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$\mathcal{P}$-topological spaces in simple graphs

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Abstract

This paper introduces and develops the concept of topological spaces within graph theory, with a particular focus on pathing vertices and $\mathcal{P}$-topological spaces. We define pathing vertices as those that facilitate the formation of paths within a graph, enabling the creation of $\mathcal{P}$-topological spaces. Our research presents key contributions, including the proof of openness properties in these topologies and the establishment of relationships between homeomorphisms in $\mathcal{P}$-topological spaces and graph isomorphisms, particularly in the context of connectedness. Furthermore, we explore the application of $\mathcal{P}$-topological spaces in the study of $\mathcal{N}$-star graphs, demonstrating their utility in understanding graphic topological structures. This work significantly advances the integration of topological concepts in graph theory, offering new insights and methodologies for future research.

Keywords: Simple graph, $\mathcal{P}$-topology, continuity, connectedness, density.

2020 MSC: 05C20, 05C99, 65C10, 54D99.

1. Introduction

The simple graphs were introduced by Leonhard Euler in 1736 [3], in order to solve some problems in discrete mathematics. In recent years many authors considered possibilities to define and study topologies in the set of vertices or in the set of edges of a given graph. In [11], the authors constructed a topology, called the graphic topology, on the set of vertices of a locally finite simple graph by using a special kind of neighborhoods. The authors of the paper [12], introduced the topologies, called compatible topologies, on the set of edges of a directed graph, while in [1] they considered bitopologies on undirected graphs. Nianga and Canoy [13], proposed a way of structuring a topology on a graph by using hop neighborhoods and described some topologies induced by the complements in simple undirected graphs. The paper [9], presented topologies induced by the corona, edge corona and tensor product of two graphs. By using monophonic eccentric neighborhoods, Abnabel et al. [8] defined certain topologies on the set of

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vertices of a graph, and Sari and Kopuzlu [17] structured also some topologies on the set of vertices of simple undirected graphs. The notion of pathless directed topological space on the set of vertices of directed graphs was introduced and studied in [14, 15, 18]. In [16], the authors introduced the L_2-directed topological spaces on the set of vertices of directed graphs. In [5, 6], Faten et al. used monophonic paths, the monophonic eccentric system, and the upper approximation system to define various topologies in graph theory, demonstrating their role in connectedness and discreteness in both the human nervous system and the representation of COVID-19 diffusion.

This paper provides further development of the concept of topological space in simple graph theory. In this introductory section we provide some basic notions and notation in graph theory that will be used throughout the paper. Section 2 gives the concept of pathing vertex in simple graphs which enables us to introduce the notion of $\mathcal{P}$-topological space for simple graphs. In Section 3 we give certain properties of the $\mathcal{P}$-topology and discuss relationships between homeomorphisms of $\mathcal{P}$-topological spaces and isomorphisms of graphs. Section 4 presents the role of $\mathcal{P}$-topological spaces in graphic topological spaces for $\mathcal{N}$-star graphs.

Throughout the paper G denotes a graph, $\mathsf{V}(\mathsf{G})$ the set of vertices of G, and $\mathsf{E}(\mathsf{G})$ the set of edges of G). The symbol ∂x is used to denote the degree of $x \in \mathsf{V}(\mathsf{G})$. If an edge $\gamma \in \mathsf{E}(\mathsf{G})$ joins vertices x_1 and x_2 in $\mathsf{V}(\mathsf{G})$, then we write $J_\gamma = \{x_1, x_2\}$. K_n, $n > 0$, denotes the complete graph with n vertices, and $K_{n,m}$, $n, m > 0$ denotes a complete bipartite graph. By C_n, $n > 2$, we denote a cc graph.

Let G be a simple graph. For a vertex $x \in \mathsf{V}(\mathsf{G})$, the set of all vertices adjacent with x is denoted by $\mathcal{ON}(x)$ and called the *open neighborhood* of x. The set $\mathcal{CN}[x] = \mathcal{ON}(x) \cup \{x\}$ is called the *closed neighborhood* of x. For $S \subseteq \mathsf{V}(\mathsf{G})$, the open neighborhood $\mathcal{ON}(S)$ of S is $\mathcal{ON}(S) = \bigcup_{x \in DS} \mathcal{ON}(x)$, and the closed neighborhood $\mathcal{CN}[S]$ of S is $\mathcal{CN}[S] = \mathcal{ON}(S) \cup S$.

For a simple graph G, the *graphic topological space* of G is defined as a pair $(\mathsf{V}(\mathsf{G}), T_{\mathcal{A}\mathsf{V}})$, where $T_{\mathcal{A}\mathsf{V}}$ is a topology on $\mathsf{V}(\mathsf{G})$ generated by the family of open neighborhoods of vertices in G as a subbase [11].

For undefined notions in graph theory, we refer the reader to [3]. All graphs in this paper are assumed to be undirected and locally finite.

2. The pathing topology

A *walk* in a graph is a way from one vertex to another and consists of a sequence of edges one following another. A walk in which no vertex appears more than once is called a *path*, and the path which starts and ends at the same vertex is called a *closed path*. A vertex x in a graph G is called a *pathing vertex* if there is no closed path containing all elements of $\mathcal{ON}(x)$. By $\mathsf{V}_{\mathcal{P}}(\mathsf{G})$ we denote the set of all pathing vertices in G. For a vertex $x \in \mathsf{V}(\mathsf{G})$, the set $\mathcal{K}_{\mathcal{P}}(x)$ of all pathing vertices adjacent with x is said to be the *open pathing neighborhood* of x; the *closed pathing neighborhood* $\mathcal{K}_{\mathcal{P}}[x]$ of x is the set $\mathcal{K}_{\mathcal{P}}[x] = \mathcal{K}_{\mathcal{P}}(x) \cup \{x\}$. For $H \subseteq \mathsf{V}(\mathsf{G})$ the open pathing neighborhood of H is the set $\mathcal{K}_{\mathcal{P}}(H) = \bigcup_{x \in H} \mathcal{K}_{\mathcal{P}}(x)$, and the closed pathing neighborhood of H is $\mathcal{K}_{\mathcal{P}}[H] = \mathcal{K}_{\mathcal{P}}(H) \cup H$.

Definition 2.1. Let G be a graph. Consider the family $\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G}) = \{\mathcal{K}_{\mathcal{P}}[x] : x \in \mathsf{V}(\mathsf{G})\}$ of subsets of $\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G})$. The *$\mathcal{P}$-topological space* of G is a pair $(\mathsf{V}(\mathsf{G}), T_{\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G})})$, where $T_{\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G})}$ is a topology on $\mathsf{V}(\mathsf{G})$ induced by the subbase $\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G})$.

Example 2.2.

