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# Some results on existence and uniqueness of solutions for fractional boundary value problems



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## Abstract

The purpose of this work is to improve Ferreira's results [R. A. C. Ferreira, Electron J. Differ. Equ., **2016** (2016), 5 pages]. The advancement is achieved through an application of Rus's contraction mapping theorem. To this end, we derive new estimates for integrals involving Green's function. Applying these estimates in conjunction with Rus's contraction mapping theorem demonstrates that a larger class of fractional boundary value problems admit a unique solution than those obtained by Ferreira. We conclude this article with numerical validation with applications, that highlight the nature and significance of the advancements made.

**Keywords:** Fractional boundary value problem, existence and uniqueness of solutions, Rus's contraction mapping theorem, nonlinear equations, numerical estimates.

**2020 MSC:** 34A08, 47H10.



## 1. Introduction

In 2016, Ferreira [4] considered the following two-point fractional boundary value problem (FBVP)

$$ {}_{a}D^{\alpha}y(t) = -g(t, y(t)), \quad a < t < b, \quad 1 < \alpha \leqslant 2, \quad y(a) = 0, \quad y(b) = B, \quad B \in \mathbb{R}, \tag{1.1} $$

where ${}_{a}D^{\alpha}$ represents the Riemann-Liouville derivative of order $\alpha$, and $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is assumed to be a continuous function. The reader is refered to [8] for the definitions and basic results on fractional calculus.

The author of [4] established a property of the Green's function associated with the FBVP (1.1) and skilfully obtained the existence and uniqueness of solutions for FBVP (1.1) through the application of Banach fixed-point theorem [3]. Ferreira's work [4] extended the result of Peterson and Kelley [7, Theorem 7.7] by considering the FBVP (1.1) of order $1 < \alpha \leqslant 2$ instead of $\alpha = 2$.

The aim of the present work is to improve Ferreira's existence and uniqueness result for the FBVP (1.1). To this end, we derive new and insightful estimates for integrals regarding Green's function. We then apply these estimates to the FBVP (1.1) through an application of Rus's contraction mapping theorem [13] within a metric space, where two distinct metrics are used. As will be demonstrated, our new result

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sharpens Ferreira's work [4] by showing that a larger class of these kinds of problems admit a unique solution. This involves improving the Lipschitz constants in a global (unbounded) setting. Consequently, our approach offers an improvement over traditional methods, such as the use of Banach's fixed point theorem. Thus, this approach can be a great strategy to ensure the existence and uniqueness of solutions for fractional boundary value problems. For a recent discussion, comparison and for various variations of Rus's contraction mapping theorem the reader is referred to [1, 2, 16].

The following result due to Rus [13] will be a key element in our analysis to improve Ferreira's work.

**Theorem 1.1** (Rus, [13]). *Let $\mathcal{Y}$ be a nonempty set and let $d$ and $\rho$ be two metrics on $\mathcal{Y}$ such that $(\mathcal{Y}, d)$ forms a complete metric space. If the mapping $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ is continuous with respect to $d$ on $\mathcal{Y}$ and:*

$$d(\mathcal{T}x, \mathcal{T}y) \leqslant c\rho(x, y), \text{ for some } c > 0 \text{ and all } x, y \in \mathcal{Y}, \tag{1.2}$$

$$\rho(\mathcal{T}x, \mathcal{T}y) \leqslant \beta\rho(x, y), \text{ for some } 0 < \beta < 1 \text{ and all } x, y \in \mathcal{Y}, \tag{1.3}$$

*then there is a unique $z \in \mathcal{Y}$ such that $\mathcal{T}z = z$.*

In the context of the present work, we will consider the following notation, sets, metrics, definitions and some useful results. Consider the set of continuous functions defined on $[a, b]$ and we denote this space by $\mathcal{Y} := C([a, b])$. For functions $x, y \in \mathcal{Y}$ consider the following two metrics on $\mathcal{Y}$:

$$d(x, y) := \max_{t\in[a,b]} |x(t) - y(t)|, \tag{1.4}$$

and

$$\rho(x, y) := \left(\int_a^b |x(t) - y(t)|^p \, dt\right)^{1/p}, \quad p > 1. \tag{1.5}$$

