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Analytical properties of Δ_h-hybrid Laguerre-Appell polynomials and their applications in computer modeling

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Abstract

This work thoroughly investigates a new class of polynomials, the Δ_h Laguerre-Appell polynomials, which combine certain operational strategies with the principle of monomiality. By using an innovative approach, this study adds to previous research on the subject and presents discoveries. Since these polynomials are essential for simulating entropy in quantum systems, their significance is especially clear in quantum mechanics. In-depth derivations of explicit formulas and an analysis of important characteristics and links to well-known polynomial families are given. Through an examination of the special characteristics and uses of Δ_h Laguerre-Appell polynomials, this work greatly expands their theoretical knowledge and increases their prospective applications in a variety of mathematical and scientific contexts.

Keywords: Δ_h polynomials, monomiality principle, Laguerre-Appell polynomials, explicit forms, determinant form.

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1. Introduction

Special polynomials are essential to mathematical theory because of their many uses and complex structural characteristics. Among these are the Jacobi, Laguerre, Hermite, and Chebyshev polynomials; each has special properties that make it useful for resolving issues in a variety of domains. In quantum mechanics, for example, Hermite polynomials are essential, especially when constructing wavefunctions for the quantum harmonic oscillator. Laguerre polynomials are important in quantum physics, particularly in the hydrogen atom's radial portion of the Schrödinger equation. Famous for their minimizing capabilities, Chebyshev polynomials are essential to approximation theory and numerical algorithms. In

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contrast, classical mechanics difficulties are associated with Jacobi polynomials, which emerge in the context of orthogonal polynomials. Because of their shared characteristic of orthogonality, which makes it easier to manipulate and apply them in complex equations, these polynomials are essential tools in both theoretical and applied mathematics.

The utility of these special polynomials extends significantly into the realm of computer modeling. They are frequently employed in numerical methods for approximating solutions to differential equations, such as finite element analysis, providing a basis for representing complex functions and fields. Furthermore, their properties, particularly orthogonality and efficient computation via recurrence relations, make them ideal for spectral methods used in computational fluid dynamics and weather forecasting. This allows for accurate and efficient simulations of physical phenomena that would be intractable using other approaches.

Recent studies have focused a lot of emphasis on two-variable special polynomials because of their many uses in mathematics and related fields. This class of polynomials, which includes bivariate Chebyshev, Hermite, and Laguerre polynomials as examples, has special qualities that make it useful in fields like approximation theory, algebraic geometry, and numerical analysis. Bivariate Chebyshev polynomials are frequently used in least squares fitting and interpolation methods because of their symmetry. The classical Laguerre polynomials are also extended to two variables by bivariate Laguerre polynomials, which also fulfill a corresponding bivariate differential equation. Especially in two-degree-of-freedom systems, these polynomials are used in quantum physics, random matrix theory, and potential theory. In probability theory mathematical physics, and other fields, two-variable polynomials are often researched because they are orthogonal concerning a particular weight function. They are important because they offer a strong foundation for multivariate function analysis and the resolution of challenging issues in a variety of fields, see [4, 14–16, 21, 22, 24, 26, 29–32, 35, 36].

Laguerre polynomials in two-variable, denoted as $\mathcal{W}_\lambda(u,v)$ [18, 19], are defined through the generating function:

$$e^{v\zeta}C_0(u\zeta) = \sum_{\lambda=0}^{\infty} \mathcal{W}_\lambda(u,v)\frac{\zeta^\lambda}{\lambda!},$$

where $C_0(u\zeta)$ represents the 0^{th} Bessel-Tricomi function [5]. Operationally, it is given as:

$$C_0(-u\zeta) = e^{D_u^{-1}\zeta}. \tag{1.1}$$

The Bessel-Tricomi function $C_\lambda(u)$ can also be expanded as:

$$C_\lambda(u) = u^{-\lambda/2}J_\lambda(2\sqrt{u}) = \sum_{i=0}^{\infty}\frac{(-1)^i u^i}{k!(\lambda+i)!},$$

where $J_\lambda(u)$ denotes the cylindrical Bessel function of the first kind [5]. Additionally, the two-variable Laguerre polynomials $\mathcal{W}_\lambda(u,v)$ are expressed via the series:

$$\mathcal{W}_\lambda(u,v) = \lambda!\sum_{k=0}^{\lambda}\frac{(-1)^k u^k v^{\lambda-k}}{(k!)^2(\lambda-k)!}.$$

