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Choquet integral-based GRA method with linguistic interval-valued q-Rung orthopair fuzzy sets

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Abstract

Linguistic interval-valued q-rung orthopair fuzzy (LIVq-ROF) sets offer a powerful framework for modeling uncertainty and vagueness in complex decision-making environments. This study leverages the expressive strength of LIVq-ROF sets to develop novel aggregation operators-specifically, the LIVq-ROF Choquet integral averaging and geometric operators-which are designed to capture interdependencies among attributes in multiple attribute group decision-making (MAGDM) scenarios. The theoretical properties of these operators are rigorously established. Building on this foundation, we propose a Choquet integral-based grey relational analysis (GRA) method tailored for MAGDM under uncertainty. The proposed model is applied to a real-world case study involving the selection of the optimal neural network model for predicting crop yields. Results demonstrate the model's effectiveness in identifying the best-performing alternative. A thorough sensitivity analysis and comparison with existing approaches confirm the robustness and superior performance of the proposed method.

Keywords: Linguistic interval-valued q-rung orthopair fuzzy set, Choquet integral, decision-making, software development, GRA method.

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1. Introduction

Multi-attribute group decision-making (MAGDM) is a fundamental aspect of modern decision science, focusing on selecting the optimal alternative from a set of potential options based on multiple attributes. In real-world decision-making scenarios, decision-makers (DMs) often encounter uncertainty and hesitation due to asymmetric information and time constraints. As a result, they may struggle to provide precise evaluation values, leading to ambiguity in the decision-making process [6]. Since its inception, fuzzy set theory has demonstrated its effectiveness in addressing such uncertainties across various domains, making it a widely adopted approach in decision science [2, 21, 22, 26, 29, 43].

The integration of fuzzy set theory into MAGDM has proven highly effective in addressing the inherent uncertainty and imprecision in decision-making processes. Early research suggested that DMs

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uncertainty could be represented by a pair of numerical values: a membership degree (MD) and a non-membership degree (NMD). This foundational concept led to the successive development of advanced fuzzy set models, including the intuitionistic fuzzy set [5], the Pythagorean fuzzy set [41], and the q-rung orthopair fuzzy set [42], all of which have been widely applied to solve MAGDM problems [1, 25, 30, 38, 44]. In recent years, to better capture DMs' uncertainty from both qualitative and quantitative perspectives, Chen et al. [7] introduced the linguistic intuitionistic fuzzy set (LIFS), which integrates linguistic variables into the membership and non-membership structure of intuitionistic fuzzy sets. While LIFS and its extensions-such as Pythagorean fuzzy sets-have been widely applied in MAGDM due to their improved capability in handling ambiguity, they still impose restrictive conditions on the sum of membership and non-membership degrees, limiting their flexibility in more complex scenarios. To address these limitations, Lin et al. [19] proposed the linguistic q-rung orthopair fuzzy set (Lq-ROFS), which relaxes these constraints and allows for a broader expressive capacity by enabling higher values of the sum of q-powered membership and non-membership degrees. This generalization provides more nuanced modeling of uncertainty. Building on this, the linguistic interval-valued q-rung orthopair fuzzy set (LIVq-ROFS) model was introduced by Khan et al. [15] to further improve representational flexibility in real-world applications. LIVq-ROFSs enhance the descriptive power of Lq-ROFSs by incorporating linguistic membership degree (LMD) and linguistic non-membership degree (LNMD), thus enabling the modeling of a wider range of imprecise expert judgments. By capturing the variability and hesitation in expert assessments more effectively than intuitionistic or Pythagorean models, LIVq-ROFSs provide a more robust and comprehensive framework for decision-making under uncertainty. Following this development, a growing body of research has emerged on LIVq-ROFSs. For example, Gong [10] proposed entropy-based attribute weighting for decision analysis, while Gurmani et al. [11] introduced operational rules and a new score function, culminating in a VIKOR-based MAGDM model that demonstrates the practical strength of this framework.

Aggregation operators (AOs) play a pivotal role in MAGDM by integrating individual preference information into a collective decision [3, 13, 23, 28]. To improve decision-making under uncertainty, several scholars have contributed to the development and application of AOs for LIVq-ROFSs. For instance, Khan et al. [15] introduced fundamental AOs and explored their theoretical properties. Additionally, Qi et al. [24] investigated Heronian mean operators and their specific cases in a complex LIVq-ROFS setting to better capture interdependencies between attributes. Furthermore, Gurmani et al. [11] developed operational laws for LIVq-ROFSs and proposed the LIVq-ROF weighted averaging operator, along with an improved score function to enhance decision accuracy. Despite these advancements, most existing LIVq-ROFS AOs operate under the assumption that the criteria are independent-except for Qi et al. [24], whose work remains restricted to pairwise relationships. However, in many real-world MAGDM problems, attributes are not independent; rather, they exhibit correlations that influence decision outcomes. In such cases, the conventional weighted AOs fail to capture the intricate relationships among attributes. Fuzzy measures provide a more flexible framework by accommodating interdependencies, including complementary, redundant, and independent relationships among criteria. Unlike traditional weight-based methods, fuzzy measures allow the sum of individual measures to exceed one, while ensuring that the total measure remains constrained to unity, thereby offering greater adaptability. The Choquet integral has emerged as a powerful tool for aggregating evaluations in scenarios where attributes are interdependent [9]. Given its ability to model complex interactions, integrating the Choquet integral with LIVq-ROFSs presents a promising research direction. Surprisingly, despite its potential, this integration has not yet been explored. Addressing this gap, our study aims to develop a novel Choquet integral-based aggregation approach for LIVq-ROFSs, effectively capturing attribute interdependencies in MAGDM problems. The key advantage of this approach lies in its ability to reflect the real-world dependency structure among attributes, which traditional weighted AOs fail to capture. By utilizing fuzzy measures and the Choquet integral, the proposed method enables more accurate, adaptable, and interpretable aggregation of uncertain linguistic information in MAGDM problems.

The grey relational analysis (GRA) method, originally developed by Deng [14], is a widely used approach in MADM. It is designed to select the optimal alternative by identifying the option with the highest grey relational degree to the positive ideal solution (PIS) and the lowest grey relational degree to the negative ideal solution (NIS). Over the years, GRA has been extensively applied in various MAGDM scenarios. For instance, Kuo et al. [18] applied the GRA method to facility layout and dispatching rule selection, demonstrating its effectiveness in optimizing industrial decision processes. Similarly, Kung and Wen [17] employed GRA to classify financial indicators, ranking twenty financial ratio items to evaluate the overall performance of venture capital enterprises. Beyond financial and industrial applications, GRA has also been leveraged for environmental and sustainability assessments. Alptekin et al. [4] utilized GRA to evaluate the low-carbon development of European Union countries and Turkey, emphasizing its adaptability in energy policy analysis. Tan et al. [39] integrated GRA with the analytical hierarchy process (AHP) to assess green design alternatives, reinforcing its applicability in sustainable product development. Sun et al. [37] introduced the hesitant fuzzy sets (HFSs) slope grey relational degree, constructing a synthetic grey relational model that considers both closeness and linearity in decision-making. The authors in [27] presented a GRA-based holistic framework developed to evaluate e-procurement readiness in hospitals. Additionally, Malek et al. [20] proposed an improved hybrid GRA approach for optimizing green resilient supply chains, while Tian et al. [40] developed a grey-correlation-based hybrid MADM method for selecting green decoration materials based on interior environmental characteristics. Despite these developments, no existing studies have applied the GRA method within the framework of MAGDM under LIVq-ROFSs. Given the increasing complexity of decision environments that require handling both linguistic uncertainty and q-rung orthopair fuzziness, there is a pressing need to extend GRA to the LIVq-ROFS domain. Addressing this gap, our study aims to develop a novel GRA-based approach under LIVq-ROFSs, facilitating more robust and flexible decision-making models in complex MAGDM scenarios. The integration of GRA with LIVq-ROFSs brings several advantages. It allows decision-makers to simultaneously handle linguistic uncertainty, interval-based hesitation, and the high-order fuzziness inherent in q-rung orthopair structures. Moreover, by leveraging GRA's ability to compute relational closeness to both the PIS and the NIS, the proposed method enables precise ranking of alternatives even under partial or incomplete data conditions. This combination enhances the model's robustness and interpretability, making it well-suited for real-world MAGDM problems where attribute interdependence and imprecise judgments are prevalent.

Despite notable progress in fuzzy set theories and decision-making models, several pressing challenges remain unresolved in practical applications such as neural network model selection for crop yield prediction. First, most existing AOs assume independence among evaluation attributes, which rarely reflects real-world complexity where factors like data preprocessing, model architecture, and training parameters often interact. This assumption leads to oversimplified outcomes that fail to capture key interdependencies. Second, current models are often limited in their capacity to represent linguistic uncertainty, particularly when expert judgments exhibit ambiguity or hesitation that cannot be fully expressed using conventional fuzzy or intuitionistic frameworks. Third, although methods like GRA are widely adopted in multiple attribute decision-making, they have not yet been fully integrated with the expressive power of LIVq-ROFSs, which offer superior flexibility in modeling interval-valued, high-order fuzzy information. These limitations highlight the need for a more adaptable decision-making framework-one capable of handling complex attribute interactions, incorporating richer forms of uncertainty, and aligning with real-world decision contexts.

To address these research gaps, this paper proposes a novel decision-making method that integrates the Choquet integral and GRA methodology within the LIVq-ROFS framework. The key contributions of this study are as follows.

- We introduce LIVq-ROF Choquet integral averaging (LIVq-ROF-CIA) operator and LIVq-ROF Choquet integral geometric (LIVq-ROF-CIG) operator, allowing decision-makers to incorporate qualitative linguistic assessments while capturing attribute interactions during the aggregation process.
- Given the Choquet integral's ability to model complex attribute interactions, we propose a novel

MAGDM technique that combines LIVq-ROFS with the Choquet integral, enhancing decision-making accuracy in scenarios where criteria exhibit interdependencies.

- When attribute weights are not fully known, we formulate an optimization model to determine the optimal weight distribution, ensuring a more reliable aggregation process.
- We develop a GRA-based MAGDM methodology under LIVq-ROFS, capable of handling partially known attribute weights while accommodating linguistic uncertainty and q-rung orthopair fuzziness in decision-making scenarios.

The remainder of this article is organized as follows: Section 2 reviews fundamental concepts necessary for understanding the novelty of the proposed approach. Section 3 introduces the LIVq-ROFCIA and LIVq-ROFCIG operators and explores their key properties. Section 4 presents the GRA method for solving MAGDM problems under LIVq-ROF data. Section 5 includes a case study on selecting the optimal neural network model for predicting crop yields. Section 6 provides a comparative analysis to highlight the advantages of the proposed GRA-based model. Finally, Section 7 concludes the study.

2. Preliminaries

To establish a strong foundation for our proposed methodology, we present essential definitions and fundamental concepts related to linguistic term sets, linguistic q-rung orthopair fuzzy sets, fuzzy measure, and associated mathematical tools.

2.1. Linguistic term sets and its extensions

In the following, the concepts of linguistic term sets and Lq-ROFS are discussed to facilitate a better understanding of the subsequent section.

Definition 2.1 ([12]). Let $S = \{T_\alpha \mid \alpha = 0, 1, 2, \ldots, t\}$ be a finite linguistic term set with an odd cardinality, where each T_α corresponds to a possible linguistic term associated with a linguistic variable. For example, a linguistic term set S comprising five terms can be expressed as follows:

$$S = \{T_0 = \text{very poor}, T_1 = \text{poor}, T_2 = \text{medium}, T_3 = \text{good}, T_4 = \text{very good}\}.$$

If $T_i, T_l \in S$, then the linguistic term set adheres to the following properties:

1. $T_i > T_l \iff i > l$;
2. $Neg(T_i) = T_l$, such that $l = t - i$;
3. $\max(T_i, T_l) = T_l \iff l \geqslant i$;
4. $\min(T_i, T_l) = T_l \iff l \leqslant i$.

Definition 2.2 ([19]). Let X be a finite universal set, and let $\bar{S} = \{T_\alpha \mid T_0 \leqslant T_\alpha \leqslant T_t, \alpha \in [0, t]\}$ represents a continuous linguistic term set. A linguistic q-rung orthopair fuzzy set (Lq-ROFS) A in X is defined as

$$A = \{(x, T_\theta(x), T_\sigma(x)) \mid x \in X\},$$

where $T_\theta(x), T_\sigma(x) \in \bar{S}$ denote the linguistic membership and nonmembership degrees of the element x in A. The ordered pair $(T_\theta(x), T_\sigma(x))$ is represented as $A = (T_\theta, T_\sigma)$ and is referred to as the linguistic q-rung orthopair fuzzy value (Lq-ROFV) or linguistic q-rung orthopair fuzzy number (Lq-ROFN).

For any $x \in X$, the following condition holds: $0 \leqslant \theta^q + \sigma^q \leqslant t^q$. Furthermore, the linguistic indeterminacy degree of x with respect to A is given by: $\pi(x) = T_{\sqrt[q]{t^q - \theta^q - \sigma^q}}$.

2.2. *Fuzzy measure and Choquet integral*

Fuzzy measures and aggregation operators are fundamental to information fusion and decision-making in uncertain environments. This section revisits the essential concepts of fuzzy measures and the Choquet integral.

Definition 2.3 ([8])**.** Let $Z = \{z_1, z_2, \dots, z_n\}$ be a universal set, and let $P(Z)$ denote its power set. A function $\yen : P(Z) \to [0, 1]$ is considered a fuzzy measure on Z if it satisfies the following conditions:

1. $\yen(\emptyset) = 0$ and $\yen(Z) = 1$;
2. If $\Virgo_1, \Virgo_2 \in P(Z)$ and $\Virgo_1 \subseteq \Virgo_2$, then $\yen(\Virgo_1) \leqslant \yen(\Virgo_2)$.

Although the axiom of continuity is often required, in practical applications where Z is finite, this assumption can be relaxed. The value $\yen(\{z_1, z_2, \dots, z_n\})$ represents the subjective significance of a given set of decision attributes. Furthermore, it allows for the assignment of weights to individual attributes as well as their combinations.

For any two criteria sets $\Virgo_1, \Virgo_2 \in P(Z)$, where $\Virgo_1 \cap \Virgo_2 = \emptyset$, their interaction can be classified as follows.

- If they are independent, then: $\yen(\Virgo_1 \cup \Virgo_2) = \yen(\Virgo_1) + \yen(\Virgo_2)$, which defines an additive measure.
- If they exhibit a positive synergistic interaction (i.e., they complement each other), then:
$$\yen(\Virgo_1 \cup \Virgo_2) > \yen(\Virgo_1) + \yen(\Virgo_2), \tag{2.1}$$
making it a superadditive measure.
- If they demonstrate a negative synergistic interaction (i.e., they are redundant or substitutive), then:
$$\yen(\Virgo_1 \cup \Virgo_2) < \yen(\Virgo_1) + \yen(\Virgo_2),$$
which is referred to as a subadditive measure.

Determining a suitable fuzzy measure in MAGDM can be challenging. To address this, Sugeno [36] introduced the λ-fuzzy measure, which is defined as:

$$\yen(\Virgo_1 \cup \Virgo_2) = \yen(\Virgo_1) + \yen(\Virgo_2) + \Aries\yen(\Virgo_1)\yen(\Virgo_2),$$

where $\Aries \in [-1, \infty)$ and $\Virgo_1 \cap \Virgo_2 = \emptyset$. The parameter $\Aries$ determines the degree of interaction between attributes. Specifically:

- if $\Aries = 0$, the measure reduces to an additive form;
- if $\Aries > 0$, the measure is superadditive;
- if $\Aries < 0$, the measure is subadditive.

Additionally, if all elements in Z are independent, we have

$$\yen\left(\Virgo\right) = \sum_{1=1}^{n} \yen\left(\{z_1\}\right).$$

If Z is a finite set, then $\bigcup\limits_{1=1}^{n} z_1 = Z$. The $\Aries$-fuzzy measure $\yen$ satisfies

$$\yen\left(Z\right) = \yen\left(\bigcup_{1=1}^{n} z_1\right) = \begin{cases} \frac{1}{\Aries}\left(\prod\limits_{1=1}^{n}\left[1 + \Aries\yen(z_1)\right] - 1\right), & \text{if } \Aries \neq 0, \\ \sum\limits_{1=1}^{n} \yen(z_1), & \text{if } \Aries = 0, \end{cases}$$

where $z_i \cap z_l = \emptyset$ for all $i, l = 1, 2, \ldots, n$ and $i \neq l$. It is important to note that the function $¥(z_l)$, when applied to a single-element subset z_l, is referred to as the fuzzy density and can be expressed as $¥_l = ¥(z_l)$. Specifically, for any subset $\mathfrak{M}_l \in P(Z)$, the fuzzy measure is given by:

$$¥(\mathfrak{M}_l) = \begin{cases} \frac{1}{\Upsilon}\left(\prod_{l=1}^{n}[1+\Upsilon ¥(z_l)]-1\right), & \text{if } \Upsilon \neq 0, \\ \sum_{l=1}^{n} ¥(z_l), & \text{if } \Upsilon = 0. \end{cases} \tag{2.2}$$

From Eq. (2.1), the parameter Υ can be uniquely determined by setting $¥(Z) = 1$, which leads to solving:

$$\Upsilon + 1 = \prod_{l=1}^{n}[1+\Upsilon ¥_l]. \tag{2.3}$$

This ensures that the value of Υ is uniquely determined by the constraint $¥(Z) = 1$.

Definition 2.4. Let $¥$ be a fuzzy measure on Z, and let g be a positive real-valued function defined on Z. The discrete Choquet integral of g with respect to $¥$ is formulated as:

$$C_{¥}(g) = \sum_{l=1}^{n} g_{\tau(l)}\left[¥(\mathfrak{M}_{\tau(l)}) - ¥(\mathfrak{M}_{\tau(l-1)})\right],$$

where $\tau(l)$ represents a permutation of Z such that $g_{\tau(1)} \geqslant g_{\tau(2)} \geqslant \cdots \geqslant g_{\tau(n)}$, and the subsets are defined as $\mathfrak{M}_{\tau(l)} = \{1, 2, \ldots, n\}$, with $\mathfrak{M}_{\tau(0)} = \emptyset$.

2.3. *Linguistic interval-valued q-Rung orthopair fuzzy sets and their associated concepts*

In this subsection, we define the fundamental properties and basic operations of LIVq-ROFS, which are essential for formulating aggregation operators and decision-making models.

Definition 2.5 ([15]). Let $Z = \{z_1, z_2, \ldots, z_n\}$ be a finite universal set, and let $\bar{S} = \{T_\alpha \mid T_0 \leqslant T_\alpha \leqslant T_t,\ \alpha \in [0, t]\}$ be a continuous linguistic term set. A LIVq-ROFS A in Z is defined as:

$$A = \left\{\left(z, \left[T_{\xi^L}(z), T_{\xi^U}(z)\right], \left[T_{\eta^L}(z), T_{\eta^U}(z)\right]\right) \mid z \in Z\right\},$$

where $\left[T_{\xi^L}(z), T_{\xi^U}(z)\right]$, $\left[T_{\eta^L}(z), T_{\eta^U}(z)\right]$ are subsets of $[T_0, T_t]$ and are referred to as the LMD and LNMD of element $z_l \in Z$ for A. For any $z_l \in Z$, it holds that: $\left(T_{\xi^U}(z)\right)^q + \left(T_{\eta^U}(z)\right)^q \leqslant (T_t)^q$, (i.e., $(\xi^U)^q + (\eta^U)^q \leqslant t^q$). We denote the pair $\left(\left[T_{\xi^L}(z), T_{\xi^U}(z)\right], \left[T_{\eta^L}(z), T_{\eta^U}(z)\right]\right)$ by $\beta = ([T_a, T_b], [T_c, T_d])$, referred to as the linguistic interval-valued q-rung orthopair fuzzy value (LIVq-ROFV). The linguistic indeterminacy degree of $z_l \in Z$ for A is denoted by $\left[\pi^L(z_l), \pi^U(z_l)\right]$ and defined as: $\pi^L(z_l) = T_{\sqrt[q]{t^q-(a(z_l))^q-(c(z_l))^q}}$, $\pi^U(z_l) = T_{\sqrt[q]{t^q-(b(z_l))^q-(d(z_l))^q}}$.