Thus, we can see that for the metric $d$ defined in (1.4), the pair $(\mathcal{Y}, d)$ is a complete metric space. In contrast, under the metric $\rho$ defined in (1.5), the pair $(\mathcal{Y}, \rho)$ is a metric space, but it is not complete. An important relationship between these two metrics on $\mathcal{Y}$, which we will utilize, is the following

$$\rho(x, y) \leqslant (b - a)^{1/p} d(x, y), \quad \text{for all } \ x, y \in \mathcal{Y}. \tag{1.6}$$

Let us now present a set of symbols and notations.

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of non-negative integers, integers, real numbers, and complex numbers, respectively.
- $\mathcal{R}(z)$ denotes the real part of $z$.

The following is the definition of the Gauss hypergeometric function

**Definition 1.2** ([8]). The Gauss hypergeometric function ${}_2F_1(a_1, b_1; c_1; z)$ is defined in the unit disk as the sum of the hypergeometric series as follows:

$${}_2F_1(a_1, b_1; c_1; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (b_1)_k}{(c_1)_k} \frac{z^k}{k!} \quad (a_1, b_1 \in \mathbb{C}; c_1 \in \mathbb{C} \setminus \mathbb{Z}_0^-, \ \ z \in \overline{B} := \{z \in \mathbb{C} \mid |z| \leqslant 1\}, \ \ \mathcal{R}(c_1 - b_1 - a_1) > 0),$$

where, $(z)_k$ is the Pochhammer symbol, defined for $z \in \mathbb{C}$ and $k \in \mathbb{N}^*$ by

$$(z)_0 = 1 \ \text{ and } \ (z)_k = z(z + 1) \cdots (z + k - 1), \ \ \forall k \in \mathbb{N}.$$

A natural extension of ${}_2F_1$ to ${}_3F_2$ is defined by

$$ {}_3F_2(a_1, b_1, c_1; d_1, e_1; z) = \sum_{k=0}^{+\infty} \frac{(a_1)_k (b_1)_k (c_1)_k}{(d_1)_k (e_1)_k} \frac{z^k}{k!} \quad (z \in \overline{B};\ \ \mathcal{R}(d_1 + e_1 - a_1 - b_1 - c_1) > 0). $$

We now present the following result recently obtained by Abdelhamid and Almuthaybiri [17, Theorem 4.3], which will play a central role in the development of our analysis. This result is particularly important for establishing integral estimates involving the Green's function associated with the fractional boundary value problem (1.1), and will serve as a foundational tool in the results that follow.

**Theorem 1.3** ([17, Theorem 4.3]). *For all* $a \leqslant t \leqslant b$, *and* $\alpha > 1/2$, *the following integral representations for the Gauss hypergeometric function hold true.*

1.

$$ \begin{aligned} I_3^{\alpha}(t) &= \int_a^t (t-a)^{\alpha-1} (t-s)^{\alpha-1} (b-s)^{\alpha-1} ds \\ &= \left[ \frac{(b-a)^{\alpha-1} (t-a)^{2\alpha-1}}{\alpha} \right] {}_2F_1(1-\alpha, 1, \alpha+1; g(t)), \end{aligned} \tag{1.7} $$

*where* $g(t) := \frac{t-a}{b-a}$.

2.

$$ \int_a^b I_3^{\alpha}(t) dt = \left[ \frac{(b-a)^{3\alpha-1}}{2\alpha^2} \right] {}_3F_2(1-\alpha, 1, 2\alpha; \alpha+1, 2\alpha+1; 1). \tag{1.8} $$

The authors of [17] also succeeded in determining the explicit expressions of the following hypergeometric functions ${}_3F_2$, see [17, Corollary 2.1], and [17, Lemma 3.3]. These results will be important in establishing specific cases of integral estimates involving the Green's function.