(1) The graph G in Figure 1-A is given by $\mathsf{V}(\mathsf{G}) = \{1, 2, \cdots, 9\}$ and $\mathsf{E}(\mathsf{G}) = \{\gamma_1, \gamma_2, \cdots, \gamma_9\}$. Note that $\mathsf{V}_{\mathcal{P}}(\mathsf{G}) = \mathsf{V}(\mathsf{G})$ and the subbase $\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G})$ is the collection

$$\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G}) = \{\mathcal{K}_{\mathcal{P}}[i] : i = 1, 2, \ldots, 9\},$$

where $\mathcal{K}_{\mathcal{P}}[1] = \{1\}$, $\mathcal{K}_{\mathcal{P}}[2] = \mathcal{K}_{\mathcal{P}}[3] = \{1, 2, 3\}$, $\mathcal{K}_{\mathcal{P}}[4] = \mathcal{K}_{\mathcal{P}}[5] = \{1, 4, 5\}$, $\mathcal{K}_{\mathcal{P}}[6] = \mathcal{K}_{\mathcal{P}}[7] = \{1, 6, 7\}$, $\mathcal{K}_{\mathcal{P}}[8] = \mathcal{K}_{\mathcal{P}}[9] = \{1, 8, 9\}$. The $\mathcal{P}$-topology on $\mathsf{V}(\mathsf{G})$ is given by

$$\begin{aligned} T_{\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G})} = \{&\emptyset, \mathsf{V}(\mathsf{G}), \{1\}, \{1,2,3\}, \{1,4,5\}, \{1,6,7\}, \{1,8,9\}, \{1,2,3,4,5\}, \{1,2,3,6,7\}, \{1,2,3,8,9\}, \\ &\{1,2,3,4,5,6,7\}, \{1,2,3,4,5,8,9\}, \{1,4,5,6,7\}, \{1,4,5,8,9\}, \{1,4,5,6,7,8,9\}, \{1,6,7,8,9\}\}. \end{aligned}$$

(2) The graph G in Figure 1-B is given by $\mathsf{V}(\mathsf{G}) = \{1,2,3,4,5,6\}$ and $\mathsf{E}(\mathsf{G}) = \{\gamma_1,\gamma_2,\dots,\gamma_7\}$. Then $\mathsf{V}_{\mathcal{P}}(\mathsf{G}) = \{1,2,5,6\}$, and the subbase $\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G})$ is

$$\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G}) = \{\mathcal{K}_{\mathcal{P}}[i] : i = 1,2,3,4,5,6\},$$

where $\mathcal{K}_{\mathcal{P}}[1] = \mathcal{K}_{\mathcal{P}}[2] = \{1,2\}$, $\mathcal{K}_{\mathcal{P}}[3] = \{3\}$, $\mathcal{K}_{\mathcal{P}}[4] = \{4\}$, $\mathcal{K}_{\mathcal{P}}[5] = \mathcal{K}_{\mathcal{P}}[6] = \{5,6\}$. The $\mathcal{P}$-topology of the graph G is given by

$$\begin{aligned} \mathsf{T}_{\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G})} = \{&\emptyset, \mathsf{V}(\mathsf{G}), \{3\}, \{4\}, \{3,4\}, \{1,2\}, \{5,6\}, \{1,2,3\}, \{3,5,6\}, \{1,2,4\}, \{4,5,6\}, \\ &\{1,2,5,6\}, \{1,2,3,4\}, \{3,4,5,6\}, \{1,2,3,5,6\}, \{1,2,4,5,6\}\}. \end{aligned}$$

(3) The graph G in Figure 1-C is given by $\mathsf{V}(\mathsf{G}) = \{1,2,3,4,5,6,7,8\}$ and $\mathsf{E}(\mathsf{G}) = \{\gamma_1,\gamma_2,\cdots,\gamma_8\}$. Here, we have $\mathsf{V}_{\mathcal{P}}(\mathsf{G}) = \{5,6,7,8\}$, and the $\mathcal{P}$-topology of this graph is induced by the subbase $\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G})$ given by

$$\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G}) = \{\mathcal{K}_{\mathcal{P}}[i] : i = 1,2,3,4,5,6,7,8\},$$

where $\mathcal{K}_{\mathcal{P}}[1] = \{1\}$, $\mathcal{K}_{\mathcal{P}}[2] = \{2\}$, $\mathcal{K}_{\mathcal{P}}[3] = \{3\}$, $\mathcal{K}_{\mathcal{P}}[4] = \{4\}$, $\mathcal{K}_{\mathcal{P}}[5] = \{1,5\}$, $\mathcal{K}_{\mathcal{P}}[6] = \{2,6\}$, $\mathcal{K}_{\mathcal{P}}[7] = \{3,7\}$, $\mathcal{K}_{\mathcal{P}}[8] = \{4,8\}$.

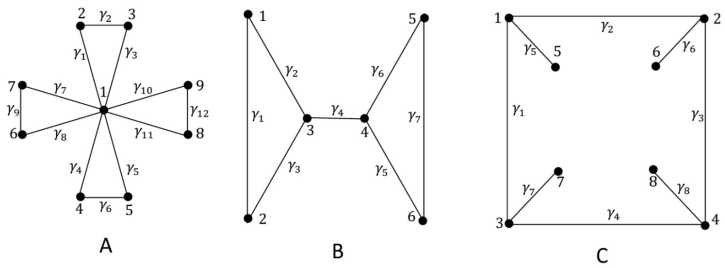

Figure 1:

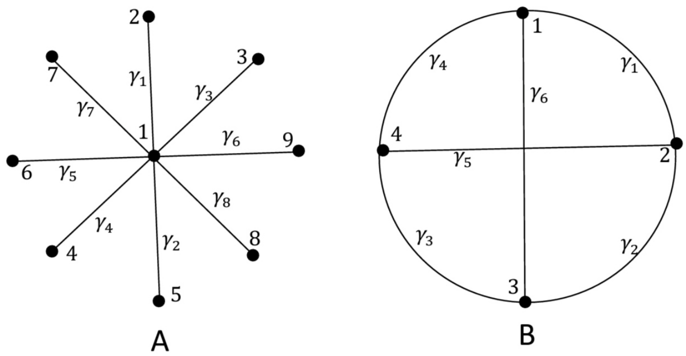

Figure 2:

Example 2.3.

(a) The graph G in Figure 2-A is given by $\mathsf{V}(\mathsf{G}) = \{1,2,3,4,5,6,7,8,9\}$ and $\mathsf{E}(\mathsf{G}) = \{\gamma_1,\gamma_2,\cdots,\gamma_8\}$. Observe that $\mathsf{V}_{\mathcal{P}}(\mathsf{G}) = \emptyset$, and the subbase $\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G})$ is given by

$$\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G}) = \{\mathcal{K}_{\mathcal{P}}[i] = \{i\} : i = 1,2,3,4,5,6,7,8,9\}.$$

The $\mathcal{P}$-topology of G is discrete.

(b) The graph G in Figure 2-B is given by $V(G) = \{1, 2, 3, 4\}$ and $E(G) = \{\gamma_1, \gamma_2, \dots, \gamma_6\}$. Now, $V_{\mathcal{P}}(G) = V(G)$, and the subbase $\mathcal{K}V_{\mathcal{P}}(G)$ is given by

$$\mathcal{K}V_{\mathcal{P}}(G) = \{\mathcal{K}_{\mathcal{P}}[i] = V(G) : i = 1, 2, 3, 4\}.$$

The $\mathcal{P}$-topology of this graph is indiscrete.

Remark 2.4. Let G be a graph. It is clear that for any isolated vertex $x \in V(G)$, $\{x\}$ is an open set in the $\mathcal{P}$-topological space $(V(G), T_{\mathcal{K}V_{\mathcal{P}}(G)})$, and if $\gamma \in E(G)$ is an isolated edge with $J_\gamma = \{x, y\}$, then $\{x\}$ and $\{y\}$ are open sets in $(V(G), T_{\mathcal{K}V_{\mathcal{P}}(G)})$.

Theorem 2.5. *Let G be a graph and $H, F \subseteq V(G)$. Then*

(a) *If $F \subseteq H$, then $\mathcal{K}_{\mathcal{P}}(F) \subseteq \mathcal{K}_{\mathcal{P}}(H)$;*
(b) $\mathcal{K}_{\mathcal{P}}(H \cap F) \subseteq \mathcal{K}_{\mathcal{P}}(H) \cap \mathcal{K}_{\mathcal{P}}(F)$;
(c) $\mathcal{K}_{\mathcal{P}}(H \cup F) = \mathcal{K}_{\mathcal{P}}(H) \cup \mathcal{K}_{\mathcal{P}}(F)$.

