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Subfamilies of bi-univalent functions governed by Bernoulli polynomials

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Abstract

In the context of univalent function theory, special functions play an important role and have been studied by a number of researchers earlier. This article presents and examines two subfamilies of bi-univalent functions that are governed by Bernoulli polynomials in the open unit disk. We obtain limits on initial coefficients for functions in the specified subfamilies. The Fekete-Szegö problem is also addressed for the elements of the subfamilies that have been defined. We also present some new results and discuss pertinent links to earlier findings.

Keywords: Regular functions, subordination, bi-univalent functions, Bernoulli polynomials.

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1. Preliminaries

One important area of mathematics in complex analysis is Geometric Function Theory (GFT). Researchers' attention has been successfully drawn to this subbranch in recent years. Let $\mathfrak{U} = \{\sigma \in \mathbb{C} : |\sigma| < 1\}$, where $\mathbb{C}$ is the complex plane. The class of regular functions ϕ in $\mathfrak{U}$ is identified by $\mathcal{A}$ and the elements of $\mathcal{A}$ are of the form

$$\phi(\sigma) = \sigma + d_2\sigma^2 + d_3\sigma^3 + \cdots = \sigma + \sum_{j=2}^{\infty} d_j\sigma^j, \quad \sigma \in \mathfrak{U}, \tag{1.1}$$

and let $\mathcal{S} = \{\phi \in \mathcal{A} : \phi \text{ be univalent in } \mathfrak{U}\}$. In [2], Biebereach conjectured that $|d_j| \leqslant j, j \geqslant 2$ for every function $\phi \in \mathcal{S}$. Numerous new subclasses of $\mathcal{S}$ were defined to settle the Biebereach conjecture, and a number of results were established. Reserchers worked on this conjecture's proof for many years, and finally, Luis De Branges solved this conjecture for every $j \geqslant 2$ in [8]. Another problem in GFT is Fekete-Szegö functional $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, for every function $\phi \in \mathcal{S}$ (see [10]). Numerous papers were published on above mentioned problem for functions belonging to subclasses of $\mathcal{S}$ (for example, see [9, 15, 16, 25, 34]).

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One of the remarkable subclasses of $\mathcal{S}$ is the bi-univalent function class σ. Levin introduced the concept of σ of bi-univalent functions in his work [18]. Let ϕ represent these analytic functions, where ϕ and $\phi^{-1} = \psi$ are univalent in $\mathfrak{U}$. The Koebe theorem (see [9]) states that each function $\phi \in \mathcal{S}$ of the form (1.1) has an inverse given by

$$\phi^{-1}(w) = w - d_2w^2 + (2d_2^2 - d_3)w^3 - (5d_2^3 - 5d_2d_3 + d_4)w^4 + \cdots = \psi(w) \tag{1.2}$$

satisfying $\sigma = \psi(\phi(\sigma))$ and $w = \phi(\psi(w))$, $|w| < r_0(\phi)$, $1/4 \leqslant r_0(\phi)$, $\sigma, w \in \mathfrak{U}$. The class σ is not an empty set since the functions $\frac{1}{2}\log\left(\frac{1+\sigma}{1-\sigma}\right)$, $-\log(1-\sigma)$, and $\frac{\sigma}{1-\sigma}$ are members in the σ family. However, $\sigma - \frac{\sigma^2}{2}$, $\frac{\sigma}{1-\sigma^2}$, and the Koebe function $\frac{\sigma}{(1-\sigma)^2}$ are not elements of σ, even though they are in $\mathcal{S}$. For a succinct examination and to learn about some of the traits of the σ family, see [3, 4, 21, 31]. The current explosion in research on the bi-univalent function family was sparked by the article by Srivastava and his coauthors [27]. Several fascinating special σ families have since been studied by a number of researchers; see [1, 7, 11, 12, 32] as well as the citation provided in these articles.

In many fields, including number theory, numerical analysis, combinatorics, computer science, physics, and engineering, special polynomials like Lucas, Chebyshev, Horadam, Bernoulli, Gegenbauer, Lucas-Lehmer, Pell-Lucas, Fibonacci, and their generalizations are crucial. Numerous studies on functions that are governed by known special polynomials and belong to a particular σ subfamily have been conducted in recent years. For elements of σ subfamilies that are subordinate to special polynomials, a number of researchers have discovered coefficient estmates and the Fekete-Szegö functional $|d_3 - \xi d_2^2|$, $\xi \in \mathbb{R}$ (See [13, 14, 22, 26, 28, 30, 33]). Researchers have recently focused attention on a specific type of polynomials called Bernoulli polynomials. Named for Jacob Bernoulli [23], the Bernoulii polynomials have been applied in new ways to numerically solve Fredholm fractional integro-differential equations [19] and fractional-order differential equations. Research papers [5, 6, 17, 24, 29] have some interesting results regarding coefficient estimates and the Fekete-Szegö functional for members of specific subfamilies of σ associated with Bernoulli polynomials.

Bernoulli polynomials play a significant role in GFT due to their analytical properties and connections to various areas of mathematics. Here's an overview of their applications and importance in GFT. Estimating the coefficients of analytic functions is crucial, in GFT. Bernoulli polynomials assist in deriving bounds and relationships for these coefficients, especially in classes of functions like starlike or convex functions. Bernoulli polynomials appear in the series expansions of functions related to conformal mappings. Their presence facilitates the study of how functions map domains in the complex plane, preserving angles and shapes locally. These polynomials are linked to special functions such as the Riemann zeta function and the Hurwitz zeta function. Such connections are valuable in GFT for analyzing complex functions and their properties.