Definition 2.6 ([15]). Let $\beta_1 = ([T_{a_1}, T_{b_1}], [T_{c_1}, T_{d_1}])$ and $\beta_2 = ([T_{a_2}, T_{b_2}], [T_{c_2}, T_{d_2}])$ be two LIVq-ROFNs and $\psi > 0$. The following operational laws hold:

1. $\beta_1 \oplus \beta_2 = \left(\left[T_{t\sqrt[q]{\frac{a_1^q}{t^q}+\frac{a_2^q}{t^q}-\frac{a_1^q}{t^q}\cdot\frac{a_2^q}{t^q}}}, T_{t\sqrt[q]{\frac{b_1^q}{t^q}+\frac{b_2^q}{t^q}-\frac{b_1^q}{t^q}\cdot\frac{b_2^q}{t^q}}}\right], \left[T_{t\left(\frac{c_1c_2}{t^2}\right)}, T_{t\left(\frac{d_1d_2}{t^2}\right)}\right]\right);$
2. $\beta_1 \otimes \beta_2 = \left(\left[T_{t\left(\frac{a_1a_2}{t^2}\right)}, T_{t\left(\frac{b_1b_2}{t^2}\right)}\right], \left[T_{t\sqrt[q]{\frac{c_1^q}{t^q}+\frac{c_2^q}{t^q}-\frac{c_1^q}{t^q}\cdot\frac{c_2^q}{t^q}}}, T_{t\sqrt[q]{\frac{d_1^q}{t^q}+\frac{d_2^q}{t^q}-\frac{d_1^q}{t^q}\cdot\frac{d_2^q}{t^q}}}\right]\right);$

3. $\psi\beta_1 = \left(\left[\mathsf{T}_{t\sqrt[q]{1-\left(1-\frac{a_1^q}{t^q}\right)^{\psi}}}, \mathsf{T}_{t\sqrt[q]{1-\left(1-\frac{b_1^q}{t^q}\right)^{\psi}}}\right], \left[\mathsf{T}_{t\left(\frac{c_1}{t}\right)^{\psi}}, \mathsf{T}_{t\left(\frac{d_1}{t}\right)^{\psi}}\right]\right)$;

4. $\beta_1^{\psi} = \left(\left[\mathsf{T}_{t\left(\frac{a_1}{t}\right)^{\psi}}, \mathsf{T}_{t\left(\frac{b_1}{t}\right)^{\psi}}\right], \left[\mathsf{T}_{t\sqrt[q]{1-\left(1-\frac{c_1^q}{t^q}\right)^{\psi}}}, \mathsf{T}_{t\sqrt[q]{1-\left(1-\frac{d_1^q}{t^q}\right)^{\psi}}}\right]\right)$.

To rank the LIVq-ROFNs, we define the score and accuracy functions as follows.

Definition 2.7 ([15]). Let $\beta = ([\mathsf{T}_a, \mathsf{T}_b], [\mathsf{T}_c, \mathsf{T}_d])$ be a LIVq-ROFN. The score function is given by: $Sc(\beta) = \mathsf{T}_{\sqrt[q]{\frac{t^q+a^q-c^q+b^q-d^q}{4}}}$, and the accuracy function is given by: $Ac(\beta) = \mathsf{T}_{\sqrt[q]{\frac{a^q+b^q+c^q+d^q}{2}}}$.

For two LIVq-ROFNs β_1 and β_2, the following comparison laws are used for ranking.

1. if $Sc(\beta_1) < Sc(\beta_2)$, then $\beta_1 < \beta_2$;
2. if $Sc(\beta_1) > Sc(\beta_2)$, then $\beta_1 > \beta_2$;
3. if $Sc(\beta_1) = Sc(\beta_2)$, then:
   (a) if $Ac(\beta_1) = Ac(\beta_2)$, then $\beta_1 = \beta_2$;
   (b) if $Ac(\beta_1) < Ac(\beta_2)$, then $\beta_1 < \beta_2$;
   (c) if $Ac(\beta_1) > Ac(\beta_2)$, then $\beta_1 > \beta_2$.

3. LIVq-ROF Choquent integral operators

This section presents the LIVq-ROFCIA and LIVq-ROFCIG operators and explores some of their key characteristics.

Definition 3.1. Let $\beta_l = ([\mathsf{T}_{a_l}, \mathsf{T}_{b_l}], [\mathsf{T}_{c_l}, \mathsf{T}_{d_l}])$ for $l = 1, 2, \ldots, n$ represent a collection of LIVq-ROFNs, and let Υ be a fuzzy measure on Z. The linguistic interval-valued q-rung orthopair fuzzy Choquet integral averaging (LIVq-ROFCIA) operator of dimension n is defined as a mapping: LIVq-ROFCIA : $\Omega^n \longrightarrow \Omega$ such that

$$\begin{aligned}\text{LIVq-ROFCIA}(\beta_1, \beta_2, \ldots, \beta_n) &= \beta_1\left(\Upsilon(\mathfrak{M}_{\tau(1)}) - \Upsilon(\mathfrak{M}_{\tau(0)})\right) \oplus \beta_2\left(\Upsilon(\mathfrak{M}_{\tau(2)}) - \Upsilon(\mathfrak{M}_{\tau(1)})\right) \\ &\quad \oplus \ldots \oplus \beta_n\left(\Upsilon(\mathfrak{M}_{\tau(n)}) - \Upsilon(\mathfrak{M}_{\tau(n-1)})\right) \\ &= \bigoplus_{l=1}^{n} \beta_l\left(\Upsilon(\mathfrak{M}_{\tau(l)}) - \Upsilon(\mathfrak{M}_{\tau(l-1)})\right),\end{aligned}$$

where $\{\tau(1), \tau(2), \ldots, \tau(n)\}$ is a permutation of $\{1, 2, \ldots, n\}$ such that $\beta_{\tau(1)} \geqslant \beta_{\tau(2)} \geqslant \cdots \geqslant \beta_{\tau(n)}$, and $\mathfrak{M}_{\tau(k)} = \left\{z_{\tau(k)} \mid l \leqslant k\right\}$ for $k \geqslant 1$, with $\mathfrak{M}_{\tau(0)} = \emptyset$.

Theorem 3.2. *Let $\beta_l = ([\mathsf{T}_{a_l}, \mathsf{T}_{b_l}], [\mathsf{T}_{c_l}, \mathsf{T}_{d_l}])$ for $l = 1, 2, \ldots, n$ represent a collection of LIVq-ROFNs on Z, and let Υ be a fuzzy measure on Z. The aggregated value of these LIVq-ROFNs using the LIVq-ROFCIA operator is also an LIVq-ROFN, and it is given by:*

$$\begin{aligned}&\text{LIVq-ROFCIA}(\beta_1, \beta_2, \ldots, \beta_n) \\ &= \left(\begin{array}{c}\left[\mathsf{T}_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{a_{\tau(l)}^q}{t^q}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}}, \mathsf{T}_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{b_{\tau(l)}^q}{t^q}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}}\right], \\ \left[\mathsf{T}_{t\prod_{l=1}^{n}\left(\frac{c_{\tau(l)}}{t}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}, \mathsf{T}_{t\prod_{l=1}^{n}\left(\frac{d_{\tau(l)}}{t}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}\right]\end{array}\right), \quad (3.1)\end{aligned}$$

where $\{\tau(1), \tau(2), \ldots, \tau(n)\}$ *is a permutation of* $\{1, 2, \ldots, n\}$ *such that* $\beta_{\tau(1)} \geqslant \beta_{\tau(2)} \geqslant \cdots \geqslant \beta_{\tau(n)}$*, and* $\mathcal{M}_{\tau(k)} = \{z_{\tau(k)} \mid 1 \leqslant k\}$ *for* $k \geqslant 1$*, with* $\mathcal{M}_{\tau(0)} = \emptyset$*.*

Proof. We will prove Eq. (3.1) using mathematical induction. First, assume the result holds for $n = 1$. Now, we demonstrate it for $n = 2$ by applying the operational laws from Definition 2.6. This gives us:

$$\left(\Upsilon\left(\mathcal{M}_{\tau(1)}\right) - \Upsilon\left(\mathcal{M}_{\tau(0)}\right)\right)\beta_{\tau(1)} = \left(\begin{array}{c}\left[T_{t\sqrt[q]{1-\left(1-\frac{a^q_{\tau(1)}}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(1)})-\Upsilon(\mathcal{M}_{\tau(0)})}}}, T_{t\sqrt[q]{1-\left(1-\frac{b^q_{\tau(1)}}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(1)})-\Upsilon(\mathcal{M}_{\tau(0)})}}}\right], \\ \left[T_{t\left(\frac{c_{\tau(1)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(1)})-\Upsilon(\mathcal{M}_{\tau(0)})}}, T_{t\left(\frac{d_{\tau(1)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(1)})-\Upsilon(\mathcal{M}_{\tau(0)})}}\right]\end{array}\right),$$

$$\left(\Upsilon\left(\mathcal{M}_{\tau(2)}\right) - \Upsilon\left(\mathcal{M}_{\tau(1)}\right)\right)\beta_{\tau(2)} = \left(\begin{array}{c}\left[T_{t\sqrt[q]{1-\left(1-\frac{a^q_{\tau(2)}}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(2)})-\Upsilon(\mathcal{M}_{\tau(1)})}}}, T_{t\sqrt[q]{1-\left(1-\frac{b^q_{\tau(2)}}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(2)})-\Upsilon(\mathcal{M}_{\tau(1)})}}}\right], \\ \left[T_{t\left(\frac{c_{\tau(2)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(2)})-\Upsilon(\mathcal{M}_{\tau(1)})}}, T_{t\left(\frac{d_{\tau(2)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(2)})-\Upsilon(\mathcal{M}_{\tau(1)})}}\right]\end{array}\right).$$

Since $\beta_1 \oplus \beta_2 = \left(\left[T_{t\sqrt[q]{\frac{a_1^q}{t^q}+\frac{a_2^q}{t^q}-\frac{a_1^q}{t^q}\frac{a_2^q}{t^q}}}, T_{t\sqrt[q]{\frac{b_1^q}{t^q}+\frac{b_2^q}{t^q}-\frac{b_1^q}{t^q}\frac{b_2^q}{t^q}}}\right], \left[T_{t\left(\frac{c_1c_2}{t^2}\right)}, T_{t\left(\frac{d_1d_2}{t^2}\right)}\right]\right)$, therefore

$$\begin{aligned}&\text{LIVq-ROFCIA}\left(\beta_1, \beta_2\right) \\ &= \left(\Upsilon\left(\mathcal{M}_{\tau(1)}\right) - \Upsilon\left(\mathcal{M}_{\tau(0)}\right)\right)\beta_{\tau(1)} \oplus \left(\Upsilon\left(\mathcal{M}_{\tau(2)}\right) - \Upsilon\left(\mathcal{M}_{\tau(1)}\right)\right)\beta_{\tau(2)} \\ &= \left(\begin{array}{c}\left[T_{t\sqrt[q]{1-\prod_{l=1}^{2}\left(1-\frac{a^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}}, T_{t\sqrt[q]{1-\prod_{l=1}^{2}\left(1-\frac{b^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}}\right], \\ \left[T_{t\prod_{l=1}^{2}\left(\frac{c_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}, T_{t\prod_{l=1}^{2}\left(\frac{d_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}\right]\end{array}\right).\end{aligned}$$

Thus, Eq. (3.1) holds for $n = 2$. Now, we demonstrate that Eq. (3.1) holds for $n = k$.

$$\text{LIVq-ROFCIA}\left(\beta_1, \beta_2, \ldots, \beta_k\right)$$

$$= \left(\begin{array}{c} \left[\mathsf{T}_{t\sqrt[q]{1-\prod_{l=1}^{k}\left(1-\frac{a^{q}_{\tau(l)}}{t^{q}}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}}, \mathsf{T}_{t\sqrt[q]{1-\prod_{l=1}^{k}\left(1-\frac{b^{q}_{\tau(l)}}{t^{q}}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}} \right], \\ \left[\mathsf{T}_{t\prod_{l=1}^{k}\left(\frac{c_{\tau(l)}}{t}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}, \mathsf{T}_{t\prod_{l=1}^{k}\left(\frac{d_{\tau(l)}}{t}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}} \right] \end{array} \right).$$

Assume that Eq. (3.1) holds for $n = k$. We will now prove that Eq. (3.1) holds for $n = k+1$. Since

$$\bigoplus_{l=1}^{k} \left(\Upsilon(\mathfrak{M}_{\tau(l)}) - \Upsilon(\mathfrak{M}_{\tau(l-1)})\right) \beta_l$$
$$= \left(\begin{array}{c} \left[\mathsf{T}_{t\sqrt[q]{1-\prod_{l=1}^{k}\left(1-\frac{a^{q}_{\tau(l)}}{t^{q}}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}}, \mathsf{T}_{t\sqrt[q]{1-\prod_{l=1}^{k}\left(1-\frac{b^{q}_{\tau(l)}}{t^{q}}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}} \right], \\ \left[\mathsf{T}_{t\prod_{l=1}^{k}\left(\frac{c_{\tau(l)}}{t}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}, \mathsf{T}_{t\prod_{l=1}^{k}\left(\frac{d_{\tau(l)}}{t}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}} \right] \end{array} \right),$$

and

$$\left(\Upsilon\left(\mathfrak{M}_{\tau(k+1)}\right) - \Upsilon\left(\mathfrak{M}_{\tau(k)}\right)\right) \beta_{\tau(k+1)}$$
$$= \left(\begin{array}{c} \left[\mathsf{T}_{t\sqrt[q]{1-\left(1-\frac{a^{q}_{\tau(k+1)}}{t^{q}}\right)^{\Upsilon(\mathfrak{M}_{\tau(k+1)})-\Upsilon(\mathfrak{M}_{\tau(k)})}}}, \mathsf{T}_{t\sqrt[q]{1-\left(1-\frac{b^{q}_{\tau(k+1)}}{t^{q}}\right)^{\Upsilon(\mathfrak{M}_{\tau(k+1)})-\Upsilon(\mathfrak{M}_{\tau(k)})}}} \right], \\ \left[\mathsf{T}_{t\left(\frac{c_{\tau(k+1)}}{t}\right)^{\Upsilon(\mathfrak{M}_{\tau(k+1)})-\Upsilon(\mathfrak{M}_{\tau(k)})}}, \mathsf{T}_{t\left(\frac{d_{\tau(k+1)}}{t}\right)^{\Upsilon(\mathfrak{M}_{\tau(k+1)})-\Upsilon(\mathfrak{M}_{\tau(k)})}} \right] \end{array} \right).$$

Therefore

$$\text{LIVq-ROFCIA}\,(\beta_1, \beta_2, \ldots, \beta_k, \beta_{k+1})$$
$$= \bigoplus_{l=1}^{k} \left(\Upsilon(\mathfrak{M}_{\tau(l)}) - \Upsilon(\mathfrak{M}_{\tau(l-1)})\right) \beta_l \oplus \left(\Upsilon(\mathfrak{M}_{\tau(k+1)}) - \Upsilon(\mathfrak{M}_{\tau(k)})\right) \beta_{k+1}$$
$$= \left(\begin{array}{c} \left[\mathsf{T}_{t\sqrt[q]{1-\prod_{l=1}^{k}\left(1-\frac{a^{q}_{\tau(l)}}{t^{q}}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}}, \mathsf{T}_{t\sqrt[q]{1-\prod_{l=1}^{k}\left(1-\frac{b^{q}_{\tau(l)}}{t^{q}}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}} \right], \\ \left[\mathsf{T}_{t\prod_{l=1}^{k}\left(\frac{c_{\tau(l)}}{t}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}, \mathsf{T}_{t\prod_{l=1}^{k}\left(\frac{d_{\tau(l)}}{t}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}} \right] \end{array} \right)$$

$$\oplus\left(\begin{array}{c}\left[T_{t\sqrt[q]{1-\left(1-\frac{a^{q}_{\tau(k+1)}}{t^{q}}\right)^{\Upsilon(♍_{\tau(k+1)})-\Upsilon(♍_{\tau(k)})}}},\ T_{t\sqrt[q]{1-\left(1-\frac{b^{q}_{\tau(k+1)}}{t^{q}}\right)^{\Upsilon(♍_{\tau(k+1)})-\Upsilon(♍_{\tau(k)})}}}\right],\\ \left[T_{t\left(\frac{c_{\tau(k+1)}}{t}\right)^{\Upsilon(♍_{\tau(k+1)})-\Upsilon(♍_{\tau(k)})}},\ T_{t\left(\frac{d_{\tau(k+1)}}{t}\right)^{\Upsilon(♍_{\tau(k+1)})-\Upsilon(♍_{\tau(k)})}}\right]\end{array}\right)$$

$$=\left(\begin{array}{c}\left[T_{t\sqrt[q]{1-\prod_{\imath=1}^{k+1}\left(1-\frac{a^{q}_{\tau(\imath)}}{t^{q}}\right)^{\Upsilon(♍_{\tau(\imath)})-\Upsilon(♍_{\tau(\imath-1)})}}},\ T_{t\sqrt[q]{1-\prod_{\imath=1}^{k+1}\left(1-\frac{b^{q}_{\tau(\imath)}}{t^{q}}\right)^{\Upsilon(♍_{\tau(\imath)})-\Upsilon(♍_{\tau(\imath-1)})}}}\right],\\ \left[T_{t\prod_{\imath=1}^{k+1}\left(\frac{c_{\tau(\imath)}}{t}\right)^{\Upsilon(♍_{\tau(\imath)})-\Upsilon(♍_{\tau(\imath-1)})}},\ T_{t\prod_{\imath=1}^{k+1}\left(\frac{d_{\tau(\imath)}}{t}\right)^{\Upsilon(♍_{\tau(\imath)})-\Upsilon(♍_{\tau(\imath-1)})}}\right]\end{array}\right).$$

Thus, Eq. (3.1) holds for $n = k+1$, and consequently, it is true for all values of n. □

Theorem 3.3. *Let $\beta_\imath = ([T_{a_\imath}, T_{b_\imath}], [T_{c_\imath}, T_{d_\imath}])$ for $\imath = 1, 2, \ldots, n$ be a collection of LIVq-ROFNs on Z, and let Υ be a fuzzy measure on Z. If all $\beta_\imath$ ($\imath = 1, 2, \ldots, n$) are equal, i.e., for all $\beta = ([T_a, T_b], [T_c, T_d])$, then the following holds:*

$$\text{LIVq-ROFCIA}\left(\beta_1, \beta_2, \ldots, \beta_n\right) = \beta.$$

Proof. Since

$$\text{LIVq-ROFCIA}\left(\beta_1, \beta_2, \ldots, \beta_n\right)$$
$$=\left(\begin{array}{c}\left[T_{t\sqrt[q]{1-\prod_{\imath=1}^{n}\left(1-\frac{a^{q}_{\tau(\imath)}}{t^{q}}\right)^{\Upsilon(♍_{\tau(\imath)})-\Upsilon(♍_{\tau(\imath-1)})}}},\ T_{t\sqrt[q]{1-\prod_{\imath=1}^{n}\left(1-\frac{b^{q}_{\tau(\imath)}}{t^{q}}\right)^{\Upsilon(♍_{\tau(\imath)})-\Upsilon(♍_{\tau(\imath-1)})}}}\right],\\ \left[T_{t\prod_{\imath=1}^{n}\left(\frac{c_{\tau(\imath)}}{t}\right)^{\Upsilon(♍_{\tau(\imath)})-\Upsilon(♍_{\tau(\imath-1)})}},\ T_{t\prod_{\imath=1}^{n}\left(\frac{d_{\tau(\imath)}}{t}\right)^{\Upsilon(♍_{\tau(\imath)})-\Upsilon(♍_{\tau(\imath-1)})}}\right]\end{array}\right).$$

