.1, *then*

$$\Pi\left(\frac{(m\mathfrak{x}_1+\mathfrak{x}_2)}{2}\right)\leqslant\frac{1}{\mathfrak{x}_2-m\mathfrak{x}_1}\int_{m\mathfrak{x}_1}^{\mathfrak{x}_2}\Pi(x)dx\leqslant\frac{m\Pi(\mathfrak{x}_1)+\Pi(\mathfrak{x}_2)}{2}.$$

Theorem 4.6. *Suppose* $\mathbb{A}^{\circ}\subseteq\mathbb{R}$ *be an open generalized* m*-convex subset for some fixed* $m\in(0,1]$. *Let* $\mathfrak{x}_1,\mathfrak{x}_2\in\mathbb{A}^{\circ}$, $\mathfrak{x}_1<\mathfrak{x}_2$ *with* $m\mathfrak{x}_1\leqslant m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)$. *Suppose* $\Pi:[m\mathfrak{x}_1,m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)]\to\mathbb{R}$ *is* $PG_m CRF$, *then*

$$\frac{2^{-1}n}{n+2^{-n}-1}\Pi\left(\frac{2m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\right)$$

$$\leqslant \frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{\mathfrak{m}\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})-\frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi(\mathfrak{v})\right]$$
$$\leqslant \frac{\mathfrak{m}\Pi(\mathfrak{x}_1)+\Pi(\mathfrak{x}_2)}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\frac{\mathfrak{z}}{\mathfrak{z}+1},$$

where $\sigma\in[0,1]$ *and* $\mathfrak{v}\in[\mathfrak{m}\mathfrak{x}_1,\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)]$.

Proof. Considering inequality (4.1) and utilizing $\mathrm{PG}_{\mathfrak{m}}\mathrm{CRF}$, we have

$$\begin{aligned}&\frac{2^{-1}\mathfrak{n}}{\mathfrak{n}+2^{-\mathfrak{n}}-1}\Pi\left(\frac{2\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\right)\\&\leqslant\frac{2}{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\int_{\mathfrak{m}\mathfrak{x}_1}^{\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(x)dx\\&=\frac{2}{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\left(\int_{\mathfrak{m}\mathfrak{x}_1}^{\mathfrak{v}}\Pi(x)dx+\int_{\mathfrak{v}}^{\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(x)dx\right).\end{aligned}\tag{4.2}$$

Multiplying both sides of (4.2) by $\frac{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2\mathcal{B}(\sigma)}$ gives

$$\begin{aligned}&\frac{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2\mathcal{B}(\sigma)}\frac{2^{-1}\mathfrak{n}}{\mathfrak{n}+2^{-\mathfrak{n}}-1}\Pi\left(\frac{2\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\right)\\&\leqslant\frac{\sigma}{\mathcal{B}(\sigma)}\left(\int_{\mathfrak{m}\mathfrak{x}_1}^{k}\Pi(x)dx+\int_{k}^{\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(x)dx\right).\end{aligned}\tag{4.3}$$

Adding $\frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi(\mathfrak{v})$ to both sides of (4.3),

$$\begin{aligned}&\frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi(\mathfrak{v})+\frac{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2\mathcal{B}(\sigma)}\frac{2^{-1}\mathfrak{n}}{\mathfrak{n}+2^{-\mathfrak{n}}-1}\Pi\left(\frac{2\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\right)\\&\leqslant\frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi(\mathfrak{v})+\frac{\sigma}{\mathcal{B}(\sigma)}\left(\int_{\mathfrak{m}\mathfrak{x}_1}^{\mathfrak{v}}\Pi(x)dx+\int_{\mathfrak{v}}^{\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(x)dx\right)\\&=\left(\frac{(1-\sigma)}{\mathcal{B}(\sigma)}\Pi(\mathfrak{v})+\frac{\sigma}{\mathcal{B}(\sigma)}\int_{\mathfrak{m}\mathfrak{x}_1}^{\mathfrak{v}}\Pi(x)dx\right)+\left(\frac{(1-\sigma)}{\mathcal{B}(\sigma)}\Pi(\mathfrak{v})+\frac{\sigma}{\mathcal{B}(\sigma)}\int_{\mathfrak{v}}^{\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(x)dx\right)\\&=\left({}^{\mathcal{CF}}_{\mathfrak{m}\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi\right)(\mathfrak{v}).\end{aligned}\tag{4.4}$$

On the other hand, from inequality (4.1), we have

$$\frac{2}{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\int_{\mathfrak{m}\mathfrak{x}_1}^{\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(x)dx\leqslant\frac{\mathfrak{m}\Pi(\mathfrak{x}_1)+\Pi(\mathfrak{x}_2)}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\frac{2\mathfrak{z}}{\mathfrak{z}+1}.\tag{4.5}$$

If we multiply (4.5) by $\frac{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2\mathcal{B}(\sigma)}$ and add $\frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi(\mathfrak{v})$ to the resulting inequality, we obtain

$$\begin{aligned}&\left({}^{\mathcal{CF}}_{\mathfrak{m}\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\\&\leqslant\frac{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{\mathcal{B}(\sigma)}\frac{\mathfrak{m}\Pi(\mathfrak{x}_1)+\Pi(\mathfrak{x}_2)}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\frac{\mathfrak{z}}{\mathfrak{z}+1}+\frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi(\mathfrak{v}).\end{aligned}\tag{4.6}$$

Combining the inequalities (4.4) and (4.6), we attain the required result. □

Remark 4.7. Considering Theorem 4.6, we establish the following new mathematical approach of H-H inequality pertaining to the classical Mittag-Leffler function via C-FFIO if we pick $\rho = (1,1,\ldots)$ with $\epsilon = \alpha$, $\theta = 1$:

$$\begin{aligned}
&\frac{2^{-1}n}{n+2^{-n}-1}\Pi\left(\frac{2m\mathfrak{x}_1+\mathfrak{E}_\alpha(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\right)\\
&\leqslant \frac{\mathcal{B}(\sigma)}{\sigma\mathfrak{E}_\alpha(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1+\mathfrak{E}_\alpha(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})-\frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi(\mathfrak{v})\right]\\
&\leqslant \frac{m\Pi(\mathfrak{x}_1)+\Pi(\mathfrak{x}_2)}{n}\sum_{\mathfrak{z}=1}^{n}\frac{\mathfrak{z}}{\mathfrak{z}+1}.
\end{aligned}$$

Remark 4.8. Considering Theorem 4.6, we establish the following new mathematical approach of H-H inequality pertaining to the classical Mittag-Leffler function via C-FFIO if we pick $\rho = (1,1,\ldots)$ with $\epsilon = \alpha$, $\theta = 1$ and $n = 1$:

$$\begin{aligned}
&\Pi\left(\frac{2m\mathfrak{x}_1+\mathfrak{E}_\alpha(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\right)\\
&\leqslant \frac{\mathcal{B}(\sigma)}{\sigma\mathfrak{E}_\alpha(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1+\mathfrak{E}_\alpha(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})-\frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi(\mathfrak{v})\right]\\
&\leqslant \frac{m\Pi(\mathfrak{x}_1)+\Pi(\mathfrak{x}_2)}{2}.
\end{aligned}$$

Corollary 4.9. *Assume that* $\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)=\mathfrak{x}_2-m\mathfrak{x}_1$. *Then Theorem 4.6 becomes,*

$$\begin{aligned}
\frac{2^{-1}n}{n+2^{-n}-1}\Pi\left(\frac{m\mathfrak{x}_1+\mathfrak{x}_2}{2}\right) &\leqslant \frac{\mathcal{B}(\sigma)}{\sigma(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{x}_2}\Pi\right)(\mathfrak{v})-\frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi(\mathfrak{v})\right]\\
&\leqslant \frac{m\Pi(\mathfrak{x}_1)+\Pi(\mathfrak{x}_2)}{n}\sum_{\mathfrak{z}=1}^{n}\frac{\mathfrak{z}}{\mathfrak{z}+1}.
\end{aligned}$$

Corollary 4.10. *Assume that* $n=1$ *and* $\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)=\mathfrak{x}_2-m\mathfrak{x}_1$. *Then Theorem 4.6 becomes,*

$$\Pi\left(\frac{m\mathfrak{x}_1+\mathfrak{x}_2}{2}\right)\leqslant\frac{\mathcal{B}(\sigma)}{\sigma(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{x}_2}\Pi\right)(\mathfrak{v})-\frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi(\mathfrak{v})\right]\leqslant\frac{m\Pi(\mathfrak{x}_1)+\Pi(\mathfrak{x}_2)}{2}.$$

5. New refinements of H-H type inequalities via n-polynomial m-convex involving Raina's function with the help of power mean and Hölder inequality pertaining to Caputo-Fabrizo fractional operator

In the subject of convex theory, several scholars have recently contributed novel ways to this problem from different perspectives. Recent studies on H-H inequalities involving convexity have generated a broad range of research and enhancements.

At first, in this section, we consider the lemma on the basis of the newly introduced definition. We shall incorporate results using power-mean and Hölder inequality following the recently introduced lemma. In addition, few remarks pertaining to the Mittag-Leffler function involving Raina'sfunction and corollaries to enhance the significance and level of this section.

Lemma 5.1. *Let* $\Pi:\mathbb{I}=[m\mathfrak{x}_1,m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)]\to\mathbb{R}$ *be a differentiable mapping on* $\mathbb{I}^{\circ}$, $m\mathfrak{x}_1<m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)$. *If* $\Pi'\in\mathcal{L}[m\mathfrak{x}_1,m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)]$, *then*

$$\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\int_0^1(1-2\mathfrak{e})\Pi'(m\mathfrak{x}_1+\mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))d\mathfrak{e}+\frac{2(1-\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})$$

$$= \frac{-\Pi(m\mathfrak{x}_1)+\Pi(m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))}{2}+\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[{}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi(\mathfrak{v})+{}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})\right],$$

where $\sigma\in[0,1]$ *and* $\mathfrak{v}\in[m\mathfrak{x}_1, m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)]$.