The Bernoulli polynomials $B_j(x), x \in \mathbb{R}$, and j is non-negative integer, are frequently specified (see [20]) using the generating function:

$$\mathfrak{B}(x,\sigma) = \frac{\sigma e^{x\sigma}}{e^{\sigma}-1} = B_0(x) + B_1(x)\sigma + B_2(x)\frac{\sigma^2}{2!} + \cdots, |\sigma| < 2|\pi. \tag{1.3}$$

With the following recursion, the Bernoulli polynomials can be easily calculated:

$$\sum_{n=0}^{j-1} \binom{j}{n} B_n(x) = jx^{j-1}, \quad j = 2,3,4,\ldots,$$

with the initial condition $B_0(x) = 1$. The following are the first few Bernoulli polynomials:

$$B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \ldots. \tag{1.4}$$

For $\mathfrak{a}_1$, $\mathfrak{a}_2 \in \mathcal{A}$ analytic in $\mathfrak{U}$, $\mathfrak{a}_1$ is subordinate to $\mathfrak{a}_2$, if there is a Schwarz function $\theta(\sigma)$ that is analytic in $\mathfrak{U}$ with $\theta(0) = 0$ and $|\theta(\sigma)| < 1$, such that $\mathfrak{a}_1(\sigma) = \mathfrak{a}_2(\theta(\sigma))$, $\sigma \in \mathfrak{U}$. This is symbolized as

$$\mathfrak{a}_1 \prec \mathfrak{a}_2 \text{ or } \mathfrak{a}_1(\sigma) \prec \mathfrak{a}_2(\sigma), \sigma \in \mathfrak{U}.$$

In case, if $\mathfrak{a}_2 \in \mathcal{S}$, then $\mathfrak{a}_1(\sigma) \prec \mathfrak{a}_2(\sigma) \Leftrightarrow \mathfrak{a}_1(0) = \mathfrak{a}_2(0)$ and $\mathfrak{a}_1(\mathfrak{U}) \subset \mathfrak{a}_2(\mathfrak{U})$. We define two subfamilies $\mathfrak{T}_\sigma(\rho, \nu, \varkappa)$ and $\mathfrak{W}_\sigma(\beta, \tau, \nu, x)$ of σ that are governed by Bernoulli polynomials as follows.

Definition 1.1. Let $\rho \geqslant 0$ and $0 < \nu \leqslant 1$. If $\phi \in \sigma$ satisfies

$$\frac{1}{2}\left(\left(\frac{(\sigma\phi'(\sigma) + \rho\sigma^2\phi''(\sigma))'}{\phi'(\sigma)}\right) + \left(\frac{(\sigma\phi'(\sigma) + \rho\sigma^2\phi''(\sigma))'}{\phi'(\sigma)}\right)^{\frac{1}{\nu}}\right) \prec \mathfrak{B}(x, \sigma), \tag{1.5}$$

and

$$\frac{1}{2}\left(\left(\frac{(w\psi'(w) + \rho w^2\psi''(w))'}{\psi'(w)}\right) + \left(\frac{(w\psi'(w) + \rho w^2\psi''(w))'}{\psi'(w)}\right)^{\frac{1}{\nu}}\right) \prec \mathfrak{B}(x, w), \tag{1.6}$$

then we say that $\phi \in \mathfrak{T}_\sigma(\rho, \nu, x)$, where $\mathfrak{B}(x, \sigma)$ is as given by (1.3), $\psi(w) = \phi^{-1}(w)$ is as in (1.2), and $\sigma, w \in \mathfrak{U}$.

Definition 1.2. Let $0 \leqslant \beta \leqslant 1, \tau \geqslant 1$, and $0 < \nu \leqslant 1$. If $\phi \in \sigma$ satisfies

$$\frac{1}{2}\left(\left(\frac{\sigma(\phi'(\sigma))^\tau}{(1-\beta)\sigma + \beta\phi(\sigma)}\right) + \left(\frac{\sigma(\phi'(\sigma))^\tau}{(1-\beta)\sigma + \beta\phi(\sigma)}\right)^{\frac{1}{\nu}}\right) \prec \mathfrak{B}(x, \sigma), \tag{1.7}$$

and

$$\frac{1}{2}\left(\left(\frac{w(\psi'(w))^\tau}{(1-\beta)w + \beta\psi(w)}\right) + \left(\frac{w(\psi'(w))^\tau}{(1-\beta)w + \beta\psi(w)}\right)^{\frac{1}{\nu}}\right) \prec \mathfrak{B}(x, w), \tag{1.8}$$

then we say that $\phi \in \mathfrak{W}_\sigma(\beta, \tau, \nu, x)$, where $\mathfrak{B}(x, \sigma)$ is as given by (1.3), $\psi(w) = \phi^{-1}(w)$ is as in (1.2), and $\sigma, w \in \mathfrak{U}$.

The content of this article is organized as follows. The estimates for $|d_2|$, $|d_3|$, and $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$ for functions in the families $\mathfrak{T}_\sigma(\rho, \nu, x)$ and $\mathfrak{W}_\sigma(\beta, \tau, \nu, x)$ are found in Section 2. We draw attention to pertinent links between some of the specific cases and the main findings in Section 3. We wrap up the study with a few observations in Section 4.

2. Principal findings

Section 2 begins with bounds for $|d_2|$, $|d_3|$, and $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$ for functions in the families $\mathfrak{T}_\sigma(\rho, \nu, x)$ and $\mathfrak{W}_\sigma(\beta, \tau, \nu, x)$.