**Lemma 1.4** ([17]). *We have*

$$ {}_3F_2(-1, 1, 4; 3, 5; 1) = \frac{11}{15}, \tag{1.9} $$

$$ {}_3F_2(-\frac{1}{2}, 1, 3; \frac{5}{2}, 4; 1) = \frac{33}{35} - \frac{6}{35} \ln(2). \tag{1.10} $$

The reader is referred to [10, 11, 14, 15, 17] for more expressions of the hypergeometric function ${}_3F_2$, and relation between ${}_2F_1$ and ${}_3F_2$. We also remark that the (1.7) is special case from [14, (2.46), p. 41]. We sincerely thank the referee for drawing our attention to this reference, which allows us to properly acknowledge the original source.

The rest of our work is outlined as follows. In Section 2 we establish new and insightful estimates for integrals regarding Green's function. In Section 3 these estimates are applied to FBVP (1.1) using Rus's fixed point theorems, which yields improved results on existence and uniqueness of solutions. To fully illustrate the contributions of this work, Section 4 concludes with numerical validation with applications, that highlight the nature and significance of the advancements made.

## 2. Fractional integral estimates involving the Green's function

In this section we derive some properties of the Green's function associated with the FBVP (1.1). The following result [4, Lemma 2.1] shows that the FBVP (1.1) can be equivalently formulated as an integral equation.

**Lemma 2.1** ([4, Lemma 2.1]). *Suppose that $g$ is a continuous function. A function $y \in C[a,b]$ is a solution of (1.1) if and only if $y$ satisfies the integral equation*

$$y(t) = \phi(t) + \int_a^b \mathfrak{G}(t,s)g(s,y(s))ds, \quad t \in [a,b],$$

*where $\mathfrak{G}$ is given explicitly by*

$$\mathfrak{G}(t,s) = \begin{cases} \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{\Gamma(\alpha)(b-a)^{\alpha-1}} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{for } a \leqslant s \leqslant t \leqslant b, \\ \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{\Gamma(\alpha)(b-a)^{\alpha-1}}, & \text{for } a \leqslant t \leqslant s \leqslant b, \end{cases}$$

*and $\phi$ is defined by*

$$\phi(t) := \frac{B(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}.$$

We now establish the following result, which provides estimates for the integral of the Green's function associated with the FBVP (1.1). These estimates will be particularly useful for the analysis presented in Section 3.

**Lemma 2.2.** *Let $1 < \alpha \leqslant 2$, then the function $G(t,s)$ satisfies the following properties:*

(a)

$$\max_{t\in[a,b]} \left( \int_a^b |\mathfrak{G}(t,s)|^2 ds \right) \leqslant \left( \frac{2(b-a)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)} \right), \quad \text{for all } \ t \in [a,b],$$

(b)

$$\int_a^b \left( \int_a^b |\mathfrak{G}(t,s)|^2 ds \right) dt = N_1(a,b,\alpha), \quad \text{for all } \ t \in [a,b], \tag{2.1}$$

*where*

$$N_1(a,b,\alpha) = \frac{(b-a)^{2\alpha}}{\Gamma^2(\alpha)(2\alpha-1)^2} + \frac{(b-a)^{2\alpha}}{\Gamma^2(\alpha)(4\alpha^2-2\alpha)} - \left[ \frac{(b-a)^{2\alpha}}{\Gamma^2(\alpha)\alpha^2} \right] {}_3F_2(1-\alpha, 1, 2\alpha; \alpha+1, 2\alpha+1; 1). \tag{2.2}$$

*Proof.*

(a) Let $1 < \alpha \leqslant 2$. For convenience we set $U(t) := \frac{(t-a)^{2\alpha-2}}{\Gamma^2(\alpha)(b-a)^{2\alpha-2}}$, $M := \frac{2}{\Gamma^2(\alpha)(b-a)^{\alpha-1}}$. For all $t \in [a,b]$, we have