Proof.

(a) Let $z \in \mathcal{K}_{\mathcal{P}}(F)$. Then there is $x \in F$ such that $z \in \mathcal{K}_{\mathcal{P}}(x)$. Since $F \subseteq H$, then $x \in H$, that is, $z \in \mathcal{K}_{\mathcal{P}}(H)$. Hence $\mathcal{K}_{\mathcal{P}}(F) \subseteq \mathcal{K}_{\mathcal{P}}(H)$.

(b) Since $F \cap H \subseteq F$ and $F \cap H \subseteq H$, then by (a), we get that $\mathcal{K}_{\mathcal{P}}(F \cap H) \subseteq \mathcal{K}_{\mathcal{P}}(F)$ and $\mathcal{K}_{\mathcal{P}}(F \cap H) \subseteq \mathcal{K}_{\mathcal{P}}(H)$, that is, $\mathcal{K}_{\mathcal{P}}(F \cap H) \subseteq \mathcal{K}_{\mathcal{P}}(F) \cap \mathcal{K}_{\mathcal{P}}(H)$.

(c) Since $F \subseteq F \cup H$ and $H \subseteq F \cup H$, then by (a), we get that $\mathcal{K}_{\mathcal{P}}(F) \subseteq \mathcal{K}_{\mathcal{P}}(F \cup H)$ and $\mathcal{K}_{\mathcal{P}}(H) \subseteq \mathcal{K}_{\mathcal{P}}(F \cup H)$, that is, $\mathcal{K}_{\mathcal{P}}(F) \cup \mathcal{K}_{\mathcal{P}}(H) \subseteq \mathcal{K}_{\mathcal{P}}(F \cup H)$.

Conversely, let $z \in \mathcal{K}_{\mathcal{P}}(F \cup H)$. Then there is $x \in F \cup H$ such that $z \in \mathcal{K}_{\mathcal{P}}(x)$. Then $x \in F$ or $x \in H$. Hence $z \in \mathcal{K}_{\mathcal{P}}(F)$ or $z \in \mathcal{K}_{\mathcal{P}}(H)$, that is, $z \in \mathcal{K}_{\mathcal{P}}(F) \cup \mathcal{K}_{\mathcal{P}}(H)$. Hence, $\mathcal{K}_{\mathcal{P}}(F \cup H) \subseteq \mathcal{K}_{\mathcal{P}}(F) \cup \mathcal{K}_{\mathcal{P}}(H)$. Therefore, $\mathcal{K}_{\mathcal{P}}(H \cup F) = \mathcal{K}_{\mathcal{P}}(H) \cup \mathcal{K}_{\mathcal{P}}(F)$. □

In the item (b) of the above theorem, $\mathcal{K}_{\mathcal{P}}(H \cap F)$ may be a proper subset of $\mathcal{K}_{\mathcal{P}}(H) \cap \mathcal{K}_{\mathcal{P}}(F)$. For example, for the graph G in Figure 3 with $V(G) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $E(G) = \{\gamma_1, \gamma_2, \dots, \gamma_{10}\}$, we have $V_{\mathcal{P}}(G) = V(G) - \{9\}$. If we take $F = \{3, 4, 7, 8\}$ and $H = \{1, 2, , 5, 6\}$, then $F \cap H = \emptyset$ and

$$\mathcal{K}_{\mathcal{P}}(F) \cap \mathcal{K}_{\mathcal{P}}(H) = \{3, 4, 7, 8, 9\} \cap \{1, 2, 5, 6, 9\} = \{9\}.$$

It is clear that the $\mathcal{P}$-topology of the cycle graph C_3 is indiscrete. However, for $n > 3$ the situation is quite different.

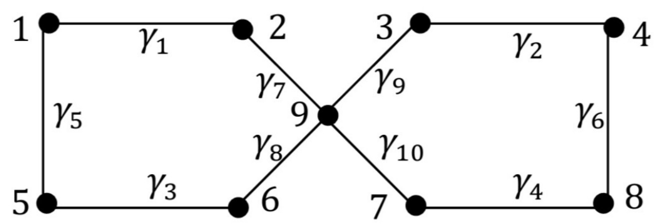

Figure 3:

Theorem 2.6. *For all $n > 3$, the $\mathcal{P}$-topology of the cycle graph C_n is discrete.*

Proof. It is clear that for all $n > 3$, $V_{\mathcal{P}}(C_n) = V(C_n)$. First, we prove the case $n = 4$. Let $V(C_4) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and

$$E(C_4) : \alpha_1 \to \alpha_2 \to \alpha_3 \to \alpha_4 \to \alpha_1.$$

The subbase $\mathcal{K}V_{\mathcal{P}}(C_4)$ is given by

$$\mathcal{K}V_{\mathcal{P}}(C_4) = \{\mathcal{K}_{\mathcal{P}}[\alpha_1] = \{\alpha_4, \alpha_1, \alpha_2\}, \ \ \mathcal{K}_{\mathcal{P}}[\alpha_2] = \{\alpha_1, \alpha_2, \alpha_3\}, \ \ \mathcal{K}_{\mathcal{P}}[\alpha_3] = \{\alpha_2, \alpha_3, \alpha_4\}, \ \ \mathcal{K}_{\mathcal{P}}[\alpha_4] = \{\alpha_3, \alpha_4, \alpha_1\}\}.$$

For $n > 4$, let $V(C_n) = \{\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_n\}$ and

$$E(C_n) : \alpha_1 \to \alpha_2 \to \cdots \to \alpha_{n-1} \to \alpha_n \to \alpha_1.$$

The subbase $\mathcal{K}V_{\mathcal{P}}(C_n)$ is given by

$$\mathcal{K}V_{\mathcal{P}}(C_n) = \{\mathcal{K}_{\mathcal{P}}[\alpha_1] = \{\alpha_n, \alpha_1, \alpha_2\}, \dots, \mathcal{K}_{\mathcal{P}}[\alpha_k] = \{\alpha_{k-1}, \alpha_k, \alpha_{k+1}\}, \dots \mathcal{K}_{\mathcal{P}}[\alpha_n] = \{\alpha_{n-1}, \alpha_n, \alpha_1\}\}$$

for all $k = 2, 3, \dots, n-1$. Note that

$$\{\alpha_1\} = \mathcal{K}_{\mathcal{P}}[\alpha_2] \cap \mathcal{K}_{\mathcal{P}}[\alpha_n] \in T_{\mathcal{K}V_{\mathcal{P}}(C_n)}, \ \ \{\alpha_n\} = \mathcal{K}_{\mathcal{P}}[\alpha_{n-1}] \cap \mathcal{K}_{\mathcal{P}}[\alpha_1] \in T_{\mathcal{K}V_{\mathcal{P}}(C_n)},$$

and

$$\{\alpha_k\} = \mathcal{K}_{\mathcal{P}}[\alpha_{k-1}] \cap \mathcal{K}_{\mathcal{P}}[\alpha_{k+1}] \in T_{\mathcal{K}V_{\mathcal{P}}(C_n)},$$

for all $k = 2, 3, \dots, n-1$. One concludes that the $\mathcal{P}$-topology of the graph C_n is discrete. □

It is not difficult to see that for $n = m = 1$ or $n = 2$ and $m = 1$, and $n = m = 2$ the $\mathcal{P}$-topology of a complete bipartite graph $K_{n,m}$ is discrete.