Theorem 2.1. *If a function $\phi \in \sigma$ is a member of the family $\mathfrak{T}_\sigma(\rho, \nu, x)$, $(\rho \geqslant 0, 0 < \nu \leqslant 1)$, then*

$$|d_2| \leqslant \sqrt{\frac{\nu^2|2x-1|^3}{|(\nu(\nu+1)(1+5\rho) + (1-\nu)(1+2\rho)^2)(2x-1)^2 - 2(\nu+1)^2(1+2\rho)^2(x^2 - x + \frac{1}{6})|}}, \tag{2.1}$$

$$|d_3| \leqslant \frac{|\nu(2x-1)|^2}{4(\nu+1)^2(1+2\rho)^2} + \frac{|\nu(2x-1)|}{6(\nu+1)(1+3\rho)}, \tag{2.2}$$

and for $\xi \in \mathbb{R}$

$$|d_3 - \xi d_2^2| \leqslant \begin{cases} \frac{|\nu(2x-1)|}{6(\nu+1)(1+3\rho)}, & |1-\xi| \leqslant \daleth, \\ \frac{|\nu|^2|2x-1|^3|1-\xi|}{|(\nu(\nu+1)(1+5\rho)+(1-\nu)(1+2\rho)^2)(2x-1)^2 - 2(\nu+1)^2(1+2\rho)^2(x^2-x+\frac{1}{6})|}, & |1-\xi| \geqslant \daleth, \end{cases} \tag{2.3}$$

where

$$\daleth = \left| \frac{(\nu(\nu+1)(1+5\rho)+(1-\nu)(1+2\rho)^2)(2x-1)^2 - 2(\nu+1)^2(1+2\rho)^2(x^2-x+\frac{1}{6})}{3\nu(\nu+1)(1+3\rho)(2x-1)^2} \right|. \tag{2.4}$$

Proof. Let $\phi \in \mathfrak{T}_\sigma(\rho,\nu,x)$. Then, from subordinations (1.5) and (1.6), we can write

$$\frac{1}{2}\left(\left(\frac{(\sigma\phi'(\sigma)+\rho\sigma^2\phi''(\sigma))'}{\phi'(\sigma)}\right)+\left(\frac{(\sigma\phi'(\sigma)+\rho\sigma^2\phi''(\sigma))'}{\phi'(\sigma)}\right)^{\frac{1}{\nu}}\right) = \mathfrak{B}(x,\mathfrak{l}(\sigma)), \tag{2.5}$$

and

$$\frac{1}{2}\left(\left(\frac{(w\psi'(w)+\rho w^2\psi''(w))'}{\psi'(w)}\right)+\left(\frac{(w\psi'(w)+\rho w^2\psi''(w))'}{\psi'(w)}\right)^{\frac{1}{\nu}}\right) = \mathfrak{B}(x,\mathfrak{m}(w)), \tag{2.6}$$

where Schwarz functions $\mathfrak{l}(\sigma) = \sum_{j=1}^{\infty} \mathfrak{l}_j\sigma^j$ and $\mathfrak{m}(w) = \sum_{j=1}^{\infty} \mathfrak{m}_j w^j$, $\sigma, w \in \mathfrak{U}$ satisfy the property (see [9])

$$|\mathfrak{l}_j| \leqslant 1 \quad \text{and} \quad |\mathfrak{m}_j| \leqslant 1\,(j \in \mathbb{N}). \tag{2.7}$$

A few basic mathematical methods allow us to write equations (2.5) and (2.6) as

$$\begin{aligned}
&\frac{1}{2}\left(\left(\frac{(\sigma\phi'(\sigma)+\rho\sigma^2\phi''(\sigma))'}{\phi'(\sigma)}\right)+\left(\frac{(\sigma\phi'(\sigma)+\rho\sigma^2\phi''(\sigma))'}{\phi'(\sigma)}\right)^{\frac{1}{\nu}}\right)\\
&\quad = 1+\left(\frac{\nu+1}{\nu}\right)(1+2\rho)d_2\sigma+\left(\left(\frac{\nu+1}{\nu}\right)(3(1+3\rho)d_3-2(1+2\rho)d_2^2)\right.\\
&\qquad \left.+\left(\frac{1-\nu}{\nu^2}\right)(1+2\rho)^2d_2^2\right)\sigma^2+\cdots,\\
&\mathfrak{B}(x,\mathfrak{l}(\sigma)) = 1+B_1(x)\mathfrak{l}_1\sigma+\left[B_1(x)\mathfrak{l}_2+\frac{B_2(x)}{2!}\mathfrak{l}_1^2\right]\sigma^2+\cdots,
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
&\frac{1}{2}\left(\left(\frac{(w\psi'(w)+\rho w^2\psi''(w))'}{\psi'(w)}\right)+\left(\frac{(w\psi'(w)+\rho w^2\psi''(w))'}{\psi'(w)}\right)^{\frac{1}{\nu}}\right)\\
&\quad = 1-\left(\frac{\nu+1}{\nu}\right)(1+2\rho)d_2w+\left(\left(\frac{\nu+1}{\nu}\right)(2(2+7\rho)d_2^2-3(1+3\rho)d_3)\right.\\
&\qquad \left.+\left(\frac{1-\nu}{\nu^2}\right)(1+2\rho)^2d_2^2\right)w^2+\cdots,\\
&\mathfrak{B}(x,\mathfrak{m}(w)) = 1+B_1(x)\mathfrak{m}_1w+\left[B_1(x)\mathfrak{m}_2+\frac{B_2(x)}{2!}\mathfrak{m}_1^2\right]w^2+\cdots.
\end{aligned} \tag{2.9}$$

We draw the following conclusions by contrasting the terms with the same degree in (2.8) because of (2.5),

$$\left(\frac{\nu+1}{\nu}\right)(1+2\rho)d_2 = B_1(x)\mathfrak{l}_1, \tag{2.10}$$

$$\left(\frac{\nu+1}{\nu}\right)(3(1+3\rho)d_3-2(1+2\rho)d_2^2)+\left(\frac{1-\nu}{\nu^2}\right)(1+2\rho)^2d_2^2 = B_1(x)\mathfrak{l}_2+\frac{B_2(x)}{2!}\mathfrak{l}_1^2. \tag{2.11}$$