Proof. It is easy to see that

$$\int_0^1(1-2\mathfrak{e})\Pi'(m\mathfrak{x}_1+\mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))d\mathfrak{e}=-\frac{\Pi(m\mathfrak{x}_1)+\Pi(m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))}{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}+\frac{2}{(\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))^2}$$
$$\times\left(\int_{m\mathfrak{x}_1}^{\mathfrak{v}}\Pi(x)dx+\int_{\mathfrak{v}}^{m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(x)dx\right).$$

Multiplying both sides with $\frac{\sigma(\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))^2}{2\mathcal{B}(\sigma)}$ and adding $\frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi(\mathfrak{v})$, we have

$$\begin{aligned}
&\frac{\sigma(\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))^2}{2\mathcal{B}(\sigma)}\int_0^1(1-2\mathfrak{e})\Pi'(m\mathfrak{x}_1+\mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))d\mathfrak{e}+\frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi(\mathfrak{v})\\
&=-\frac{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}{\mathcal{B}(\sigma)}\frac{\Pi(m\mathfrak{x}_1)+\Pi(m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))}{2}\\
&\quad+\left(\frac{(1-\sigma)}{\mathcal{B}(\sigma)}\Pi(\mathfrak{v})+\frac{\sigma}{\mathcal{B}(\sigma)}\int_{m\mathfrak{x}_1}^{\mathfrak{v}}\Pi(x)dx\right)+\left(\frac{(1-\sigma)}{\mathcal{B}(\sigma)}\Pi(\mathfrak{v})+\frac{\sigma}{\mathcal{B}(\sigma)}\int_{\mathfrak{v}}^{m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(x)dx\right)\\
&=\frac{-\Pi(m\mathfrak{x}_1)+\Pi(m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))}{2}+\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[{}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi(\mathfrak{v})+{}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})\right].
\end{aligned}$$

This completes the proof. □

Theorem 5.2. *Let* $\Pi:\mathbb{I}=[m\mathfrak{x}_1, m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)]\to\mathbb{R}$ *be a differentiable mapping on* $\mathbb{I}^{\circ}$, $m\mathfrak{x}_1<m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)$ *and* $|\Pi'|$ *is* $\mathrm{PG}_{\mathfrak{m}}\mathrm{CRF}$ *on* $[m\mathfrak{x}_1, m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)]$. *If* $\Pi'\in\mathcal{L}[m\mathfrak{x}_1, m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)]$, *then*

$$\begin{aligned}
&\Bigg|-\frac{\Pi(m\mathfrak{x}_1)+\Pi(m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})\\
&\quad+\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\Bigg|\\
&\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\left[\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}\mathfrak{z}-2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}}\right]\left(\frac{\mathfrak{m}|\Pi'(\mathfrak{x}_1)|+|\Pi'(\mathfrak{x}_2)|}{2}\right),
\end{aligned}$$

where $\sigma\in[0,1]$, $\mathfrak{m}\in(0,1]$ *and* $\mathfrak{v}\in[m\mathfrak{x}_1, m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)]$.

Proof. Employing the Lemma 5.1, we have

$$\begin{aligned}
&\Bigg|-\frac{\Pi(m\mathfrak{x}_1)+\Pi(m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})\\
&\quad+\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\Bigg|\\
&\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\int_0^1|1-2\mathfrak{e}||\Pi'(m\mathfrak{x}_1+\mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))|d\mathfrak{e}.
\end{aligned}$$

Employing the property of $PG_m CRF$, we have

$$\begin{aligned}
&\Bigg| -\frac{\Pi(m\mathfrak{x}_1)+\Pi(m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})\\
&\quad+\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\Bigg|\\
&\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\int_0^1|1-2\mathfrak{e}|\left(\frac{m}{n}\sum_{\mathfrak{z}=1}^{n}(1-\mathfrak{e}^{\mathfrak{z}})|\Pi'(\mathfrak{x}_1)|+\frac{1}{n}\sum_{\mathfrak{z}=1}^{n}(1-(1-\mathfrak{e})^{\mathfrak{z}})|\Pi'(\mathfrak{x}_2)|\right)d\mathfrak{e}\\
&\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2n}\left[m|\Pi'(\mathfrak{x}_1)|\sum_{\mathfrak{z}=1}^{n}\int_0^1|1-2\mathfrak{e}|(1-\mathfrak{e}^{\mathfrak{z}})d\mathfrak{e}+|\Pi'(\mathfrak{x}_2)|\sum_{\mathfrak{z}=1}^{n}\int_0^1|1-2\mathfrak{e}|(1-(1-\mathfrak{e})^{\mathfrak{z}})d\mathfrak{e}\right]\\
&=\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2n}\left[m|\Pi'(\mathfrak{x}_1)|\sum_{\mathfrak{z}=1}^{n}\left[\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}}\right]+|\Pi'(\mathfrak{x}_2)|\sum_{\mathfrak{z}=1}^{n}\left[\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}}\right]\right]\\
&=\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}{n}\sum_{\mathfrak{z}=1}^{n}\left[\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}}\right]\left(\frac{m|\Pi'(\mathfrak{x}_1)|+|\Pi'(\mathfrak{x}_2)|}{2}\right).
\end{aligned}$$

This completes the proof. □

Remark 5.3. Considering Theorem 5.2, we establish the following new refinement of H-H inequality pertaining to the classical Mittag-Leffler function via C-FFIO with the help of newly investigated idea if we pick $\rho=(1,1,\dots)$ with $\epsilon=\alpha$, $\theta=1$:

$$\begin{aligned}
&\Bigg| -\frac{\Pi(m\mathfrak{x}_1)+\Pi(m\mathfrak{x}_1+\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})\\
&\quad+\frac{\mathcal{B}(\sigma)}{\sigma\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1+\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\Bigg|\\
&\leqslant\frac{\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}{n}\sum_{\mathfrak{z}=1}^{n}\left[\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}\mathfrak{z}-2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}}\right]\left(\frac{m|\Pi'(\mathfrak{x}_1)|+|\Pi'(\mathfrak{x}_2)|}{2}\right).
\end{aligned}$$

Remark 5.4. Considering Theorem 5.2, we establish the following new refinement of H-H inequality pertaining to the classical Mittag-Leffler function via C-FFIO with the help of a newly investigated idea if we pick $\rho=(1,1,\dots)$ with $\epsilon=\alpha$, $\theta=1$ and $n=1$:

$$\begin{aligned}
&\Bigg| -\frac{\Pi(m\mathfrak{x}_1)+\Pi(m\mathfrak{x}_1+\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})\\
&\quad+\frac{\mathcal{B}(\sigma)}{\sigma\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1+\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\Bigg|\leqslant\frac{\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)\left(m|\Pi'(\mathfrak{x}_1)|+|\Pi'(\mathfrak{x}_2)|\right)}{8}.
\end{aligned}$$

Remark 5.5. Considering Theorem 5.2, we establish the following new refinement of H-H inequality pertaining to the classical Mittag-Leffler function via C-FFIO with the help of a newly investigated idea if we pick $\rho=(1,1,\dots)$ with $\epsilon=\alpha$, $\theta=1$ and $m=1$:

$$\begin{aligned}
&\Bigg| -\frac{\Pi(\mathfrak{x}_1)+\Pi(\mathfrak{x}_1+\mathfrak{E}_{\alpha}(\mathfrak{x}_2-\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathfrak{E}_{\alpha}(\mathfrak{x}_2-\mathfrak{x}_1)}\Pi(\mathfrak{v})\\
&\quad+\frac{\mathcal{B}(\sigma)}{\sigma\mathfrak{E}_{\alpha}(\mathfrak{x}_2-\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{x}_1+\mathfrak{E}_{\alpha}(\mathfrak{x}_2-\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\Bigg|\\
&\leqslant\frac{\mathfrak{E}_{\alpha}(\mathfrak{x}_2-\mathfrak{x}_1)}{n}\sum_{\mathfrak{z}=1}^{n}\left[\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}\mathfrak{z}-2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}}\right]\left(\frac{|\Pi'(\mathfrak{x}_1)|+|\Pi'(\mathfrak{x}_2)|}{2}\right).
\end{aligned}$$

Corollary 5.6. *If we put* $\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)=\mathfrak{x}_2-m\mathfrak{x}_1$, *then Theorem 5.2 becomes*

$$\left|-\frac{\Pi(m\mathfrak{x}_1)+\Pi(\mathfrak{x}_2)}{2}-\frac{2(1-\sigma)}{\sigma(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})+\frac{\mathcal{B}(\sigma)}{\sigma(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{x}_2}\Pi\right)(\mathfrak{v})\right]\right|$$
$$\leqslant\frac{\mathfrak{x}_2-m\mathfrak{x}_1}{n}\sum_{\mathfrak{z}=1}^{n}\left[\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}}\right]\left(\frac{m|\Pi'(\mathfrak{x}_1)|+|\Pi'(\mathfrak{x}_2)|}{2}\right).$$

Corollary 5.7. *If* $n=1$ *and* $\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)=\mathfrak{x}_2-m\mathfrak{x}_1$, *then Theorem 5.2 becomes*

$$\left|-\frac{\Pi(m\mathfrak{x}_1)+\Pi(\mathfrak{x}_2)}{2}-\frac{2(1-\sigma)}{\sigma(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})+\frac{\mathcal{B}(\sigma)}{\sigma(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{x}_2)}\Pi\right)(\mathfrak{v})\right]\right|$$
$$\leqslant\frac{(\mathfrak{x}_2-m\mathfrak{x}_1)\left(m|\Pi'(\mathfrak{x}_1)|+|\Pi'(\mathfrak{x}_2)|\right)}{8}.$$

Theorem 5.8. *Suppose that* $\Pi:\mathbb{I}=[m\mathfrak{x}_1,m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)]\to\mathbb{R}$ *is a differentiable mapping on* $\mathbb{I}^{\circ}$, $m\mathfrak{x}_1<m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)$ *and* $|\Pi'|^{q}$ *is* $\mathrm{PG}_m\mathrm{CRF}$ *on* $[m\mathfrak{x}_1,m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)]$. *If* $\Pi'\in\mathcal{L}[m\mathfrak{x}_1,m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)]$, *then*

$$\begin{aligned}&\left|-\frac{\Pi(m\mathfrak{x}_1)+\Pi(m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})\right.\\&\left.+\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\right|\\&\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{\mathfrak{z}=1}^{n}\frac{\mathfrak{z}}{\mathfrak{z}+1}\right)^{\frac{1}{q}}\left(\frac{m|\Pi'(\mathfrak{x}_1)|^{q}+|\Pi'(\mathfrak{x}_2)|^{q}}{2}\right)^{\frac{1}{q}},\end{aligned}\tag{5.1}$$

where $\sigma\in[0,1]$, $m\in(0,1]$ *and* $\mathfrak{v}\in[m\mathfrak{x}_1,m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)]$.