These polynomials have important physical applications in addition to their theoretical significance, especially as partial differential equation solutions. Here is an illustration of such an equation:

$$\frac{\partial}{\partial v}\mathcal{W}_\lambda(u,v) = -\left(\frac{\partial}{\partial u}u\frac{\partial}{\partial u}\right)\mathcal{W}_\lambda(u,v), \quad \mathcal{W}_\lambda(u,0) = \frac{(-u)^\lambda}{\lambda!},$$

This helps research beam durations impacted by quantum fluctuations in storage rings and closely resembles a Fokker-Planck-type heat diffusion equation [34]. Moreover, $\mathcal{W}_n(u,v)$ exhibit quasi-monomial behavior under specific multiplicative and derivative operators:

$$\Theta_\lambda^+ = v - D_u^{-1}, \quad \Theta_\lambda^- = -D_u u D_u.$$

The creation of Δ_h variants of special polynomials has garnered significant attention due to their versatility and computational efficiency. These polynomials, formulated using the classical finite difference operator Δ_h, have been studied extensively in various mathematical frameworks, leading to new families of Δ_h special polynomials [1, 2, 4–6, 8–11, 17, 23, 25, 32, 33]. Their inherent structural properties make them particularly valuable in numerical analysis, applied mathematics, and computer-aided simulations. In numerical computation, Δ_h special polynomials serve as foundational tools for approximating solutions to differential and integral equations, especially in discretized domains. Their recurrence relations and explicit closed-form expressions enable efficient computation of function values over large datasets, reducing computational complexity and enhancing stability in numerical simulations. These polynomials are instrumental in modelling dynamic systems governed by discrete and continuous equations. They play a critical role in simulating wave propagation, heat transfer, and fluid dynamics, where the discrete nature of Δ_h operators allows for more accurate representation of physical phenomena in digital environments. In particular, finite difference methods employing Δ_h polynomials are widely used in computational physics and engineering to solve partial difference equations that arise in real-world problems.

Also, in statistical computing and machine learning, Δ_h polynomials contribute to data approximation and regression techniques. Their ability to interpolate and approximate discrete datasets makes them useful in constructing predictive models, time-series analysis, and pattern recognition algorithms. Additionally, they play a crucial role in probabilistic modeling, where discrete analogs of special functions are required for statistical inference and uncertainty quantification. In symbolic computation and computer algebra systems (CAS), these polynomials aid in the efficient manipulation of large-scale algebraic expressions. Their structured operational rules allow for automated simplifications and symbolic differentiation, making them valuable in software implementations for solving discrete versions of differential equations, such as in image processing, digital signal analysis, and network modelling. Furthermore, in graph theory and combinatorial optimization, Δ_h polynomials provide analytical tools for studying discrete structures and optimizing network algorithms. Their application extends to shortest path problems, spanning tree calculations, and network flow optimizations, which are crucial in computer science and logistics.

Given their extensive utility in numerical simulations, predictive modelling, and computational optimization, Δ_h special polynomials represent a powerful mathematical framework bridging discrete and continuous computation. Their continued development holds promise for advancing various scientific and engineering disciplines, particularly in high-performance computing and artificial intelligence-driven modelling. The Δ_h-Appell polynomial's mathematical form is as follows:

$$\mathbb{A}_\lambda^{[h]}(u) := \mathbb{A}_\lambda(u), \quad \lambda \in \mathbb{N}_0,$$

and defined by

$$\Delta_u\{\mathbb{A}_\lambda^{[h]}(u)\} = \lambda h \mathcal{A}_{\lambda-1}(u), \quad \lambda \in \mathbb{N}, \tag{1.2}$$

where Δ_h is the finite difference operator:

$$\Delta_h \mathbb{H}^{[h]}(u) = \mathbb{H}^{[h]}(u+h) - \mathbb{H}^{[h]}(u).$$

The Δ_h-Appell polynomials $\mathbb{A}_\lambda(u)$ are specified by the following generating function [11]:

$$\gamma(\zeta)(1+h\zeta)^{\frac{u}{h}} = \sum_{\lambda=0}^{\infty} \mathbb{A}_\lambda^{[h]}(u)\frac{\zeta^\lambda}{\lambda!},$$

where

$$\gamma(\zeta) = \sum_{\lambda=0}^{\infty} \gamma_{\lambda,h}\frac{\zeta^\lambda}{\lambda!}, \quad \gamma_{0,h} \neq 0.$$

Motivated by Costabile [11], here we introduced the two variable Δ_h Laguerre-Appell polynomials:

$$\gamma(t)(1+h\zeta)^{\frac{\nu}{h}}(1+h\zeta^2)^{\frac{D_u^{-1}}{h}} = \sum_{\lambda=0}^{\infty} {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,\nu)\frac{\zeta^{\lambda}}{\lambda!} \tag{1.3}$$

through the generating function concept. Table 1 outlines the appropriate choices for individuals entering the class of Δ_h Appell polynomials.