Theorem 2.7. *If* $V(K_{n,m}) = V_n \cup V_m$, $n, m \geqslant 2$, $m \neq n$, *is a bipartite graph, then*

$$T_{\mathcal{K}V_{\mathcal{P}}(K_{n,m})} = \{\emptyset, V(K_{n,m}), V_n, V_m\}.$$

Proof. It is clear that for all $n, m \geqslant 2$, $m \neq n$, $V_{\mathcal{P}}(K_{n,m}) = V(K_{n,m})$. Let $\alpha \in V(K_{n,m})$ be any vertex. Since $V_n \cap V_m = \emptyset$, then $\alpha \in V_n$ and $\alpha \notin V_m$ or $\alpha \in V_m$ and $\alpha \notin V_m$. Let $\alpha \in V_n$. Then $\mathcal{K}_{\mathcal{P}}(\alpha) = V_m$.

Similarly, if $\alpha \in V_m$ (and $\alpha \notin V_m$), then $\mathcal{K}_{\mathcal{P}}(\alpha) = V_n$. Hence, the $\mathcal{P}$-topological space $T_{\mathcal{K}V_{\mathcal{P}}(K_{n,m})} = \{\emptyset, V(K_{n,m}), V_n, V_m\}$. □

It is not difficult to see that the $\mathcal{P}$-topology of the complete graphs K_1 and K_2 is discrete.

Theorem 2.8. *For all* $n > 2$, *the* $\mathcal{P}$*-topology of the complete graph* K_n *is indiscrete.*

Proof. It is easily seen that for all $n > 2$, $V_{\mathcal{P}}(K_n) = V(K_n)$. Hence, for any $\alpha \in V(K_n)$, $\mathcal{K}_{\mathcal{P}}[\alpha] = V(K_n)$. It follows that the $\mathcal{P}$-topological space $(V(K_n), T_{\mathcal{K}V_{\mathcal{P}}(K_n)})$ is indiscrete. □

Recall that a topological space is said to be an *Alexandroff space* if the intersection of an arbitrary collection of open sets in it is also an open set.

Theorem 2.9. *Let* G *be a locally finite simple graph. The* $\mathcal{P}$*-topological space* $(V(G), T_{\mathcal{K}V_{\mathcal{P}}(G)})$ *is an Alexandroff space.*

Proof. Let $\mathcal{U} = \{\mathcal{K}_{\mathcal{P}}[\alpha] : \alpha \in A \subseteq V(G)\}$ be the collection of elements of $\mathcal{K}V_{\mathcal{P}}(G)$. We show that $\bigcap_{\alpha \in A} \mathcal{K}_{\mathcal{P}}[\alpha]$ is an open set. If $\beta \in \bigcap_{\alpha \in A} \mathcal{K}_{\mathcal{P}}[\alpha]$, then $\beta \in \mathcal{K}_{\mathcal{P}}[\alpha]$ for all $\alpha \in A$. Hence, $\alpha \in \mathcal{K}_{\mathcal{P}}[\beta]$ for all $\alpha \in A$. That is, $A \subseteq \mathcal{K}_{\mathcal{P}}[\beta]$. Since G is locally finite, then A is finite. Hence, $\bigcap_{\alpha \in A} \mathcal{K}_{\mathcal{P}}[\alpha]$ is an open set. □

By the above theorem, the intersection of all open sets containing a vertex α is the smallest open set containing $\alpha \in V(G)$. It is denoted by $\mathcal{MO}(\alpha)$.

Remark 2.10. If a vertex $\alpha \in V(G)$ is an isolated vertex of a graph G, then $\mathcal{MO}(\alpha) = \{\alpha\}$. On the other side, if γ is an isolated edge of G with $J_\gamma = \{\alpha, \beta\}$, then $\mathcal{MO}(\alpha) = \mathcal{MO}(\beta) = J_\gamma$.

Theorem 2.11. *For any simple graph G and for each $\alpha \in V(G)$, we have $\mathcal{MO}(\alpha) = \bigcap_{\beta \in \mathcal{K}_\mathcal{P}[\alpha]} \mathcal{K}_\mathcal{P}[\beta]$.*

Proof. It is clear by the definition of $\mathcal{K}V_\mathcal{P}(G)$ that $\mathcal{K}_\mathcal{P}[\alpha]$ is an open set in the $\mathcal{P}$-topological space $(V(G), T_{\mathcal{K}V_\mathcal{P}(G)})$. By Theorem 2.9, $\bigcap_{\beta \in \mathcal{K}_\mathcal{P}[\alpha]} \mathcal{K}_\mathcal{P}[\beta]$ is an open set. If $\beta \in \mathcal{K}_\mathcal{P}[\alpha]$, then clearly $\alpha \in \mathcal{K}_\mathcal{P}[\beta]$. Hence, $\bigcap_{\beta \in \mathcal{K}_\mathcal{P}[\alpha]} \mathcal{K}_\mathcal{P}[\beta]$ is an open neighborhood of α. As $\mathcal{MO}(\alpha)$ is the smallest open neighborhood of α, $\mathcal{MO}(\alpha) \subseteq \bigcap_{\beta \in \mathcal{K}_\mathcal{P}[\alpha]} \mathcal{K}_\mathcal{P}[\beta]$. On the other side, $\mathcal{MO}(\alpha)$ is the intersection of all open sets containing α. Let

$$\mathcal{MO}(\alpha) = \bigcap_{\beta \in A} \mathcal{K}_\mathcal{P}[\beta]$$

for all $\beta \in A$. This implies $\beta \in \mathcal{K}_\mathcal{P}[\alpha]$ for all $\beta \in A$. This means $A \subseteq \mathcal{K}_\mathcal{P}[\alpha]$. Therefore,

$$\bigcap_{\beta \in \mathcal{K}_\mathcal{P}[\alpha]} \mathcal{K}_\mathcal{P}[\beta] \subseteq \bigcap_{\beta \in A} \mathcal{K}_\mathcal{P}[\beta] = \mathcal{MO}(\alpha).$$

So, $\mathcal{MO}(\alpha) = \bigcap_{\beta \in \mathcal{K}_\mathcal{P}[\alpha]} \mathcal{K}_\mathcal{P}[\beta]$. □

Theorem 2.12. *Let G be a simple graph and $\alpha, \beta \in V(G)$. Then $\alpha \in \mathcal{K}_\mathcal{P}[\beta]$ if and only if $\mathcal{K}_\mathcal{P}[\beta] \subseteq \mathcal{K}_\mathcal{P}[\alpha]$.*

Proof. Let $\alpha \in \mathcal{K}_\mathcal{P}[\beta]$. Then, by the above, $\mathcal{K}_\mathcal{P}[\beta] = \bigcap_{z \in \mathcal{K}_\mathcal{P}[\beta]} \mathcal{K}_\mathcal{P}[z]$. Then $\alpha \in \bigcap_{z \in \mathcal{K}_\mathcal{P}[\beta]} \mathcal{K}_\mathcal{P}[z]$. So, $\alpha \in \mathcal{K}_\mathcal{P}[z]$ for all $z \in \mathcal{K}_\mathcal{P}[\beta]$. This implies $z \in \mathcal{K}_\mathcal{P}[\alpha]$ for all $z \in \mathcal{K}_\mathcal{P}[\beta]$, which means $\mathcal{K}_\mathcal{P}[\beta] \subseteq \mathcal{K}_\mathcal{P}[\alpha]$.