Proof. Employing Lemma 5.1, we have

$$\begin{aligned}&\left|-\frac{\Pi(m\mathfrak{x}_1)+\Pi(m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})\right.\\&\left.+\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\right|\\&\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\int_0^1|1-2\mathfrak{e}||\Pi'(m\mathfrak{x}_1+\mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))|d\mathfrak{e}.\end{aligned}$$

Employing Hölder inequality, we have

$$\begin{aligned}&\left|-\frac{\Pi(m\mathfrak{x}_1)+\Pi(m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})\right.\\&\left.+\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\right|\\&\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\left(\int_0^1|1-2\mathfrak{e}|^{p}d\mathfrak{e}\right)^{\frac{1}{p}}\left(\int_0^1|\Pi'(m\mathfrak{x}_1+\mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))|^{q}d\mathfrak{e}\right)^{\frac{1}{q}}.\end{aligned}$$

Employing the property of $\mathrm{PG}_m\mathrm{CRF}$ of $|\Pi'|^{q}$, we have

$$\left|-\frac{\Pi(m\mathfrak{x}_1)+\Pi(m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})\right.$$

$$
\begin{aligned}
&+\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\Bigg| \\
&\leqslant \frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{m|\Pi'(\mathfrak{x}_1)|^q}{n}\int_0^1(1-\epsilon^{\mathfrak{z}})d\epsilon+\frac{|\Pi'(\mathfrak{x}_2)|^q}{n}\int_0^1(1-(1-\epsilon)^{\mathfrak{z}})d\epsilon\right]^{\frac{1}{q}} \\
&=\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{m|\Pi'(\mathfrak{x}_1)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{\mathfrak{z}}{\mathfrak{z}+1}+\frac{|\Pi'(\mathfrak{x}_2)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{\mathfrak{z}}{\mathfrak{z}+1}\right]^{\frac{1}{q}} \\
&=\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{\mathfrak{z}=1}^{n}\frac{\mathfrak{z}}{\mathfrak{z}+1}\right)^{\frac{1}{q}}\left(\frac{m|\Pi'(\mathfrak{x}_1)|^q+|\Pi'(\mathfrak{x}_2)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}
$$

This is the required proof. □

Remark 5.9. Considering Theorem 5.8, we establish the following new refinement of H-H inequality pertaining to the classical Mittag-Leffler function via C-FFIO with the help of Hölder inequality and newly investigated idea if we pick $\rho=(1,1,\dots)$ with $\epsilon=\alpha$, $\theta=1$:

$$
\begin{aligned}
&\Bigg|-\frac{\Pi(m\mathfrak{x}_1)+\Pi(m\mathfrak{x}_1+\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v}) \\
&\quad+\frac{\mathcal{B}(\sigma)}{\sigma\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1+\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\Bigg| \\
&\leqslant\frac{\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{\mathfrak{z}=1}^{n}\frac{\mathfrak{z}}{\mathfrak{z}+1}\right)^{\frac{1}{q}}\left(\frac{m|\Pi'(\mathfrak{x}_1)|^q+|\Pi'(\mathfrak{x}_2)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Remark 5.10. Considering Theorem 5.8, we establish the following new refinement of H-H inequality pertaining to the classical Mittag-Leffler function via C-FFIO with the help of Hölder inequality and newly investigated idea if we pick $\rho=(1,1,\dots)$ with $\epsilon=\alpha$, $\theta=1$ and $n=1$:

$$
\begin{aligned}
&\Bigg|-\frac{\sigma(m\mathfrak{x}_1)+\Pi(m\mathfrak{x}_1+\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v}) \\
&\quad+\frac{\mathcal{B}(\sigma)}{\sigma\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1+\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\Bigg| \\
&\leqslant\frac{\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{q}}\left(\frac{m|\Pi'(\mathfrak{x}_1)|^q+|\Pi'(\mathfrak{x}_2)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Remark 5.11. Considering Theorem 5.8, we establish the following new refinement of H-H inequality pertaining to the classical Mittag-Leffler function via C-FFIO with the help of Hölder inequality and newly investigated ideat if we pick $\rho=(1,1,\dots)$ with $\epsilon=\alpha$, $m=\theta=1$:

$$
\begin{aligned}
&\Bigg|-\frac{\sigma(\mathfrak{x}_1)+\Pi(\mathfrak{x}_1+\mathfrak{E}_{\alpha}(\mathfrak{x}_2-\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathfrak{E}_{\alpha}(\mathfrak{x}_2-\mathfrak{x}_1)}\Pi(\mathfrak{v}) \\
&\quad+\frac{\mathcal{B}(\sigma)}{\sigma\mathfrak{E}_{\alpha}(\mathfrak{x}_2-\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{x}_1+\mathfrak{E}_{\alpha}(\mathfrak{x}_2-\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\Bigg| \\
&\leqslant\frac{\mathfrak{E}_{\alpha}(\mathfrak{x}_2-\mathfrak{x}_1)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{q}}\left(\frac{2}{n}\sum_{\mathfrak{z}=1}^{n}\frac{\mathfrak{z}}{\mathfrak{z}+1}\right)^{\frac{1}{q}}\left(\frac{|\Pi'(\mathfrak{x}_1)|^q+|\Pi'(\mathfrak{x}_2)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Corollary 5.12. *Choosing* $\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)=\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1$, *then Theorem 5.8 becomes*

$$\begin{aligned}&\left|-\frac{\Pi(\mathfrak{m}\mathfrak{x}_1)+\Pi(\mathfrak{x}_2)}{2}-\frac{2(1-\sigma)}{\sigma(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(\mathfrak{v})+\frac{\mathcal{B}(\sigma)}{\sigma(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{\mathfrak{m}\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{x}_2}\Pi\right)(\mathfrak{v})\right]\right|\\&\leqslant\frac{\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{\mathfrak{z}=1}^{n}\frac{\mathfrak{z}}{\mathfrak{z}+1}\right)^{\frac{1}{q}}\left(\frac{\mathfrak{m}|\Pi'(\mathfrak{x}_1)|^q+|\Pi'(\mathfrak{x}_2)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Corollary 5.13. *Choosing* $\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)=\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1$ *and* $\mathfrak{n}=1$, *then Theorem 5.8 becomes*

$$\begin{aligned}&\left|-\frac{\Pi(\mathfrak{m}\mathfrak{x}_1)+\Pi(\mathfrak{x}_2)}{2}-\frac{2(1-\sigma)}{\sigma(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(\mathfrak{v})+\frac{\mathcal{B}(\sigma)}{\sigma(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{\mathfrak{m}\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{x}_2}\Pi\right)(\mathfrak{v})\right]\right|\\&\leqslant\frac{\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{\mathfrak{m}|\Pi'(\mathfrak{x}_1)|^q+|\Pi'(\mathfrak{x}_2)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Theorem 5.14. *Let* $\Pi:\mathbb{I}=[\mathfrak{m}\mathfrak{x}_1,\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)]\to\mathbb{R}$ *be a differentiable mapping on* $\mathbb{I}^{\circ}$ *and* $\mathfrak{x}_1,\mathfrak{x}_2\in\mathbb{I}$ *with* $\mathfrak{m}\mathfrak{x}_1<\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)$ *and assume that* $\Pi'\in\mathcal{L}[\mathfrak{m}\mathfrak{x}_1,\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)]$. *If* $|\Pi'|^q$ *is* $\mathrm{PG}_{\mathfrak{m}}\mathrm{CRF}$ *on* $[\mathfrak{m}\mathfrak{x}_1,\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)]$, *then we obtain*

$$\begin{aligned}&\left|-\frac{\Pi(\mathfrak{m}\mathfrak{x}_1)+\Pi(\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(\mathfrak{v})\right.\\&\quad\left.+\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{\mathfrak{m}\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\right|\\&\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}}\right)^{\frac{1}{q}}\left(\frac{\mathfrak{m}|\Pi'(\mathfrak{x}_1)|^q+|\Pi'(\mathfrak{x}_2)|^q}{2}\right)^{\frac{1}{q}},\end{aligned}\tag{5.2}$$

where $\sigma\in[0,1]$, $\mathfrak{m}\in(0,1]$ *and* $\mathfrak{v}\in[\mathfrak{m}\mathfrak{x}_1,\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)]$.

Proof. Employing Lemma 5.1, we have

$$\begin{aligned}&\left|\frac{-\Pi(\mathfrak{m}\mathfrak{x}_1)+\Pi(\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(\mathfrak{v})\right.\\&\quad\left.+\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\left[{}^{\mathcal{CF}}_{\mathfrak{m}\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi(\mathfrak{v})+{}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(\mathfrak{v})\right]\right|\\&\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\int_0^1|1-2\mathfrak{e}|\left|\Pi'(\mathfrak{m}\mathfrak{x}_1+\mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1))\right|d\mathfrak{e}.\end{aligned}$$

Employing power mean inequality, we have

$$\begin{aligned}&\left|\frac{-\Pi(\mathfrak{m}\mathfrak{x}_1)+\Pi(\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(\mathfrak{v})\right.\\&\quad\left.+\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\left[{}^{\mathcal{CF}}_{\mathfrak{m}\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi(\mathfrak{v})+{}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(\mathfrak{v})\right]\right|\\&\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\left(\int_0^1|1-2\mathfrak{e}|d\mathfrak{e}\right)^{1-\frac{1}{q}}\left(|1-2\mathfrak{e}|\left|\Pi'(\mathfrak{m}\mathfrak{x}_1+\mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1))\right|^q d\mathfrak{e}\right)^{\frac{1}{q}}.\end{aligned}$$

Employing the property of $PG_m CRF$ of $|\Pi'|^q$, we have

$$\begin{aligned}
&\left|\frac{-\Pi(m\mathfrak{x}_1)+\Pi(m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})\right.\\
&\quad\left.+\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[{}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi(\mathfrak{v})+{}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})\right]\right|\\
&\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1|1-2\mathfrak{e}|\left[\frac{m}{n}\sum_{\mathfrak{z}=1}^{n}[1-\mathfrak{e}^{\mathfrak{z}}]|\Pi'(\mathfrak{x}_1)|^q+\frac{1}{n}\sum_{\mathfrak{z}=1}^{n}[1-(1-\mathfrak{e})^{\mathfrak{z}}]|\Pi'(\mathfrak{x}_2)|^q\right]d\mathfrak{e}\right)^{\frac{1}{q}}\\
&\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{m|\Pi'(\mathfrak{x}_1)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\int_0^1|1-2\mathfrak{e}|[1-\mathfrak{e}^{\mathfrak{z}}]d\mathfrak{e}+\frac{|\Pi'(\mathfrak{x}_2)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\int_0^1|1-2\mathfrak{e}|[1-(1-\mathfrak{e})^{\mathfrak{z}}]d\mathfrak{e}\right)^{\frac{1}{q}}\\
&\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{m|\Pi'(\mathfrak{x}_1)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}}+\frac{|\Pi'(\mathfrak{x}_2)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}}\right)^{\frac{1}{q}}.
\end{aligned}$$