Similarly, due to (2.6), we draw our conclusion by contrasting the terms of the same degree in (2.9),

$$-\left(\frac{\nu+1}{\nu}\right)(1+2\rho)d_2 = B_1(x)m_1, \tag{2.12}$$

$$\left(\frac{\nu+1}{\nu}\right)(2(2+7\rho)d_2^2 - 3(1+3\rho)d_3) + \left(\frac{1-\nu}{\nu^2}\right)(1+2\rho)^2 d_2^2 = B_1(x)m_2 + \frac{B_2(x)}{2!}m_1^2. \tag{2.13}$$

From (2.10) and (2.12), we get $l_1 = -m_1$ and

$$2\left(\frac{\nu+1}{\nu}\right)^2(1+2\rho)^2 d_2^2 = (l_1^2 + m_1^2)B_1^2(x). \tag{2.14}$$

Addition of (2.11) and (2.13) yields

$$\left(\left(\frac{1+\nu}{\nu}\right)(1+5\rho) + \left(\frac{1-\nu}{\nu^2}\right)(1+2\rho)^2\right)2d_2^2 = B_1(x)(l_2+m_2) + \frac{B_2(x)}{2}(l_1^2+m_1^2). \tag{2.15}$$

Replacing $l_1^2 + m_1^2$ from (2.14) in (2.15) we get

$$d_2^2 = \frac{\nu^2 B_1^3(x)(l_2+m_2)}{2(\nu(\nu+1)(1+5\rho) + (1-\nu)(1+2\rho)^2)B_1^2(x) - (\nu+1)^2(1+2\rho)^2 B_2(x)}. \tag{2.16}$$

Utilizing (1.4) for $B_1(x), B_2(x)$ and applying (2.7) to l_2, m_2 produces (2.1). From (2.11) we subtract (2.13) to get the bound on $|d_3|$:

$$d_3 = d_2^2 + \frac{\nu B_1(x)(l_2 - m_2)}{6(\nu+1)(1+3\rho)}. \tag{2.17}$$

If we replace d_2^2 from (2.14) in (2.17) we get

$$d_3 = \frac{\nu^2 B_1^2(x)(l_1^2+m_2^2)}{2(\nu+1)^2(1+2\rho)^2} + \frac{\nu B_1(x)(l_2-m_2)}{6(\nu+1)(1+3\rho)}. \tag{2.18}$$

We deduce (2.2) from (2.18) by applying (1.4) and (2.7). Finally, we compute the bound on $|d_3 - \xi d_2^2|$ using the values of d_2^2 and d_3 from (2.16) and (2.17), respectively. Consequently, we have

$$|d_3 - \xi d_2^2| = \nu|B_1(x)|\left|\left(\frac{1}{6(\nu+1)(1+3\rho)} + \mathcal{V}(\xi,x)\right)l_2 - \left(\frac{1}{6(\nu+1)(1+3\rho)} - \mathcal{V}(\xi,x)\right)m_2\right|,$$

where

$$\mathcal{V}(\xi,x) = \frac{(1-\xi)\nu B_1^2(x)}{2(\nu(\nu+1)(1+5\rho) + (1-\nu)(1+2\rho)^2)B_1^2(x) - (\nu+1)^2(1+2\rho)^2 B_2(x)}.$$

Clearly

$$|d_3 - \xi d_2^2| \leqslant \begin{cases} \frac{\nu|B_1(x)|}{3(\nu+1)(1+3\rho)}, & |\mathcal{V}(\xi,x)| \leqslant \frac{1}{6(\nu+1)(1+3\rho)}, \\ 2\nu|B_1(x)||\mathcal{V}(\xi,x)|, & |\mathcal{V}(\xi,x)| \geqslant \frac{1}{6(\nu+1)(1+3\rho)}. \end{cases} \tag{2.19}$$

We derive (2.3) from (2.19), where $\daleth$ is the same as in (2.4). □

The following is obtained by applying $\xi = 1$ in Theorem 2.1.

Corollary 2.2. *If a function $\phi \in \sigma$ is a member of the family $\mathfrak{T}_\sigma(\rho,\nu,x)$, $(\rho \geqslant 0, 0 < \nu \leqslant 1)$, then $|d_3 - d_2^2| \leqslant \frac{|\nu(2x-1)|}{6(\nu+1)(1+3\rho)}$.*

Theorem 2.3. *If a function $\phi \in \sigma$ is a member of the family $\mathfrak{W}_\sigma(\beta,\tau,\nu,x)$, $(0 \leqslant \beta \leqslant 1, \tau \geqslant 1, 0 < \nu \leqslant 1)$, then*

$$|d_2| \leqslant \sqrt{\frac{\nu^2|2x-1|^3}{|(\nu(\nu+1)(X+S)+(1-\nu)Y^2)(2x-1)^2-(\nu+1)^2Y^2(x^2-x+\frac{1}{6})|}}, \tag{2.20}$$

$$|d_3| \leqslant \frac{\nu^2|2x-1|^2}{(\nu+1)^2Y^2}+\frac{\nu|2x-1|}{(\nu+1)X}, \tag{2.21}$$

and for $\xi \in \mathbb{R}$,

$$|d_3-\xi d_2^2| \leqslant \begin{cases} \frac{\nu|2x-1|}{(\nu+1)X}, & |1-\xi| \leqslant \Upsilon, \\ \frac{\nu^2|2x-1|^3|1-\xi|}{|(\nu(\nu+1)(X+S)+(1-\nu)Y^2)(2x-1)^2-(\nu+1)^2Y^2(x^2-x+\frac{1}{6})|}, & |1-\xi| \geqslant \Upsilon, \end{cases} \tag{2.22}$$

where

$$\Upsilon = \left|\frac{(\nu(\nu+1)(X+S)+(1-\nu)Y^2)(2x-1)^2-(\nu+1)^2Y^2(x^2-x+\frac{1}{6})}{\nu(\nu+1)X(2x-1)^2}\right|, \tag{2.23}$$

and

$$X = 3\tau-\beta, \quad Y = 2\tau-\beta \quad \textit{and} \quad S = 2\tau(\tau-1-\beta)+\beta^2. \tag{2.24}$$