Conversely, let $\mathcal{K}_\mathcal{P}[\beta] \subseteq \mathcal{K}_\mathcal{P}[\alpha]$. In other words,

$$\alpha \in \bigcap_{z \in \mathcal{K}_\mathcal{P}[\alpha]} \mathcal{K}_\mathcal{P}[z] \subseteq \bigcap_{z \in \mathcal{K}_\mathcal{P}[\beta]} \mathcal{K}_\mathcal{P}[z] = \mathcal{K}_\mathcal{P}[\beta].$$

It means $\alpha \in \mathcal{K}_\mathcal{P}[\beta]$. □

Theorem 2.13. *Let G be a simple graph. Then the space $(V(G), T_{\mathcal{K}V_\mathcal{P}(G)})$ is discrete if and only if $\mathcal{K}_\mathcal{P}[\beta] \nsubseteq \mathcal{K}_\mathcal{P}[\alpha]$ and $\mathcal{K}_\mathcal{P}[\alpha] \nsubseteq \mathcal{K}_\mathcal{P}[\beta]$ for all $\alpha, \beta \in V(G)$.*

Proof. Let $(V(G), T_{\mathcal{K}V_\mathcal{P}(G)})$ be discrete. Then $\mathcal{K}_\mathcal{P}[\alpha] = \{\alpha\}$ for all $\alpha \in V(G)$. Thus for two different vertices $\alpha, \beta \in V(G)$, $\beta \in \mathcal{K}_\mathcal{P}[\alpha]$ and $\alpha \in \mathcal{K}_\mathcal{P}[\beta]$. By Theorem 2.12, $\mathcal{K}_\mathcal{P}[\beta] \nsubseteq \mathcal{K}_\mathcal{P}[\alpha]$ and $\mathcal{K}_\mathcal{P}[\alpha] \nsubseteq \mathcal{K}_\mathcal{P}[\beta]$.

Conversely, let α be any vertex in $V(G)$. It is clear that $\alpha \in \mathcal{K}_\mathcal{P}[\alpha]$. If $\beta \neq \alpha \in V(G)$ and $\beta \in \mathcal{K}_\mathcal{P}[\alpha]$, then by Theorem 2.12, $\mathcal{K}_\mathcal{P}[\alpha] \subseteq \mathcal{K}_\mathcal{P}[\beta]$, which is a contradiction. Hence, $\mathcal{MO}(\alpha) = \{\alpha\}$ for all $\alpha \in V(G)$. The set $\{\alpha\}$ is an open set for all $\alpha \in V(G)$. That is, $(E(G), T_{\mathcal{K}V_\mathcal{P}(G)})$ is discrete. □

Theorem 2.14. *Let G be a simple graph and $\alpha \in V(G)$. Then for all $\beta \in \mathcal{K}_\mathcal{P}[\alpha]$, $\mathcal{K}_\mathcal{P}[\alpha] \subseteq \mathcal{K}_\mathcal{P}[\beta]$ (and $\overline{\mathcal{K}_\mathcal{P}[\alpha]} \subseteq \overline{\mathcal{K}_\mathcal{P}[\beta]}$).*

Proof. By Theorem 2.12, $\mathcal{K}_\mathcal{P}[\alpha] = \bigcap_{\beta \in \mathcal{K}_\mathcal{P}[\alpha]} \mathcal{K}_\mathcal{P}[\beta]$. That is, $\mathcal{K}_\mathcal{P}[\alpha] \subseteq \mathcal{K}_\mathcal{P}[\beta]$ for all $\beta \in \mathcal{K}_\mathcal{P}[\alpha]$.

Let $\mu \in \overline{\mathcal{K}_\mathcal{P}[\alpha]}$. Then for every open set U containing μ, $U \cap \mathcal{K}_\mathcal{P}[\alpha] \neq \emptyset$. Since $\mathcal{K}_\mathcal{P}[\alpha] \subseteq \mathcal{K}_\mathcal{P}[\beta]$ for all $\beta \in \mathcal{K}_\mathcal{P}[\alpha]$, then $U \cap \mathcal{K}_\mathcal{P}[\beta] \neq \emptyset$ for all $\beta \in \mathcal{K}_\mathcal{P}[\alpha]$. Then $\mu \in \overline{\mathcal{K}_\mathcal{P}[\beta]}$ for all $\beta \in \mathcal{K}_\mathcal{P}[\alpha]$. Hence, $\overline{\mathcal{K}_\mathcal{P}[\alpha]} \subseteq \overline{\mathcal{K}_\mathcal{P}[\beta]}$ for all $\beta \in \mathcal{K}_\mathcal{P}[\alpha]$. □

Corollary 2.15. *Let G be a simple graph and $\alpha \in V(G)$. Then $\overline{\{\alpha\}} \subseteq \overline{\mathcal{K}_\mathcal{P}[\alpha]} \subseteq \overline{\mathcal{K}_\mathcal{P}[\beta]}$ for all $\beta \in \mathcal{K}_\mathcal{P}[\alpha]$.*

Proof. Let $\mu \in \overline{\{\alpha\}}$. Then for every open set U containing μ, $U \cap \{\alpha\} \neq \emptyset$. Since $\alpha \in \mathcal{K}_\mathcal{P}[\alpha]$, then $U \cap \mathcal{K}_\mathcal{P}[\alpha] \neq \emptyset$. Then $\mu \in \overline{\mathcal{K}_\mathcal{P}[\alpha]}$. Hence, $\overline{\{\alpha\}} \subseteq \overline{\mathcal{K}_\mathcal{P}[\alpha]}$.

The second part is clear from Theorem 2.14. □

Corollary 2.16. *For a simple graph G and $\alpha, \beta \in V(G)$, $\alpha \in \overline{\{\beta\}}$ if and only if $\mathcal{K}_\mathcal{P}[\alpha] \subseteq \mathcal{K}_\mathcal{P}[\beta]$.*

3. Continuity in the $\mathcal{P}$-topology

In this section we consider continuity properties in pathing topological spaces of graphs. Topological terminology follows the book [7].

If γ is an edge in a simple graph G with $J_\gamma = \{x, y\}$, then we write γ as xy or yx, where $x, y \in V(G)$. Graphs G_1 and G_2 are called isomorphic, denoted by $G_1 \cong G_2$, if there is a bijective mapping $f : V(G_1) \to V(G_2)$ such that $xy \in V(G_1)$ if and only if $f(x)f(y) \in V(G_1)$.

Theorem 3.1. *Let G_1 and G_2 be two simple graphs and $f : V(G_1) \to V(G_2)$ be a mapping. The mapping f is continuous if and only if for all $x, y \in V(G_1)$, $\mathcal{K}_\mathcal{P}[x] \subseteq \mathcal{K}_\mathcal{P}[y]$ implies $\mathcal{K}_\mathcal{P}[f(x)] \subseteq \mathcal{K}_\mathcal{P}[f(y)]$.*

Proof. Suppose f is continuous. Let $x, y \in V(G_1)$ be such that $\mathcal{K}_\mathcal{P}[x] \subseteq \mathcal{K}_\mathcal{P}[y]$. Then by Corollary 2.16, we get $x \in \overline{\{y\}}$, and by the continuity of f, $f(x) \in f(\overline{\{y\}}) \subseteq \overline{\{f(y)\}}$. By Corollary 2.16 we get $\mathcal{K}_\mathcal{P}[f(x)] \subseteq \mathcal{K}_\mathcal{P}[f(y)]$.