If $\beta_\imath = \beta$ for all $\imath\,(\imath = 1, 2, \ldots, n)$, then

$$\text{LIVq-ROFCIA}\left(\beta_1, \beta_2, \ldots, \beta_n\right)$$
$$=\left(\begin{array}{c}\left[T_{t\sqrt[q]{1-(1-\frac{a^{q}}{t^{q}})^{\sum_{\imath=1}^{n}\Upsilon(♍_{\tau(\imath)})-\Upsilon(♍_{\tau(\imath-1)})}}},\ T_{t\sqrt[q]{1-(1-\frac{b^{q}}{t^{q}})^{\sum_{\imath=1}^{n}\Upsilon(♍_{\tau(\imath)})-\Upsilon(♍_{\tau(\imath-1)})}}}\right],\\ \left[T_{t(\frac{c}{t})^{\sum_{\imath=1}^{n}\Upsilon(♍_{\tau(\imath)})-\Upsilon(♍_{\tau(\imath-1)})}},\ T_{t(\frac{d}{t})^{\sum_{\imath=i}^{n}\Upsilon(♍_{\tau(\imath)})-\Upsilon(♍_{\tau(\imath-1)})}}\right]\end{array}\right).$$

Since, $\sum_{\imath=1}^{n}\Upsilon\left(♍_{\tau(\imath)}\right)-\Upsilon\left(♍_{\tau(\imath-1)}\right)=1$. So $\text{LIVq-ROFCIA}\left(\beta_1, \beta_2, \ldots, \beta_n\right) = ([T_a, T_b], [T_c, T_d]) = \beta$. □

Theorem 3.4. *Let $\beta_l = ([\mathsf{T}_{a_l}, \mathsf{T}_{b_l}], [\mathsf{T}_{c_l}, \mathsf{T}_{d_l}])$ for $l = 1, 2, \ldots, n$ be a collection of LIVq-ROFNs on Z, and let Υ be a fuzzy measure on Z. If $\beta = ([\mathsf{T}_a, \mathsf{T}_b], [\mathsf{T}_c, \mathsf{T}_d])$ is a LINq-ROFN on Z, then the following equation holds:*

$$\textit{LIVq-ROFCIA}\,(\beta_1 \oplus \beta, \beta_2 \oplus \beta, \ldots, \beta_n \oplus \beta) = \textit{LIVq-ROFCIA}\,(\beta_1, \beta_2, \ldots, \beta_n) \oplus \beta. \tag{3.2}$$

Proof. For any $l\,(l = 1, 2, \ldots, n)$, since

$$\beta_l \oplus \beta = \left(\left[\mathsf{T}_{t\sqrt[q]{\frac{a_l^q}{t^q}+\frac{a^q}{t^q}-\frac{a_l^q}{t^q}\frac{a^q}{t^q}}}, \mathsf{T}_{t\sqrt[q]{\frac{b_l^q}{t^q}+\frac{b^q}{t^q}-\frac{b_l^q}{t^q}\frac{b^q}{t^q}}}\right], \left[\mathsf{T}_{t\left(\frac{c_l c}{t^2}\right)}, \mathsf{T}_{t\left(\frac{d_l d}{t^2}\right)}\right]\right)$$

or

$$\beta_l \oplus \beta = \left(\left[\mathsf{T}_{t\sqrt[q]{1-\left(1-\frac{a_l^q}{t^q}\right)\left(1-\frac{a^q}{t^q}\right)}}, \mathsf{T}_{t\sqrt[q]{1-\left(1-\frac{b_l^q}{t^q}\right)\left(1-\frac{b^q}{t^q}\right)}}\right], \left[\mathsf{T}_{t\left(\frac{c_l c}{t^2}\right)}, \mathsf{T}_{t\left(\frac{d_l d}{t^2}\right)}\right]\right).$$

By using Theorem 3.2, we have

$$\text{LIVq-ROFCIA}\,(\beta_1 \oplus \beta, \beta_2 \oplus \beta, \ldots, \beta_n \oplus \beta) = \left(\begin{array}{c}\left[\begin{array}{l}\mathsf{T}_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(\left(1-\frac{a_{\tau(l)}^q}{t^q}\right)\left(1-\frac{a^q}{t^q}\right)\right)^{\Upsilon(\mathscr{M}_{\tau(l)})-\Upsilon(\mathscr{M}_{\tau(l-1)})}}},\\ \mathsf{T}_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(\left(1-\frac{b_{\tau(l)}^q}{t^q}\right)\left(1-\frac{b^q}{t^q}\right)\right)^{\Upsilon(\mathscr{M}_{\tau(l)})-\Upsilon(\mathscr{M}_{\tau(l-1)})}}}\end{array}\right],\\ \left[\begin{array}{l}\mathsf{T}_{t\prod_{l=1}^{n}\left(\left(\frac{c_{\tau(l)}}{t}\right)\left(\frac{c}{t}\right)\right)^{\Upsilon(\mathscr{M}_{\tau(l)})-\Upsilon(\mathscr{M}_{\tau(l-1)})}},\\ \mathsf{T}_{t\prod_{l=1}^{n}\left(\left(\frac{d_{\tau(l)}}{t}\right)\left(\frac{d}{t}\right)\right)^{\Upsilon(\mathscr{M}_{\tau(l)})-\Upsilon(\mathscr{M}_{\tau(l-1)})}}\end{array}\right]\end{array}\right)$$

$$= \left(\begin{array}{c}\left[\begin{array}{l}\mathsf{T}_{t\sqrt[q]{1-\left(1-\frac{a^q}{t^q}\right)\prod_{l=1}^{n}\left(1-\frac{a_{\tau(l)}^q}{t^q}\right)^{\Upsilon(\mathscr{M}_{\tau(l)})-\Upsilon(\mathscr{M}_{\tau(l-1)})}}},\\ \mathsf{T}_{t\sqrt[q]{1-\left(1-\frac{b^q}{t^q}\right)\prod_{l=1}^{n}\left(1-\frac{b_{\tau(l)}^q}{t^q}\right)^{\Upsilon(\mathscr{M}_{\tau(l)})-\Upsilon(\mathscr{M}_{\tau(l-1)})}}}\end{array}\right],\\ \left[\begin{array}{l}\mathsf{T}_{t\left(\frac{c}{t}\right)\prod_{l=1}^{n}\left(\frac{c_{\tau(l)}}{t}\right)^{\Upsilon(\mathscr{M}_{\tau(l)})-\Upsilon(\mathscr{M}_{\tau(l-1)})}},\\ \mathsf{T}_{t\left(\frac{d}{t}\right)\prod_{l=1}^{n}\left(\frac{d_{\tau(l)}}{t}\right)^{\Upsilon(\mathscr{M}_{\tau(l)})-\Upsilon(\mathscr{M}_{\tau(l-1)})}}\end{array}\right]\end{array}\right).$$

By Definition 2.6, we have

$$\text{LIVq-ROFCIA}\,(\beta_1, \beta_2, \ldots, \beta_n) \oplus \beta = \left(\begin{array}{c}\left[\begin{array}{l}\mathsf{T}_{t\sqrt[q]{1-\left(1-\frac{a^q}{t^q}\right)\prod_{l=1}^{n}\left(1-\frac{a_{\tau(l)}^q}{t^q}\right)^{\Upsilon(\mathscr{M}_{\tau(l)})-\Upsilon(\mathscr{M}_{\tau(l-1)})}}},\\ \mathsf{T}_{t\sqrt[q]{1-\left(1-\frac{b^q}{t^q}\right)\prod_{l=1}^{n}\left(1-\frac{b_{\tau(l)}^q}{t^q}\right)^{\Upsilon(\mathscr{M}_{\tau(l)})-\Upsilon(\mathscr{M}_{\tau(l-1)})}}}\end{array}\right],\\ \left[\begin{array}{l}\mathsf{T}_{t\left(\frac{c}{t}\right)\prod_{l=1}^{n}\left(\frac{c_{\tau(l)}}{t}\right)^{\Upsilon(\mathscr{M}_{\tau(l)})-\Upsilon(\mathscr{M}_{\tau(l-1)})}},\\ \mathsf{T}_{t\left(\frac{d}{t}\right)\prod_{l=1}^{n}\left(\frac{d_{\tau(l)}}{t}\right)^{\Upsilon(\mathscr{M}_{\tau(l)})-\Upsilon(\mathscr{M}_{\tau(l-1)})}}\end{array}\right]\end{array}\right).$$

Hence,

$$\text{LIVq-ROFCIA}\left(\beta_1 \oplus \beta, \beta_2 \oplus \beta, \ldots, \beta_n \oplus \beta\right) = \text{LIVq-ROFCIA}\left(\beta_1, \beta_2, \ldots, \beta_n\right) \oplus \beta. \tag{3.3}$$

Thus, the proof is complete. □

Theorem 3.5. *Let $\beta_l = ([T_{a_l}, T_{b_l}], [T_{c_l}, T_{d_l}])$ $(l = 1, 2, \ldots, n)$ be a collection of LIVq-ROFNs on Z, and Υ be a fuzzy measure on Z. If $\gamma > 0$, then*

$$\textit{LIVq-ROFCIA}\left(\beta_1^\gamma, \beta_2^\gamma, \ldots, \beta_n^\gamma\right) = \textit{LIVq-ROFCIA}\left(\beta_1, \beta_2, \ldots, \beta_n\right)^\gamma.$$

Proof. For any $l\,(l = 1, 2, \ldots, n)$, since

$$\beta_l^\gamma = \left(\left[T_{t\left(\frac{a_l}{t}\right)^\gamma}, T_{t\left(\frac{b_l}{t}\right)^\gamma}\right], \left[T_{t\sqrt[q]{1-\left(1-\frac{c_l^q}{t^q}\right)^\gamma}}, T_{t\sqrt[q]{1-\left(1-\frac{d_l^q}{t^q}\right)^\gamma}}\right]\right),$$

by using Theorem 3.2, we have

$$\begin{aligned}
&\text{LIVq-ROFCIA}\left(\beta_1^\gamma, \beta_2^\gamma, \ldots, \beta_n^\gamma\right)\\
&= \left(\begin{array}{c}\left[T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{a_{\tau(l)}^q}{t^q}\right)^{\gamma\left(\Upsilon\left(\mathfrak{M}_{\tau(l)}\right)-\Upsilon\left(\mathfrak{M}_{\tau(l-1)}\right)\right)}}}, T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{b_{\tau(l)}^q}{t^q}\right)^{\gamma\left(\Upsilon\left(\mathfrak{M}_{\tau(l)}\right)-\Upsilon\left(\mathfrak{M}_{\tau(l-1)}\right)\right)}}}\right],\\
\left[T_{t\prod_{l=1}^{n}\left(\frac{c_{\tau(l)}}{t}\right)^{\gamma\left(\Upsilon\left(\mathfrak{M}_{\tau(l)}\right)-\Upsilon\left(\mathfrak{M}_{\tau(l-1)}\right)\right)}}, T_{t\prod_{l=1}^{n}\left(\frac{d_{\tau(l)}}{t}\right)^{\gamma\left(\Upsilon\left(\mathfrak{M}_{\tau(l)}\right)-\Upsilon\left(\mathfrak{M}_{\tau(l-1)}\right)\right)}}\right]\end{array}\right).
\end{aligned}$$

Since,

$$\begin{aligned}
&\text{LIVq-ROFCIA}\left(\beta_1, \beta_2, \ldots, \beta_n\right)^\gamma\\
&= \left(\begin{array}{c}\left[T_{t\sqrt[q]{1-\left(\prod_{l=1}^{n}\left(1-\frac{a_{\tau(l)}^q}{t^q}\right)^{\Upsilon\left(\mathfrak{M}_{\tau(l)}\right)-\Upsilon\left(\mathfrak{M}_{\tau(l-1)}\right)}\right)^\gamma}}, T_{t\sqrt[q]{1-\left(\prod_{l=1}^{n}\left(1-\frac{b_{\tau(l)}^q}{t^q}\right)^{\Upsilon\left(\mathfrak{M}_{\tau(l)}\right)-\Upsilon\left(\mathfrak{M}_{\tau(l-1)}\right)}\right)^\gamma}}\right],\\
\left[T_{t\left(\prod_{l=1}^{n}\left(\frac{c_{\tau(l)}}{t}\right)^{\Upsilon\left(\mathfrak{M}_{\tau(l)}\right)-\Upsilon\left(\mathfrak{M}_{\tau(l-1)}\right)}\right)^\gamma}, T_{t\left(\prod_{l=1}^{n}\left(\frac{d_{\tau(l)}}{t}\right)^{\Upsilon\left(\mathfrak{M}_{\tau(l)}\right)-\Upsilon\left(\mathfrak{M}_{\tau(l-1)}\right)}\right)^\gamma}\right]\end{array}\right)\\
&= \left(\begin{array}{c}\left[T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{a_{\tau(l)}^q}{t^q}\right)^{\gamma\left(\Upsilon\left(\mathfrak{M}_{\tau(l)}\right)-\Upsilon\left(\mathfrak{M}_{\tau(l-1)}\right)\right)}}}, T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{b_{\tau(l)}^q}{t^q}\right)^{\gamma\left(\Upsilon\left(\mathfrak{M}_{\tau(l)}\right)-\Upsilon\left(\mathfrak{M}_{\tau(l-1)}\right)\right)}}}\right],\\
\left[T_{t\prod_{l=1}^{n}\left(\frac{c_{\tau(l)}}{t}\right)^{\gamma\left(\Upsilon\left(\mathfrak{M}_{\tau(l)}\right)-\Upsilon\left(\mathfrak{M}_{\tau(l-1)}\right)\right)}}, T_{t\prod_{l=1}^{n}\left(\frac{d_{\tau(l)}}{t}\right)^{\gamma\left(\Upsilon\left(\mathfrak{M}_{\tau(l)}\right)-\Upsilon\left(\mathfrak{M}_{\tau(l-1)}\right)\right)}}\right]\end{array}\right).
\end{aligned}$$

Thus, $\text{LIVq-ROFCIA}\left(\beta_1^\gamma, \beta_2^\gamma, \ldots, \beta_n^\gamma\right) = \text{LIVq-ROFCIA}\left(\beta_1, \beta_2, \ldots, \beta_n\right)^\gamma$. Hence, the proof is completed. □

The following result of Theorem 3.6 is obtained using Theorem 3.2.

Theorem 3.6. *Let* $\beta_l = ([T_{a_l}, T_{b_l}], [T_{c_l}, T_{d_l}])$ *for* $l = 1, 2, \ldots, n$ *be a collection of LIVq-ROFNs on* Z*, and* Υ *be a fuzzy measure on* Z*. Then we have following.*

1. *If* $\Upsilon(\mathfrak{M}_1) = 1$ *for any* $\mathfrak{M}_1 \in P(Z)$*, then LIVq-ROFCIA* $(\beta_1, \beta_2, \ldots, \beta_n) = \max(\beta_1, \beta_2, \ldots, \beta_n) = \beta_{(n)}$.
2. *If* $\Upsilon(\mathfrak{M}_1) = 0$ *for any* $\mathfrak{M}_1 \in P(Z)$*, then LIVq-ROFCIA* $(\beta_1, \beta_2, \ldots, \beta_n) = \min(\beta_1, \beta_2, \ldots, \beta_n) = \beta_{(1)}$.
3. *For any* $\mathfrak{M}_1, \mathfrak{M}_2 \in P(Z)$*, such that* $|\mathfrak{M}_1| = |\mathfrak{M}_2|$*,* $\Upsilon(\mathfrak{M}_1) = \Upsilon(\mathfrak{M}_2)$ *and* $\Upsilon((l) \ldots, (n)) = \frac{n-l+1}{n}$*,* $1 \leqslant l \leqslant n$*, then*

$$\text{LIVq-ROFCIA}(\beta_1, \beta_2, \ldots, \beta_n) = \left(\begin{array}{c} \left[T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{a^q_{\tau(l)}}{t^q}\right)^{\frac{1}{n}}}}, T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{b^q_{\tau(l)}}{t^q}\right)^{\frac{1}{n}}}} \right], \\ \left[T_{t\prod_{l=1}^{n}\left(\frac{c_{\tau(l)}}{t}\right)^{\frac{1}{n}}}, T_{t\prod_{l=1}^{n}\left(\frac{d_{\tau(l)}}{t}\right)^{\frac{1}{n}}} \right] \end{array} \right)$$

and

$$\text{LIVq-ROFCIA}(\beta_1, \beta_2, \ldots, \beta_n) = \max(\beta_1, \beta_2, \ldots, \beta_n) = \beta_{(n)}.$$

Theorem 3.7. *Let* $\beta_l = ([T_{a_l}, T_{b_l}], [T_{c_l}, T_{d_l}])$ *for* $l = 1, 2, \ldots, n$ *and* $p_l = \left(\left[T_{a'_l}, T_{b'_l}\right], \left[T_{c'_l}, T_{d'_l}\right]\right)$ *for* $l = 1, 2, \ldots, n$ *be two collections of LIVq-ROFNs on* Z*, and* Υ *be a fuzzy measure on* Z*, where* $\{\tau(1), \tau(2), \ldots, \tau(n)\}$ *is a permutation of* $\{1, 2, \ldots, n\}$ *such that* $\beta_{\tau(1)} \geqslant \beta_{\tau(2)} \geqslant \cdots \geqslant \beta_{\tau(n)}$ *and* $\mathfrak{M}_{\tau(k)} = \{z_{\tau(k)} | l \leqslant k\}$ *for* $k \geqslant 1$*, and* $\mathfrak{M}_{\tau(0)} = \phi$*. If* $T_{a_{(l)}} \leqslant T_{a'_{(l)}}$, $T_{b_{(l)}} \leqslant T_{b'_{(l)}}$, $T_{c_{(l)}} \geqslant T_{c'_{(l)}}$*, and* $T_{d_{(l)}} \geqslant T_{d'_{(l)}}$ *for all* l*, that is,* $\beta_l \leqslant p_l$*, then*

$$\text{LIVq-ROFCIA}(\beta_1, \beta_2, \ldots, \beta_n) \leqslant \text{LIVq-ROFCIA}(p_1, p_2, \ldots, p_n).$$

Proof. Since $\mathfrak{M}_{\tau(l)} \subseteq \mathfrak{M}_{\tau(l-1)}$, then, $\Upsilon\left(\mathfrak{M}_{\tau(l)}\right) - \Upsilon\left(\mathfrak{M}_{\tau(l-1)}\right) \geqslant 0$. For all l, $T_{a_{\tau(l)}} \leqslant T_{a'_{\tau(l)}}$; $T_{b_{\tau(l)}} \leqslant T_{b'_{\tau(l)}}$ and $T_{c_{\tau(l)}} \geqslant T_{c'_{\tau(l)}}$; $T_{d_{\tau(l)}} \geqslant T_{d'_{\tau(l)}}$, we have

$$\begin{aligned}
a_{\tau(l)} \leqslant a'_{\tau(l)} &\Longrightarrow 1 - \frac{a^q_{\tau(l)}}{t^q} \geqslant 1 - \frac{(a')^q_{\tau(l)}}{t^q} \\
&\Longrightarrow \prod_{l=1}^{n}\left(1 - \frac{a^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})} \geqslant \prod_{l=1}^{n}\left(1 - \frac{(a')^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})} \\
&\Longrightarrow 1 - \prod_{l=1}^{n}\left(1 - \frac{a^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})} \leqslant 1 - \prod_{l=1}^{n}\left(1 - \frac{(a')^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})} \\
&\Longrightarrow t\sqrt[q]{1 - \prod_{l=1}^{n}\left(1 - \frac{a^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}} \\
&\qquad \leqslant t\sqrt[q]{1 - \prod_{l=1}^{n}\left(1 - \frac{(a')^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}} \\
&\Longrightarrow T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{a^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}} \leqslant T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{(a')^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}}.
\end{aligned}$$