Proof. Let $\phi \in \mathfrak{W}_\sigma(\beta,\tau,\nu,x)$. Then, from subordinations (1.7) and (1.8), we can write

$$\frac{1}{2}\left(\left(\frac{\sigma(\phi'(\sigma))^\tau}{(1-\beta)\sigma+\beta\phi(\sigma)}\right)+\left(\frac{\sigma(\phi'(\sigma))^\tau}{(1-\beta)\sigma+\beta\phi(\sigma)}\right)^{\frac{1}{\nu}}\right) = \mathfrak{B}(x,\mathfrak{l}(\sigma)), \tag{2.25}$$

and

$$\frac{1}{2}\left(\left(\frac{w(\psi'(w))^\tau}{(1-\beta)w+\beta\psi(w)}\right)+\left(\frac{w(\psi'(w))^\tau}{(1-\beta)w+\beta\psi(w)}\right)^{\frac{1}{\nu}}\right) = \mathfrak{B}(x,\mathfrak{m}(w)), \tag{2.26}$$

where Schwarz functions $\mathfrak{l}(\sigma) = \sum_{j=1}^{\infty} \mathfrak{l}_j\sigma^j$ and $\mathfrak{m}(w) = \sum_{j=1}^{\infty} \mathfrak{m}_j w^j$, $\sigma, w \in \mathfrak{U}$ satisfy the property (2.7). Equations (2.25) and (2.26) can be written as follows by employing a few basic mathematical techniques:

$$\begin{aligned} &\frac{1}{2}\left(\left(\frac{\sigma(\phi'(\sigma))^\tau}{(1-\beta)\sigma+\beta\phi(\sigma)}\right)+\left(\frac{\sigma(\phi'(\sigma))^\tau}{(1-\beta)\sigma+\beta\phi(\sigma)}\right)^{\frac{1}{\nu}}\right) \\ &\quad = 1+\left(\frac{\nu+1}{2\nu}\right)Yd_2\sigma+\left(\left(\frac{\nu+1}{2\nu}\right)(Xd_3+Sd_2^2)+\left(\frac{1-\nu}{4\nu^2}\right)Y^2d_2^2\right)\sigma^2+\cdots, \\ &\mathfrak{B}(x,\mathfrak{l}(\sigma)) = 1+B_1(x)\mathfrak{l}_1\sigma+\left[B_1(x)\mathfrak{l}_2+\frac{B_2(x)}{2!}\mathfrak{l}_1^2\right]\sigma^2+\cdots, \end{aligned} \tag{2.27}$$

and

$$\begin{aligned} &\frac{1}{2}\left(\left(\frac{w(\psi'(w))^\tau}{(1-\beta)w+\beta\psi(w)}\right)+\left(\frac{w(\psi'(w))^\tau}{(1-\beta)w+\beta\psi(w)}\right)^{\frac{1}{\nu}}\right) \\ &\quad = 1-\left(\frac{\nu+1}{2\nu}\right)Yd_2w+\left(\left(\frac{\nu+1}{2\nu}\right)(X(2d_2^2-d_3)+Sd_2^2)+\left(\frac{1-\nu}{4\nu^2}\right)Y^2d_2^2\right)w^2+\cdots, \\ &\mathfrak{B}(x,\mathfrak{m}(w)) = 1+B_1(x)\mathfrak{m}_1w+\left[B_1(x)\mathfrak{m}_2+\frac{B_2(x)}{2!}\mathfrak{m}_1^2\right]w^2+\cdots. \end{aligned} \tag{2.28}$$

By comparing terms of the same degree in (2.27), and because of (2.25), we arrive at

$$\left(\frac{\nu+1}{2\nu}\right)Yd_2 = B_1(x)\mathfrak{l}_1, \tag{2.29}$$

$$\left(\frac{\nu+1}{2\nu}\right)(Xd_3+Sd_2^2)+\left(\frac{1-\nu}{4\nu^2}\right)Y^2d_2^2=B_1(x)\mathfrak{l}_2+\frac{B_2(x)}{2!}\mathfrak{l}_1. \tag{2.30}$$

where X, Y, and S are as in (2.24). Likewise, because of equality (2.26), we compare terms of the same degree in (2.28) to arrive at

$$-\left(\frac{\nu+1}{2\nu}\right)Yd_2=B_1(x)\mathfrak{m}_1,$$

and

$$\left(\frac{\nu+1}{2\nu}\right)(X(2d_2^2-d_3)+Sd_2^2)+\left(\frac{1-\nu}{4\nu^2}\right)Y^2d_2^2=B_1(x)\mathfrak{m}_2+\frac{B_2(x)}{2!}\mathfrak{m}_1^2, \tag{2.31}$$

where X, Y, and S are as in (2.24). From (2.29) and (2.12), we get $\mathfrak{l}_1=-\mathfrak{m}_1$, and

$$\frac{1}{2}\left(\frac{\nu+1}{\nu}\right)^2Y^2d_2^2=(\mathfrak{l}_1^2+\mathfrak{m}_1^2)B_1^2(x). \tag{2.32}$$

Addition of (2.30) and (2.31) yields

$$\left(\left(\frac{1+\nu}{\nu}\right)(X+S)+\left(\frac{1-\nu}{2\nu^2}\right)Y^2\right)d_2^2=B_1(x)(\mathfrak{l}_2+\mathfrak{m}_2)+\frac{B_2(x)}{2}(\mathfrak{l}_1^2+\mathfrak{m}_1^2). \tag{2.33}$$