Conversely, suppose that for all $x, y \in V(G_1)$, $\mathcal{K}_\mathcal{P}[x] \subseteq \mathcal{K}_\mathcal{P}[y]$ implies $\mathcal{K}_\mathcal{P}[f(x)] \subseteq \mathcal{K}_\mathcal{P}[f(y)]$. Let $A \subseteq V(G_1)$. If $x \in \overline{A}$, then $x \in \overline{\{y\}}$ for some $y \in A$. Hence, $\mathcal{K}_\mathcal{P}[x] \subseteq \mathcal{K}_\mathcal{P}[y]$. By our assumption we get $\mathcal{K}_\mathcal{P}[f(x)] \subseteq \mathcal{K}_\mathcal{P}[f(y)]$. This implies $f(x) \in \overline{\{f(y)\}} \subseteq \overline{f(A)}$, which means that f is continuous. □

Theorem 3.2. *Let G_1 and G_2 be simple graphs and $f : V(G_1) \to V(G_2)$ be a mapping. If f is onto and for all $x, y \in V(G_1)$, $\mathcal{K}_\mathcal{P}[f(x)] \subseteq \mathcal{K}_\mathcal{P}[f(y)]$ implies $\mathcal{K}_\mathcal{P}[x] \subseteq \mathcal{K}_\mathcal{P}[y]$, then f is a closed mapping.*

Proof. Let $F \subseteq V(G_1)$ be a closed set. Since f is a surjection, there exists a mapping $g : V(G_2) \to V(G_1)$ such that $f \circ g = id_{V(G_2)}$. We will prove that g is a continuous mapping. Let $x, y \in V(G_2)$ be such that $\mathcal{K}_\mathcal{P}[x] \subseteq \mathcal{K}_\mathcal{P}[y]$. Therefore, $\mathcal{K}_\mathcal{P}[f(g(x))] \subseteq \mathcal{K}_\mathcal{P}[f(g(y))]$. By the hypothesis, we get $\mathcal{K}_\mathcal{P}[g(x)] \subseteq \mathcal{K}_\mathcal{P}[g(y)]$. Then by Theorem 3.1, g is continuous. Hence, $f(F) = g^{\leftarrow}(F)$ is a closed set, which means that f is a closed mapping. □

Theorem 3.3. *Let G_1 and G_2 be simple graphs and $f : V(G_1) \to V(G_2)$ be a mapping. If f is a closed injection, then for all $x, y \in V(G_1)$, $\mathcal{K}_\mathcal{P}[f(x)] \subseteq \mathcal{K}_\mathcal{P}[f(y)]$ implies $\mathcal{K}_\mathcal{P}[x] \subseteq \mathcal{K}_\mathcal{P}[y]$.*

Proof. Let $x, y \in V(G_2)$ be arbitrary vertices such that $\mathcal{K}_\mathcal{P}[f(x)] \subseteq \mathcal{K}_\mathcal{P}[f(y)]$. Since f is 1-1, there is a mapping $g : V(G_2) \to V(G_1)$ such that $g \circ f = id_{V(G_1)}$. Since f is 1-1 and closed, it is not difficult to prove that g is continuous. This implies that $\mathcal{K}_\mathcal{P}[g(f(x))] \subseteq \mathcal{K}_\mathcal{P}[g(f(y))]$. So, $\mathcal{K}_\mathcal{P}[x] \subseteq \mathcal{K}_\mathcal{P}[y]$. □

Lemma 3.4. *A bijective mapping $f : V(G_1) \to V(G_2)$ between simple graphs G_1 and G_2 is a homeomorphism if and only if for all $x, y \in V(G_1)$, $\mathcal{K}_\mathcal{P}[x] \subseteq \mathcal{K}_\mathcal{P}[y]$ if and only if $\mathcal{K}_\mathcal{P}[f(x)] \subseteq \mathcal{K}_\mathcal{P}[f(y)]$.*

Theorem 3.5. *Let G_1 and G_2 be simple graphs without isolated vertices. If $G_1 \cong G_2$, then there is a homeomorphism between spaces $(V(G_1), T_{\mathcal{K}V_\mathcal{P}(G_1)})$ and $(V(G_1), T_{\mathcal{K}V_\mathcal{P}(G_1)})$.*

Proof. Let $f : V(G_1) \to V(G_2)$ be a bijective mapping such that $xy \in E(G_1)$ if and only if $f(x)f(y) \in E(G_2)$ for $x, y \in V(G_1)$. Clearly, if $x \in V(G_1)$ is a pathing vertex, then the subgraph of G_2 induced by the open neighborhood $\mathcal{ON}(f(x))$ is complete and $f(z) \in \mathcal{K}_\mathcal{P}(\mathcal{ON}(f(x)))$. So, $f(x) \in V(G_2)$ is a pathing vertex in G_2. So, by Lemma 3.4, f is a homeomorphism between spaces $(V(G_1), T_{\mathcal{K}V_\mathcal{P}(G_1)})$ and $(V(G_2), T_{\mathcal{K}V_\mathcal{P}(G_2)})$. □

If the $\mathcal{P}$-topological spaces $(V(G_1), T_{\mathcal{K}V_\mathcal{P}(G_1)})$ and $(V(G_2), T_{\mathcal{K}V_\mathcal{P}(G_2)})$ are homeomorphic, then $G_1 \cong G_2$ is not always true. The graph G_1 in Figure 4 given by $V(G_1) = \{1, 2, 3, 4, 5, 6\}$ and $E(G_1) = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ with the subbase

$$\mathcal{K}V_\mathcal{P}(G_1) = \{\mathcal{K}_\mathcal{P}[i] = \{i\} : i = 1, 2, 3, 4, 5, 6\}$$

has the discrete $\mathcal{P}$-topology. Also, the graph G_2 given by

$$V(G_2) = \{v_1, x_2, x_3, x_4, x_5, x_6\}, \quad E(G_2) = \{\gamma'_1, \gamma'_2, \gamma'_3, \gamma'_4, \gamma'_5\}$$

with the subbase

$$\mathcal{K}V_\mathcal{P}(G_2) = \{\mathcal{K}_\mathcal{P}[x_i] = \{x_i\} : i = 1, 2, 3, 4, 5, 6\}$$

has the discrete $\mathcal{P}$-topology. However, G_1 and G_2 are not isomorphic graphs since $|E(G_1)|$ and $|E(G_2)|$ are finite and $|E(G_2)| \neq |E(G_1)|$.

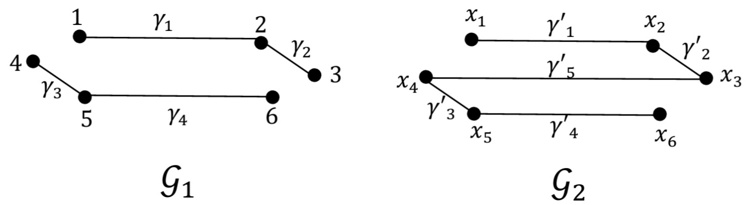

Figure 4:

Theorem 3.6. *Let* G *be a simple graph without isolated vertex. If* G *is a disconnected graph, then the* $\mathcal{P}$*-topological space* $(\mathsf{V}(\mathsf{G}), \mathsf{T}_{\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G})})$ *is a disconnected space.*

Proof. Let G be a disconnected graph. Consider the collection $\Sigma := \{\mathsf{G}_i : i \in I\}$ of all components in G, where $\mathsf{G}_i = (\mathsf{V}(\mathsf{G}_i), \mathsf{E}(\mathsf{G}_i))$ for all $i \in I$. For all $i \in I$, $\mathsf{E}(\mathsf{G}_i) = \bigcup_{x \in \mathsf{V}(\mathsf{G}_i)} \mathcal{K}_{\mathcal{P}}[x]$. Then $U := \mathsf{V}(\mathsf{G}_j)$, $j \in I$, is a nonempty proper open subset of $\mathsf{V}(\mathsf{G})$. It follows that

$$U^c = [\mathsf{V}(\mathsf{G}_j)]^c = \bigcup_{i \in I-j} \mathsf{V}(\mathsf{G}_i)$$

is also a nonempty proper open subset of $\mathsf{V}(\mathsf{G})$. That is, $(\mathsf{V}(\mathsf{G}), \mathsf{T}_{\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G})})$ is a disconnected space. □

In Figure 5 the simple graph G is connected. Here

$$\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G}) = \{\mathcal{K}_{\mathcal{P}}[i] = \{i\} : i = 1,2,3,4,5,6,7,8\},$$

while the space $(\mathsf{V}(\mathsf{G}), \mathsf{T}_{\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G})})$ is discrete, hence disconnected.