By simplifying, we achieve the required result. □

Remark 5.15. Considering Theorem 5.14, we establish the following new refinement of H-H inequality pertaining to the classical Mittag-Leffler function via C-FFIO in the mode of power mean inequality and newly investigated idea if we pick $\rho=(1,1,\dots)$ with $\epsilon=\alpha$, $\theta=1$:

$$\begin{aligned}
&\left|-\frac{\Pi(m\mathfrak{x}_1)+\Pi(m\mathfrak{x}_1+\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})\right.\\
&\quad\left.+\frac{\mathcal{B}(\sigma)}{\sigma\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1+\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\right|\\
&\leqslant\frac{\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}}\right)^{\frac{1}{q}}\left(\frac{m|\Pi'(\mathfrak{x}_1)|^q+|\Pi'(\mathfrak{x}_2)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Remark 5.16. Considering Theorem 5.14, we establish the following new refinement of H-H inequality pertaining to the classical Mittag-Leffler function via C-FFIO in the mode of power mean inequality and newly investigated idea if we pick $\rho=(1,1,\dots)$ with $\epsilon=\alpha$, $\theta=1$ and $n=1$:

$$\begin{aligned}
&\left|\frac{-\Pi(m\mathfrak{x}_1)+\Pi(m\mathfrak{x}_1+\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})\right.\\
&\quad\left.+\frac{\mathcal{B}(\sigma)}{\sigma\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[{}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi(\mathfrak{v})+{}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1+\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})\right]\right|\\
&\leqslant\frac{\mathfrak{E}_{\alpha}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\left(\frac{m|\Pi'(\mathfrak{x}_1)|^q+|\Pi'(\mathfrak{x}_2)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Remark 5.17. Considering Theorem 5.14, we establish the following new refinement of H-H inequality pertaining to the classical Mittag-Leffler function via C-FFIO in the mode of power mean inequality and newly investigated idea if we pick $\rho=(1,1,\dots)$ with $\epsilon=\alpha$, $\theta=m=1$:

$$\begin{aligned}
&\left|-\frac{\Pi(\mathfrak{x}_1)+\Pi(\mathfrak{x}_1+\mathfrak{E}_{\alpha}(\mathfrak{x}_2-\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathfrak{E}_{\alpha}(\mathfrak{x}_2-\mathfrak{x}_1)}\Pi(\mathfrak{v})\right.\\
&\quad\left.+\frac{\mathcal{B}(\sigma)}{\sigma\mathfrak{E}_{\alpha}(\mathfrak{x}_2-\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{x}_1+\mathfrak{E}_{\alpha}(\mathfrak{x}_2-\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\right|
\end{aligned}$$

$$\leqslant \frac{\mathfrak{E}_\alpha(\mathfrak{x}_2-\mathfrak{x}_1)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}}\right)^{\frac{1}{q}}\left(\frac{|\Pi'(\mathfrak{x}_1)|^q+|\Pi'(\mathfrak{x}_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 5.18. *Choosing* $\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)=\mathfrak{x}_2-m\mathfrak{x}_1$, *then Theorem 5.14 becomes*

$$\left|\frac{\Pi(m\mathfrak{x}_1)+\Pi(\mathfrak{x}_2)}{2}-\frac{2(1-\sigma)}{\sigma(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})+\frac{\mathcal{B}(\sigma)}{\sigma(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[{}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi(\mathfrak{v})+{}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{x}_2}\Pi(\mathfrak{v})\right]\right|$$
$$\leqslant \frac{\mathfrak{x}_2-m\mathfrak{x}_1}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}}\right)^{\frac{1}{q}}\left(\frac{m|\Pi'(\mathfrak{x}_1)|^q+|\Pi'(\mathfrak{x}_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 5.19. *Choosing* $n=1$ *and* $\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)=\mathfrak{x}_2-m\mathfrak{x}_1$, *then Theorem 5.14 becomes*

$$\left|\frac{\Pi(m\mathfrak{x}_1)+\Pi(\mathfrak{x}_2)}{2}-\frac{2(1-\sigma)}{\sigma(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})+\frac{\mathcal{B}(\sigma)}{\sigma(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[{}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi(\mathfrak{v})+{}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{x}_2}\Pi(\mathfrak{v})\right]\right|$$
$$\leqslant \frac{\mathfrak{x}_2-m\mathfrak{x}_1}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{m|\Pi'(\mathfrak{x}_1)|^q+|\Pi'(\mathfrak{x}_2)|^q}{2}\right)^{\frac{1}{q}}.$$

6. New refinements of H-H type inequalities using Hölder İşcan and improved power mean inequality involving Caputo-Fabrizo operator via n-polynomial generalized m-convex involving Raina's funcion

Recently, a number of mathematicians have concentrated on new approaches to the subject of integral inequalities from different perspectives and various points of view. İşcan [13] introduced Hölder Iscan integral inequality and Kadakal [15] presented improved power mean inequality.

The primary objective of this part is to obtain novel approaches of the H-H type integral inequalities via C-FFIO involving $\mathrm{PG}_m\mathrm{CRF}$ in the manner of improved power mean inequality and Hölder-Iscan inequality. We provide some comments and corollaries for worth and value.

Theorem 6.1. *Let* $\Pi:\mathbb{I}=[m\mathfrak{x}_1,m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)]\to\mathbb{R}$ *be a differentiable mapping on* $\mathbb{I}^\circ$, *with* $m\mathfrak{x}_1<m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ *and suppose that* $\Pi'\in\mathcal{L}[m\mathfrak{x}_1,m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)]$. *If* $|\Pi'|^q$ *is* $\mathrm{PG}_m\mathrm{CRF}$ *on* $[m\mathfrak{x}_1,m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)]$, *then*

$$\left|-\frac{\Pi(m\mathfrak{x}_1)+\Pi(m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi(\mathfrak{v})\right.$$
$$\left.+\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\right|$$
$$\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{m|\Pi'(\mathfrak{x}_1)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{\mathfrak{z}(\mathfrak{z}+3)}{2(\mathfrak{z}+1)(\mathfrak{z}+2)}+\frac{|\Pi'(\mathfrak{x}_2)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{\mathfrak{z}}{2(\mathfrak{z}+2)}\right)^{\frac{1}{q}}$$
$$+\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Pi'(\mathfrak{x}_2)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{\mathfrak{z}(\mathfrak{z}+3)}{2(\mathfrak{z}+1)(\mathfrak{z}+2)}+\frac{|\Pi'(\mathfrak{x}_1)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{\mathfrak{z}}{2(\mathfrak{z}+2)}\right)^{\frac{1}{q}},$$

where $\sigma\in[0,1]$, $m\in(0,1]$ *and* $\mathfrak{v}\in[m\mathfrak{x}_1,m\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-m\mathfrak{x}_1)]$.

Proof. Employing Lemma 5.1, we have

$$\begin{aligned}
&\Bigg| -\frac{\Pi(\mathfrak{m}\mathfrak{x}_1)+\Pi(\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(\mathfrak{v})\\
&\quad+\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{\mathfrak{m}\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\Bigg|\\
&\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\int_0^1|1-2\mathfrak{e}||\Pi'(\mathfrak{m}\mathfrak{x}_1+\mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1))|\mathrm{d}\mathfrak{e}.
\end{aligned}$$

Employing Hölder-İşcan inequality, we have

$$\begin{aligned}
&\Bigg| -\frac{\Pi(\mathfrak{m}\mathfrak{x}_1)+\Pi(\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(\mathfrak{v})\\
&\quad+\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{\mathfrak{m}\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\Bigg|\\
&\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\left(\int_0^1(1-\mathfrak{e})|1-2\mathfrak{e}|^{\mathfrak{p}}\mathrm{d}\mathfrak{e}\right)^{\frac{1}{\mathfrak{p}}}\left(\int_0^1(1-\mathfrak{e})|\Pi'(\mathfrak{m}\mathfrak{x}_1+\mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1))|^{\mathfrak{q}}\mathrm{d}\mathfrak{e}\right)^{\frac{1}{\mathfrak{q}}}\\
&\quad+\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\left(\int_0^1\mathfrak{e}|1-2\mathfrak{e}|^{\mathfrak{p}}\mathrm{d}\mathfrak{e}\right)^{\frac{1}{\mathfrak{p}}}\left(\int_0^1\mathfrak{e}|\Pi'(\mathfrak{m}\mathfrak{x}_1+\mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1))|^{\mathfrak{q}}\mathrm{d}\mathfrak{e}\right)^{\frac{1}{\mathfrak{q}}}.
\end{aligned}$$

Employing $\mathrm{PG}_{\mathfrak{m}}\mathrm{CRF}$ of $|\Pi'|^{\mathfrak{q}}$, we have

$$\begin{aligned}
&\Bigg| -\frac{\Pi(\mathfrak{m}\mathfrak{x}_1)+\Pi(\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(\mathfrak{v})\\
&\quad+\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{\mathfrak{m}\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\Bigg|\\
&\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\left(\frac{1}{2(\mathfrak{p}+1)}\right)^{\frac{1}{\mathfrak{p}}}\\
&\quad\times\left(\frac{\mathfrak{m}|\Pi'(\mathfrak{x}_1)|^{\mathfrak{q}}}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\int_0^1(1-\mathfrak{e})(1-\mathfrak{e}^{\mathfrak{z}})\mathrm{d}\mathfrak{e}+\frac{|\Pi'(\mathfrak{x}_2)|^{\mathfrak{q}}}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\int_0^1(1-\mathfrak{e})(1-(1-\mathfrak{e})^{\mathfrak{z}})\mathrm{d}\mathfrak{e}\right)^{\frac{1}{\mathfrak{q}}}\\
&\quad+\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\left(\frac{1}{2(\mathfrak{p}+1)}\right)^{\frac{1}{\mathfrak{p}}}\left(\frac{\mathfrak{m}|\Pi'(\mathfrak{x}_1)|^{\mathfrak{q}}}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\int_0^1\mathfrak{e}(1-\mathfrak{e}^{\mathfrak{z}})\mathrm{d}\mathfrak{e}+\frac{|\Pi'(\mathfrak{x}_2)|^{\mathfrak{q}}}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\int_0^1\mathfrak{e}(1-(1-\mathfrak{e})^{\mathfrak{z}})\mathrm{d}\mathfrak{e}\right)^{\frac{1}{\mathfrak{q}}}\\
&=\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\left(\frac{1}{2(\mathfrak{p}+1)}\right)^{\frac{1}{\mathfrak{p}}}\left(\frac{\mathfrak{m}|\Pi'(\mathfrak{x}_1)|^{\mathfrak{q}}}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\frac{\mathfrak{z}(\mathfrak{z}+3)}{2(\mathfrak{z}+1)(\mathfrak{z}+2)}+\frac{|\Pi'(\mathfrak{x}_2)|^{\mathfrak{q}}}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\frac{\mathfrak{z}}{2(\mathfrak{z}+2)}\right)^{\frac{1}{\mathfrak{q}}}\\
&\quad+\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\left(\frac{1}{2(\mathfrak{p}+1)}\right)^{\frac{1}{\mathfrak{p}}}\left(\frac{|\Pi'(\mathfrak{x}_2)|^{\mathfrak{q}}}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\frac{\mathfrak{z}(\mathfrak{z}+3)}{2(\mathfrak{z}+1)(\mathfrak{z}+2)}+\frac{\mathfrak{m}|\Pi'(\mathfrak{x}_1)|^{\mathfrak{q}}}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\frac{\mathfrak{z}}{2(\mathfrak{z}+2)}\right)^{\frac{1}{\mathfrak{q}}}.
\end{aligned}$$