$$\begin{aligned}
&\int_a^b |\mathfrak{G}(t,s)|^2 ds \\
&= \int_a^t |\mathfrak{G}(t,s)|^2 ds + \int_t^b |\mathfrak{G}(t,s)|^2 ds \\
&= \int_a^t \left( U(t)(b-s)^{2\alpha-2} - M(t-a)^{\alpha-1}(b-s)^{\alpha-1}(t-s)^{\alpha-1} + \frac{(t-s)^{2\alpha-2}}{\Gamma^2(\alpha)} \right) ds + \int_t^b U(t)(b-s)^{2\alpha-2} ds \\
&= \frac{U(t)\left[-(b-t)^{2\alpha-1} + (b-a)^{2\alpha-1}\right]}{(2\alpha-1)} \\
&\quad - \left[ \frac{M(b-a)^{\alpha-1}(t-a)^{2\alpha-1}}{\alpha} \right] {}_2F_1(1-\alpha, 1; \alpha+1; g(t)) + \frac{(t-a)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)} + \frac{U(t)(b-t)^{2\alpha-1}}{(2\alpha-1)} \qquad (2.3) \\
&= \frac{U(t)(b-a)^{2\alpha-1}}{(2\alpha-1)} + \frac{(t-a)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)} - \left[ \frac{M(b-a)^{\alpha-1}(t-a)^{2\alpha-1}}{\alpha} \right] {}_2F_1(1-\alpha, 1; \alpha+1; g(t)) \qquad (2.4)
\end{aligned}$$

$$\leqslant \frac{U(t)(b-a)^{2\alpha-1}}{(2\alpha-1)} + \frac{(t-a)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)} \leqslant \frac{U(b)(b-a)^{2\alpha-1}}{(2\alpha-1)} + \frac{(b-a)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)} = \frac{2(b-a)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)},$$

where above we applied Theorem 1.3 (1.7) to obtain (2.3). Thus, we have

$$\max_{t\in[a,b]} \left( \int_a^b |\mathfrak{G}(t,s)|^2 ds \right) \leqslant \left( \frac{2(b-a)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)} \right),$$

which yields the desired result of (a).

(b) From (2.4) we have

$$\begin{aligned} &\int_a^b \left( \int_a^b |\mathfrak{G}(t,s)|^2 ds \right) dt \\ &= \int_a^b \left( \frac{(b-a)^{2\alpha-1}U(t)}{(2\alpha-1)} + \frac{(t-a)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)} \right) dt \\ &\quad - M\int_a^b \left( \left[ \frac{(b-a)^{\alpha-1}(t-a)^{2\alpha-1}}{\alpha} \right] {}_2F_1(1-\alpha,1;\alpha+1;g(t)) \right) dt \\ &= \frac{(b-a)^{2\alpha}}{\Gamma^2(\alpha)(2\alpha-1)^2} + \frac{(b-a)^{2\alpha}}{\Gamma^2(\alpha)(4\alpha^2-2\alpha)} - \left[ \frac{(b-a)^{2\alpha}}{\Gamma^2(\alpha)\alpha^2} \right] {}_3F_2(1-\alpha,1,2\alpha;\alpha+1,2\alpha+1;1), \end{aligned} \tag{2.5}$$

where above we applied Theorem 1.3 (1.8) to (2.5) to obtain (2.5), which yields the desired result of (b). □

We now evaluate three special cases of (2.2) for $\alpha = 2$ and $\alpha = 3/2$. First case, if $\alpha = 2$, then

$$N_1(a,b,2) = \frac{(b-a)^4}{9} + \frac{(b-a)^4}{12} - \left[ \frac{(b-a)^4}{4} \right] {}_3F_2(-1,1,4;3,5;1) \tag{2.6}$$

$$= \frac{(b-a)^4}{90}, \tag{2.7}$$

above we have applied (1.9) to (2.6) to obtain (2.7). Second case, if $\alpha = 3/2$, then

$$N_1(a,b,3/2) = \frac{(b-a)^3}{4\Gamma^2(3/2)} + \frac{(b-a)^3}{6\Gamma^2(3/2)} - \left[ \frac{4(b-a)^3}{9\Gamma^2(3/2)} \right] {}_3F_2(-\frac{1}{2},1,3;\frac{5}{2},4;1), \tag{2.8}$$

$$= (b-a)^3 \left( \frac{32\ln(2)-1}{105\pi} \right), \tag{2.9}$$

above we have applied (1.10) to (2.8) to obtain (2.9).

## 3. Main results

In this section we establish our new results for the existence and uniqueness of solutions to (1.1) via applications of Rus's theorem where two distinct metrics are employed.