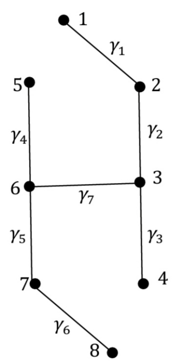

Figure 5:

Remark 3.7. It is clear that the $\mathcal{P}$-topological space $(\mathsf{E}(\mathsf{G}), \mathsf{T}_{\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G})})$ of any simple graph G is a compact space if $\mathsf{E}(\mathsf{G})$ is a finite set. However, the converse is not always true; the $\mathcal{P}$-topology of a complete graph with an infinite set of vertices is compact.

4. $\mathcal{P}$-topological spaces for $\mathcal{N}$-star graphs

A vertex in a simple graph G is said to be a *star vertex* if it is adjacent to all the other vertices. A (simple) graph G is called an $\mathcal{N}$*-star graph* if the vertices of G are not stars and not isolated vertices. The graph G in Figure 6-A is an $\mathcal{N}$-star graph, where its subgraph with the vertices set $\mathsf{V}_{\mathcal{P}}(\mathsf{G}) = \{1, 2, \dots, 12\}$ has no stars and isolated vertices. This graph G is given by

$$\mathsf{V}(\mathsf{G}) = \{1,2,3,4,5,6,7\} \text{ and } \mathsf{E}(\mathsf{G}) = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8\}.$$

The subbase $\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G})$ is given by

$$\mathcal{K}\mathsf{V}_{\mathcal{P}}(\mathsf{G}) = \{\mathcal{K}_{\mathcal{P}}[i] : i = 1, 2, 3, 4, 5, 6, 7\},$$

where $\mathcal{K}_{\mathcal{P}}[1] = \{1\}$, $\mathcal{K}_{\mathcal{P}}[2] = \{2,3\}$, $\mathcal{K}_{\mathcal{P}}[3] = \{2,3,4\}$, $\mathcal{K}_{\mathcal{P}}[4] = \{3,4\}$, $\mathcal{K}_{\mathcal{P}}[5] = \{5,6\}$, $\mathcal{K}_{\mathcal{P}}[6] = \{5,6,7\}$, $\mathcal{K}_{\mathcal{P}}[7] = \{6,7\}$. In Figure 6-B, the subgraph $\mathsf{G}_{\mathcal{P}} = (\mathsf{V}_{\mathcal{P}}(\mathsf{G}), \mathsf{E}_{\mathcal{P}}(\mathsf{G}))$ is given by

$$\mathsf{V}_{\mathcal{P}}(\mathsf{G}) = \{2,3,4,5,6,7\} \text{ and } \mathsf{E}_{\mathcal{P}}(\mathsf{G}) = \{\gamma_2, \gamma_3, \gamma_6, \gamma_7\}.$$

The subbase $\mathcal{A}\mathsf{V}_{\mathcal{P}}$ of the graphic topological space $(\mathsf{V}_{\mathcal{P}}(\mathsf{G}), \mathsf{T}_{\mathcal{A}\mathsf{V}_{\mathcal{P}}})$ is given by $\mathcal{K}_{\mathcal{P}}[2] = \{2,3\}$, $\mathcal{K}_{\mathcal{P}}[3] = \{2,3,4\}$, $\mathcal{K}_{\mathcal{P}}[4] = \{3,4\}$, $\mathcal{K}_{\mathcal{P}}[5] = \{5,6\}$, $\mathcal{K}_{\mathcal{P}}[6] = \{5,6,7\}$, $\mathcal{K}_{\mathcal{P}}[7] = \{6,7\}$. Note that $\mathcal{A}\mathsf{V}_{\mathcal{P}} = \{\mathcal{K}_{\mathcal{P}}(i) : i \in \mathsf{V}_{\mathcal{P}}(\mathsf{G})\}$.

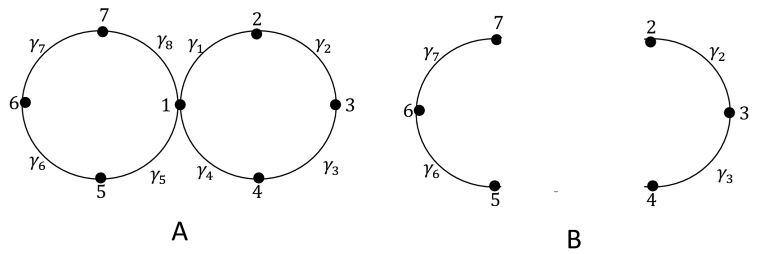

Figure 6:

Since the topological space $(\mathsf{V}_{\mathcal{P}}(\mathsf{G}), \mathsf{T}_{\mathcal{A}\mathsf{V}_{\mathcal{P}}})$ of any $\mathcal{N}$-star graph G is an Alexandroff space, then for any $\alpha \in \mathsf{V}(\mathsf{G})$, $\mathcal{P}\mathcal{O}_\alpha$ denotes the smallest open set containing α (which is the intersection of all open sets containing α).

Theorem 4.1. *Let $\mathcal{G} = (\mathsf{V}(\mathsf{G}), \mathsf{E}(\mathsf{G}))$ be an $\mathcal{N}$-star graph and $\mathsf{G}_{\mathcal{P}} = (\mathsf{V}_{\mathcal{P}}(\mathsf{G}), \mathsf{E}_{\mathcal{P}}(\mathsf{G}))$ be a connected graph. Then the set $\mathsf{D} = \{\alpha \in \mathsf{V}_{\mathcal{P}}(\mathsf{G}) : \partial\alpha > 1\}$ is dense in the graphic topological space $(\mathsf{V}_{\mathcal{P}}(\mathsf{G}), \mathsf{T}_{\mathcal{A}\mathsf{V}_{\mathcal{P}}})$.*

Proof. Since $\mathcal{P}\mathcal{O}_\alpha$ is the smallest open set containing α for all $\alpha \in \mathsf{V}_{\mathcal{P}}(\mathsf{G})$, to prove $\mathsf{D} \cap \mathsf{O} \neq \emptyset$ for any open set O in $(\mathsf{V}_{\mathcal{P}}(\mathsf{G}), \mathsf{T}_{\mathcal{A}\mathsf{V}_{\mathcal{P}}})$, it is enough to prove that $\mathsf{D} \cap \mathcal{P}\mathcal{O}_\alpha \neq \emptyset$ for all $\alpha \in \mathsf{V}_{\mathcal{P}}(\mathsf{G}) \setminus \mathsf{D}$. Let $\alpha \in \mathsf{V}_{\mathcal{P}}(\mathsf{G}) \setminus \mathsf{D}$. Since α is not isolated, there is $\beta \in \mathsf{V}_{\mathcal{P}}(\mathsf{G})$ such that $\mathcal{O}\mathcal{N}(\alpha) = \{\beta\}$. Hence, $\mathcal{P}\mathcal{O}_\alpha = \mathcal{O}\mathcal{N}(\beta)$. Since G is not a star, then $\partial(\beta) > 1$. So, there exists some $w \in \mathsf{D}$ such that $w \in \mathcal{O}\mathcal{N}(\beta)$. Then $w \in \mathsf{D} \cap \mathcal{O}\mathcal{N}(\beta) = \mathsf{D} \cap \mathcal{P}\mathcal{O}_\alpha$, that is, $\mathsf{D} \cap \mathcal{P}\mathcal{O}_\alpha \neq \emptyset$ for all $\alpha \in \mathsf{V}_{\mathcal{P}}(\mathsf{G}) \setminus \mathsf{D}$. □