Table 1: Various members of the Δ_h Appell polynomial family.

S. No.	Δ_h Appell polynomials	Generating function	$\mathbb{A}_{\lambda}^{[h]}(u)$
I.	The Bernoulli polynomials [6]	$\frac{\zeta}{(1+\lambda h)^{\frac{1}{h}}-1}(1+\lambda h)^{\frac{u}{h}} = \sum_{\lambda=0}^{\infty} \mathbb{B}_{\lambda}^{[h]}(u)\frac{\zeta^{\lambda}}{\lambda!}$	$\mathbb{A}(\zeta) = \frac{\zeta}{(1+\lambda h)^{\frac{1}{h}}-1}$
II.	The Euler polynomials [6]	$\frac{2}{(1+\lambda h)^{\frac{1}{h}}+1}(1+\lambda h)^{\frac{u}{h}} = \sum_{\lambda=0}^{\infty} \mathbb{E}_{\lambda}^{[h]}(u)\frac{\zeta^{\lambda}}{\lambda!}$	$\mathbb{A}(\zeta) = \frac{2}{(1+\lambda h)^{\frac{1}{h}}+1}$
III.	The Genocchi polynomials [6]	$\frac{2\zeta}{(1+\lambda h)^{\frac{1}{h}}+1}(1+\lambda h)^{\frac{u}{h}} = \sum_{\lambda=0}^{\infty} \mathbb{G}_{\lambda}^{[h]}(u)\frac{\zeta^{\lambda}}{\lambda!}$	$\mathbb{A}(\zeta) = \frac{\zeta}{(1+\lambda h)^{\frac{1}{h}}+1}$

The article is organised as follows. Section 2 examines the recurrence relations that characterize the behavior of Δ_h Laguerre-Appell polynomials and talks about how they are generated. These Δ_h Laguerre-Appell polynomials can be evaluated over certain ranges or under specific situations using the summation formulae and methods given in Section 3, which offers effective approaches to calculate their values. In Section 4, the monomiality principle is introduced, which creates the determinant form of these polynomials and looks at how they behave under particular operations. Section 5 provides instances of these polynomials together with the results that follow. In the conclusion, the article's main conclusions are outlined, together with their consequences, possible uses, and research directions for Δ_h Laguerre-Appell polynomials.

2. Bivariate Δ_h Laguerre-Appell polynomial systems

Let us consider a novel class of bivariate polynomial systems, specifically the Δ_h Laguerre-Appell polynomials, denoted by $\mathbb{L}\mathbb{A}^{[h]}\lambda(u,\nu)$. This section is devoted to the formal construction and characterization of these polynomials, thereby extending the existing theory of polynomial systems and establishing a framework for further mathematical investigation.

A central contribution of this section is the derivation of the generating function for the Δ_h Laguerre-Appell polynomials. Generating functions, being fundamental objects in analytical combinatorics and mathematical physics, provide a powerful analytical tool for studying the structural properties and asymptotic behavior of sequences. The establishment of this generating function serves as a foundation for subsequent analysis of various analytical properties, including but not limited to,

- orthogonality conditions;
- recurrence relations;
- connections to special function identities.

The correspondence between the Δ_h Laguerre-Appell polynomials and their associated generating function enriches our understanding of polynomial families and their applications in mathematical analysis. The results presented herein not only elucidate the intrinsic properties of these polynomials but also demonstrate their potential applications across diverse mathematical domains. We proceed by establishing the following theorem concerning the generating function of the Δ_h Laguerre-Appell polynomials ${}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,\nu)$.