Similarly, we can obtain

$$T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{b^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}} \leqslant T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{(b')^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}}$$

and

$$\begin{aligned} c_{\tau(l)} \geqslant c'_{\tau(l)} &\Longrightarrow \prod_{l=1}^{n}\left(\frac{c_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})} \geqslant \prod_{l=1}^{n}\left(\frac{(c')_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})} \\ &\Longrightarrow t\prod_{l=1}^{n}\left(\frac{c_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})} \geqslant t\prod_{l=1}^{n}\left(\frac{(c')_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})} \\ &\Longrightarrow T_{t\prod_{l=1}^{n}\left(\frac{c_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}} \geqslant T_{t\prod_{l=1}^{n}\left(\frac{(c')_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}. \end{aligned}$$

Similarly, for each l, $d_l \geqslant d'_l$, we get

$$T_{t\prod_{l=1}^{n}\left(\frac{d_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}} \geqslant T_{t\prod_{l=1}^{n}\left(\frac{(d')_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}.$$

Thus, by Theorem 3.2 and Definition 2.7, we have

$$\text{LIVq-ROFCIA}\left(\beta_1,\beta_2,\ldots,\beta_n\right) \leqslant \text{LIVq-ROFCIA}\left(p_1,p_2,\ldots,p_n\right).$$

Hence, the proof is completed. □

Theorem 3.8. *Let $\beta_l = ([T_{a_l}, T_{b_l}], [T_{c_l}, T_{d_l}])$, $l = 1,2,\ldots,n$ be a collection of LIVq-ROFNs on Z, and let Υ be a fuzzy measure on Z. Suppose that $\{\tau(1),\tau(2),\ldots,\tau(n)\}$ is a permutation of $\{1,2,\ldots,n\}$ such that $\beta_{\tau(1)} \geqslant \beta_{\tau(2)} \geqslant \cdots \geqslant \beta_{\tau(n)}$ and define $\mathcal{M}_{\tau(k)} = \{z_{\tau(k)} \mid l \leqslant k\}$, $k \geqslant 1$, with $\mathcal{M}_{\tau(0)} = \phi$. If*

$$\beta^- = \left(\left[\min_l T_{a_l}, \min_l T_{b_l}\right], \left[\max_l T_{c_l}, \max_l T_{d_l}\right]\right), \qquad \beta^+ = \left(\left[\max_l T_{a_l}, \max_l T_{b_l}\right], \left[\min_l T_{c_l}, \min_l T_{d_l}\right]\right),$$

then

$$\beta^- \leqslant \text{LIVq-ROFCIA}\left(\beta_1,\beta_2,\ldots,\beta_n\right) \leqslant \beta^+.$$

Proof. For any $\beta_l = ([T_{a_l}, T_{b_l}], [T_{c_l}, T_{d_l}])$, $l = 1,2,\ldots,n$, it is evident that β^- and β^+ are LIVq-ROFNs. Since $\mathcal{M}_{\tau(l)} \subseteq \mathcal{M}_{\tau(l-1)}$, it follows that $\Upsilon\left(\mathcal{M}_{\tau(l)}\right) - \Upsilon\left(\mathcal{M}_{\tau(l-1)}\right) \geqslant 0$. Let $\{\tau(1),\tau(2),\ldots,\tau(n)\}$ be a permutation of $\{1,2,\ldots,n\}$ such that $\beta_{\tau(1)} \geqslant \beta_{\tau(2)} \geqslant \ldots \geqslant \beta_{\tau(n)}$. Then, we have the following inequalities: $\min_l T_{a_{\tau(l)}} \leqslant T_{a_{\tau(l)}} \leqslant \max_l T_{a_{\tau(l)}}$, $\min_l T_{b_{\tau(l)}} \leqslant T_{b_{\tau(l)}} \leqslant \max_l T_{b_{\tau(l)}}$, $\min_l T_{c_{\tau(l)}} \leqslant T_{c_{\tau(l)}} \leqslant \max_l T_{c_{\tau(l)}}$, $\min_l T_{d_{\tau(l)}} \leqslant T_{d_{\tau(l)}} \leqslant \max_l T_{d_{\tau(l)}}$.

$$\begin{aligned} T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{\min_l a^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}} &\leqslant T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{a^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}} \\ &\leqslant T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{\max_l a^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}}, \end{aligned}$$

$$T_{t\sqrt[q]{1-\left(1-\frac{\min\limits_{l} a^{q}_{\tau(l)}}{t^{q}}\right)^{\sum_{l=1}^{n}\left(\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})\right)}}} \leqslant T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{a^{q}_{\tau(l)}}{t^{q}}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}}$$
$$\leqslant T_{t\sqrt[q]{1-\left(1-\frac{\max\limits_{l} a^{q}_{\tau(l)}}{t^{q}}\right)^{\sum_{l=1}^{n}\left(\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})\right)}}}.$$

Since, $\sum_{l=1}^{n}\left(\Upsilon\left(\mathcal{M}_{\tau(l)}\right)-\Upsilon\left(\mathcal{M}_{\tau(l-1)}\right)\right)=1$. So we have,

$$\frac{\min\limits_{l} a^{q}_{\tau(l)}}{t^{q}} \leqslant t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{a^{q}_{\tau(l)}}{t^{q}}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}} \leqslant \frac{\max\limits_{l} a^{q}_{\tau(l)}}{t^{q}}.$$

Similarly, we can obtain

$$\frac{\min\limits_{l} b^{q}_{\tau(l)}}{t^{q}} \leqslant t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{b^{q}_{\tau(l)}}{t^{q}}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}} \leqslant \frac{\max\limits_{l} b^{q}_{\tau(l)}}{t^{q}}$$

and

$$T_{t\prod_{l=1}^{n}\left(\frac{\min_{l} c_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}} \leqslant T_{t\prod_{l=1}^{n}\left(\frac{c_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}} \leqslant T_{t\prod_{l=1}^{n}\left(\frac{\max_{l} c_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}},$$
$$T_{\left(\frac{\min_{l} c_{\tau(l)}}{t}\right)^{\sum_{l=1}^{n}\left(\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})\right)}} \leqslant T_{t\prod_{l=1}^{n}\left(\frac{c_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}} \leqslant T_{\left(\frac{\max_{l} c_{\tau(l)}}{t}\right)^{\sum_{l=1}^{n}\left(\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})\right)}}.$$

Since $\sum_{l=1}^{n}\left(\Upsilon\left(\mathcal{M}_{\tau(l)}\right)-\Upsilon\left(\mathcal{M}_{\tau(l-1)}\right)\right)=1$, so we have

$$\frac{\min\limits_{l} c_{\tau(l)}}{t} \leqslant t\prod_{l=1}^{n}\left(\frac{c_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})} \leqslant \frac{\max\limits_{l} c_{\tau(l)}}{t}.$$

Similarly $\frac{\min_{l} d_{\tau(l)}}{t} \leqslant t\prod_{l=1}^{n}\left(\frac{d_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})} \leqslant \frac{\max_{l} d_{\tau(l)}}{t}$. Thus, by Theorem 3.2 and Definition 2.7, we have

$$\left(\left[\min_{l} T_{a_l}, \min_{l} T_{b_l}\right], \left[\max_{l} T_{c_l}, \max_{l} T_{d_l}\right]\right) \leqslant \text{LIVq-ROFCIA}\left(\beta_1, \beta_2, \ldots, \beta_n\right)$$
$$\leqslant \left(\left[\max_{l} T_{a_l}, \max_{l} T_{b_l}\right], \left[\min_{l} T_{c_l}, \min_{l} T_{d_l}\right]\right).$$

Hence, $\beta^{-} \leqslant \text{LIVq-ROFCIA}\left(\beta_1, \beta_2, \ldots, \beta_n\right) \leqslant \beta^{+}$. Thus, the proof of the theorem is complete. □

In the following, we will define and study the LIVq-ROFCIG operator.

Definition 3.9. Let $\beta_l = ([T_{a_l}, T_{b_l}], [T_{c_l}, T_{d_l}])$, $l = 1, 2, \ldots, n$ be a set of LIVq-ROFNs, and let Υ denote a fuzzy measure on Z. Utilizing this fuzzy measure, we define the LIVq-ROF Choquet Integral Geometric (LIVq-ROFCIG) operator, which maps LIVq-ROFCIG : $\Omega^{n} \longrightarrow \Omega$. This operator is given by

$$\text{LIVq-ROFCIG}\left(\beta_1, \beta_2, \ldots, \beta_n\right) = \bigotimes_{l=1}^{n} \beta_l^{\left(\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})\right)},$$

where $\{\tau(1), \tau(2), \ldots, \tau(n)\}$ is a permutation of $\{1, 2, \ldots, n\}$ such that $\beta_{\tau(1)} \geqslant \beta_{\tau(2)} \geqslant \cdots \geqslant \beta_{\tau(n)}$. Additionally, the sets $\mathcal{M}_{\tau(k)}$ are defined as follows: $\mathcal{M}_{\tau(k)} = \left\{z_{\tau(k)} \mid l \leqslant k\right\}$, $k \geqslant 1$, with $\mathcal{M}_{\tau(0)} = \phi$.

Theorem 3.10. *Let $\beta_l = ([T_{a_l}, T_{b_l}], [T_{c_l}, T_{d_l}])$, $l = 1, 2, \ldots, n$ be a collection of LIVq-ROFNs on Z, and let Υ be a fuzzy measure on Z. Then, the aggregated value obtained using the LIVq-ROFCIG operator is also an LIVq-ROFN, and it is given by*

$$\begin{aligned}&\mathit{LIVq\text{-}ROFCIG}(\beta_1, \beta_2, \ldots, \beta_n)\\ &= \left(\begin{array}{c} \left[T_{t\prod_{l=1}^{n}\left(\frac{a_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}, \quad T_{t\prod_{l=1}^{n}\left(\frac{b_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}} \right], \\ \left[T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{c^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}}, \quad T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{d^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}} \right] \end{array} \right),\end{aligned} \tag{3.4}$$

where $\{\tau(1), \tau(2), \ldots, \tau(n)\}$ is a permutation of $\{1, 2, \ldots, n\}$ such that $\beta_{\tau(1)} \geqslant \beta_{\tau(2)} \geqslant \cdots \geqslant \beta_{\tau(n)}$ and $\mathcal{M}_{\tau(k)} = \{z_{\tau(k)} \mid l \leqslant k\}$, $k \geqslant 1$, with $\mathcal{M}_{\tau(0)} = \phi$.

Proof. We prove Eq. (3.4) using mathematical induction. First, we assume that the statement holds for $n = 1$. Now, we prove that the result is true for $n = 2$. Using the operational laws from Definition 2.6, we obtain

$$\beta_{\tau(1)}^{\Upsilon(\mathcal{M}_{\tau(1)})-\Upsilon(\mathcal{M}_{\tau(0)})} = \left(\begin{array}{c} \left[T_{t\left(\frac{a_{\tau(1)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(1)})-\Upsilon(\mathcal{M}_{\tau(0)})}}, T_{t\left(\frac{b_{\tau(1)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(1)})-\Upsilon(\mathcal{M}_{\tau(0)})}} \right], \\ \left[T_{t\sqrt[q]{1-\left(1-\frac{c^q_{\tau(1)}}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(1)})-\Upsilon(\mathcal{M}_{\tau(0)})}}}, T_{t\sqrt[q]{1-\left(1-\frac{d^q_{\tau(1)}}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(1)})-\Upsilon(\mathcal{M}_{\tau(0)})}}} \right] \end{array} \right),$$

$$\beta_{\tau(2)}^{\Upsilon(\mathcal{M}_{\tau(2)})-\Upsilon(\mathcal{M}_{\tau(1)})} = \left(\begin{array}{c} \left[T_{t\left(\frac{a_{\tau(2)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(2)})-\Upsilon(\mathcal{M}_{\tau(1)})}}, T_{t\left(\frac{b_{\tau(2)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(2)})-\Upsilon(\mathcal{M}_{\tau(1)})}} \right], \\ \left[T_{t\sqrt[q]{1-\left(1-\frac{c^q_{\tau(2)}}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(2)})-\Upsilon(\mathcal{M}_{\tau(1)})}}}, T_{t\sqrt[q]{1-\left(1-\frac{d^q_{\tau(2)}}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(2)})-\Upsilon(\mathcal{M}_{\tau(1)})}}} \right] \end{array} \right).$$

Since

$$\beta_1 \otimes \beta_2 = \left(\left[T_{t\left(\frac{a_1 a_2}{t^2}\right)}, T_{t\left(\frac{b_1 b_2}{t^2}\right)} \right], \left[T_{t\sqrt[q]{\frac{c_1^q}{t^q}+\frac{c_2^q}{t^q}-\frac{c_1^q}{t^q}\frac{c_2^q}{t^q}}}, T_{t\sqrt[q]{\frac{d_1^q}{t^q}+\frac{d_2^q}{t^q}-\frac{d_1^q}{t^q}\frac{d_2^q}{t^q}}} \right] \right).$$

Therefore,

$$\begin{aligned}&\text{LIVq-ROFCIG}(\beta_1, \beta_2)\\ &= \beta_{\tau(1)}^{\Upsilon(\mathcal{M}_{\tau(1)})-\Upsilon(\mathcal{M}_{\tau(0)})} \otimes \beta_{\tau(2)}^{\Upsilon(\mathcal{M}_{\tau(2)})-\Upsilon(\mathcal{M}_{\tau(1)})}\\ &= \left(\begin{array}{c} \left[T_{t\prod_{l=1}^{2}\left(\frac{a_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}, T_{t\prod_{l=1}^{2}\left(\frac{b_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}} \right], \\ \left[T_{t\sqrt[q]{1-\prod_{l=1}^{2}\left(1-\frac{c^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}}, T_{t\sqrt[q]{1-\prod_{l=1}^{2}\left(1-\frac{d^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}} \right] \end{array} \right).\end{aligned}$$

Hence, Eq. (3.4) holds for $n = 2$. Next, we demonstrate that Eq. (3.4) is valid for $n = k$.

$$
\begin{aligned}
&\text{LIVq-ROFCIG}\left(\beta_1, \beta_2, \ldots, \beta_k\right)\\
&= \left(\begin{array}{c}
\left[\mathsf{T}_{t\prod_{l=1}^{k}\left(\frac{a_{\tau(l)}}{t}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}, \mathsf{T}_{t\prod_{l=1}^{k}\left(\frac{b_{\tau(l)}}{t}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}} \right],\\
\left[\mathsf{T}_{t\sqrt[q]{1-\prod_{l=1}^{k}\left(1-\frac{c^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}}, \mathsf{T}_{t\sqrt[q]{1-\prod_{l=1}^{k}\left(1-\frac{d^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}} \right]
\end{array} \right).
\end{aligned}
$$

Assuming that Eq. (3.4) holds for $n = k$, we now prove that it also holds for $n = k+1$. Since

$$
\begin{aligned}
&\bigotimes_{l=1}^{k} \beta_l^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}\\
&= \left(\begin{array}{c}
\left[\mathsf{T}_{t\prod_{l=1}^{k}\left(\frac{a_{\tau(l)}}{t}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}, \mathsf{T}_{t\prod_{l=1}^{k}\left(\frac{b_{\tau(l)}}{t}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}} \right],\\
\left[\mathsf{T}_{t\sqrt[q]{1-\prod_{l=1}^{k}\left(1-\frac{c^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}}, \mathsf{T}_{t\sqrt[q]{1-\prod_{l=1}^{k}\left(1-\frac{d^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}} \right]
\end{array} \right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\beta_{\tau(k+1)}^{\Upsilon(\mathfrak{M}_{\tau(k+1)})-\Upsilon(\mathfrak{M}_{\tau(k)})}\\
&= \left(\begin{array}{c}
\left[\mathsf{T}_{t\left(\frac{a_{\tau(k+1)}}{t}\right)^{\Upsilon(\mathfrak{M}_{\tau(k+1)})-\Upsilon(\mathfrak{M}_{\tau(k)})}}, \mathsf{T}_{t\left(\frac{b_{\tau(k+1)}}{t}\right)^{\Upsilon(\mathfrak{M}_{\tau(k+1)})-\Upsilon(\mathfrak{M}_{\tau(k)})}} \right],\\
\left[\mathsf{T}_{t\sqrt[q]{1-\left(1-\frac{c^q_{\tau(k+1)}}{t^q}\right)^{\Upsilon(\mathfrak{M}_{\tau(k+1)})-\Upsilon(\mathfrak{M}_{\tau(k)})}}}, \mathsf{T}_{t\sqrt[q]{1-\left(1-\frac{d^q_{\tau(k+1)}}{t^q}\right)^{\Upsilon(\mathfrak{M}_{\tau(k+1)})-\Upsilon(\mathfrak{M}_{\tau(k)})}}} \right]
\end{array} \right).
\end{aligned}
$$

Now

$$
\begin{aligned}
&\text{LIVq-ROFCIG}\left(\beta_1, \beta_2, \ldots, \beta_k, \beta_{k+1}\right)\\
&= \bigotimes_{l=1}^{k} \beta_l^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})} \otimes \beta_{k+1}^{\Upsilon(\mathfrak{M}_{\tau(k+1)})-\Upsilon(\mathfrak{M}_{\tau(k)})}\\
&= \left(\begin{array}{c}
\left[\mathsf{T}_{t\prod_{l=1}^{k}\left(\frac{a_{\tau(l)}}{t}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}, \mathsf{T}_{t\prod_{l=1}^{k}\left(\frac{b_{\tau(l)}}{t}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}} \right],\\
\left[\mathsf{T}_{t\sqrt[q]{1-\prod_{l=1}^{k}\left(1-\frac{c^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}}, \mathsf{T}_{t\sqrt[q]{1-\prod_{l=1}^{k}\left(1-\frac{d^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\mathfrak{M}_{\tau(l)})-\Upsilon(\mathfrak{M}_{\tau(l-1)})}}} \right]
\end{array} \right)
\end{aligned}
$$

$$\otimes\left(\begin{array}{c}\left[T_{t\left(\frac{a_{\tau(k+1)}}{t}\right)^{\Upsilon(\text{♍}_{\tau(k+1)})-\Upsilon(\text{♍}_{\tau(k)})}}, T_{t\left(\frac{b_{\tau(k+1)}}{t}\right)^{\Upsilon(\text{♍}_{\tau(k+1)})-\Upsilon(\text{♍}_{\tau(k)})}}\right],\\ \left[T_{t\sqrt[q]{1-\left(1-\frac{c^q_{\tau(k+1)}}{t^q}\right)^{\Upsilon(\text{♍}_{\tau(k+1)})-\Upsilon(\text{♍}_{\tau(k)})}}}, T_{t\sqrt[q]{1-\left(1-\frac{d^q_{\tau(k+1)}}{t^q}\right)^{\Upsilon(\text{♍}_{\tau(k+1)})-\Upsilon(\text{♍}_{\tau(k)})}}}\right]\end{array}\right)$$

$$=\left(\begin{array}{c}\left[T_{t\prod_{l=1}^{k+1}\left(\frac{a_{\tau(l)}}{t}\right)^{\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}}, T_{t\prod_{l=1}^{k+1}\left(\frac{b_{\tau(l)}}{t}\right)^{\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}}\right],\\ \left[T_{t\sqrt[q]{1-\prod_{l=1}^{k+1}\left(1-\frac{c^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}}}, T_{t\sqrt[q]{1-\prod_{l=1}^{k+1}\left(1-\frac{d^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}}}\right]\end{array}\right).$$