**Theorem 3.1.** *Suppose that* $1 < \alpha \leqslant 2$. *Let* $g : [a,b] \times \mathbb{R} \to \mathbb{R}$ *be continuous, and let* $K > 0$ *such that*

$$|g(t,u) - g(t,v)| \leqslant K|u-v|, \quad \textit{for all} \quad (t,u),(t,v) \in [a,b] \times \mathbb{R}. \tag{3.1}$$

*If there are constants* $p > 1$ *and* $q > 1$ *such that* $1/p + 1/q = 1$ *with*

$$K \left( \int_a^b \left( \int_a^b |\mathfrak{G}(t,s)|^q \, ds \right)^{p/q} dt \right)^{1/p} < 1, \tag{3.2}$$

*then the FBVP* (1.1) *admits a unique solution in* $C([a,b])$.

*Proof.* Consider the pair $(\mathcal{Y}, d) = (C([a, b]), d)$, which is a complete metric space. In addition, consider the metric $\rho$ on $\mathcal{Y}$ where $q > 1$. By Lemma 2.1, we define the operator $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ by

$$(\mathcal{T}y)(t) := \int_a^b \mathcal{G}(t, s)g(s, y(s))ds + \phi(t), \quad t \in [a, b].$$

We will show that the operator $\mathcal{T}$ has a unique fixed point by showing that the conditions of Theorem 1.1 hold, and we do this in three steps.

Step one: we show the inequality (1.2) of Theorem 1.1 holds. For $x, y \in \mathcal{Y}$ and $t \in [a, b]$, consider

$$\begin{aligned} |(\mathcal{T}x)(t) - (\mathcal{T}y)(t)| &\leqslant \int_a^b |\mathcal{G}(t, s)||g(s, x(s)) - f(s, y(s)|ds \\ &\leqslant \int_a^b |\mathcal{G}(t, s)|K|x(s) - y(s)|ds \\ &\leqslant \left(\int_a^b |\mathcal{G}(t, s)|^q ds\right)^{1/q} K \left(\int_a^b |x(s) - y(s)|^p ds\right)^{1/p} \qquad (3.3) \\ &\leqslant K \max_{t\in[a,b]} \left(\int_a^b |\mathcal{G}(t, s)|^q ds\right)^{1/q} \rho(x, y). \end{aligned}$$

Hoelder's inequality [6, 12] and (3.1) have used above to obtain (3.3) and by defining

$$c := K \max_{t\in[a,b]} \left(\int_a^b |\mathcal{G}(t, s)|^q ds\right)^{1/q},$$

we obtain

$$d(\mathcal{T}x, \mathcal{T}y) \leqslant c\rho(x, y), \quad \text{for some } \ c > 0 \text{ and all } x, y \in \mathcal{Y}. \qquad (3.4)$$

This shows the inequality (1.2) holds.

Step two: we show that the operator $\mathcal{T}$ is continuous on $\mathcal{Y}$ with respect to the $d$ metric. Now, for all $x, y \in \mathcal{Y}$ we apply (1.6) to (3.4) to obtain

$$d(\mathcal{T}x, \mathcal{T}y) \leqslant c\rho(x, y) \leqslant c(b - a)^{1/p} d(x, y).$$

Thus, for any $\varepsilon > 0$ we can choose $\Theta = \frac{\varepsilon}{c(b-a)^{1/p}}$ such that $d(\mathcal{T}x, \mathcal{T}y) < \varepsilon$ whenever $d(x, y) < \Theta$. Hence we have obtained the desired result.