Replacing $\mathfrak{l}_1^2+\mathfrak{m}_1^2$ from (2.32) in (2.33) we get

$$d_2^2=\frac{4\nu^2B_1^3(x)(\mathfrak{l}_2+\mathfrak{m}_2)}{2(\nu(\nu+1)(X+S)+(1-\nu)Y^2)4B_1^2(x)-(\nu+1)^2Y^2B_2(x)}. \tag{2.34}$$

Utilizing (1.4) for $B_1(x), B_2(x)$ and applying (2.7) to $\mathfrak{l}_2$, $\mathfrak{m}_2$ produces (2.20). From (2.30) we subtract (2.31) to get the bound on $|d_3|$:

$$d_3=d_2^2+\frac{\nu B_1(x)(\mathfrak{l}_2-\mathfrak{m}_2)}{(\nu+1)X}. \tag{2.35}$$

If we replace d_2^2 from (2.32) in (2.35) we get

$$d_3=\frac{2\nu^2B_1^2(x)(\mathfrak{l}_1^2+\mathfrak{m}_2^2)}{(\nu+1)^2Y^2}+\frac{\nu B_1(x)(\mathfrak{l}_2-\mathfrak{m}_2)}{(\nu+1)X}. \tag{2.36}$$

We deduce (2.21) from (2.36) by applying (1.4) and (2.7). Finally, we compute the bound on $|d_3-\xi d_2^2|$ using the value of d_2^2 from (2.34) in (2.35). Consequently, we have

$$|d_3-\xi d_2^2|=|B_1(x)|\left|\left(\frac{\nu}{(\nu+1)X}+\mathcal{V}_2(\xi,x)\right)\mathfrak{l}_2-\left(\frac{\nu}{(\nu+1)X}-\mathcal{V}_2(\xi,x)\right)\mathfrak{m}_2\right|,$$

where

$$\mathcal{V}_2(\xi,x)=\frac{4(1-\xi)\nu^2B_1^2(x)}{(\nu(\nu+1)(X+S)+(1-\nu)Y^2)4B_1^2(x)-(\nu+1)^2Y^2B_2(x)}.$$

Clearly

$$|d_3-\xi d_2^2|\leqslant\begin{cases}\frac{2\nu|B_1(x)|}{(\nu+1)X}, & |\mathcal{V}_2(\xi,x)|\leqslant\frac{\nu}{(\nu+1)X},\\ 2\nu|B_1(x)||\mathcal{V}_2(\xi,x)|, & |\mathcal{V}_2(\xi,x)|\geqslant\frac{\nu}{(\nu+1)X}.\end{cases} \tag{2.37}$$

We derive (2.22) from (2.37), with Υ is the same as in (2.23). □

3. Specific instances

We derive the following instances by specializing the parameters ρ, and ν in Theorem 2.1 demonstrated in Section 2.

Example 3.1. Letting $\rho = 0$ in the class $\mathfrak{T}_\sigma(\rho,\nu,x)$, we get a subclass $\mathfrak{C}_\sigma(\nu,x)\equiv\mathfrak{T}_\sigma(0,\nu,x)$ of functions $\phi \in \sigma$ satisfying

$$\frac{1}{2}\left(\left(\frac{(\sigma\phi'(\sigma))'}{\phi'(\sigma)}\right)+\left(\frac{(\sigma\phi'(\sigma))'}{\phi'(\sigma)}\right)^{\frac{1}{\nu}}\right) \prec \mathfrak{B}(x,\sigma),$$

and

$$\frac{1}{2}\left(\left(\frac{(w\psi'(w))'}{\psi'(w)}\right)+\left(\frac{(w\psi'(w))'}{\psi'(w)}\right)^{\frac{1}{\nu}}\right) \prec \mathfrak{B}(x,w),$$

where $0<\nu\leqslant 1$, $\mathfrak{B}(x,\sigma)$ is as in (1.3), $\psi(w)=\phi^{-1}(w)$ is as in (1.2), and $\sigma, w \in \mathfrak{U}$.

Corollary 3.2. *If a function $\phi \in \sigma$ is a member of the family $\mathfrak{C}_\sigma(\nu,x)$, $(0<\nu\leqslant 1)$, then*

$$|d_2| \leqslant \nu|(2x-1)|\sqrt{\frac{|2x-1|}{|(\nu^2+1)(2x-1)^2-2(\nu+1)^2(x^2-x+\frac{1}{6})|}}, \quad |d_3| \leqslant \frac{\nu^2(2x-1)^2}{4(\nu+1)^2}+\frac{\nu|(2x-1)|}{6(\nu+1)},$$

and for $\xi \in \mathbb{R}$,

$$|d_3-\xi d_2^2| \leqslant \begin{cases} \frac{\nu|2x-1|}{6(\nu+1)}, & |1-\xi| \leqslant \left|\frac{(\nu^2+1)(2x-1)^2-2(\nu+1)^2(x^2-x+\frac{1}{6})}{3\nu(\nu+1)(2x-1)^2}\right|, \\ \frac{\nu^2|2x-1|^3|1-\xi|}{|(\nu^2+1)(2x-1)^2-2(\nu+1)^2(x^2-x+\frac{1}{6})|}, & |1-\xi| \geqslant \left|\frac{(\nu^2+1)(2x-1)^2-2(\nu+1)^2(x^2-x+\frac{1}{6})}{3\nu(\nu+1)(2x-1)^2}\right|. \end{cases}$$

By taking $\xi = 1$ in Corollary 3.2 we get the corresponding result on Fekete-Szegö type inequality.