Hence, Eq. (3.4) holds for $n = k+1$, and therefore, it is true for all n. □

Theorem 3.11. *Let* $\beta_l = ([T_{a_l}, T_{b_l}], [T_{c_l}, T_{d_l}])$ *for* $l = 1, 2, \ldots, n$ *be a collection of LIVq-ROFNs on* Z*, and let* Υ *be a fuzzy measure on* Z*. If all* β_l $(l = 1, 2, \ldots, n)$ *are identical, meaning that for all* l*, we have* $\beta = ([T_a, T_b], [T_c, T_d])$*, then*

$$\textit{LIVq-ROFCIG}(\beta_1, \beta_2, \ldots, \beta_n) = \beta.$$

Proof. Since

$$\text{LIVq-ROFCIG}(\beta_1, \beta_2, \ldots, \beta_n)$$
$$=\left(\begin{array}{c}\left[T_{t\prod_{l=1}^{n}\left(\frac{a_{\tau(l)}}{t}\right)^{\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}}, T_{t\prod_{l=1}^{n}\left(\frac{b_{\tau(l)}}{t}\right)^{\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}}\right],\\ \left[T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{c^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}}}, T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{d^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}}}\right]\end{array}\right).$$

If $\beta_l = \beta$ for all $l\,(l = 1, 2, \ldots, n)$, then

$$\text{LIVq-ROFCIG}(\beta_1, \beta_2, \ldots, \beta_n)$$
$$=\left(\begin{array}{c}\left[T_{t\left(\frac{a}{t}\right)^{\sum_{l=1}^{n}\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}}, T_{t\left(\frac{b}{t}\right)^{\sum_{l=i}^{n}\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}}\right],\\ \left[T_{t\sqrt[q]{1-\left(1-\frac{c^q}{t^q}\right)^{\sum_{l=1}^{n}\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}}}, T_{t\sqrt[q]{1-\left(1-\frac{d^q}{t^q}\right)^{\sum_{l=1}^{n}\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}}}\right]\end{array}\right).$$

Since $\sum_{l=1}^{n}\Upsilon\left(\text{♍}_{\tau(l)}\right)-\Upsilon\left(\text{♍}_{\tau(l-1)}\right) = 1$, it follows that

$$\text{LIVq-ROFCIG}(\beta_1, \beta_2, \ldots, \beta_n) = ([T_a, T_b], [T_c, T_d]) = \beta.$$

□

Theorem 3.12. *Let $\beta_l = ([T_{a_l}, T_{b_l}], [T_{c_l}, T_{d_l}])$ for $l = 1, 2, \ldots, n$ be a collection of LIVq-ROFNs on Z, and let Υ be a fuzzy measure on Z. If $\beta = ([T_a, T_b], [T_c, T_d])$ is a LINq-ROFN on Z, then*

$$\textit{LIVq-ROFCIG}(\beta_1 \otimes \beta, \beta_2 \otimes \beta, \ldots, \beta_n \otimes \beta) = \textit{LIVq-ROFCIG}(\beta_1, \beta_2, \ldots, \beta_n) \otimes \beta.$$

Proof. For any l $(l = 1, 2, \ldots, n)$, we have

$$\beta_l \otimes \beta = \left(\left[T_{t\left(\frac{a_l a}{t^2}\right)}, T_{t\left(\frac{b_l b}{t^2}\right)}\right], \left[T_{t\sqrt[q]{\frac{c_l^q}{t^q}+\frac{c^q}{t^q}-\frac{c_l^q}{t^q}\frac{c^q}{t^q}}}, T_{t\sqrt[q]{\frac{d_l^q}{t^q}+\frac{d^q}{t^q}-\frac{d_l^q}{t^q}\frac{d^q}{t^q}}}\right]\right).$$

Alternatively,

$$\beta_l \otimes \beta = \left(\left[T_{t\left(\frac{a_l a}{t^2}\right)}, T_{t\left(\frac{b_l b}{t^2}\right)}\right], \left[T_{t\sqrt[q]{1-\left(1-\frac{c_l^q}{t^q}\right)\left(1-\frac{c^q}{t^q}\right)}}, T_{t\sqrt[q]{1-\left(1-\frac{d_l^q}{t^q}\right)\left(1-\frac{d^q}{t^q}\right)}}\right]\right).$$

By applying Theorem 3.10, we obtain

$$\text{LIVq-ROFCIA}(\beta_1 \otimes \beta, \beta_2 \otimes \beta, \ldots, \beta_n \otimes \beta) = \left(\begin{array}{c} \left[\begin{array}{l} T_{t\prod_{l=1}^{n}\left(\left(\frac{a_{\tau(l)}}{t}\right)\left(\frac{a}{t}\right)\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}, \\ T_{t\prod_{l=1}^{n}\left(\left(\frac{b_{\tau(l)}}{t}\right)\left(\frac{b}{t}\right)\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}} \end{array}\right], \\ \left[\begin{array}{l} T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(\left(1-\frac{c_{\tau(l)}^q}{t^q}\right)\left(1-\frac{c^q}{t^q}\right)\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}}, \\ T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(\left(1-\frac{d_{\tau(l)}^q}{t^q}\right)\left(1-\frac{d^q}{t^q}\right)\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}} \end{array}\right] \end{array}\right)$$

$$= \left(\begin{array}{c} \left[\begin{array}{l} T_{t\left(\frac{a}{t}\right)\prod_{l=1}^{n}\left(\frac{a_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}, \\ T_{t\left(\frac{d}{t}\right)\prod_{l=1}^{n}\left(\frac{b_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}} \end{array}\right], \\ \left[\begin{array}{l} T_{t\sqrt[q]{1-\left(1-\frac{c^q}{t^q}\right)\prod_{l=1}^{n}\left(1-\frac{c_{\tau(l)}^q}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}}, \\ T_{t\sqrt[q]{1-\left(1-\frac{d^q}{t^q}\right)\prod_{l=1}^{n}\left(1-\frac{d_{\tau(l)}^q}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}} \end{array}\right] \end{array}\right).$$

According to Definition 2.6, we have

$$\text{LIVq-ROFCIG}(\beta_1, \beta_2, \ldots, \beta_n) \otimes \beta = \left(\begin{array}{c} \left[\begin{array}{l} T_{t\left(\frac{a}{t}\right)\prod_{l=1}^{n}\left(\frac{a_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}, \\ T_{t\left(\frac{b}{t}\right)\prod_{l=1}^{n}\left(\frac{b_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}} \end{array}\right], \\ \left[\begin{array}{l} T_{t\sqrt[q]{1-\left(1-\frac{c^q}{t^q}\right)\prod_{l=1}^{n}\left(1-\frac{c_{\tau(l)}^q}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}}, \\ T_{t\sqrt[q]{1-\left(1-\frac{d^q}{t^q}\right)\prod_{l=1}^{n}\left(1-\frac{d_{\tau(l)}^q}{t^q}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}} \end{array}\right] \end{array}\right).$$

Therefore, we have LIVq-ROFCIG $(\beta_1 \otimes \beta, \beta_2 \otimes \beta, \ldots, \beta_n \otimes \beta) =$ LIVq-ROFCIG $(\beta_1, \beta_2, \ldots, \beta_n) \otimes \beta$. Thus, the proof is complete. □

Theorem 3.13. *Let* $\beta_l = ([T_{a_l}, T_{b_l}], [T_{c_l}, T_{d_l}])$ *for* $l = 1, 2, \ldots, n$ *be a collection of LIVq-ROFNs on* Z*, and let* Υ *be a fuzzy measure on* Z*. If* $\gamma > 0$*, then*

$$\text{LIVq-ROFCIG}\left(\beta_1^{\gamma}, \beta_2^{\gamma}, \ldots, \beta_n^{\gamma}\right) = \text{LIVq-ROFCIG}\,(\beta_1, \beta_2, \ldots, \beta_n)^{\gamma}.$$

Proof. For any $l\,(l = 1, 2, \ldots, n)$, since

$$\beta_l^{\gamma} = \left(\left[T_{t\left(\frac{a_l}{t}\right)^{\gamma}}, T_{t\left(\frac{b_l}{t}\right)^{\gamma}}\right], \left[T_{t\sqrt[q]{1-\left(1-\frac{c_l^q}{t^q}\right)^{\gamma}}}, T_{t\sqrt[q]{1-\left(1-\frac{d_l^q}{t^q}\right)^{\gamma}}}\right]\right),$$

by using Theorem 3.10, we have

$$\text{LIVq-ROFCIA}\left(\beta_1^{\gamma}, \beta_2^{\gamma}, \ldots, \beta_n^{\gamma}\right) = \left(\begin{array}{c} \left[\begin{array}{c} T_{t\prod_{l=1}^{n}\left(\frac{a_{\tau(l)}}{t}\right)^{\gamma(\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)}))}}, \\ T_{t\prod_{l=1}^{n}\left(\frac{b_{\tau(l)}}{t}\right)^{\gamma(\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)}))}} \end{array}\right], \\ \left[\begin{array}{c} T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{c_{\tau(l)}^q}{t^q}\right)^{\gamma(\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)}))}}}, \\ T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{d_{\tau(l)}^q}{t^q}\right)^{\gamma(\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)}))}}} \end{array}\right] \end{array}\right).$$

Since

$$\text{LIVq-ROFCIG}\,(\beta_1, \beta_2, \ldots, \beta_n)^{\gamma} = \left(\begin{array}{c} \left[\begin{array}{c} T_{t\left(\prod_{l=1}^{n}\left(\frac{a_{\tau(l)}}{t}\right)^{\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}\right)^{\gamma}}, \\ T_{t\left(\prod_{l=1}^{n}\left(\frac{b_{\tau(l)}}{t}\right)^{\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}\right)^{\gamma}} \end{array}\right], \\ \left[\begin{array}{c} T_{t\sqrt[q]{1-\left(\prod_{l=1}^{n}\left(1-\frac{c_{\tau(l)}^q}{t^q}\right)^{\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}\right)^{\gamma}}}, \\ T_{t\sqrt[q]{1-\left(\prod_{l=1}^{n}\left(1-\frac{d_{\tau(l)}^q}{t^q}\right)^{\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}\right)^{\gamma}}} \end{array}\right] \end{array}\right)$$

$$= \left(\begin{array}{c} \left[\begin{array}{c} T_{t\prod_{l=1}^{n}\left(\frac{a_{\tau(l)}}{t}\right)^{\gamma(\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)}))}}, \\ T_{t\prod_{l=1}^{n}\left(\frac{b_{\tau(l)}}{t}\right)^{\gamma(\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)}))}} \end{array}\right], \\ \left[\begin{array}{c} T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{c_{\tau(l)}^q}{t^q}\right)^{\gamma(\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)}))}}}, \\ T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{d_{\tau(l)}^q}{t^q}\right)^{\gamma(\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)}))}}} \end{array}\right] \end{array}\right),$$

thus, LIVq-ROFCIG $\left(\beta_1^{\gamma}, \beta_2^{\gamma}, \ldots, \beta_n^{\gamma}\right) =$ LIVq-ROFCIG $(\beta_1, \beta_2, \ldots, \beta_n)^{\gamma}$. Hence, the proof is complete. □

The following result of Theorem 3.14 is obtained using Theorem 3.10.

Theorem 3.14. *Let* $\beta_l = ([T_{a_l}, T_{b_l}], [T_{c_l}, T_{d_l}])$ *for* $l = 1, 2, \ldots, n$ *be a collection of LIVq-ROFNs on* Z*, and let* Υ *be a fuzzy measure on* Z*. Then*

1. *if* $\Upsilon(\mathcal{M}_l) = 1$ *for any* $\mathcal{M}_l \in P(Z)$*, then LIVq-ROFCIG*$(\beta_1, \beta_2, \ldots, \beta_n) = \max(\beta_1, \beta_2, \ldots, \beta_n) = \beta_{(n)}$;
2. *if* $\Upsilon(\mathcal{M}_l) = 0$ *for any* $\mathcal{M}_l \in P(Z)$*, then LIVq-ROFCIG*$(\beta_1, \beta_2, \ldots, \beta_n) = \min(\beta_1, \beta_2, \ldots, \beta_n) = \beta_{(1)}$;
3. *for any* $\mathcal{M}_1, \mathcal{M}_2 \in P(Z)$*, such that* $|\mathcal{M}_1| = |\mathcal{M}_2|$, $\Upsilon(\mathcal{M}_1) = \Upsilon(\mathcal{M}_2)$ *and* $\Upsilon((l)\ldots,(n)) = \frac{n-l+1}{n}$, $1 \leqslant l \leqslant n$*, then*

$$\text{LIVq-ROFCIG}(\beta_1, \beta_2, \ldots, \beta_n) = \left(\begin{array}{c} \left[T_{t\prod_{l=1}^{n}\left(\frac{a_{\tau(l)}}{t}\right)^{\frac{1}{n}}}, T_{t\prod_{l=1}^{n}\left(\frac{b_{\tau(l)}}{t}\right)^{\frac{1}{n}}} \right], \\ \left[T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{c^q_{\tau(l)}}{t^q}\right)^{\frac{1}{n}}}}, T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{d^q_{\tau(l)}}{t^q}\right)^{\frac{1}{n}}}} \right] \end{array} \right),$$

$$\text{LIVq-ROFCIG}(\beta_1, \beta_2, \ldots, \beta_n) = \max(\beta_1, \beta_2, \ldots, \beta_n) = \beta_{(n)}.$$

Theorem 3.15. *Let* $\beta_l = ([T_{a_l}, T_{b_l}], [T_{c_l}, T_{d_l}])$ *for* $l = 1, 2, \ldots, n$ *and* $p_l = \left(\left[T_{a'_l}, T_{b'_l}\right], \left[T_{c'_l}, T_{d'_l}\right]\right)$ *for* $l = 1, 2, \ldots, n$ *be two collections of LIVq-ROFNs on* Z*, and let* Υ *be a fuzzy measure on* Z*. Let* $\{\tau(1), \tau(2), \ldots, \tau(n)\}$ *be a permutation of* $\{1, 2, \ldots, n\}$ *such that* $\beta_{\tau(1)} \geqslant \beta_{\tau(2)} \geqslant \cdots \geqslant \beta_{\tau(n)}$*, and let* $\mathcal{M}_{\tau(k)} = \left\{z_{\tau(k)} \mid l \leqslant k\right\}$ *for* $k \geqslant 1$*, with* $\mathcal{M}_{\tau(0)} = \phi$*. If* $T_{a_{(l)}} \leqslant T_{a'_{(l)}}$; $T_{b_{(l)}} \leqslant T_{b'_{(l)}}$; $T_{c_{(l)}} \geqslant T_{c'_{(l)}}$; *and* $T_{d_{(l)}} \geqslant T_{d'_{(l)}}$ *for all* l*, i.e.,* $\beta_l \leqslant p_l$*, then*

$$\text{LIVq-ROFCIG}(\beta_1, \beta_2, \ldots, \beta_n) \leqslant \text{LIVq-ROFCIG}(p_1, p_2, \ldots, p_n).$$

Proof. Since, $\mathcal{M}_{\tau(l)} \subseteq \mathcal{M}_{\tau(l-1)}$, then, $\Upsilon\left(\mathcal{M}_{\tau(l)}\right) - \Upsilon\left(\mathcal{M}_{\tau(l-1)}\right) \geqslant 0$. For all l, $T_{a_{\tau(l)}} \leqslant T_{a'_{\tau(l)}}$; $T_{b_{\tau(l)}} \leqslant T_{b'_{\tau(l)}}$ and $T_{c_{\tau(l)}} \geqslant T_{c'_{\tau(l)}}$; $T_{d_{\tau(l)}} \geqslant T_{d'_{\tau(l)}}$, we have $a_{\tau(l)} \leqslant a'_{\tau(l)}$,

$$\prod_{l=1}^{n}\left(\frac{a_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})} \leqslant \prod_{l=1}^{n}\left(\frac{(a')_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})},$$

$$t\prod_{l=1}^{n}\left(\frac{a_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})} \leqslant t\prod_{l=1}^{n}\left(\frac{(a')_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})},$$

$$T_{t\prod_{l=1}^{n}\left(\frac{a_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}} \leqslant T_{t\prod_{l=1}^{n}\left(\frac{(a')_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}.$$

Similarly, for each l, $b_l \leqslant b'_l$, we get

$$T_{t\prod_{l=1}^{n}\left(\frac{b_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}} \leqslant T_{t\prod_{l=1}^{n}\left(\frac{(b')_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}$$

and from $c_{\tau(l)} \geqslant c'_{\tau(l)}$, we have

$$1 - \frac{c^q_{\tau(l)}}{t^q} \leqslant 1 - \frac{(c')^q_{\tau(l)}}{t^q},$$

$$\prod_{l=1}^{n}\left(1-\frac{c^{q}_{\tau(l)}}{t^{q}}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})} \leqslant \prod_{l=1}^{n}\left(1-\frac{(c')^{q}_{\tau(l)}}{t^{q}}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})},$$

$$1-\prod_{l=1}^{n}\left(1-\frac{c^{q}_{\tau(l)}}{t^{q}}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})} \geqslant 1-\prod_{l=1}^{n}\left(1-\frac{(c')^{q}_{\tau(l)}}{t^{q}}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})},$$

$$t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{c^{q}_{\tau(l)}}{t^{q}}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}} \geqslant t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{(c')^{q}_{\tau(l)}}{t^{q}}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}},$$

$$\mathsf{T}_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{c^{q}_{\tau(l)}}{t^{q}}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}} \geqslant \mathsf{T}_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{(c')^{q}_{\tau(l)}}{t^{q}}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}}.$$

Similarly, we can obtain

$$\mathsf{T}_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{d^{q}_{\tau(l)}}{t^{q}}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}} \geqslant \mathsf{T}_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{(d')^{q}_{\tau(l)}}{t^{q}}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}}}.$$

Thus, by Theorem 3.10 and Definition 2.7, we have

$$\text{LIVq-ROFCIG}\left(\beta_1,\beta_2,\ldots,\beta_n\right) \leqslant \text{LIVq-ROFCIG}\left(p_1,p_2,\ldots,p_n\right).$$

Hence, the proof is complete. □

Theorem 3.16. *Let* $\beta_l = ([\mathsf{T}_{a_l},\mathsf{T}_{b_l}],[\mathsf{T}_{c_l},\mathsf{T}_{d_l}])$ *for* $l=1,2,\ldots,n$ *represent a collection of LIVq-ROFNs on* Z*, and let* Υ *denote a fuzzy measure on* Z*.*

Let $\{\tau(1),\tau(2),\ldots,\tau(n)\}$ *be a permutation of the set* $\{1,2,\ldots,n\}$ *such that* $\beta_{\tau(1)} \geqslant \beta_{\tau(2)} \geqslant \cdots \geqslant \beta_{\tau(n)}$*, and define* $\mathcal{M}_{\tau(k)} = \{z_{\tau(k)} \mid l \leqslant k\}$ *for* $k \geqslant 1$*, with* $\mathcal{M}_{\tau(0)} = \phi$*. If* $\beta^- = ([\min_l \mathsf{T}_{a_l},\min_l \mathsf{T}_{b_l}],[\max_l \mathsf{T}_{c_l},\max_l \mathsf{T}_{d_l}])$ *and* $\beta^+ = ([\max_l \mathsf{T}_{a_l},\max_l \mathsf{T}_{b_l}],[\min_l \mathsf{T}_{c_l},\min_l \mathsf{T}_{d_l}])$*, then the following inequality holds:*

$$\beta^- \leqslant \textit{LIVq-ROFCIG}\left(\beta_1,\beta_2,\ldots,\beta_n\right) \leqslant \beta^+.$$