Final step: we show the inequality (1.3) in Theorem 1.1 holds. From (3.3), we consider for each $x, y \in \mathcal{Y}$

$$\left(\int_a^b |(\mathcal{T}x)(t) - (\mathcal{T}y)(t)|^p dt\right)^{1/p} \leqslant K \left(\int_a^b \left(\int_a^b |\mathcal{G}(t, s)|^q \, ds\right) dt\right)^{1/q} \rho(x, y),$$

that is

$$\rho(\mathcal{T}x, \mathcal{T}y) \leqslant K \left(\int_a^b \left(\int_a^b |\mathcal{G}(t, s)|^q \, ds\right) dt\right)^{1/q} \rho(x, y).$$

Thus, from (3.2), we obtain $\rho(\mathcal{T}x, \mathcal{T}y) \leqslant \beta\rho(x, y)$, for some $0 < \beta < 1$ and all $x, y \in \mathcal{Y}$, where $\beta$ is the left hand side of (3.2). Thus we have shown that $\mathcal{T}$ is contractive on $\mathcal{Y}$ with respect to the $\rho$ metric.

This concludes that Theorem 1.1 is applicable and the operator $\mathcal{T}$ has a unique fixed point in $\mathcal{Y}$, that is equivalently shown the FBVP (1.1) admins a unique solution for $1 < \alpha \leqslant 2$ . □

If we choose $p = 2$ and $q = 2$, then our Theorem 3.1 reduces to the following new result.

**Theorem 3.2.** *Suppose that* $1 < \alpha \leqslant 2$. *Let* $g : [a, b] \times \mathbb{R} \to \mathbb{R}$ *be continuous, and let* $K > 0$ *such that*

$$|g(t, u) - g(t, v)| \leqslant K|u - v|, \quad \text{for all} \quad (t, u), (t, v) \in [a, b] \times \mathbb{R}. \tag{3.5}$$

*If*

$$K\sqrt{N_1(a, b, \alpha)} < 1, \tag{3.6}$$

*where* $N_1(a, b, \alpha)$ *is defined in* (2.2), *then the FBVP* (1.1) *admits a unique solution in* $C([a, b])$.

*Proof.* The proof proceeds along similar lines to that of Theorem 3.1; therefore, we only show that $\mathcal{T}$ is contractive on $\mathcal{Y}$. From (3.3), we consider for each $x, y \in \mathcal{Y}$,

$$\left(\int_a^b |(\mathcal{T}x)(t) - (\mathcal{T}y)(t)|^2 dt\right)^{1/2} \leqslant K\left(\int_a^b \left(\int_a^b |\mathcal{G}(t, s)|^2 \, ds\right) dt\right)^{1/2} \rho(x, y) = K\sqrt{N_1(a, b, \alpha)}\rho(x, y); \tag{3.7}$$

above, we have used (2.1) to obtain (3.7), that is

$$\rho(\mathcal{T}x, \mathcal{T}y) \leqslant K\sqrt{N_1(a, b, \alpha)}\rho(x, y).$$

Thus, from (3.6), we obtain $\rho(\mathcal{T}x, \mathcal{T}y) \leqslant \beta_1\rho(x, y)$, for some $0 < \beta_1 < 1$ and all $x, y \in \mathcal{Y}$ with $\beta_1 := K\sqrt{N_1(a, b, \alpha)}$. We conclude that Theorem 1.1 is applicable and the operator $\mathcal{T}$ has a unique fixed point in $\mathcal{Y}$, that is equivalently shown the FBVP (1.1) admins a unique solution for $1 < \alpha \leqslant 2$. □

## 4. Numerical validation with applications

This section is devoted to illustrating through numerical validation with applications, that the condition outlined in our Theorem 3.2 (3.6) is less restrictive than those obtained by Ferreira.

*Remark* 4.1. Consider $1 < \alpha \leqslant 2$ and $a < t < b$, Ferreira's result [4, Theorem 2.3] for FBVP (1.1) assumes (3.5) holds with Lipschitz constant $K$ such that

$$KM_1(a, b, \alpha) < 1, \tag{4.1}$$

where

$$M_1(a, b, \alpha) := \frac{1}{\Gamma(\alpha)} \frac{(\alpha - 1)^{\alpha-1}}{\alpha^{\alpha+1}} (b - a)^{\alpha}. \tag{4.2}$$