Corollary 3.3. *If a function $\phi \in \sigma$ is a member of the family $\mathfrak{C}_\sigma(\nu,x)$, then $|d_3-d_2^2| \leqslant \frac{\nu|2x-1|}{6(\nu+1)}$.*

Example 3.4. Letting $\nu = 1$ in the family $\mathfrak{T}_\sigma(\rho,\nu,x)$, we get a subfamily $\mathfrak{F}_\sigma(\rho,x) \equiv \mathfrak{T}_\sigma(\rho,1,x)$ of functions $\phi \in \sigma$ satisfying

$$\frac{(\sigma\phi'(\sigma)+\rho\sigma^2\phi''(\sigma))'}{\phi'(\sigma)} \prec \mathfrak{B}(x,\sigma), \quad \frac{(w\psi'(w)+\rho w^2\psi''(w))'}{\psi'(w)} \prec \mathfrak{B}(x,w),$$

where $\rho \geqslant 0$, $\mathfrak{B}(x,\sigma)$ is as in (1.3), $\psi(w)=\phi^{-1}(w)$ is as in (1.2), and $\sigma, w \in \mathfrak{U}$.

Corollary 3.5. *If a function $\phi \in \sigma$ is a member of the family $\mathfrak{F}_\sigma(\rho,x)$, then*

$$|d_2| \leqslant |2x-1|\sqrt{\frac{|2x-1|}{2|(1+5\rho)(2x-1)^2-4(1+2\rho)^2(x^2-x+\frac{1}{6})|}}, \quad |d_3| \leqslant \frac{(2x-1)^2}{16(1+2\rho)^2}+\frac{|2x-1|}{12(1+3\rho)},$$

and

$$|d_3-\xi d_2^2| \leqslant \begin{cases} \frac{|2x-1|}{12(1+3\rho)}, & |1-\xi| \leqslant \left|\frac{(1+5\rho)(2x-1)^2-4(1+2\rho)^2(x^2-x+\frac{1}{6})}{3(1+3\rho)(2x-1)^2}\right|, \\ \frac{|2x-1|^3|1-\xi|}{|2|(1+5\rho)(2x-1)^2-4(1+2\rho)^2(x^2-x+\frac{1}{6})|}, & |1-\xi| \geqslant \left|\frac{(1+5\rho)(2x-1)^2-4(1+2\rho)^2(x^2-x+\frac{1}{6})}{3(1+3\rho)(2x-1)^2}\right|. \end{cases}$$

We derive the following corollaries by specializing the parameters ν, β, and τ in Theorems 2.3 demonstrated in Section 2.

Example 3.6. Letting $\beta = 0$ in the class $\mathfrak{W}_\sigma(\beta,\tau,\nu,x)$, we get a subclass $\mathfrak{G}_\sigma(\tau,\nu,x) \equiv \mathfrak{W}_\sigma(0,\tau,\nu,x)$ of functions $\phi \in \sigma$ satisfying

$$\frac{1}{2}\left((\phi'(\sigma))^\tau + (\phi'(\sigma))^{\frac{\tau}{\nu}}\right) \prec \mathfrak{B}(x,\sigma), \quad \frac{1}{2}\left((\psi'(w))^\tau + (\psi'(w))^{\frac{\tau}{\nu}}\right) \prec \mathfrak{B}(x,w),$$

where $\mathfrak{B}(x,\sigma)$ is as in (1.3), the inverse function $\phi^{-1}(w) = \psi(w)$ as in (1.2), $\tau \geqslant 1$, $0 < \nu \leqslant 1$, and $\sigma, w \in \mathfrak{U}$.

The following is obtained by allowing $\beta = 0$ in Theorem 2.3.

Corollary 3.7. *For any function $\phi \in \mathfrak{G}_\sigma(\tau,\nu,x)$, the upper bounds of $|d_2|$, $|d_3|$, and $|d_3 - \xi d_2^2|$, $\xi \in \mathbb{R}$, are given by (2.20), (2.21), and (2.22), respectively, with $X = X_1 = 3\tau$, $Y = Y_1 = 2\tau$, and $S = S_1 = 2\tau(\tau-1)$. X_1, Y_1, and S_1 are to be used in place of X, Y, and S for Υ in (2.23).*

Example 3.8. Letting $\beta = 1$ in $\mathfrak{W}_\sigma(\beta,\tau,\nu,x)$, we get a subclass $\mathfrak{H}_\sigma(\tau,\nu,x) \equiv \mathfrak{W}_\sigma(1,\tau,\nu,x)$ of functions $\phi \in \sigma$ satisfying

$$\frac{1}{2}\left(\left(\frac{\sigma(\phi'(\sigma))^\tau}{\phi(\sigma)}\right) + \left(\frac{\sigma(\phi'(\sigma))^\tau}{\phi(\sigma)}\right)^{\frac{1}{\nu}}\right) \prec \mathfrak{B}(x,\sigma),$$

and

$$\frac{1}{2}\left(\left(\frac{w(\psi'(w))^\tau}{\psi(w)}\right) + \left(\frac{w(\psi'(w))^\tau}{\psi(w)}\right)^{\frac{1}{\nu}}\right) \prec \mathfrak{B}(x,w),$$

where $\mathfrak{B}(x,\sigma)$ is as in (1.3), $\psi(w) = \phi^{-1}(w)$ is as in (1.2), $\tau \geqslant 1$, $0 < \nu \leqslant 1$, and $\sigma, w \in \mathfrak{U}$.

Taking β=1 in Theorem 2.3, we have the following.