Proof. For any $\beta_l = ([\mathsf{T}_{a_l},\mathsf{T}_{b_l}],[\mathsf{T}_{c_l},\mathsf{T}_{d_l}])$ for $l=1,2,\ldots,n$, it is clear that β^- and β^+ are LIVq-ROFNs. Since $\mathcal{M}_{\tau(l)} \subseteq \mathcal{M}_{\tau(l-1)}$, it follows that $\Upsilon\left(\mathcal{M}_{\tau(l)}\right) - \Upsilon\left(\mathcal{M}_{\tau(l-1)}\right) \geqslant 0$. Let $\{\tau(1),\tau(2),\ldots,\tau(n)\}$ be a permutation of $\{1,2,\ldots,n\}$ such that $\beta_{\tau(1)} \geqslant \beta_{\tau(2)} \geqslant \cdots \geqslant \beta_{\tau(n)}$, then we have $\min_l \mathsf{T}_{a_{\tau(l)}} \leqslant \mathsf{T}_{a_{\tau(l)}} \leqslant \max_l \mathsf{T}_{a_{\tau(l)}}$, $\min_l \mathsf{T}_{b_{\tau(l)}} \leqslant \mathsf{T}_{b_{\tau(l)}} \leqslant \max_l \mathsf{T}_{b_{\tau(l)}}$, and $\min_j \mathsf{T}_{c_{\tau(j)}} \leqslant \mathsf{T}_{c_{\tau(j)}} \leqslant \max_j \mathsf{T}_{c_{\tau(j)}}$, $\min_j \mathsf{T}_{d_{\tau(j)}} \leqslant \mathsf{T}_{d_{\tau(j)}} \leqslant \max_j \mathsf{T}_{d_{\tau(j)}}$. Then, we have

$$\mathsf{T}_{t\prod_{l=1}^{n}\left(\frac{\min_l a_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}} \leqslant \mathsf{T}_{t\prod_{l=1}^{n}\left(\frac{a_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}} \leqslant \mathsf{T}_{t\prod_{l=1}^{n}\left(\frac{\max_l a_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}},$$

$$\mathsf{T}_{\left(\frac{\min_l a_{\tau(l)}}{t}\right)^{\sum_{l=1}^{n}\left(\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})\right)}} \leqslant \mathsf{T}_{t\prod_{l=1}^{n}\left(\frac{a_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})}} \leqslant \mathsf{T}_{\left(\frac{\max_l a_{\tau(l)}}{t}\right)^{\sum_{l=1}^{n}\left(\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})\right)}}.$$

Since, $\sum_{l=1}^{n}\left(\Upsilon\left(\mathcal{M}_{\tau(l)}\right)-\Upsilon\left(\mathcal{M}_{\tau(l-1)}\right)\right)=1$. So, we have

$$\frac{\min_l a_{\tau(l)}}{t} \leqslant t\prod_{l=1}^{n}\left(\frac{a_{\tau(l)}}{t}\right)^{\Upsilon(\mathcal{M}_{\tau(l)})-\Upsilon(\mathcal{M}_{\tau(l-1)})} \leqslant \frac{\max_l a_{\tau(l)}}{t}.$$

Similarly

$$\frac{\min\limits_{l} b_{\tau(l)}}{t} \leqslant t\prod_{l=1}^{n}\left(\frac{b_{\tau(l)}}{t}\right)^{\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})} \leqslant \frac{\max\limits_{l} b_{\tau(l)}}{t}$$

and

$$T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{\min_l c^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}}} \leqslant T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{c^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}}} \leqslant T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{\max_l c^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}}},$$

$$T_{t\sqrt[q]{1-\left(1-\frac{\min_l c^q_{\tau(l)}}{t^q}\right)^{\sum_{l=1}^{n}\left(\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})\right)}}} \leqslant T_{t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{c^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}}} \leqslant T_{t\sqrt[q]{1-\left(1-\frac{\max_l c^q_{\tau(l)}}{t^q}\right)^{\sum_{l=1}^{n}\left(\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})\right)}}}.$$

Since $\sum\limits_{l=1}^{n}\left(\Upsilon\left(\text{♍}_{\tau(l)}\right)-\Upsilon\left(\text{♍}_{\tau(l-1)}\right)\right)=1$, so we have

$$\frac{\min\limits_{j} c^q_{\tau(j)}}{t^q} \leqslant t\sqrt[q]{1-\prod_{j=1}^{n}\left(1-\frac{c^q_{\tau(j)}}{t^q}\right)^{\Upsilon(\text{♍}_{\tau(j)})-\Upsilon(\text{♍}_{\tau(j-1)})}} \leqslant \frac{\max\limits_{j} c^q_{\tau(j)}}{t^q}.$$

Similarly, we can obtain

$$\frac{\min\limits_{j} d^q_{\tau(j)}}{t^q} \leqslant t\sqrt[q]{1-\prod_{l=1}^{n}\left(1-\frac{d^q_{\tau(l)}}{t^q}\right)^{\Upsilon(\text{♍}_{\tau(l)})-\Upsilon(\text{♍}_{\tau(l-1)})}} \leqslant \frac{\max\limits_{l} d^q_{\tau(l)}}{t^q}.$$

Thus, by Theorem 3.10 and Definition 2.7, we have

$$\left(\left[\min_{l} T_{a_l}, \min_{l} T_{b_l}\right], \left[\max_{l} T_{c_l}, \max_{l} T_{d_l}\right]\right) \leqslant \text{LIVq-ROFCIG}\left(\beta_1,\beta_2,\ldots,\beta_n\right)$$
$$\leqslant \left(\left[\max_{l} T_{a_l}, \max_{l} T_{b_l}\right], \left[\min_{l} T_{c_l}, \min_{l} T_{d_l}\right]\right).$$

Hence, $\beta^- \leqslant \text{LIVq-ROFCIG}\left(\beta_1,\beta_2,\ldots,\beta_n\right) \leqslant \beta^+$. Thus, the proof of the theorem is completed. □

4. Proposed linguistic interval-valued q-Rung orthopair fuzzy GRA method

In this section, we develop a GRA method based on LIVq-ROFSs, utilizing Choquet integral operators to handle interactions among arguments.

Assume that $A=\{a_1,a_2,\ldots,a_m\}$ represents the m alternatives, and $C=\{c_1,c_2,\ldots,c_n\}$ represents the n criteria. The weight of each criterion is denoted as $w=(w_1,w_2,\ldots,w_n)^T$, where $w_l \geqslant 0$ for $l=1,2,\ldots,n$ and $\sum\limits_{l=1}^{n} w_l=1$.

We assume that the DM provides information about the weights of the criteria in the following forms [16]. For $i \neq l$, the ranking can be classified as:

1. if $\{w_i \geqslant w_l\}$, the ranking is weak;
2. if $\{w_i - w_l \geqslant \Upsilon_i(> 0)\}$, the ranking is strict;
3. if $\{w_i \geqslant \Upsilon_i w_l\}$, where $0 \leqslant \Upsilon_i \leqslant 1$, the ranking is multiple ranking.
4. if $\{\Upsilon_i \leqslant w_i \leqslant \Upsilon_i + \delta_i\}$, where $0 \leqslant \Upsilon_i \leqslant \Upsilon_i + \delta_i \leqslant 1$, the ranking is an interval ranking.

For convenience, let Δ denote the set of known information about the criteria weights provided by the experts. Let $R^k = \left[\beta_{il}^{(k)}\right]_{m\times n}$ represent the LIVq-ROF decision matrix, as provided by decision maker $d_k = (k = 1, 2, \dots, l)$. This matrix is given in the following form

$$R^k = \left[\beta_{il}^{(k)}\right]_{m\times n} = \begin{array}{c|cccc} & c_1 & c_2 & \cdots & c_n \\ \hline a_1 & \beta_{11}^{(k)} & \beta_{12}^{(k)} & \cdots & \beta_{1n}^{(k)} \\ a_2 & \beta_{21}^{(k)} & \beta_{22}^{(k)} & \cdots & \beta_{2n}^{(k)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_m & \beta_{m1}^{(k)} & \beta_{m2}^{(k)} & \cdots & \beta_{mn}^{(k)} \end{array}$$

where $\beta_{il}^{(k)} = \left(\left[T_{a_{il}^{(k)}}, T_{b_{il}^{(k)}}\right], \left[T_{c_{il}^{(k)}}, T_{d_{il}^{(k)}}\right]\right)$ is an LIVq-ROFN that represents the performance rating of the alternative $a_i \in A$ with respect to the criterion $c_l \in C$, as provided by the decision maker d_k.

To extend the GRA method in the group decision-making process, we first need to fuse all individual decision matrices into a collective matrix by using the LIVq-ROFCIA operator.

Step 1. Suppose that for every $A = \{a_1, a_2, \dots, a_m\}$, where m is the number of alternatives, each expert d_k $(k = 1, 2, \dots, r)$ is invited to express their individual evaluation or preference according to each criterion C_l $(l = 1, 2, \dots, n)$ by an LIVq-ROFN $\beta_{il}^{(k)} = \left(\left[T_{a_{il}^{(k)}}, T_{b_{il}^{(k)}}\right], \left[T_{c_{il}^{(k)}}, T_{d_{il}^{(k)}}\right]\right)$, where $i = 1, 2, \dots, m$; $l = 1, 2, \dots, n$; $k = 1, 2, \dots, r$. In this step, we construct the LIVq-ROF decision-making matrices, $D^{(k)} = \left[\beta_{il}^{(k)}\right]_{m\times n}$ for decision-making, where $k = 1, 2, \dots, r$. If the criteria have two types, such as benefit criteria and cost criteria, then the LIVq-ROF decision matrices, $D^{(k)} = \left[\beta_{il}^{(k)}\right]_{m\times n}$ can be converted into the normalized LIVq-ROF decision matrices, $R^k = \left[\beta_{il}^{(k)}\right]_{m\times n}$, where

$$R^k = \left[\beta_{il}^{(k)}\right]_{m\times n} = \begin{cases} \beta_{il}^{(k)}, & \text{for benefit criteria,} \\ \overline{\beta_{il}^{(k)}}, & \text{for cost criteria,} \end{cases}$$

where $l = 1, 2, \dots, n$, and $\overline{\beta_{il}^{(k)}}$ is the complement of $\beta_{il}^{(k)}$. The normalization is not required if all the criteria are of the same type. Then, we obtain the decision-making matrix as follows:

$$R^k = \left[\beta_{il}^{(k)}\right]_{m\times n} = \begin{array}{c|cccc} & c_1 & c_2 & \cdots & c_n \\ \hline a_1 & \beta_{11}^{(k)} & \beta_{12}^{(k)} & \cdots & \beta_{1n}^{(k)} \\ a_2 & \beta_{21}^{(k)} & \beta_{22}^{(k)} & \cdots & \beta_{2n}^{(k)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_m & \beta_{m1}^{(k)} & \beta_{m2}^{(k)} & \cdots & \beta_{mn}^{(k)} \end{array}$$

Step 2. Confirm the fuzzy density $¥_k = ¥(a_k)$ of each expert. According to Eq. (2.3), the parameter Υ_1 of each expert can be determined.

Step 3. By Definition 2.7, reorder $\beta_{il}^{(k)}$ such that $\beta_{il}^{(k)} \geqslant \beta_{il}^{(k-1)}$. Utilize the LIVq-ROFCIA operator

$$\text{LIVq-ROFCIA}\left(\beta_{il}^{(1)}, \beta_{il}^{(2)}, \dots, \beta_{mn}^{(r)}\right) = \left(\begin{array}{c} \left[\begin{array}{l} \mathsf{T}_{t\sqrt[q]{1-\prod_{k=1}^{r}\left(1-\frac{a^q_{\tau(il)}}{t^q}\right)^{\Upsilon(\mho_{\tau(k)})-\Upsilon(\mho_{\tau(k-1)})}}}, \\ \mathsf{T}_{t\sqrt[q]{1-\prod_{k=1}^{r}\left(1-\frac{b^q_{\tau(il)}}{t^q}\right)^{\Upsilon(\mho_{\tau(k)})-\Upsilon(\mho_{\tau(k-1)})}}} \end{array} \right], \\ \left[\begin{array}{l} \mathsf{T}_{t\prod_{k=1}^{r}\left(\frac{c_{\tau(il)}}{t}\right)^{\Upsilon(\mho_{\tau(k)})-\Upsilon(\mho_{\tau(k-1)})}}, \\ \mathsf{T}_{t\prod_{k=1}^{r}\left(\frac{d_{\tau(il)}}{t}\right)^{\Upsilon(\mho_{\tau(k)})-\Upsilon(\mho_{\tau(k-1)})}} \end{array} \right] \end{array} \right), \tag{4.1}$$

to aggregate all the LIVq-ROF decision matrices $R^k = \left[\beta_{il}^{(k)}\right]_{m\times n}$ for $k = 1, 2, \dots, r$ into a collective LIVq-ROF decision matrix $R = [\beta_{il}]_{m\times n}$, where $\beta_{il} = ([\mathsf{T}_{a_{il}}, \mathsf{T}_{b_{il}}], [\mathsf{T}_{c_{il}}, \mathsf{T}_{d_{il}}])$, and $\mho_{\tau(k)} = \{1, 2, \dots, r\}$, $\mho_{\tau(0)} = \emptyset$, and ¥$(c_k)$ can be calculated by Eq. (2.2).

Step 4. The LIVq-ROF positive-ideal solution (LIVq-ROFPIS), denoted by $P^+ = \{P_1^+, P_2^+, \dots, P_m^+\}$, and the LIVq-ROF negative-ideal solution (LIVq-ROFNIS), denoted by $P^- = \{P_1^-, P_2^-, \dots, P_m^-\}$, are defined as follows:

$$P^+ = \left\{ \left(\begin{array}{c} C_l, \left[\left\{ \begin{array}{ll} \left(\max\limits_i a_{il}, \max\limits_i b_{il}\right), & \text{if } C_l \text{ is benefit type} \\ \left(\min\limits_i a_{il}, \min\limits_i b_{il}\right), & \text{if } C_l \text{ is cost type} \end{array} \right. \right], \\ \left[\left\{ \begin{array}{ll} \left(\min\limits_i c_{il}, \min\limits_i d_{il}\right), & \text{if } C_l \text{ is benefit type} \\ \left(\max\limits_i c_{il}, \max\limits_i d_{il}\right), & \text{if } C_l \text{ is cost type} \end{array} \right. \right] \end{array} \right) \right\}, \tag{4.2}$$

and

$$P^- = \left\{ \left(\begin{array}{c} C_l, \left[\left\{ \begin{array}{ll} \left(\min\limits_i a_{il}, \min\limits_i b_{il}\right), & \text{if } C_l \text{ is benefit type} \\ \left(\max\limits_i a_{il}, \max\limits_i b_{il}\right), & \text{if } C_l \text{ is cost type} \end{array} \right. \right], \\ \left[\left\{ \begin{array}{ll} \left(\max\limits_i c_{il}, \max\limits_i d_{il}\right), & \text{if } C_l \text{ is benefit type} \\ \left(\min\limits_i c_{il}, \min\limits_i d_{il}\right), & \text{if } C_l \text{ is cost type} \end{array} \right. \right] \end{array} \right) \right\}, \tag{4.3}$$

where $P^+ = \left(\left[\mathsf{T}_{a_i^+}, \mathsf{T}_{b_i^+}\right], \left[\mathsf{T}_{c_i^+}, \mathsf{T}_{d_i^+}\right]\right)$ and $P^- = \left(\left[\mathsf{T}_{a_i^-}, \mathsf{T}_{a_i^-}\right], \left[\mathsf{T}_{c_i^-}, \mathsf{T}_{d_i^-}\right]\right)$ for $i = 1, 2, \dots, m$.

Step 5. Using the LIVq-ROF distance, calculate the distance between each alternative A_i and the LIVq-ROFPIS P^+ and the LIVq-ROFNIS P^-, respectively:

$$d_i\left(\beta_i, P^+\right) = \frac{1}{4(t)^q}\left[\begin{array}{l} |a_i^q - a_{il}^q| + |b_i^q - b_{il}^q| + |c_i^q - c_{il}^q| + |d_i^q - d_{il}^q| \\ +|(\pi_i^L)^q - (\pi_{il}^L)^q| + |(\pi_i^U)^q - (\pi_{il}^U)^q| \end{array} \right]. \tag{4.4}$$

Then, construct the LIVq-ROF positive-ideal separation matrix D^+ and LIVq-ROF negative-ideal separation matrix D^- as follows:

$$D^+ = [D_{il}^+]_{m\times n} = \begin{bmatrix} d(\beta_{11}, P_1^+) & d(\beta_{12}, P_2^+) & \cdots & d(\beta_{1n}, P_n^+) \\ d(\beta_{21}, P_1^+) & d(\beta_{22}, P_2^+) & \cdots & d(\beta_{2n}, P_n^+) \\ \vdots & \vdots & \vdots & \vdots \\ d(\beta_{m1}, P_1^+) & d(\beta_{m2}, P_2^+) & \cdots & d(\beta_{mn}, P_n^+) \end{bmatrix}, \tag{4.5}$$

and

$$D^- = [D^-_{il}]_{m\times n} = \begin{bmatrix} d(\beta_{11}, P^-_1) & d(\beta_{12}, P^-_2) & \cdots & d(\beta_{1n}, P^-_n) \\ d(\beta_{21}, P^-_1) & d(\beta_{22}, P^-_2) & \cdots & d(\beta_{2n}, P^-_n) \\ \vdots & \vdots & \vdots & \vdots \\ d(\beta_{m1}, P^-_1) & d(\beta_{m2}, P^-_2) & \cdots & d(\beta_{mn}, P^-_n) \end{bmatrix}. \tag{4.6}$$

Step 6. The grey relational coefficient for each alternative is calculated from the LIVq-ROFPIS and LIVq-ROFNIS using the following equations. The grey relational coefficient for each alternative from LIVq-ROFPIS is given by:

$$\xi^+_{il} = \frac{\min\limits_{1\leqslant i\leqslant m}\min\limits_{1\leqslant l\leqslant n} d(\beta_{il}, P^+_l) + \rho \max\limits_{1\leqslant i\leqslant m}\max\limits_{1\leqslant l\leqslant n} d(\beta_{il}, P^+_l)}{\min d(\beta_{il}, P^+_l) + \rho \max\limits_{1\leqslant i\leqslant m}\max\limits_{1\leqslant l\leqslant n} d(\beta_{il}, P^+_l)}. \tag{4.7}$$

Similarly, the grey relational coefficient for each alternative from LIVq-ROFNIS is given by

$$\xi^-_{il} = \frac{\min\limits_{1\leqslant i\leqslant m}\min\limits_{1\leqslant l\leqslant n} d(\beta_{il}, P^-_l) + \rho \max\limits_{1\leqslant i\leqslant m}\max\limits_{1\leqslant l\leqslant n} d(\beta_{il}, P^-_l)}{\min d(\beta_{il}, P^-_l) + \rho \max\limits_{1\leqslant i\leqslant m}\max\limits_{1\leqslant l\leqslant n} d(\beta_{il}, P^-_l)}, \tag{4.8}$$

where $i = 1, 2, \dots, m$, $l = 1, 2, \dots, n$, and the identification coefficient $\rho = 0.5$.