Theorem 4.2. *Let $\mathsf{G} = (\mathsf{V}(\mathsf{G}), \mathsf{E}(\mathsf{G}))$ be an $\mathcal{N}$-star graph, $\psi = \{\mathcal{P}\mathcal{O}_\alpha : \alpha \in \mathsf{V}_{\mathcal{P}}(\mathsf{G})\}$, and $\mathcal{M}_\psi$ be the collection of minimal sets in ψ. If $\mathsf{D} \subseteq \mathsf{V}_{\mathcal{P}}(\mathsf{G})$ is a minimal dense set in $(\mathsf{V}_{\mathcal{P}}(\mathsf{G}), \mathsf{T}_{\mathcal{A}\mathsf{V}_{\mathcal{P}}})$, then there is an onto mapping $f : \mathcal{M}_\psi \to \mathsf{D}$ such that $f(\mathcal{P}\mathcal{O}_\alpha) \in \mathcal{P}\mathcal{O}_\alpha$ for all $\mathcal{P}\mathcal{O}_\alpha \in \mathcal{M}_\psi$.*

Proof. By the definition of $\mathcal{M}_\psi$, the elements of $\mathcal{M}_\psi$ are pairwise disjoint. Since $\overline{\mathsf{D}} = \mathsf{V}_{\mathcal{P}}(\mathsf{G})$ in $(\mathsf{V}_{\mathcal{P}}(\mathsf{G}), \mathsf{T}_{\mathcal{A}\mathsf{V}_{\mathcal{P}}})$, then there is some $\alpha \in \mathsf{O} \cap \mathsf{D}$ for all $\mathsf{O} \in \mathcal{M}_\psi$. Since $\alpha \in \mathsf{O}$ and $\mathsf{O} \in \mathcal{M}_\psi$, then $\mathsf{O} \subseteq \mathcal{P}\mathcal{O}_\alpha$ and it is clear that $\mathcal{P}\mathcal{O}_\alpha \subseteq \mathsf{O}$, that is, $\mathcal{P}\mathcal{O}_\alpha = \mathsf{O}$. If $\beta \in \mathsf{O} \cap (\mathsf{D} \setminus \{\alpha\})$, then similarly we get that $\mathcal{P}\mathcal{O}_\alpha = \mathcal{P}\mathcal{O}_\beta = \mathsf{O}$. Hence, $\overline{\{\alpha\}} = \overline{\{\beta\}}$. In this case we get that $\overline{\mathsf{D} \setminus \{\beta\}} = \mathsf{V}_{\mathcal{P}}(\mathsf{G})$ and this contradicts minimality of D. So, $\mathsf{O} \cap \mathsf{D} = \{\alpha\}$. Define a mapping $f : [\mathsf{M}]_\psi \to \mathsf{D}$ sending $\mathsf{O} \in \mathcal{M}_\psi$ into the single element of $\mathsf{O} \cap (\mathsf{D} \setminus \{\alpha\})$. Now we prove that f is onto. Let $g \in \mathsf{D}$. Let us prove that $\mathcal{P}\mathcal{O}_g \in \mathcal{M}_\psi$ is such that $f(\mathcal{P}\mathcal{O}_g) = g$. If $\mathcal{P}\mathcal{O}_g \notin \mathcal{M}_\psi$, then there is $\alpha \in \mathsf{V}_{\mathcal{P}}(\mathsf{G})$ such that $\mathcal{P}\mathcal{O}_\alpha \subset \mathcal{P}\mathcal{O}_g$ is a proper subset of $\mathcal{P}\mathcal{O}_g$. Then $\overline{\mathcal{P}\mathcal{O}_g} = \overline{\mathcal{P}\mathcal{O}_\alpha}$. In this case we get that $\overline{\mathsf{D} \setminus \{g\}} = \mathsf{V}_{\mathcal{P}}(\mathsf{G})$, and this contradicts minimality of D. Hence, $\mathcal{P}\mathcal{O}_g \in \mathcal{M}_\psi$ such that $f(\mathcal{P}\mathcal{O}_g) = g$. □

Theorem 4.3. *Let G be an $\mathcal{N}$-star graph, $\psi = \{\mathcal{P}\mathcal{O}_\alpha : \alpha \in \mathsf{V}_{\mathcal{P}}(\mathsf{G})\}$, and $\mathcal{M}_\psi$ be the collection of minimal sets in ψ. If $f : \mathcal{M}_\psi \to \mathsf{G}$ is a mapping such that $f(\mathcal{P}\mathcal{O}_\alpha) \in \mathcal{P}\mathcal{O}_\alpha$ for all $\mathcal{P}\mathcal{O}_\alpha \in \mathcal{M}_\psi$, then $f(\mathcal{M}_\psi)$ is a minimal dense set in the graphic topological space $(\mathsf{V}_{\mathcal{P}}(\mathsf{G}), \mathsf{T}_{\mathcal{A}\mathsf{V}_{\mathcal{P}}})$.*

Proof. It is clear that for all $\alpha \in V_{\mathcal{P}}(G)$ there is $\beta \in V_{\mathcal{P}}(G)$ such that $\mathcal{PO}_\beta \in \mathcal{M}_\psi$ and $\mathcal{PO}_\beta \subseteq \mathcal{PO}_\alpha$. Hence, we get $f(\mathcal{PO}_\beta) \in \mathcal{PO}_\alpha \cap f(\mathcal{M}_\psi)$, that is, $f(\mathcal{M}_\psi)$ is dense in $(V_{\mathcal{P}}(G), T_{\mathcal{A}V_{\mathcal{P}}})$. To prove that $f(\mathcal{M}_\psi)$ is a minimal dense set in $(V_{\mathcal{P}}(G), T_{\mathcal{A}V_{\mathcal{P}}})$, let $\overline{S} = V_{\mathcal{P}}(G)$ and $S \subseteq f(\mathcal{M}_\psi)$. Suppose that $\mathcal{PO}_\alpha \in \mathcal{M}_\psi$ such that $f(\mathcal{PO}_\alpha) \notin S$. Then there is $\beta \in V_{\mathcal{P}}(G)$ such that $\mathcal{PO}_\beta \in \mathcal{M}_\psi$ and $f(\mathcal{PO}_\beta) \in \mathcal{PO}_\alpha \cap S$. Since $f(\mathcal{PO}_\beta) \in \mathcal{PO}_\alpha \cap \mathcal{PO}_\beta$ and $f(\mathcal{PO}_\alpha) \notin S$, that is, $f(\mathcal{PO}_\alpha) \in \mathcal{PO}_\alpha \setminus S$, then $\mathcal{PO}_\alpha = \mathcal{PO}_\beta$ and so $f(\mathcal{PO}_\alpha) = f(\mathcal{PO}_\beta) \in S$. This is a contradiction. Hence $S = f(\mathcal{M}_\psi)$. □

5. Conclusion

Our focus in this paper was on the study of $\mathcal{P}$-topology in simple graphs by using pathing vertices of graphs. Several topological properties (such as (in)discreteness, density, continuity, connectedness) of this topology have been presented, especially for well-known classes of graphs (cycle graphs C_n, complete graphs K_n, complete bipartite graphs $K_{n,m}$). Also, special attention is paid to the study of $\mathcal{N}$-star graphs which enable us to better understand graphic topological structures.

Notice that topologies on graphs have various applications in medical and biological sciences (see, for example, recent papers [2, 4, 10]).

We hope that our study may open new directions in investigations of topological topics in graph theory and, especially, in some applications in (bio)medical sciences.

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Availability of data and material

All the data of the study are included.

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