This is the required proof. □

Remark 6.2. Considering Theorem 6.1, we establish the following new refinement of H-H inequality pertaining to the classical Mittag-Leffler function via C-FFIO in the mode of Hölder-Iscan inequality and the

newly investigated idea if we pick $\rho = (1, 1, \ldots)$ with $\epsilon = \alpha$, $\theta = 1$:

$$\begin{aligned}
&\left| -\frac{\Pi(\mathfrak{m}\mathfrak{x}_1) + \Pi(\mathfrak{m}\mathfrak{x}_1 + \mathfrak{E}_\alpha(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1))}{2} - \frac{2(1-\sigma)}{\sigma \mathfrak{E}_\alpha(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)}\Pi(\mathfrak{v}) \right. \\
&\quad \left. + \frac{\mathcal{B}(\sigma)}{\sigma \mathfrak{E}_\alpha(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)} \left[\left({}^{\mathcal{CF}}_{\mathfrak{m}\mathfrak{x}_1} \mathrm{I}^{\sigma} \Pi \right)(\mathfrak{v}) + \left({}^{\mathcal{CF}} \mathrm{I}^{\sigma}_{\mathfrak{m}\mathfrak{x}_1 + \mathfrak{E}_\alpha(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)} \Pi \right)(\mathfrak{v}) \right] \right| \\
&\leqslant \frac{\mathfrak{E}_\alpha(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{\mathfrak{m}|\Pi'(\mathfrak{x}_1)|^q}{n} \sum_{\mathfrak{z}=1}^{n} \frac{\mathfrak{z}(\mathfrak{z}+3)}{2(\mathfrak{z}+1)(\mathfrak{z}+2)} + \frac{|\Pi'(\mathfrak{x}_2)|^q}{n} \sum_{\mathfrak{z}=1}^{n} \frac{\mathfrak{z}}{2(\mathfrak{z}+2)} \right)^{\frac{1}{q}} \\
&\quad + \frac{\mathfrak{E}_\alpha(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\Pi'(\mathfrak{x}_2)|^q}{n} \sum_{\mathfrak{z}=1}^{n} \frac{\mathfrak{z}(\mathfrak{z}+3)}{2(\mathfrak{z}+1)(\mathfrak{z}+2)} + \frac{|\Pi'(\mathfrak{x}_1)|^q}{n} \sum_{\mathfrak{z}=1}^{n} \frac{\mathfrak{z}}{2(\mathfrak{z}+2)} \right)^{\frac{1}{q}}.
\end{aligned}$$

Remark 6.3. Considering Theorem 6.1, we establish the following new refinement of H-H inequality pertaining to the classical Mittag-Leffler function via C-FFIO in the mode of Hölder-Iscan inequality and the newly investigated idea if we pick $\rho = (1, 1, \ldots)$ with $\epsilon = \alpha$, $\theta = 1$ and $n = 1$:

$$\begin{aligned}
&\left| -\frac{\Pi(\mathfrak{m}\mathfrak{x}_1) + \Pi(\mathfrak{m}\mathfrak{x}_1 + \mathfrak{E}_\alpha(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1))}{2} - \frac{2(1-\sigma)}{\sigma \mathfrak{E}_\alpha(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)}\Pi(\mathfrak{v}) \right. \\
&\quad \left. + \frac{\mathcal{B}(\sigma)}{\sigma \mathfrak{E}_\alpha(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)} \left[\left({}^{\mathcal{CF}}_{\mathfrak{m}\mathfrak{x}_1} \mathrm{I}^{\sigma} \Pi \right)(\mathfrak{v}) + \left({}^{\mathcal{CF}} \mathrm{I}^{\sigma}_{\mathfrak{m}\mathfrak{x}_1 + \mathfrak{E}_\alpha(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)} \Pi \right)(\mathfrak{v}) \right] \right| \\
&\leqslant \frac{\mathfrak{E}_\alpha(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left\{ \left(\frac{2\mathfrak{m}|\Pi'(\mathfrak{x}_1)|^q + |\Pi'(\mathfrak{x}_2)|^q}{6} \right)^{\frac{1}{q}} + \left(\frac{\mathfrak{m}|\Pi'(\mathfrak{x}_1)|^q + 2|\Pi'(\mathfrak{x}_2)|^q}{6} \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Remark 6.4. Considering Theorem 6.1, we establish the following new refinement of H-H inequality pertaining to the classical Mittag-Leffler function via C-FFIO in the mode of Hölder-Iscan inequality and the newly investigated idea if we pick $\rho = (1, 1, \ldots)$ with $\epsilon = \alpha$, $\theta = m = 1$:

$$\begin{aligned}
&\left| -\frac{\Pi(\mathfrak{x}_1) + \Pi(\mathfrak{x}_1 + \mathfrak{E}_\alpha(\mathfrak{x}_2 - \mathfrak{x}_1))}{2} - \frac{2(1-\sigma)}{\sigma \mathfrak{E}_\alpha(\mathfrak{x}_2 - \mathfrak{x}_1)}\Pi(\mathfrak{v}) \right. \\
&\quad \left. + \frac{\mathcal{B}(\sigma)}{\sigma \mathfrak{E}_\alpha(\mathfrak{x}_2 - \mathfrak{x}_1)} \left[\left({}^{\mathcal{CF}}_{\mathfrak{x}_1} \mathrm{I}^{\sigma} \Pi \right)(\mathfrak{v}) + \left({}^{\mathcal{CF}} \mathrm{I}^{\sigma}_{\mathfrak{x}_1 + \mathfrak{E}_\alpha(\mathfrak{x}_2 - \mathfrak{x}_1)} \Pi \right)(\mathfrak{v}) \right] \right| \\
&\leqslant \frac{\mathfrak{E}_\alpha(\mathfrak{x}_2 - \mathfrak{x}_1)}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\Pi'(\mathfrak{x}_1)|^q}{n} \sum_{\mathfrak{z}=1}^{n} \frac{\mathfrak{z}(\mathfrak{z}+3)}{2(\mathfrak{z}+1)(\mathfrak{z}+2)} + \frac{|\Pi'(\mathfrak{x}_2)|^q}{n} \sum_{\mathfrak{z}=1}^{n} \frac{\mathfrak{z}}{2(\mathfrak{z}+2)} \right)^{\frac{1}{q}} \\
&\quad + \frac{\mathfrak{E}_\alpha(\mathfrak{x}_2 - \mathfrak{x}_1)}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\Pi'(\mathfrak{x}_2)|^q}{n} \sum_{\mathfrak{z}=1}^{n} \frac{\mathfrak{z}(\mathfrak{z}+3)}{2(\mathfrak{z}+1)(\mathfrak{z}+2)} + \frac{|\Pi'(\mathfrak{x}_1)|^q}{n} \sum_{\mathfrak{z}=1}^{n} \frac{\mathfrak{z}}{2(\mathfrak{z}+2)} \right)^{\frac{1}{q}}.
\end{aligned}$$

Theorem 6.5. *Let* $\Pi : \mathbb{I} = [\mathfrak{m}\mathfrak{x}_1, \mathfrak{m}\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)] \to \mathbb{R}$ *be a differentiable function on* $\mathbb{I}^\circ$, *with* $\mathfrak{m}\mathfrak{x}_1 < \mathfrak{m}\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)$, $q \geqslant 1$ *and assume that* $\Pi' \in \mathcal{L}[\mathfrak{m}\mathfrak{x}_1, \mathfrak{m}\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)]$. *If* $|\Pi'|^q$ *is* $\mathsf{PG}_{\mathfrak{m}}\mathsf{CRF}$ *on* $[\mathfrak{m}\mathfrak{x}_1, \mathfrak{m}\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)]$, *then*

$$\begin{aligned}
&\left| -\frac{\Pi(\mathfrak{m}\mathfrak{x}_1) + \Pi(\mathfrak{m}\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1))}{2} - \frac{2(1-\sigma)}{\sigma \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)}\Pi(\mathfrak{v}) \right. \\
&\quad \left. + \frac{\mathcal{B}(\sigma)}{\sigma \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)} \left[\left({}^{\mathcal{CF}}_{\mathfrak{m}\mathfrak{x}_1} \mathrm{I}^{\sigma} \Pi \right)(\mathfrak{v}) + \left({}^{\mathcal{CF}} \mathrm{I}^{\sigma}_{\mathfrak{m}\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)} \Pi \right)(\mathfrak{v}) \right] \right|
\end{aligned}$$

$$\leqslant \frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{\mathfrak{m}|\Pi'(\mathfrak{x}_1)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}+5)[(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2]}{2^{\mathfrak{z}+2}(\mathfrak{z}+1)(\mathfrak{z}+2)(\mathfrak{z}+3)}+\frac{|\Pi'(\mathfrak{x}_2)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{2^{\mathfrak{z}+2}(\mathfrak{z}+2)(\mathfrak{z}+3)}\right)^{\frac{1}{q}}$$

$$+\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{\mathfrak{m}|\Pi'(\mathfrak{x}_1)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{2^{\mathfrak{z}+2}(\mathfrak{z}+2)(\mathfrak{z}+3)}+\frac{|\Pi'(\mathfrak{x}_2)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}+5)[(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2]}{2^{\mathfrak{z}+2}(\mathfrak{z}+1)(\mathfrak{z}+2)(\mathfrak{z}+3)}\right)^{\frac{1}{q}},$$

where $\sigma\in[0,1]$, $\mathfrak{m}\in(0,1]$ *and* $\mathfrak{v}\in[\mathfrak{m}\mathfrak{x}_1,\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)]$.