Corollary 3.9. *For any function $\phi \in \mathfrak{H}_\sigma(\tau,\nu,x)$, the upper bounds of $|d_2|$, $|d_3|$, and $|d_3 - \xi d_2^2|$, $\xi \in \mathbb{R}$, are given by (2.20), (2.21), and (2.22), respectively, with $X = X_2 = 3\tau - 1$, $Y = Y_2 = 2\tau - 1$, and $S = S_2 = 2\tau^2 - 4\tau + 1$. X_2, Y_2, and S_2 are to be used in place of X, Y, and S for Υ in (2.23).*

Example 3.10. Letting $\nu = 1$ in $\mathfrak{W}_\sigma(\beta,\tau,\nu,x)$, we get a family $\mathfrak{K}_\sigma(\beta,\tau,x) \equiv \mathfrak{W}_\sigma(\beta,\tau,1,x)$ of functions $\phi \in \sigma$ satisfying

$$\frac{\sigma(\phi'(\sigma))^\tau}{(1-\beta)\sigma + \beta\phi(\sigma)} \prec \mathfrak{B}(x,\sigma), \quad \frac{w(\psi'(w))^\tau}{(1-\beta)w + \beta\psi(w)} \prec \mathfrak{B}(x,w),$$

where $\tau \geqslant 1$, $0 \leqslant \beta \leqslant 1$, $\mathfrak{B}(x,\sigma)$ is as in (1.3), $\psi(w) = \phi^{-1}(w)$ is as in (1.2), and $\sigma, w \in \mathfrak{U}$.

Corollary 3.11. *If a function $\phi \in \sigma$ is a member of the family $\mathfrak{K}_\sigma(\beta,\tau,x)$, $(0 \leqslant \beta \leqslant 1, \tau \geqslant 1)$, then*

$$|d_2| \leqslant \sqrt{\frac{|2x-1|^3}{2|(X+S)(2x-1)^2 - 2Y^2(x^2 - x + \frac{1}{6})|}}, \quad |d_3| \leqslant \frac{|2x-1|^2}{4Y^2} + \frac{|2x-1|}{2X},$$

and for $\xi \in \mathbb{R}$,

$$|d_3 - \xi d_2^2| \leqslant \begin{cases} \frac{|2x-1|}{2X}, & |1-\xi| \leqslant \Upsilon_1, \\ \frac{\nu^2|2x-1|^3|1-\xi|}{2|((X+S)(2x-1)^2 - 2Y^2(x^2-x+\frac{1}{6})|}, & |1-\xi| \geqslant \Upsilon_1, \end{cases}$$

where

$$\Upsilon_1 = \left|\frac{((X+S)(2x-1)^2 - 2Y^2(x^2 - x + \frac{1}{6})}{X(2x-1)^2}\right|,$$

X, Y, and S are as mentioned in (2.24).

Example 3.12. Letting $\tau = 1$ in $\mathfrak{W}_\sigma(\beta, \tau, \nu, x)$, we get a subclass $\mathfrak{M}_\sigma(\beta, \nu, x) \equiv \mathfrak{W}_\sigma(\beta, 1, \nu, x)$ of functions $\phi \in \sigma$ satisfying

$$\frac{1}{2}\left(\left(\frac{\sigma\phi'(\sigma)}{(1-\beta)\sigma+\beta\phi(\sigma)}\right)+\left(\frac{\sigma\phi'(\sigma)}{(1-\beta)\sigma+\beta\phi(\sigma)}\right)^{\frac{1}{\nu}}\right) \prec \mathfrak{B}(x,\sigma),$$

and

$$\frac{1}{2}\left(\left(\frac{w\psi'(w)}{(1-\beta)w+\beta\psi(w)}\right)+\left(\frac{w\psi'(w)}{(1-\beta)w+\beta\psi(w)}\right)^{\frac{1}{\nu}}\right) \prec \mathfrak{B}(x,w),$$

where $0 \leqslant \beta \leqslant 1$, $0 < \nu \leqslant 1$, $\mathfrak{B}(x, \sigma)$ is as in (1.3), $\psi(w) = \phi^{-1}(w)$ is as in (1.2), and $\sigma, w \in \mathfrak{U}$.

Allowing $\tau = 1$ in Theorem 2.3, we get the following.

Corollary 3.13. *For any function $\phi \in \mathfrak{M}_\sigma(\beta, \nu, x)$, the upper bounds of $|d_2|$, $|d_3|$, and $|d_3 - \xi d_2^2|$, $\xi \in \mathbb{R}$, are given by (2.20), (2.21), and (2.22), respectively, with $X = X_3 = 3 - \beta$, $Y = Y_3 = 2 - \beta$, and $S = S_3 = \beta^2 - \beta$). X_3, Y_3, and S_3 are to be used in place of X, Y, and S for Υ in (2.23).*

Remark 3.14.

i) $\mathfrak{K}_\sigma(1, \tau, x) \equiv \mathfrak{H}_\sigma(\tau, 1, x)$;
ii) $\mathfrak{K}_\sigma(0, \tau, x) \equiv \mathfrak{G}_\sigma(\tau, 1, x)$;
iii) $\mathfrak{K}_\sigma(\beta, 1, x) \equiv \mathfrak{M}_\sigma(\beta, 1, x)$.

4. Conclusion

In this presentation, we have defined two subfamilies of regular and biunivalent functions that are subordinate to Bernoulli polynomials denoted by $\mathfrak{T}_\sigma(\rho, \nu, \varkappa)$ and $\mathfrak{W}_\sigma(\beta, \tau, \nu, x)$. We have estimated the Maclaurin coefficients $|d_2|$ and $|d_3|$ for functions that belong to the defined σ subfamilies. We have also found the Fekete-Szegö functional $|d_3 - \xi d_2^2|$, $\xi \in \mathbb{R}$ for functions in these subfamilies. The specialization of the parameters applied to our findings produces novel and untested outcomes, as covered in Section 2. Finally, we note for interested readers that the defined subfamilies can be studied for Hankel determinant problems of higher order. Making use of Bernoulli polynomials one can find the estimates of the coefficients $|d_2|$ and $|d_3|$ and the Fekete-Szegö functional $|d_3 - \xi d_2^2|$, $\xi \in \mathbb{R}$ for functions belonging to new subfamilies of biunivalent functions.

Authors' contributions

Each author contributed equally to the results derivation and gave their final manuscript approval.

Conflicts of interest

There are no competing interests with regard to the publishing of this manuscript, the authors reaffirm.

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