Step 7. To calculate the grey relational coefficient for each alternative from the LIVq-ROFPIS and LIVq-ROFNIS, use the following equations, respectively:

$$\xi^+_i = \sum_{l=1}^{n} w_l \xi^+_{il}, \qquad \xi^-_i = \sum_{l=1}^{n} w_l \xi^-_{il}. \tag{4.9}$$

The fundamental principle of the GRA method is that the preferred alternative should exhibit the highest degree of grey relation with the positive-ideal solution and the lowest degree of grey relation with the negative-ideal solution. Clearly, for a given weight vector, a smaller ξ^-_i and a larger ξ^+_i indicate a better alternative A_i. However, the information regarding the criteria weights is often incomplete. To calculate ξ^-_i and ξ^+_i, it is necessary first to determine the weight information. Therefore, the following multi-objective optimization models can be established to compute the weight information:

$$\begin{cases} \min \xi^-_i = \sum\limits_{l=1}^{n} w_l \xi^-_{il}, & \text{for } i = 1, 2, \dots, m, \\ \min \xi^+_i = \sum\limits_{l=1}^{n} w_l \xi^+_{il}, & \text{for } i = 1, 2, \dots, m. \end{cases} \tag{M1}$$

Since each alternative is non-inferior, no preference relation exists among all the alternatives. Consequently, the multiple-objective optimization models can be aggregated into a single-objective optimization model with equal weights as follows:

$$\left\{ \min \xi_i = \sum_{i=1}^{m}\sum_{l=1}^{n} \left(\xi^-_{il} - \xi^+_{il}\right) w_l. \right. \tag{M2}$$

By solving the model (M2), we obtain the optimal solution $w = (w_1, w_2, \dots, w_n)$, which can then be used as the weight vector for the criteria. Using these weight values, we can compute ξ^+_i and ξ^-_i for each $i = 1, 2, \dots, m$ using Eqs. in (4.9), respectively.

Step 8. To compute the relative closeness degree for each alternative, use the following equation:

$$\xi_i = \frac{\xi^+_i}{\xi^-_i + \xi^+_i}. \tag{4.10}$$

Step 9. Based on the ξ_i values, rank the alternatives A_i and select the best alternatives.

The pseudocode for the proposed method is presented in Algorithm 1.

Algorithm 1 Pseudocode for the proposed method.

Require: The set of alternatives A and the set of criteria C
Ensure: The most desirable neural network model for crop yield prediction
Collect LIVq-ROF decision matrices $D^{(k)} = \left[\beta_{il}^{(k)}\right]_{m\times n}$ from each decision expert $k = 1, 2, \ldots, r$; apply normalization if required
for $k = 1$ to r **do**
    Compute the fuzzy density $¥_k = ¥(a_k)$ according to Eq. (2.3);
end for
for $k = 1$ to r **do**
    Reorder the LIVq-ROFNs $\beta_{il}^{(k)}$ such that $\beta_{il}^{(k)} \geqslant \beta_{il}^{(k-1)}$ as per By Definition 2.7;
end for
for $k = 1$ to r **do**
    for $i = 1$ to m **do**
        for $l = 1$ to n **do**
            Aggregate the decision matrices R^k according to Eq. (4.1);
        end for
    end for
end for
for $i = 1$ to m **do**
    Compute the PIS P^+ and NIS P^-, according to Eqs. (4.2) and (4.3), respectively;
end for
for $i = 1$ to m **do**
    Compute the distances of alternative A_i from P^+ and P^- using Eq. (4.4);
end for
for $i = 1$ to m **do**
    for $l = 1$ to n **do**
        Compute the grey relational coefficient for each alternative A_i from P^+ and P^- using Eqs. (4.7) and (4.8), respectively;
    end for
end for
for $i = 1$ to m **do**
    for $l = 1$ to n **do**
        Solve the optimization model M_2 to obtain the optimal weight vector $w = (w_1, w_2, \ldots, w_n)$;
    end for
end for
for $i = 1$ to m **do**
    Compute the relative closeness degree ξ_i using Eq. (4.10);
end for
Based on the values of ξ_i rank the alternatives in decreasing order.

5. Case study: selecting the optimal neural network model for predicting crop yields

This section applies the proposed LIVq-ROF GRA method, based on Choquet integral operators, to select the most suitable neural network model for crop yield prediction.

5.1. Problem description

Agricultural yield prediction is one of the most pressing challenges in the modern era, especially as global populations grow and demand for food rises. The ability to accurately forecast crop yields enables better resource allocation, reduces wastage, and improves agricultural productivity. Farmers benefit significantly from such predictions as they can make informed decisions about crop selection, irrigation schedules, and the use of fertilizers. This, in turn, enhances profitability and minimizes environmental degradation.

Furthermore, policymakers rely on these insights to ensure food security by regulating markets, planning subsidies, and managing food distribution systems effectively. However, the problem becomes complex due to the diverse factors influencing crop yields, such as fluctuating weather patterns, soil quality, pest infestations, and farming practices. Integrating these multi-modal datasets into a cohesive decision-making model is a significant challenge.

To address this, a research organization aims to implement neural network-based approaches that can analyze and predict crop yields for a specific region. Neural networks are particularly well-suited to this task due to their ability to capture non-linear relationships and identify patterns in complex datasets. The organization considers the following models for evaluation.

Alternatives.

- Feedforward neural network (FNN) (O_1): This basic neural network architecture processes data in a single forward direction, from input to output. It is most effective for relatively simple datasets with linear or moderately complex patterns. While it offers simplicity and ease of implementation, its limitations include difficulty in capturing non-linear dependencies and temporal relationships, both of which are crucial for crop yield prediction.

- Convolutional neural network (CNN) (O_2): CNNs are designed for spatial data analysis, making them suitable for tasks such as analyzing satellite imagery of agricultural fields. These networks can detect patterns related to crop health, soil characteristics, and water distribution. However, CNNs struggle to incorporate sequential data, such as weather trends or seasonal variations, reducing their utility for comprehensive crop yield modeling.

- Recurrent neural network (RNN) (O_3): RNNs are specialized for handling sequential data, making them effective for analyzing historical weather conditions and crop growth cycles. They are designed to remember previous inputs, enabling them to model temporal dependencies. Despite this advantage, RNNs often face the challenge of vanishing gradients, which can hinder their performance in capturing long-term dependencies.

- Long short-term memory (LSTM) network (O_4): LSTMs, a variant of RNNs, overcome the vanishing gradient issue by introducing memory cells that maintain long-term dependencies. This makes them ideal for modeling weather cycles, seasonal variations, and other factors influencing crop yields over time. However, their high computational cost can be a drawback, particularly when dealing with large datasets or requiring real-time predictions.

- Hybrid neural network (HNN) (O_5): Hybrid models combine the strengths of multiple architectures, such as CNNs for spatial data and LSTMs for sequential data. This integration enables the simultaneous analysis of diverse data sources, including satellite imagery, weather records, and soil quality measurements. While this approach is highly effective for complex datasets, it introduces additional complexity and requires significant computational resources and expertise for implementation.

Evaluation dimensions. The models are assessed based on the following dimensions.

- Data compatibility (C_1): This criterion assesses the model's ability to process heterogeneous and multi-modal datasets, which commonly include numerical weather records, soil parameters, remote sensing images, and historical yield data. A model with high data compatibility can efficiently integrate and learn from diverse data sources, improving prediction robustness.

- Predictive accuracy (C_2): This refers to the model's effectiveness in capturing complex, nonlinear dependencies between input features and crop yield outcomes. Since agricultural processes are influenced by multifactorial and dynamic interactions, high predictive accuracy is essential for reliable forecasting.

- Computational efficiency (C_3): This criterion considers both training and inference costs, including time, memory usage, and hardware requirements. Efficient models are preferred in real-world deployments, especially in resource-limited rural settings or when real-time decision support is needed.

- Scalability (C_4): Scalability evaluates how well the model generalizes across different geographical regions with varying climatic, soil, and agricultural conditions. A scalable model remains effective when applied to new datasets from different environments without requiring extensive re-training.

- Ease of implementation (C_5): This criterion reflects the level of technical expertise and infrastructure required to deploy the model. It includes factors such as architectural complexity, software dependencies, and the need for specialized training, which can affect the practical feasibility of model adoption by agricultural agencies or field experts.

The available linguistic variables are:

$$S = \left\{ \begin{array}{c} T_0 = \text{extremelypoor}, T_1 = \text{verypoor}, T_2 = \text{poor}, T_3 = \text{slightlypoor}, T_4 = \text{fair}, \\ T_5 = \text{slightlygood}, T_6 = \text{good}, T_7 = \text{verygood}, T_8 = \text{extremlygood} \end{array} \right\}.$$

Step 1. The five alternatives O_i $(i = 1, 2, 3, 4, 5)$ are to be evaluated using the LIVq-ROF information by three decision makers d_k $(k = 1, 2, 3)$, as listed in Tables 1-3.

Table 1: The decision matrix R^1 given by d_1.

	C_1	C_2	C_3	C_4	C_5
O_1	$([T_1, T_7], [T_2, T_4])$	$([T_3, T_4], [T_2, T_6])$	$([T_1, T_6], [T_3, T_5])$	$([T_2, T_5], [T_3, T_4])$	$([T_3, T_5], [T_4, T_6])$
O_2	$([T_3, T_6], [T_1, T_4])$	$([T_2, T_5], [T_3, T_5])$	$([T_4, T_7], [T_1, T_4])$	$([T_4, T_5], [T_2, T_5])$	$([T_3, T_6], [T_4, T_5])$
O_3	$([T_1, T_5], [T_5, T_6])$	$([T_3, T_4], [T_1, T_7])$	$([T_3, T_5], [T_2, T_4])$	$([T_3, T_6], [T_1, T_5])$	$([T_3, T_4], [T_3, T_6])$
O_4	$([T_1, T_7], [T_4, T_5])$	$([T_1, T_6], [T_3, T_5])$	$([T_4, T_5], [T_2, T_5])$	$([T_1, T_7], [T_3, T_5])$	$([T_1, T_5], [T_4, T_5])$
O_5	$([T_4, T_5], [T_2, T_4])$	$([T_3, T_5], [T_2, T_7])$	$([T_3, T_6], [T_1, T_6])$	$([T_2, T_5], [T_3, T_6])$	$([T_4, T_6], [T_2, T_6])$

Table 2: The decision matrix R^2 given by d_2.

	C_1	C_2	C_3	C_4	C_5
O_1	$([T_3, T_6], [T_4, T_5])$	$([T_2, T_7], [T_3, T_5])$	$([T_2, T_6], [T_3, T_6])$	$([T_1, T_6], [T_4, T_6])$	$([T_3, T_7], [T_3, T_5])$
O_2	$([T_1, T_7], [T_2, T_4])$	$([T_3, T_6], [T_4, T_6])$	$([T_4, T_5], [T_2, T_4])$	$([T_2, T_5], [T_2, T_4])$	$([T_2, T_6], [T_2, T_5])$
O_3	$([T_2, T_5], [T_4, T_6])$	$([T_2, T_6], [T_1, T_4])$	$([T_4, T_6], [T_3, T_4])$	$([T_3, T_5], [T_3, T_6])$	$([T_3, T_7], [T_2, T_4])$
O_4	$([T_3, T_7], [T_3, T_4])$	$([T_3, T_6], [T_4, T_6])$	$([T_3, T_4], [T_1, T_3])$	$([T_1, T_7], [T_4, T_5])$	$([T_2, T_5], [T_5, T_6])$
O_5	$([T_4, T_6], [T_2, T_3])$	$([T_4, T_5], [T_2, T_3])$	$([T_1, T_6], [T_2, T_5])$	$([T_3, T_6], [T_2, T_3])$	$([T_4, T_6], [T_3, T_4])$

Table 3: The decision matrix R^3 given by d_3.

	C_1	C_2	C_3	C_4	C_5
O_1	$([T_2, T_5], [T_3, T_4])$	$([T_3, T_7], [T_4, T_5])$	$([T_2, T_7], [T_3, T_4])$	$([T_3, T_5], [T_2, T_4])$	$([T_4, T_5], [T_3, T_4])$
O_2	$([T_1, T_7], [T_2, T_3])$	$([T_2, T_5], [T_2, T_7])$	$([T_4, T_6], [T_2, T_4])$	$([T_2, T_6], [T_2, T_3])$	$([T_2, T_7], [T_2, T_3])$
O_3	$([T_1, T_5], [T_4, T_5])$	$([T_4, T_6], [T_1, T_6])$	$([T_2, T_6], [T_4, T_5])$	$([T_1, T_4], [T_2, T_4])$	$([T_3, T_6], [T_2, T_4])$
O_4	$([T_2, T_6], [T_3, T_5])$	$([T_1, T_6], [T_4, T_5])$	$([T_3, T_5], [T_1, T_3])$	$([T_2, T_7], [T_4, T_5])$	$([T_3, T_5], [T_4, T_6])$
O_5	$([T_4, T_5], [T_2, T_4])$	$([T_3, T_6], [T_2, T_6])$	$([T_3, T_7], [T_2, T_3])$	$([T_2, T_7], [T_1, T_3])$	$([T_1, T_6], [T_2, T_5])$

Assume that the criteria weight information provided by experts is partially known as

$$\Delta = \left\{ \begin{array}{c} 0.2 \leqslant w_1 \leqslant 0.35, \\ 0.15 \leqslant w_2 \leqslant 0.25, \\ 0.25 \leqslant w_3 \leqslant 0.35, \\ 0.3 \leqslant w_4 \leqslant 0.4, \\ 0.05 \leqslant w_5 \leqslant 0.15 \end{array} \right\}, \; w_l \geqslant 0, \; l = 1,2,3,4,5, \; \sum_{l=1}^{5} w_l = 1.$$

Next, we apply the developed approach to identify the most preferred alternative(s).

Step 2. First, we determine the fuzzy density for each decision maker and its corresponding Υ parameter. Assume that ¥$(d_1) = 0.40$, ¥$(d_2) = 0.50$, and ¥$(d_3) = 0.60$. From this, the Υ value for the expert can be calculated as $\Upsilon = -0.77$. Using Eq. (2.2), we obtain the following values: ¥$(d_1, d_2) = 0.74$, ¥$(d_1, d_3) = 0.81$, ¥$(d_2, d_3) = 0.86$, and ¥$(d_1, d_2, d_3) = 1$.

Step 3. According to Definition 2.7, $\beta_{il}^{(k)}$ is reordered such that $\beta_{il}^{(k)} \geqslant \beta_{il}^{(k-1)}$. Then, utilize the LIVq-ROFCIA operator

$$\text{LIVq-ROFCIA}\left(\beta_{il}^{(1)}, \beta_{il}^{(2)}, \ldots, \beta_{mn}^{(r)}\right) = \left(\begin{array}{c} \left[\begin{array}{c} \mathsf{T}_{t \sqrt[q]{1 - \prod_{k=1}^{r} \left(1 - \frac{a^q_{\tau(il)}}{t^q}\right)^{\Upsilon(\mathfrak{M}_{\tau(k)}) - \Upsilon(\mathfrak{M}_{\tau(k-1)})}}}, \\ \mathsf{T}_{t \sqrt[q]{1 - \prod_{k=1}^{r} \left(1 - \frac{b^q_{\tau(il)}}{t^q}\right)^{\Upsilon(\mathfrak{M}_{\tau(k)}) - \Upsilon(\mathfrak{M}_{\tau(k-1)})}}} \end{array} \right], \\ \left[\begin{array}{c} \mathsf{T}_{t \prod_{k=1}^{r} \left(\frac{c_{\tau(il)}}{t}\right)^{\Upsilon(\mathfrak{M}_{\tau(k)}) - \Upsilon(\mathfrak{M}_{\tau(k-1)})}}, \\ \mathsf{T}_{t \prod_{k=1}^{r} \left(\frac{d_{\tau(il)}}{t}\right)^{\Upsilon(\mathfrak{M}_{\tau(k)}) - \Upsilon(\mathfrak{M}_{\tau(k-1)})}} \end{array} \right] \end{array} \right),$$

to aggregate all the LIVq-ROF decision matrices $R^k = \left[\beta_{il}^{(k)}\right]_{m \times n}$ into a collective LIVq-ROF decision matrix $R = [\beta_{il}]_{m \times n}$, the process is outlined in Table 4 (assuming $q = 3$).

Table 4: The aggregated decision matrix R by LIVq-ROFCIA opertor.

	C_1	C_2	C_3	C_4	C_5
O_1	$\left([\mathsf{T}_{2.2760}, \mathsf{T}_{7.0000}], [\mathsf{T}_{2.5508}, \mathsf{T}_{4.3153}]\right)$	$\left([\mathsf{T}_{2.1722}, \mathsf{T}_{6.6983}], [\mathsf{T}_{3.2330}, \mathsf{T}_{5.0000}]\right)$	$\left([\mathsf{T}_{3.0914}, \mathsf{T}_{5.4142}], [\mathsf{T}_{3.3083}, \mathsf{T}_{4.3153}]\right)$	$\left([\mathsf{T}_{2.7423}, \mathsf{T}_{5.4142}], [\mathsf{T}_{3.0291}, \mathsf{T}_{5.3997}]\right)$	$\left([\mathsf{T}_{2.3725}, \mathsf{T}_{5.1603}], [\mathsf{T}_{2.8130}, \mathsf{T}_{5.3997}]\right)$
O_2	$\left([\mathsf{T}_{3.0700}, \mathsf{T}_{7.0000}], [\mathsf{T}_{1.5157}, \mathsf{T}_{3.7117}]\right)$	$\left([\mathsf{T}_{3.2223}, \mathsf{T}_{6.1422}], [\mathsf{T}_{1.5157}, \mathsf{T}_{3.7117}]\right)$	$\left([\mathsf{T}_{3.5751}, \mathsf{T}_{5.6722}], [\mathsf{T}_{2.0000}, \mathsf{T}_{4.7182}]\right)$	$\left([\mathsf{T}_{2.5033}, \mathsf{T}_{5.7262}], [\mathsf{T}_{2.6390}, \mathsf{T}_{4.0583}]\right)$	$\left([\mathsf{T}_{2.4409}, \mathsf{T}_{5.4142}], [\mathsf{T}_{2.9773}, \mathsf{T}_{5.8061}]\right)$
O_3	$\left([\mathsf{T}_{3.0000}, \mathsf{T}_{6.4421}], [\mathsf{T}_{1.5157}, \mathsf{T}_{4.3734}]\right)$	$\left([\mathsf{T}_{3.6730}, \mathsf{T}_{5.6722}], [\mathsf{T}_{1.9170}, \mathsf{T}_{4.8597}]\right)$	$\left([\mathsf{T}_{2.5033}, \mathsf{T}_{5.4584}], [\mathsf{T}_{2.2253}, \mathsf{T}_{4.9853}]\right)$	$\left([\mathsf{T}_{2.1539}, \mathsf{T}_{4.7913}], [\mathsf{T}_{3.3119}, \mathsf{T}_{5.3997}]\right)$	$\left([\mathsf{T}_{2.4047}, \mathsf{T}_{4.6672}], [\mathsf{T}_{2.2974}, \mathsf{T}_{6.0862}]\right)$
O_4	$\left([\mathsf{T}_{2.2760}, \mathsf{T}_{7.0000}], [\mathsf{T}_{3.2330}, \mathsf{T}_{4.6347}]\right)$	$\left([\mathsf{T}_{1.9909}, \mathsf{T}_{6.7120}], [\mathsf{T}_{2.7895}, \mathsf{T}_{4.3781}]\right)$	$\left([\mathsf{T}_{2.2760}, \mathsf{T}_{5.5528}], [\mathsf{T}_{2.0649}, \mathsf{T}_{4.2028}]\right)$	$\left([\mathsf{T}_{3.2937}, \mathsf{T}_{5.4584}], [\mathsf{T}_{4.3734}, \mathsf{T}_{2.7945}]\right)$	$\left([\mathsf{T}_{2.1722}, \mathsf{T}_{5.0000}], [\mathsf{T}_{4.3153}, \mathsf{T}_{5.5780}]\right)$
O_5	$\left([\mathsf{T}_{3.7951}, \mathsf{T}_{6.1018}], [\mathsf{T}_{2.0000}, \mathsf{T}_{3.3659}]\right)$	$\left([\mathsf{T}_{3.3452}, \mathsf{T}_{6.3507}], [\mathsf{T}_{1.6702}, \mathsf{T}_{3.9585}]\right)$	$\left([\mathsf{T}_{3.6730}, \mathsf{T}_{5.7953}], [\mathsf{T}_{1.7398}, \mathsf{T}_{4.7043}]\right)$	$\left([\mathsf{T}_{2.9642}, \mathsf{T}_{5.3257}], [\mathsf{T}_{2.3522}, \mathsf{T}_{4.5208}]\right)$	$\left([\mathsf{T}_{2.6358}, \mathsf{T}_{5.6722}], [\mathsf{T}_{2.0000}, \mathsf{T}_{5.9980}]\right)$

Step 4. By utilizing Eqs. (4.2) and (4.3), the positive-ideal and negative-ideal solutions are respectively given as:

$$P^{+} = \left\{ \begin{array}{c} \left([\mathsf{T}_{3.7951}, \mathsf{T}_{7.0000}], [\mathsf{T}_{1.5157}, \mathsf{T}_{3.3659}]\right), \\ \left([\mathsf{T}_{3.6730}, \mathsf{T}_{6.7120}], [\mathsf{T}_{1.5157}, \mathsf{T}_{3.7117}]\right), \\ \left([\mathsf{T}_{3.6730}, \mathsf{T}_{5.7953}], [\mathsf{T}_{1.9398}, \mathsf{T}_{4.2028}]\right), \\ \left([\mathsf{T}_{3.2937}, \mathsf{T}_{5.7262}], [\mathsf{T}_{2.3522}, \mathsf{T}_{3.7945}]\right), \\ \left([\mathsf{T}_{2.6358}, \mathsf{T}_{5.6722}], [\mathsf{T}_{2.0000}, \mathsf{T}_{5.3997}]\right) \end{array} \right\}, \quad P^{-} = \left\{ \begin{array}{c} \left([\mathsf{T}_{2.2760}, \mathsf{T}_{6.1018}], [\mathsf{T}_{3.2330}, \mathsf{T}_{4.6347}]\right), \\ \left([\mathsf{T}_{1.9909}, \mathsf{T}_{5.6722}], [\mathsf{T}_{3.2330}, \mathsf{T}_{5.0000}]\right), \\ \left([\mathsf{T}_{2.2760}, \mathsf{T}_{5.4142}], [\mathsf{T}_{3.3083}, \mathsf{T}_{4.9853}]\right), \\ \left([\mathsf{T}_{2.1539}, \mathsf{T}_{4.7913}], [\mathsf{T}_{4.3734}, \mathsf{T}_{5.3997}]\right), \\ \left([\mathsf{T}_{2.1722}, \mathsf{T}_{4.6672}], [\mathsf{T}_{4.3113}, \mathsf{T}_{6.0862}]\right) \end{array} \right\}.$$

Step 5. By utilizing Eqs. (4.5) and (4.6), the positive-ideal and negative-ideal separation matrices are respectively obtained as shown in Tables 5 and 6.