Proof. For $q>1$, from Lemma 5.1, we have

$$\left|-\frac{\Pi(\mathfrak{m}\mathfrak{x}_1)+\Pi(\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(\mathfrak{v})\right.$$
$$\left.+\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\left[\left({}^{\mathcal{C}\mathcal{F}}_{\mathfrak{m}\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{C}\mathcal{F}}\mathrm{I}^{\sigma}_{\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\right|$$
$$\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\int_0^1|1-2\mathfrak{e}||\Pi'(\mathfrak{m}\mathfrak{x}_1+\mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1))|d\mathfrak{e}.$$

Employing improved power mean inequality, we have

$$\left|-\frac{\Pi(\mathfrak{m}\mathfrak{x}_1)+\Pi(\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(\mathfrak{v})\right.$$
$$\left.+\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\left[\left({}^{\mathcal{C}\mathcal{F}}_{\mathfrak{m}\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{C}\mathcal{F}}\mathrm{I}^{\sigma}_{\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\right|$$
$$\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\left(\int_0^1(1-\mathfrak{e})|1-2\mathfrak{e}|d\mathfrak{e}\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\mathfrak{e})|1-2\mathfrak{e}||\Pi'(\mathfrak{m}\mathfrak{x}_1+\mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1))|^q d\mathfrak{e}\right)^{\frac{1}{q}}$$
$$+\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\left(\int_0^1\mathfrak{e}|1-2\mathfrak{e}|d\mathfrak{e}\right)^{1-\frac{1}{q}}\left(\int_0^1\mathfrak{e}|1-2\mathfrak{e}||\Pi'(\mathfrak{m}\mathfrak{x}_1+\mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1))|^q d\mathfrak{e}\right)^{\frac{1}{q}}.$$

Employing $\mathrm{PG}_{\mathfrak{m}}\mathrm{CRF}$ of $|\Pi'|^q$, we have

$$\left|-\frac{\Pi(\mathfrak{m}\mathfrak{x}_1)+\Pi(\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(\mathfrak{v})\right.$$
$$\left.+\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\left[\left({}^{\mathcal{C}\mathcal{F}}_{\mathfrak{m}\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{C}\mathcal{F}}\mathrm{I}^{\sigma}_{\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\right|$$
$$\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{\mathfrak{m}|\Pi'(\mathfrak{x}_1)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\int_0^1(1-\mathfrak{e})|1-2\mathfrak{e}|(1-\mathfrak{e}^{\mathfrak{z}})d\mathfrak{e}\right.$$
$$\left.+\frac{|\Pi'(\mathfrak{x}_2)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\int_0^1(1-\mathfrak{e})|1-2\mathfrak{e}|(1-(1-\mathfrak{e})^{\mathfrak{z}})d\mathfrak{e}\right)^{\frac{1}{q}}+\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}$$
$$\times\left(\frac{\mathfrak{m}|\Pi'(\mathfrak{x}_1)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\int_0^1\mathfrak{e}|1-2\mathfrak{e}|(1-\mathfrak{e}^{\mathfrak{z}})d\mathfrak{e}+\frac{|\Pi'(\mathfrak{x}_2)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\int_0^1\mathfrak{e}|1-2\mathfrak{e}|(1-(1-\mathfrak{e})^{\mathfrak{z}})d\mathfrak{e}\right)^{\frac{1}{q}}$$
$$\leqslant\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{\mathfrak{m}|\Pi'(\mathfrak{x}_1)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}+5)[(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2]}{2^{\mathfrak{z}+2}(\mathfrak{z}+1)(\mathfrak{z}+2)(\mathfrak{z}+3)}+\frac{|\Pi'(\mathfrak{x}_2)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{2^{\mathfrak{z}+2}(\mathfrak{z}+2)(\mathfrak{z}+3)}\right)^{\frac{1}{q}}$$

$$+\frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{\mathfrak{m}|\Pi'(\mathfrak{x}_1)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{2^{\mathfrak{z}+2}(\mathfrak{z}+2)(\mathfrak{z}+3)}+\frac{|\Pi'(\mathfrak{x}_2)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}+5)[(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2]}{2^{\mathfrak{z}+2}(\mathfrak{z}+1)(\mathfrak{z}+2)(\mathfrak{z}+3)}\right)^{\frac{1}{q}}.$$

This is the required proof. □

Remark 6.6. Considering Theorem 6.5, we establish the following new refinement of H-H inequality pertaining to the classical Mittag-Leffler function via C-FFIO in the mode of improved power mean inequality and the newly explored idea if we pick $\rho=(1,1,\dots)$ with $\epsilon=\alpha$, $\theta=1$:

$$\begin{aligned}&\left|-\frac{\Pi(\mathfrak{m}\mathfrak{x}_1)+\Pi(\mathfrak{m}\mathfrak{x}_1+\mathfrak{E}_\alpha(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathfrak{E}_\alpha(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(\mathfrak{v})\right.\\&\quad\left.+\frac{\mathcal{B}(\sigma)}{\sigma\mathfrak{E}_\alpha(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{\mathfrak{m}\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{m}\mathfrak{x}_1+\mathfrak{E}_\alpha(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\right|\\&\leqslant\frac{\mathfrak{E}_\alpha(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{\mathfrak{m}|\Pi'(\mathfrak{x}_1)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}+5)[(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2]}{2^{\mathfrak{z}+2}(\mathfrak{z}+1)(\mathfrak{z}+2)(\mathfrak{z}+3)}+\frac{|\Pi'(\mathfrak{x}_2)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{2^{\mathfrak{z}+2}(\mathfrak{z}+2)(\mathfrak{z}+3)}\right)^{\frac{1}{q}}\\&\quad+\frac{\mathfrak{E}_\alpha(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{\mathfrak{m}|\Pi'(\mathfrak{x}_1)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{2^{\mathfrak{z}+2}(\mathfrak{z}+2)(\mathfrak{z}+3)}+\frac{|\Pi'(\mathfrak{x}_2)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}+5)[(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2]}{2^{\mathfrak{z}+2}(\mathfrak{z}+1)(\mathfrak{z}+2)(\mathfrak{z}+3)}\right)^{\frac{1}{q}}.\end{aligned}$$

Remark 6.7. Considering Theorem 6.5, we establish the following new refinement of H-H inequality pertaining to the classical Mittag-Leffler function via C-FFIO in the mode of improved power mean inequality and the newly explored idea if we pick $\rho=(1,1,\dots)$ with $\epsilon=\alpha$, $\theta=1$ and $n=1$:

$$\begin{aligned}&\left|-\frac{\Pi(\mathfrak{m}\mathfrak{x}_1)+\Pi(\mathfrak{m}\mathfrak{x}_1+\mathfrak{E}_\alpha(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathfrak{E}_\alpha(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi(\mathfrak{v})\right.\\&\quad\left.+\frac{\mathcal{B}(\sigma)}{\sigma\mathfrak{E}_\alpha(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{\mathfrak{m}\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{m}\mathfrak{x}_1+\mathfrak{E}_\alpha(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\right|\\&\leqslant\frac{\mathfrak{E}_\alpha(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left\{\left(\frac{3\mathfrak{m}|\Pi'(\mathfrak{x}_1)|^q+|\Pi'(\mathfrak{x}_2)|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{\mathfrak{m}|\Pi'(\mathfrak{x}_1)|^q+3|\Pi'(\mathfrak{x}_2)|^q}{16}\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Remark 6.8. Considering Theorem 6.5, we establish the following new refinement of H-H inequality pertaining to the classical Mittag-Leffler function via C-FFIO in the mode of improved power mean inequality and the newly explored idea if we pick $\rho=(1,1,\dots)$ with $\epsilon=\alpha$, $\theta=m=1$:

$$\begin{aligned}&\left|-\frac{\Pi(\mathfrak{x}_1)+\Pi(\mathfrak{x}_1+\mathfrak{E}_\alpha(\mathfrak{x}_2-\mathfrak{x}_1))}{2}-\frac{2(1-\sigma)}{\sigma\mathfrak{E}_\alpha(\mathfrak{x}_2-\mathfrak{x}_1)}\Pi(\mathfrak{v})\right.\\&\quad\left.+\frac{\mathcal{B}(\sigma)}{\sigma\mathfrak{E}_\alpha(\mathfrak{x}_2-\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi\right)(\mathfrak{v})+\left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{x}_1+\mathfrak{E}_\alpha(\mathfrak{x}_2-\mathfrak{x}_1)}\Pi\right)(\mathfrak{v})\right]\right|\\&\leqslant\frac{\mathfrak{E}_\alpha(\mathfrak{x}_2-\mathfrak{x}_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\Pi'(\mathfrak{x}_1)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}+5)[(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2]}{2^{\mathfrak{z}+2}(\mathfrak{z}+1)(\mathfrak{z}+2)(\mathfrak{z}+3)}+\frac{|\Pi'(\mathfrak{x}_2)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{2^{\mathfrak{z}+2}(\mathfrak{z}+2)(\mathfrak{z}+3)}\right)^{\frac{1}{q}}\\&\quad+\frac{\mathfrak{E}_\alpha(\mathfrak{x}_2-\mathfrak{x}_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\Pi'(\mathfrak{x}_1)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{2^{\mathfrak{z}+2}(\mathfrak{z}+2)(\mathfrak{z}+3)}+\frac{|\Pi'(\mathfrak{x}_2)|^q}{n}\sum_{\mathfrak{z}=1}^{n}\frac{(\mathfrak{z}+5)[(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2]}{2^{\mathfrak{z}+2}(\mathfrak{z}+1)(\mathfrak{z}+2)(\mathfrak{z}+3)}\right)^{\frac{1}{q}}.\end{aligned}$$

7. New mathematical perspective of Pachpatte type integral inequality via n-polynomial generalized m-convex involving Raina's function pertaining to Caputo-Fabrizio fractional integral operator

The word "convexity" has also gained significant attention during the last two decades because of its applicability and recognition of the concept of inequality. Owing to advancements of the convexity

in applied sciences, numerous inequalities have been suggested and discovered in the context of convex analysis. Convexity has also been addressed by numerous mathematicians, and several books were published that give new generalizations and estimatations. Due to research works and studies, the stunning Pachpatte-type inequality in the context of convexity is significantly improved.

Based on the above literature, we will research and explore the new mathematical fractional approach of Pachpatte-type inequality via $\mathrm{PG}_{\mathfrak{m}}\mathrm{CRF}$. Several remarks and corollaries involving Mittag-Leffler pertaining to Raina's function are included to enhance the significance and value of this portion.

Theorem 7.1. *Assume that* $\sigma, \mathfrak{m} \in [0,1]$ *and* $\mathfrak{v} \in [\mathfrak{m}\mathfrak{x}_1, \mathfrak{m}\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)]$. *Let* $\Pi_1, \Pi_2 : \mathbb{I} = [\mathfrak{m}\mathfrak{x}_1, \mathfrak{m}\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)] \to \mathbb{R}$ *be two functions such that* $\mathfrak{m}\mathfrak{x}_1, \mathfrak{m}\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1) \in \mathbb{I}$ *with* $\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1) > 0$ *and* $\Pi \in \mathcal{L}[\mathfrak{m}\mathfrak{x}_1, \mathfrak{m}\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)]$. *If* Π_1 *and* Π_2 *are* $\mathrm{PG}_{\mathfrak{m}}\mathrm{CRF}$ *, then*

$$\begin{aligned}&\frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)}\left[\left({}^{\mathcal{CF}}_{\mathfrak{m}\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi_1\Pi_2\right)(\mathfrak{v}) + \left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi_1\Pi_2\right)(\mathfrak{v}) - \frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi_1(\mathfrak{v})\Pi_2(\mathfrak{v})\right]\\ &\quad\leqslant \mathbb{M}(\mathfrak{x}_1,\mathfrak{x}_2) + \mathbb{N}(\mathfrak{x}_1,\mathfrak{x}_2),\end{aligned}$$

where

$$\begin{aligned}\mathbb{M}(\mathfrak{x}_1,\mathfrak{x}_2) &= \mathfrak{m}^2\Delta_1(\mathfrak{e})\Pi_1(\mathfrak{x}_1)\Pi_2(\mathfrak{x}_1) + \Delta_4(\mathfrak{e})\Pi_1(\mathfrak{x}_2)\Pi_2(\mathfrak{x}_2),\\ \mathbb{N}(\mathfrak{x}_1,\mathfrak{x}_2) &= \mathfrak{m}\Delta_2(\mathfrak{e})\Pi_1(\mathfrak{x}_2)\Pi_2(\mathfrak{x}_1) + \mathfrak{m}\Delta_3(\mathfrak{e})\Pi_1(\mathfrak{x}_1)\Pi_2(\mathfrak{x}_2),\end{aligned}$$