We now show how Ferreira's condition (4.1) is sharpened through our Theorem 3.2. Consider $\alpha = 2$, then the condition of Ferreira's result (4.1) reduces to the result of Peterson and Kelley [7, Theorem 7.7], that is

$$K\frac{(b - a)^2}{8} < 1. \tag{4.3}$$

If we compare (4.3) with our assumption (3.6), then it can be seen that (3.6) is a less restrictive condition. In particular, from (2.7) we have

$$N_1(a, b, 2) = \frac{(b - a)^4}{90},$$

and in this special case, (3.6) takes the form

$$K\frac{(b - a)^2}{\sqrt{90}} < 1.$$

Thus, we can see that our Theorem 3.2 for this case applies to a wider class of problems than [7, Theorem 7.7]. Moreover, consider $\alpha = 3/2$, then the condition of Ferreira's result (4.1) becomes

$$K(b-a)^{3/2}\frac{8}{9\sqrt{3\pi}} < 1, \tag{4.4}$$

and from (2.9) we have

$$N_1(a,b,3/2) = (b-a)^3\left(\frac{32\ln(2)-1}{105\pi}\right).$$

Thus in this special case, (3.6) takes the form

$$K(b-a)^{3/2}\sqrt{\frac{32\ln(2)-1}{105\pi}} < 1. \tag{4.5}$$

Clearly, for $\alpha = 3/2$, the inequality (4.5) is sharper than (4.4). This shows our Theorem 3.2 applies to a wider class of problems than [4, Theorem 2.3].

In what follows we give an example to illustrate how our Theorem 3.2 applies to a wider class of problems than [4, Theorem 2.3].

**Example 4.1.** Consider the following FBVP1

$$\begin{cases} {}_0D^{3/2}y(t) = -(\frac{39}{10}\sin(y(t)) + (t+1)^2), \quad 0 < t < 1, \\ y(0) = 0, \ y(1) = 0. \end{cases} \tag{4.6}$$

Then FBVP1 (4.6) has a unique solution.

In this example we have a special case of (1.1) with: $\alpha = 3/2$ and $g(t,v) := \frac{39}{10}\sin(v) + (t+1)^2$. Thus we have

$$|g_v(t,v)| \leqslant \frac{39}{10}|\cos(v)| \leqslant \frac{39}{10} = K.$$

Since $\alpha = 3/2$, it follows that our condition (3.6) is satisfied, that is from (4.5) we have

$$\frac{39}{10}\sqrt{\frac{32\ln(2)-1}{105\pi}} < 1.$$

Hence the FBVP1 (4.6) has a unique solution since all conditions of our Theorem 3.2 are satisfied. In contrast, one can show that the Ferreira's condition (4.1) is not satisfied in this example. Consequently, the result in [4, Theorem 2.3] is not applicable in this case.

For further comparison, consider $0 < t < 1$, and for convenience, we define $\lambda(\alpha) := \sqrt{N_1(0,1,\alpha)}$ (given by (2.2)) and $M_1(0,1,\alpha)$ (given by (4.2)). We now present a comprehensive numerical validation, which compares $\lambda(\alpha)$ and $M_1$. First, both $\lambda(\alpha)$ and $M_1(\alpha)$ are computed as functions of the parameter $\alpha$ in `Python 3` using the `mpmath` library to ensure accurate numerical computation. Since $\lambda(\alpha)$ involves the generalized hypergeometric function ${}_3F_2$, it is evaluated via the command `hyp3f2([1-α, 1, 2α], [α+1, 2α+1], 1)`. The results are plotted over the interval $1 < \alpha \leqslant 2$, see Figure 1, and selected values are further evaluated in Table 1, which are rounded to 10 decimal places. This numerical analysis shall underscore the significance of the advancements presented herein.

Table 1: Evaluation of $\lambda(\alpha)$ and $M_1$ for some special values of $\alpha$.

| $\alpha$ | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 | 1.6 | 1.7 | 1.8 | 1.9 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|
| $M_1(\alpha,0,1)$ | 0.6834939696 | 0.5285483897 | 0.4246649219 | 0.3483886032 | 0.2895422293 | 0.2427039128 | 0.2046281107 | 0.1732129304 | 0.1470156172 | 0.125 |
| $\lambda(\alpha)$ | 0.5611928411 | 0.4531430256 | 0.3703360902 | 0.3053313065 | 0.2533965332 | 0.211348745 | 0.1769536183 | 0.1485896594 | 0.1250483695 | 0.1054092553 |