Table 5: Positive-ideal separation matrix.

$D^+ =$

	C_1	C_2	C_3	C_4	C_5
O_1	0.0831	0.1105	0.0653	0.1472	0.0580
O_2	0.0378	0.0847	0.0338	0.0635	0.0554
O_3	0.1009	0.1205	0.0816	0.1576	0.0832
O_4	0.1018	0.0727	0.0598	0.0935	0.1268
O_5	0.1175	0.0570	0.0292	0.0784	0.0570

Table 6: Negative-ideal separation matrix.

$D^- =$

	C_1	C_2	C_3	C_4	C_5
O_1	0.1299	0.1176	0.0599	0.1021	0.1231
O_2	0.1427	0.1017	0.0563	0.1522	0.1084
O_3	0.0688	0.0507	0.0284	0.0462	0.0666
O_4	0.1131	0.1289	0.0753	0.1576	0.0507
O_5	0.1018	0.1008	0.0720	0.1325	0.1496

Step 6. By utilizing Eqs. (4.7) and (4.8), the grey relational coefficient matrices for each alternative, derived from LIVq-ROFPIS and LIVq-ROFNIS, are obtained as shown in Tables 7 and 8.

Table 7:

$[\xi_{i1}^{+}] =$

0.6669	0.5703	0.7493	0.4777	0.7895
0.9258	0.6603	0.9586	0.7586	0.8048
0.6009	0.5416	0.6732	0.4568	0.6663
0.5977	0.7127	0.7793	0.6265	0.5251
0.5499	0.7951	1.0000	0.6869	0.7951

Table 8:

$[\xi_{i1}^{-}] =$

0.5137	0.5460	0.7732	0.5926	0.5309
0.4840	0.5939	0.7934	0.4641	0.5727
0.7262	0.8281	1.0000	0.8577	0.7372
0.5587	0.5163	0.6959	0.4536	0.8282
0.5935	0.5971	0.7112	0.5074	0.4694

Step 7. Apply model (M2) to formulate the following single-objective programming model:

$$\min \xi(w) = -0.4649w_1 - 0.1986w_2 - 0.1865w_3 - 0.1312w_4 - 0.4424w_5.$$

By solving this model, we obtain the weight vector for the attributes as $w = (0.20, 0.15, 0.25, 0.30, 0.10)$. Afterwards, we can determine the degree of the grey relational coefficient for each alternative based on LIVq-ROFPIS and LIVq-ROFNIS as

$$\xi_1^+ = 0.6285,\ \xi_2^+ = 0.8319,\ \xi_3^+ = 0.5734,\ \xi_4^+ = 0.6617,\ \xi_5^+ = 0.7648,$$
$$\xi_1^- = 0.6088,\ \xi_2^- = 0.5807,\ \xi_3^- = 0.8505,\ \xi_4^- = 0.5821,\ \xi_5^- = 0.5852.$$

Step 8. Based on Eq. (4.10), the relative closeness degree of each alternative to LIVq-ROFPIS and LIVq-ROFNIS is computed as

$$\xi_1 = 0.5080,\ \xi_2 = 0.5889,\ \xi_3 = 0.4027,\ \xi_4 = 0.5320,\ \xi_5 = 0.5665.$$

Step 9. Using the relative closeness degree, the alternatives are ranked as follows: $O_2 > O_5 > O_4 > O_1 > O_3$. Thus, the most preferable alternative is O_2.

5.2. Sensitive analysis

To evaluate the stability and robustness of the framed approach, a sensitivity analysis is conducted with respect to the parameter q, which controls the flexibility of the LIVq-ROF environment.

Table 9: Results achieved by various values of q.

q	O_1	O_2	O_3	O_4	O_5	Ranking
$q = 4$	0.5837	0.6626	0.4290	0.6043	0.5925	$O_2 > O_4 > O_5 > O_1 > O_3$
$q = 5$	0.5610	0.6197	0.3974	0.5870	0.5351	$O_2 > O_4 > O_1 > O_5 > O_3$
$q = 7$	0.5654	0.5832	0.4041	0.5867	0.5119	$O_4 > O_2 > O_1 > O_5 > O_3$
$q = 8$	0.5623	0.5683	0.4071	0.5844	0.5034	$O_4 > O_2 > O_1 > O_5 > O_3$
$q = 9$	0.5606	0.5594	0.4112	0.5827	0.4977	$O_4 > O_1 > O_2 > O_5 > O_3$
$q = 11$	0.5589	0.5486	0.4184	0.5812	0.4897	$O_4 > O_1 > O_2 > O_5 > O_3$
$q = 13$	0.5580	0.5429	0.4238	0.5800	0.4834	$O_4 > O_1 > O_2 > O_5 > O_3$
$q = 16$	0.5572	0.5394	0.4304	0.5769	0.4753	$O_4 > O_1 > O_2 > O_5 > O_3$
$q = 18$	0.5566	0.5379	0.4337	0.5748	0.4708	$O_4 > O_1 > O_2 > O_5 > O_3$
$q = 20$	0.5561	0.5367	0.4362	0.5730	0.4671	$O_4 > O_1 > O_2 > O_5 > O_3$

The proposed approach allows experts to expand their decision-making assessment space based on the parameter q. The parameter q is very important for the established technique and has a significant impact on results. Therefore, in the following, we will further investigate the influence of parameters q on the ranking order of the decision results of the above example. We take the different values of q and re-calculated the score values by re-applying the steps on the proposed approach based on each change in the value of q. From Table 9, it can be observed that when $q = 4, 5$ the best alternative is O_2. When the parameter q is between 7 and 8, the ranking order is $O_4 > O_2 > O_1 > O_5 > O_3$. However, a different ranking order is obtained when the value of q is greater than 8, i.e., $O_4 > O_1 > O_2 > O_5 > O_3$. When the value of q between $q = 7$ to $q = 20$, the ranking order can be observed, and the best and worst alternative obtained are still the same.

The observed stabilization in the results for $q \geqslant 7$ can be theoretically explained by the structural behavior of the LIVq-ROFSs. In this model, the constraint $0 \leqslant \theta^q + \sigma^q \leqslant t^q$ governs the admissible region for membership and non-membership values. When q is small, this constraint is relatively tight, which limits the simultaneous expression of higher degrees of membership and non-membership, thereby restricting the representation of hesitation and ambiguity. As q increases, the admissible space expands, offering more room for flexible modeling of uncertain or hesitant evaluations. This results in a more expressive decision environment where experts can assign moderate to high values to both membership and non-membership degrees without violating the core constraint. However, once q crosses a certain threshold-approximately $q = 7$ in this case-the flexibility becomes sufficient to accommodate all meaningful variations in the input data, and further increases in q no longer significantly affect the score values or ranking outcomes. Thus, the stabilization observed for $q \geqslant 7$ is a direct consequence of the LIVq-ROFS's enhanced capacity to capture a broader uncertainty spectrum, beyond which the decision structure becomes robust to changes in q. This justifies the selection of $q \geqslant 7$ as a reasonable choice in practical decision-making scenarios, balancing expressiveness and stability.

6. Comparative investigation

To highlight the advantages of the proposed GRA method over existing approaches, a comparative investigation is carried out. Additionally, a correlation coefficient analysis is conducted to examine the consistency and validity of the rankings obtained through different methods.

The validity of the proposed method is assessed by comparing its performance with existing MAGDM approaches, including TOPSIS [15], VIKOR [11], the entropy method [10], and TODIM [45]. The results obtained from these methods are presented in Table 10, corresponding to the MAGDM problem discussed in Section 6.

Table 10: Comparison with the existing methods.

	O_1	O_2	O_3	O_4	O_5	Ranking
(R_1)− TOPSIS [15]	0.4612	0.3611	0.6602	0.5243	0.4189	$O_3 > O_4 > O_1 > O_5 > O_2$
(R_2)− VIKOR [11]	0.2564	1.0000	0.3385	0.4448	0.9007	$O_2 > O_5 > O_4 > O_3 > O_1$
(R_3)− Entropy [10]	0.9974	0.9607	0.9813	0.9867	0.9710	$O_2 > O_5 > O_3 > O_4 > O_1$
(R_4)− TODIM [45]	0.5682	1.0000	0.0000	0.5343	0.7444	$O_2 > O_5 > O_1 > O_4 > O_3$
(R_5)− Proposed GRA	0.5080	0.5889	0.4027	0.5320	0.5665	$O_2 > O_5 > O_4 > O_1 > O_3$

1. From Table 10, the proposed approach ranks O_2 as the worst alternative, whereas other existing approaches rank it as the best one. The main reason for this discrepancy is the difference in distance matrix computations. The proposed method uses a PIS-based and NIS-based distance matrix to develop grey relational matrices, followed by the calculation of grey relational coefficients from PIS and NIS, respectively. The ranking is then derived using the relative relational degree of each alternative from PIS and NIS. In contrast, TOPSIS determines the PIS and NIS using a straightforward approach, where the closeness coefficients are derived without considering additional grey relational transformations.

2. The ranking results obtained through VIKOR [11] are $O_2 > O_5 > O_4 > O_3 > O_1$, which is almost similar to the ranking obtained via the proposed approach, with a slight alteration in the positions of O_3 and O_1. VIKOR allows the inclusion of a decision-maker's preference, balancing maximum group utility and minimum individual regret, which is a positive aspect. However, the proposed approach has a distinct advantage in that it utilizes Choquet-based operators, which better account for interactions among attributes.

3. Table 10 and Fig. 1 indicate that although the best alternative (O_2) is commonly identified by TODIM [45], Entropy [10], and the proposed GRA method, a deeper analysis reveals clear methodological advantages of the proposed approach over TODIM. TODIM models behavioral aspects such as loss aversion, which is valuable for reflecting psychological preferences, but it lacks mechanisms to capture interactions among attributes and typically assumes fixed or subjectively assigned weights. In contrast, the proposed GRA method incorporates Choquet integral-based aggregation, which explicitly considers the mutual influence among criteria-making it structurally more aligned with real-world decision environments.

Furthermore, the proposed method uses an embedded optimization model to determine weights, increasing adaptability in scenarios with partially known or uncertain weight information. It also employs grey relational matrices built from both PIS and NIS distances, offering a dual perspective on proximity that enhances robustness. Therefore, the proposed method outperforms TODIM in terms of aggregation flexibility, adaptability, and its capacity to handle attribute dependencies-critical features for complex MAGDM problems.

In light of the above analysis the main merits of the framed approach are as follows.

- Unlike existing methods [10, 11, 15, 45], the proposed approach is based on Choquet integral operators, which are more suitable for real-world MAGDM problems, where attributes are not independent but instead exhibit correlations influencing decision outcomes.

- The proposed method effectively handles cases where attribute weights are partially known by determining them through an embedded optimization model, enhancing adaptability.

- The GRA methodology selects the optimal alternative by identifying the option with the highest grey relational degree to the PIS and the lowest grey relational degree to the NIS, ensuring a robust decision framework.

Thus, the proposed approach proves to be a more comprehensive and effective alternative to existing MAGDM methods by incorporating Choquet integral-based aggregation, PIS-NIS relational degrees, and grey relational matrices, leading to improved decision-making accuracy.

To further highlight the methodological gap and justify the motivation, Table 11 summarizes the limitations of the existing LIVq-ROFS-based MAGDM methods compared with the proposed approach.

Table 11: Limitation analysis of existing LIVq-ROFS-based MAGDM methods.

Method	Attribute interaction handling	Weight determination	Behavioral modeling (Risk/Loss)	Aggregation flexibility
TOPSIS [15]	× (Independent criteria)	Fixed or subjectively assigned	×	Simple vector-based
VIKOR [11]	× (Limited interaction)	Includes DM's preference partially	×	Compromise-based
Entropy [10]	×	Fully objective only	×	Statistical entropy only
TODIM [45]	×	Subjective or fixed	✓ (loss aversion modeled)	Nonlinear, psychology-inspired
Proposed GRA (Choquet-based)	✓ (via Choquet integral)	✓ (via optimization model)	× (future integration suggested)	✓ (flexible, relational)

Despite the advantages of the proposed method, several limitations should be acknowledged.

- Although the proposed model accommodates cases with unknown attribute weights using subjective expert evaluations, it relies solely on linguistic input. Integrating objective weighting mechanisms could enhance the robustness and balance of the final decision outcomes.
- The current approach does not explicitly consider the psychological behavior of DMs, such as risk aversion or loss sensitivity, which can significantly influence preferences in real-world scenarios. Future research may benefit from incorporating behavioral decision-making theories into the mathematical framework.
- The case study presented involves a relatively moderate-sized dataset. To fully assess the scalability and generalizability of the proposed model, it should be tested on larger, more complex datasets across diverse application domains.

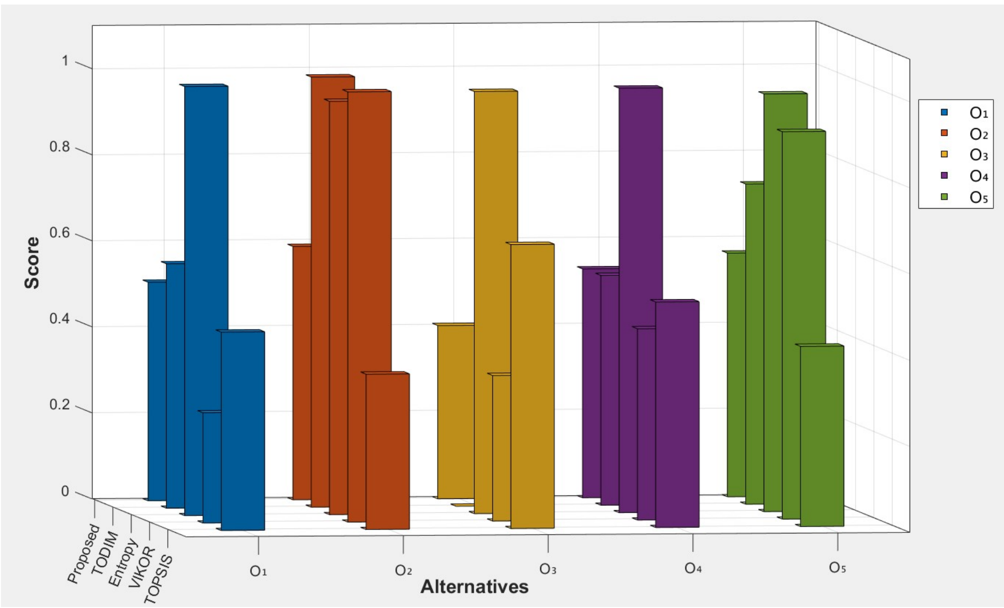

Figure 1: Graphical illustration of the comparative analysis.

To further evaluate how closely the rankings obtained from existing methods align with those of the proposed approach, the Spearman's rank correlation coefficient test is applied to the results presented in Table 10. The outcomes of this statistical analysis are illustrated in Fig. 2.

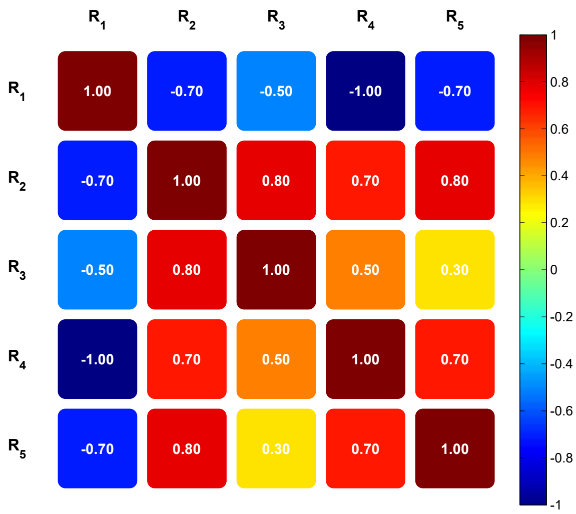

Figure 2: Spearman correlation plot.

7. Conclusion

In this study, we proposed an advanced decision-making framework based on LIVq-ROFS, incorporating newly developed AOs utilizing the Choquet integral. A series of LIVq-ROF Choquet integral averaging and LIVq-ROF Choquet integral geometric operators were introduced to enhance decision-making processes. The mathematical properties and theoretical foundations of these operators were rigorously analyzed. Building on these developments, the GRA method was extended to the LIVq-ROF setting, along with an optimization model that aids in determining criteria weight values. Our proposed approach effectively captures uncertainty, hesitation, and interdependencies among decision criteria, making it well-suited for complex MAGDM scenarios, where criteria weights are partially known. A case study on selecting the optimal neural network model for crop yield prediction validated the practical applicability of our methodology. Sensitivity analysis highlighted the significant influence of parameter q on decision outcomes, emphasizing the importance of selecting an appropriate value based on stability considerations. Furthermore, a thorough comparative analysis with existing methods demonstrated that our model offers more robust and flexible decision support by accommodating a broader range of linguistic assessments and improving the accuracy of preference aggregation. Overall, the proposed method demonstrates strong potential in handling complex decision environments characterized by uncertainty and interdependence. In addition, exploring the integration of advanced numerical approaches, as demonstrated in recent works [31–35], may broaden the applicability of the proposed framework in more dynamic or computationally intensive decision-making environments.

Data availability

All the data generated or analyzed during this study are included in this published article.

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