and

$$\begin{aligned}&\Delta_1 = \frac{1}{\mathfrak{n}_1\mathfrak{n}_2}\sum_{\mathfrak{z}=1}^{\mathfrak{n}_1}[1-\mathfrak{e}^{\mathfrak{z}}]\sum_{\mathfrak{z}=1}^{\mathfrak{n}_2}[1-\mathfrak{e}^{\mathfrak{z}}],\quad \Delta_2 = \frac{1}{\mathfrak{n}_1\mathfrak{n}_2}\sum_{\mathfrak{z}=1}^{\mathfrak{n}_1}[1-\mathfrak{e}^{\mathfrak{z}}]\sum_{\mathfrak{z}=1}^{\mathfrak{n}_2}[1-(1-\mathfrak{e})^{\mathfrak{z}}],\\ &\Delta_3 = \frac{1}{\mathfrak{n}_1\mathfrak{n}_2}\sum_{\mathfrak{z}=1}^{\mathfrak{n}_1}[1-(1-\mathfrak{e})^{\mathfrak{z}}]\sum_{\mathfrak{z}=1}^{\mathfrak{n}_2}[1-\mathfrak{e}^{\mathfrak{z}}],\quad \Delta_4 = \frac{1}{\mathfrak{n}_1\mathfrak{n}_2}\sum_{\mathfrak{z}=1}^{\mathfrak{n}_1}[1-(1-\mathfrak{e})^{\mathfrak{z}}]\sum_{\mathfrak{z}=1}^{\mathfrak{n}_2}[1-(1-\mathfrak{e})^{\mathfrak{z}}].\end{aligned}$$

Proof. Let Π_1 and Π_2 be the $\mathrm{PG}_{\mathfrak{m}}\mathrm{CRF}$, then for $\mathfrak{m}, \mathfrak{e} \in [0,1]$,

$$\Pi_1(\mathfrak{m}\mathfrak{x}_1 + \mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)) \leqslant \frac{\mathfrak{m}}{\mathfrak{n}_1}\sum_{\mathfrak{z}=1}^{\mathfrak{n}_1}[1-\mathfrak{e}^{\mathfrak{z}}]\,\Pi_1(\mathfrak{x}_1) + \frac{1}{\mathfrak{n}_1}\sum_{\mathfrak{z}=1}^{\mathfrak{n}_1}[1-(1-\mathfrak{e})^{\mathfrak{z}}]\,\Pi_1(\mathfrak{x}_2), \tag{7.1}$$

$$\Pi_2(\mathfrak{m}\mathfrak{x}_1 + \mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)) \leqslant \frac{\mathfrak{m}}{\mathfrak{n}_2}\sum_{\mathfrak{z}=1}^{\mathfrak{n}_2}[1-\mathfrak{e}^{\mathfrak{z}}]\,\Pi_2(\mathfrak{x}_1) + \frac{1}{\mathfrak{n}_2}\sum_{\mathfrak{z}=1}^{\mathfrak{n}_2}[1-(1-\mathfrak{e})^{\mathfrak{z}}]\,\Pi_2(\mathfrak{x}_2). \tag{7.2}$$

Multiplying the above inequalities (7.1) and (7.2), gives

$$\begin{aligned}&\Pi_1(\mathfrak{m}\mathfrak{x}_1 + \mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1))\Pi_2(\mathfrak{m}\mathfrak{x}_1 + \mathfrak{e}\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1))\\ &\leqslant \frac{\mathfrak{m}^2}{\mathfrak{n}_1\mathfrak{n}_2}\sum_{\mathfrak{z}=1}^{\mathfrak{n}_1}[1-\mathfrak{e}^{\mathfrak{z}}]\sum_{\mathfrak{z}=1}^{\mathfrak{n}_2}[1-\mathfrak{e}^{\mathfrak{z}}]\,\Pi(\mathfrak{x}_1)\Pi_2(\mathfrak{x}_1) + \frac{\mathfrak{m}}{\mathfrak{n}_1\mathfrak{n}_2}\sum_{\mathfrak{z}=1}^{\mathfrak{n}_1}[1-\mathfrak{e}^{\mathfrak{z}}]\sum_{\mathfrak{z}=1}^{\mathfrak{n}_2}[1-(1-\mathfrak{e})^{\mathfrak{z}}]\,\Pi_1(\mathfrak{x}_1)\Pi_2(\mathfrak{x}_2)\\ &\quad + \frac{\mathfrak{m}}{\mathfrak{n}_1\mathfrak{n}_2}\sum_{\mathfrak{z}=1}^{\mathfrak{n}_1}[1-(1-\mathfrak{e})^{\mathfrak{z}}]\sum_{\mathfrak{z}=1}^{\mathfrak{n}_2}[1-\mathfrak{e}^{\mathfrak{z}}]\,\Pi(\mathfrak{x}_2)\Pi_2(\mathfrak{x}_1) + \frac{1}{\mathfrak{n}_1\mathfrak{n}_2}\sum_{\mathfrak{z}=1}^{\mathfrak{n}_1}[1-(1-\mathfrak{e})^{\mathfrak{z}}]\sum_{\mathfrak{z}=1}^{\mathfrak{n}_2}[1-(1-\mathfrak{e})^{\mathfrak{z}}]\,\Pi_1(\mathfrak{x}_2)\Pi_2(\mathfrak{x}_2)\\ &= \mathfrak{m}^2\Delta_1(\mathfrak{e})\Pi_1(\mathfrak{x}_1)\Pi_2(\mathfrak{x}_1) + \mathfrak{m}\Delta_2(\mathfrak{e})\Pi_1(\mathfrak{x}_2)\Pi_2(\mathfrak{x}_1) + \mathfrak{m}\Delta_3(\mathfrak{e})\Pi_1(\mathfrak{x}_1)\Pi_2(\mathfrak{x}_2) + \Delta_4(\mathfrak{e})\Pi_1(\mathfrak{x}_2)\Pi_2(\mathfrak{x}_2).\end{aligned} \tag{7.3}$$

Taking integration over $[0,1]$ on both sides of inequality (7.3), we have

$$\frac{1}{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)}\int_{\mathfrak{m}\mathfrak{x}_1}^{\mathfrak{m}\mathfrak{x}_1+\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2-\mathfrak{m}\mathfrak{x}_1)}\Pi_1(x)\Pi_2(x)\,dx$$

$$\leqslant \int_0^1 \Big[m^2 \Delta_1(\mathfrak{e}) \Pi_1(\mathfrak{x}_1)\Pi_2(\mathfrak{x}_1) + m\Delta_2(\mathfrak{e})\Pi_1(\mathfrak{x}_2)\Pi_2(\mathfrak{x}_1) + m\Delta_3(\mathfrak{e})\Pi_1(\mathfrak{x}_1)\Pi_2(\mathfrak{x}_2) + \Delta_4(\mathfrak{e})\Pi_1(\mathfrak{x}_2)\Pi_2(\mathfrak{x}_2)\Big] d\mathfrak{e} = \mathbb{M}(\mathfrak{x}_1,\mathfrak{x}_2) + \mathbb{N}(\mathfrak{x}_1,\mathfrak{x}_2).$$

That is,

$$\frac{1}{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1)} \left[\int_{m\mathfrak{x}_1}^{k} \Pi_1(x)\Pi_2(x)dx + \int_{k}^{m\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1)} \Pi_1(x)\Pi_2(x)dx \right] \leqslant \mathbb{M}(\mathfrak{x}_1,\mathfrak{x}_2) + \mathbb{N}(\mathfrak{x}_1,\mathfrak{x}_2). \tag{7.4}$$

Now multiplying (7.4) by $\frac{\sigma \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1)}{\mathcal{B}(\sigma)}$ and adding $\frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi_1(\mathfrak{v})\Pi_2(\mathfrak{v})$ to the resulting inequality, we obtain

$$\frac{\sigma}{\mathcal{B}(\sigma)} \left[\int_{m\mathfrak{x}_1}^{k} \Pi_1(x)\Pi_2(x)dx + \int_{k}^{m\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1)} \Pi_1(x)\Pi_2(x)dx \right] + \frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi_1(\mathfrak{v})\Pi_2(\mathfrak{v}) \leqslant \frac{\sigma \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1)}{\mathcal{B}(\sigma)} [\mathbb{M}(\mathfrak{x}_1,\mathfrak{x}_2) + \mathbb{N}(\mathfrak{x}_1,\mathfrak{x}_2)] + \frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi_1(\mathfrak{v})\Pi_2(\mathfrak{v}).$$

Hence,

$$ {}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi_1(\mathfrak{v})\Pi_2(\mathfrak{v}) + {}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1)}\Pi_1(\mathfrak{v})\Pi_2(\mathfrak{v}) \leqslant \frac{\sigma \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1)}{\mathcal{B}(\sigma)} [\mathbb{M}(\mathfrak{x}_1,\mathfrak{x}_2) + \mathbb{N}(\mathfrak{x}_1,\mathfrak{x}_2)] + \frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi_1(\mathfrak{v})\Pi_2(\mathfrak{v}).$$

This is the required result. □

Remark 7.2. Considering Theorem 7.1, we establish the following new mathematical fractional approach of Pachpatte-type inequality pertaining to the classical Mittag-Leffler function with the aid of the newly introduced concept if we pick $\rho = (1, 1, \ldots)$ with $\epsilon = \alpha$, $\theta = 1$:

$$\frac{\mathcal{B}(\sigma)}{\sigma \mathfrak{E}_{\alpha}(\mathfrak{x}_2 - m\mathfrak{x}_1)} \left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi_1\Pi_2 \right)(\mathfrak{v}) + \left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\mathfrak{x}_1 + \mathfrak{E}_{\alpha}(\mathfrak{x}_2 - m\mathfrak{x}_1)}\Pi_1\Pi_2 \right)(\mathfrak{v}) - \frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi_1(\mathfrak{v})\Pi_2(\mathfrak{v}) \right] \leqslant \mathbb{M}(\mathfrak{x}_1,\mathfrak{x}_2) + \mathbb{N}(\mathfrak{x}_1,\mathfrak{x}_2).$$

Corollary 7.3. *Assume that* $n_1 = n_2 = m = 1$ *in Theorem* 7.1. *Then*

$$\frac{2\mathcal{B}(\sigma)}{\sigma \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{x}_1)} \left[\left({}^{\mathcal{CF}}_{\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi_1\Pi_2 \right)(\mathfrak{v}) + \left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{x}_1)}\Pi_1\Pi_2 \right)(\mathfrak{v}) - \frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi_1(\mathfrak{v})\Pi_2(\mathfrak{v}) \right] \leqslant \frac{2}{3}\mathbb{M}(\mathfrak{x}_1,\mathfrak{x}_2) + \frac{1}{3}\mathbb{N}(\mathfrak{x}_1,\mathfrak{x}_2).$$

Corollary 7.4. *If we put* $\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1) = \mathfrak{x}_2 - m\mathfrak{x}_1$ *in Theorem* 7.1, *then*

$$\frac{\mathcal{B}(\sigma)}{\sigma(\mathfrak{x}_2 - m\mathfrak{x}_1)} \left[\left({}^{\mathcal{CF}}_{m\mathfrak{x}_1}\mathrm{I}^{\sigma}\Pi_1\Pi_2 \right)(\mathfrak{v}) + \left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{\mathfrak{x}_2}\Pi_1\Pi_2 \right)(\mathfrak{v}) - \frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\Pi_1(\mathfrak{v})\Pi_2(\mathfrak{v}) \right] \leqslant \mathbb{M}(\mathfrak{x}_1,\mathfrak{x}_2) + \mathbb{N}(\mathfrak{x}_1,\mathfrak{x}_2).$$

8. Applications to means

Both topics, fractional calculus and convex analysis, are used in applied science. Owing to the literature, there are certainly an extensive number of intriguing implementations of these notions across a range of research domains, from fluid mechanics to optimization. More precisely, we will utilize C-FFIO

incorporating $PG_m CRF$ to employ several mean-type inequalities, such as harmonic, geometric, and arithmetic mean inequalities, to the H-H inequality. The following inequalities have amazing applications in the field of probability, circuit theory, statistics, machine learning, numerical approximations, stochastic processes, and engineering.