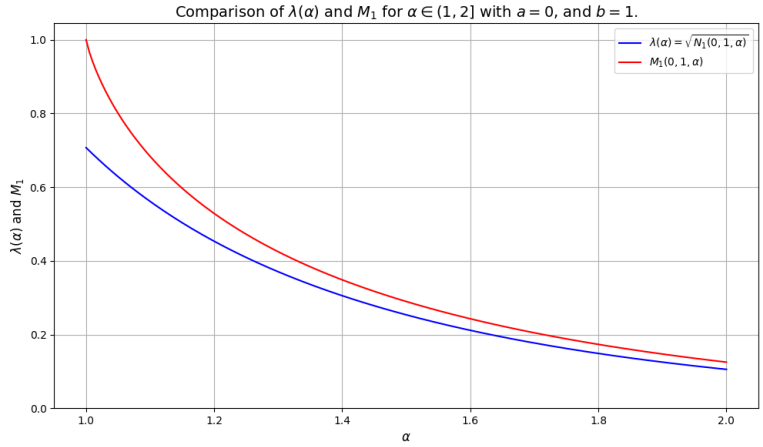


Figure 1: Comparison of $\lambda(\alpha)$ and $M_1$ for $\alpha \in (1, 2]$ with $a = 0$, and $b = 1$.

*Remark* 4.2. It is worth noting that the explicit evaluations of $\lambda(\alpha)$ and $M_1$ at $\alpha = 3/2$ and $\alpha = 2$ coincide precisely with their numerical computations.

As observed in both the Figure 1 and the Table 1, we have $\lambda(\alpha) < M_1(0, 1, \alpha)$, for all $1 < \alpha \leqslant 2$. Therefore, our Theorem 3.2 offers a significant improvement over the previous result that relies on $M_1(a, b, \alpha)$.

*Remark* 4.3. It is noteworthy that hypergeometric functions naturally arise on the left-hand side of condition (3.6). Specifically, the application of Rus's fixed point theorem to the fractional boundary value problem (1.1) leads directly to the appearance of such functions. This connection reveals a deeper interplay between the theory of boundary value problems (BVPs) and the hypergeometric functions. In particular, Rus's theorem (Theorem 1.1) provides a versatile framework for analyzing BVPs through its natural compatibility with hypergeometric function theory. This perspective can be great strategy for investigating the existence and uniqueness of solutions to fractional boundary value problems, offering an alternative to more classical approaches such as Banach's fixed point theorem.

Moreover, the expression on the left-hand side of condition (3.6) is generally not straightforward to compute analytically since hypergeometric functions usually do not have explicit expressions and are instead represented by power series or integrals. Nevertheless, in our results we have shed some light on the situation by discussing some special cases explicitly and then establish a comprehensive numerical evaluations via `Python 3`, supported by the `mpmath` library for accurate computation. For further insights and recent developments regarding explicit evaluations of hypergeometric functions, we refer the reader to [17].

*Remark* 4.4. In this work, we consider the Riemann–Liouville fractional boundary value problem (1.1), which naturally requires the restriction $y(a) = 0$ in order to ensure the continuity of solutions on the interval $[a, b]$. This condition is essential due to the inherent singularity of the Riemann–Liouville derivative at the initial point when $y(a) \neq 0$. This observation is also noted in [4, Remark 2.4].

## 5. Conclusion

In our work, we achieved a meaningful improvement over the results presented in [4] through the application of Rus's theorem (Theorem 1.1), which yields a bound that is a sharper alternative to the one derived from the classical contraction mapping principle.

Therefore, in the context of fractional differential equations, both in pure theory and real-world applications, there are many promising directions for future research based on the ideas presented in this work. For example, forced fractional differential equations play a central role in modelling various phenomena in physics, biology, and engineering see [5, 7–9]; thus, establishing the existence and uniqueness

of solutions to such problems is essential to ensure that the models are mathematically well-posed. In this context, the appliaction of Theorem 1.1 offers a valuable theoretical framework that enhances the understanding of these equations for which well-posedness can be guaranteed.

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