Here, we present the means as applications for two positive number $\mathfrak{x}_1, \mathfrak{x}_2$ with $\mathfrak{x}_1 < \mathfrak{x}_2$, which are given as follows.

(1) The arithmetic mean: $\mathcal{A} = \mathcal{A}(\mathfrak{x}_1, \mathfrak{x}_2) = \frac{\mathfrak{x}_1+\mathfrak{x}_2}{2}$, $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathbb{R}$.

(2) The generalized logarithmic mean: $\mathcal{L} = \mathcal{L}_r(\mathfrak{x}_1, \mathfrak{x}_2) = \frac{\mathfrak{x}_2^{r+1} - \mathfrak{x}_1^{r+1}}{(r+1)(\mathfrak{x}_2 - \mathfrak{x}_1)}$.

Here, we demonstrate a number of inequalities related to special means utilizing the findings in Section 5.

Proposition 8.1. *Assume that* $m\mathfrak{x}_1, m\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1) \in \mathbb{R}^+$ *and* $m\mathfrak{x}_1 < m\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1)$, *then*

$$\begin{aligned} &\left| -\mathcal{A}(m\mathfrak{x}_1^2, (m\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1))^2) + \mathcal{L}_2^2(m\mathfrak{x}_1, m\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1)) \right| \\ &\leqslant \frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1)}{n} \sum_{\mathfrak{z}=1}^{n} \left[\frac{(\mathfrak{z}^2 + \mathfrak{z} + 2)2^{\mathfrak{z}} - 2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}} \right] [|\mathfrak{x}_1| + |\mathfrak{x}_2|]. \end{aligned} \tag{8.1}$$

Proof. Suppose $\Pi(z) = z^2$ with $\mathcal{B}(\sigma) = \mathcal{B}(1) = 1$ and $\sigma = 1$ in Theorem 5.2, then we achieve the above inequality (8.1). □

Remark 8.2. Considering Proposition 8.1, we establish the following inequality pertaining to the classical Mittag-Leffler function if we pick $\rho = (1, 1, \ldots)$ with $\epsilon = \alpha$, $\theta = 1$:

$$\begin{aligned} &\left| -\mathcal{A}(m\mathfrak{x}_1^2, (m\mathfrak{x}_1 + \mathfrak{E}_{\alpha}(\mathfrak{x}_2 - m\mathfrak{x}_1))^2) + \mathcal{L}_2^2(m\mathfrak{x}_1, m\mathfrak{x}_1 + \mathfrak{E}_{\alpha}(\mathfrak{x}_2 - m\mathfrak{x}_1)) \right| \\ &\leqslant \frac{\mathfrak{E}_{\alpha}(\mathfrak{x}_2 - m\mathfrak{x}_1)}{n} \sum_{\mathfrak{z}=1}^{n} \left[\frac{(\mathfrak{z}^2 + \mathfrak{z} + 2)2^{\mathfrak{z}} - 2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}} \right] [|\mathfrak{x}_1| + |\mathfrak{x}_2|]. \end{aligned}$$

Proposition 8.3. *Assume that* $m\mathfrak{x}_1, m\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1) \in \mathbb{R}^+$ *and* $m\mathfrak{x}_1 < m\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1)$, *then*

$$\begin{aligned} &\left| -\mathcal{A}(e^{m\mathfrak{x}_1}, e^{(m\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1))}) + \mathcal{L}(e^{m\mathfrak{x}_1}, e^{(m\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1))}) \right| \\ &\leqslant \frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - m\mathfrak{x}_1)}{n} \sum_{\mathfrak{z}=1}^{n} \left[\frac{(\mathfrak{z}^2 + \mathfrak{z} + 2)2^{\mathfrak{z}} - 2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}} \right] \left[\frac{e_1^{\mathfrak{x}} + e_2^{\mathfrak{x}}}{2} \right]. \end{aligned} \tag{8.2}$$

Proof. Suppose $\Pi(z) = e^z$ with $\mathcal{B}(\sigma) = \mathcal{B}(1) = 1$ and $\sigma = 1$ in Theorem 5.2, then we achieve the above inequality (8.2). □

Remark 8.4. Considering Proposition 8.3, we establish the following inequality pertaining to the classical Mittag-Leffler function if we pick $\rho = (1, 1, \ldots)$ with $\epsilon = \alpha$, $\theta = 1$:

$$\begin{aligned} &\left| -\mathcal{A}(e^{m\mathfrak{x}_1}, e^{(m\mathfrak{x}_1 + \mathfrak{E}_{\alpha}(\mathfrak{x}_2 - m\mathfrak{x}_1))}) + \mathcal{L}(e^{m\mathfrak{x}_1}, e^{(m\mathfrak{x}_1 + \mathfrak{E}_{\alpha}(\mathfrak{x}_2 - m\mathfrak{x}_1))}) \right| \\ &\leqslant \frac{\mathfrak{E}_{\alpha}(\mathfrak{x}_2 - m\mathfrak{x}_1)}{n} \sum_{\mathfrak{z}=1}^{n} \left[\frac{(\mathfrak{z}^2 + \mathfrak{z} + 2)2^{\mathfrak{z}} - 2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}} \right] \left[\frac{e_1^{\mathfrak{x}} + e_2^{\mathfrak{x}}}{2} \right]. \end{aligned}$$

Proposition 8.5. *Assume that* $\mathfrak{m}\mathfrak{x}_1, \mathfrak{m}\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1) \in \mathbb{R}^+, \mathfrak{m}\mathfrak{x}_1 < \mathfrak{m}\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)$, *then*

$$\begin{aligned}&\left| -\mathcal{A}((\mathfrak{m}\mathfrak{x}_1)^n, (\mathfrak{m}\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1))^n) - \mathcal{L}^n_n(\mathfrak{m}\mathfrak{x}_1, \mathfrak{m}\mathfrak{x}_1 + \mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)) \right| \\ &\leqslant \frac{\mathcal{R}^{\rho}_{\epsilon,\theta}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)}{n} \sum_{\mathfrak{z}=1}^{n} \left[\frac{(\mathfrak{z}^2 + \mathfrak{z} + 2)2^{\mathfrak{z}} - 2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}} \right] \left[n \frac{|\mathfrak{x}_1^{n-1}| + |\mathfrak{x}_2^{n-1}|}{2} \right].\end{aligned} \tag{8.3}$$

Proof. Suppose $\Pi(z) = z^n$ with $\mathcal{B}(\sigma) = \mathcal{B}(1) = 1$ and $\sigma = 1$ in Theorem 5.2, then we achieve the above inequality (8.3). □

Remark 8.6. Considering Proposition 8.5, we establish the following inequality pertaining to the classical Mittag-Leffler function if we pick $\rho = (1, 1, \ldots)$ with $\epsilon = \alpha$, $\theta = 1$:

$$\begin{aligned}&\left| -\mathcal{A}((\mathfrak{m}\mathfrak{x}_1)^n, (\mathfrak{m}\mathfrak{x}_1 + \mathfrak{E}_{\alpha}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1))^n) - \mathcal{L}^n_n(\mathfrak{m}\mathfrak{x}_1, \mathfrak{m}\mathfrak{x}_1 + \mathfrak{E}_{\alpha}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)) \right| \\ &\leqslant \frac{\mathfrak{E}_{\alpha}(\mathfrak{x}_2 - \mathfrak{m}\mathfrak{x}_1)}{n} \sum_{\mathfrak{z}=1}^{n} \left[\frac{(\mathfrak{z}^2 + \mathfrak{z} + 2)2^{\mathfrak{z}} - 2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}} \right] \left[n \frac{|\mathfrak{x}_1^{n-1}| + |\mathfrak{x}_2^{n-1}|}{2} \right].\end{aligned}$$

9. Conclusions

Researchers and scholars from a wide range of scientific backgrounds are very interested in studying fractional calculus because they see how much it can be used to solve complicated problems and advance knowledge. Integral inequalities possess profound consequences across various domains of study such as physics, functional analysis, optimization and statistical theory. Therefore, inequalities and convex analysis have become more popular among researchers, as the subject continues to receive attention through several modifications, advancements, expansions, and applications. In this manuscript.

1) First, we investigated a novel type of convex function, i.e., $\mathrm{PG}_{\mathfrak{m}}\mathrm{CRF}$. Subsequently, we have appended certain algebraic properties on the basis of the recently introduced idea.
2) We established a new mathematical perspective of the H-H and Pachpatte type inequalities through $\mathrm{PG}_{\mathfrak{m}}\mathrm{CRF}$ in terms of the C-FFIO.
3) In addition, a new lemma is introduced and some results via C-FFIO are obtained through a newly established idea.
4) In order to enhance the researcher's attention and overall worth, we presented the developments of H-H inequality in the context of the investigated new equality through the support of Hölder, power, power mean improved, and Hölder Iscan inequality.
5 Certain remarks and corollaries about literature validity and Mittag-Leffler functions including Raina's function are included.
6) In the end, some important applications pertaining to newly investigated concept are examined.

The novel methods and new ideas introduced in this paper can be utilized to study interval analysis, gaining new insights and new points of view. Additionally, the inequalities addressed here can be investigated in the context of quantum calculus, uncovering possible directions for new research.

Acknowledgment

This research was funded by the National Science, Research and Innovation Fund (NSRF) and King Mongkut's University of Technology North Bangkok, under Contract No. KMUTNB-FF-68-B-04.

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