Theorem 2.1. *The two-variable Δ_h Laguerre-Appell polynomials, satisfy the generating relation:*

$$\gamma(t)(1+h\zeta)^{\frac{v}{h}}(1+h\zeta^2)^{\frac{D_u^{-1}}{h}} = \sum_{\lambda=0}^{\infty} {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)\frac{\zeta^{\lambda}}{\lambda!}. \tag{2.1}$$

Proof. By expanding the expression $\gamma(\zeta)(1+h\zeta)^{\frac{v}{h}}(1+h\zeta)^{\frac{D_u^{-1}}{h}}$ at $u=v=0$ using a Newton series for finite differences, and organizing the resulting products in terms of powers of ζ, we identify the polynomials ${}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)$ (as expressed in equation (2.1)) as the coefficients of $\frac{\zeta^{\lambda}}{\lambda!}$. This reveals the generating function of the two-variable Δ_h Laguerre-Appell polynomials ${}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)$. □

Theorem 2.2. *The following relations hold for the two-variable Δ_h Laguerre-Appell polynomials, denoted by ${}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)$:*

$$\frac{{}_{v}\Delta_h}{h}{}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v) = \lambda\,{}_{\mathbb{L}}\mathbb{A}_{\lambda-1}^{[h]}(u,v), \quad \frac{{}_{u}\Delta_h}{h}{}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v) = \lambda(\lambda-1)\,{}_{\mathbb{L}}\mathbb{A}_{\lambda-2}^{[h]}(u,v), \quad D_u^{-1}\to u. \tag{2.2}$$

Proof. By differentiating equation (2.1) with respect to v, while taking into account the expression given in (1.1), we obtain:

$$\begin{aligned}{}_{v}\Delta_h\left\{\gamma(t)(1+h\zeta)^{\frac{v}{h}}(1+h\zeta)^{\frac{D_u^{-1}}{h}}\right\} &= (1+h\zeta)^{\frac{v+h}{h}}(1+h\zeta)^{\frac{D_u^{-1}}{h}} - \gamma(\zeta)(1+h\zeta)^{\frac{v}{h}}(1+h\zeta)^{\frac{D_u^{-1}}{h}}\\ &= (1+h\zeta-1)\gamma(\zeta)(1+h\zeta)^{\frac{v}{h}}(1+h\zeta)^{\frac{D_u^{-1}}{h}}\\ &= h\zeta\,\gamma(\zeta)(1+h\zeta)^{\frac{v}{h}}(1+h\zeta)^{\frac{D_u^{-1}}{h}}.\end{aligned} \tag{2.3}$$

By inserting the right-hand side of expression (2.1) into (2.3), we find:

$${}_{v}\Delta_h\sum_{\lambda=0}^{\infty}{}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)\frac{\zeta^{\lambda}}{\lambda!} = h\sum_{\lambda=0}^{\infty}{}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)\frac{\zeta^{\lambda+1}}{\lambda!}.$$

By substituting $\lambda\to\lambda-1$ in the right-hand side of the previous expression (2.3) and comparing the coefficients of corresponding powers of λ, we can derive assertion (2.2). □

Next, we establish the explicit form for the two-variable Δ_h Laguerre-Appell polynomials ${}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)$ by presenting the following result.

Theorem 2.3. *For the two-variable Δ_h Laguerre-Appell polynomials ${}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)$, the following explicit relation holds:*

$${}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v) = \sum_{d=0}^{\frac{v}{h}}\binom{\lambda}{d}\binom{\frac{v}{h}}{d}h^d\,{}_{\mathbb{L}}\mathbb{A}_{\lambda-d}^{[h]}(u). \tag{2.4}$$

Proof. Expand the generating relation (2.1) in the described fashion:

$$\gamma(\lambda)(1+h\lambda)^{\frac{v}{h}}(1+h\lambda)^{\frac{D_u^{-1}}{h}} = \sum_{d=0}^{\frac{v}{h}}\binom{\frac{v}{h}}{d}\frac{(h\zeta)^d}{d!}\sum_{\lambda=0}^{\infty}{}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,0)\frac{\zeta^{\lambda}}{\lambda!},$$

which can be further expressed as

$$\sum_{\lambda=0}^{\infty}{}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)\frac{\zeta^{\lambda}}{\lambda!} = \sum_{\lambda=0}^{\infty}\sum_{d=0}^{\frac{v}{h}}\binom{\frac{v}{h}}{d}h^d\,{}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,0)\frac{\zeta^{\lambda+d}}{\lambda!\,d!}.$$

By substituting $\lambda \to \lambda - d$ into the right-hand side of the previous expression, it follows that

$$\sum_{\lambda=0}^{\infty} {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)\frac{\zeta^{\lambda}}{\lambda!} = \sum_{\lambda=0}^{\infty}\sum_{d=0}^{\frac{v}{h}} \binom{\frac{v}{h}}{d} h^{d}\ {}_{\mathbb{L}}\mathbb{A}_{\lambda-d}^{[h]}(u,0)\frac{\zeta^{\lambda}}{(\lambda-d)!d!}. \tag{2.5}$$

By multiplying and dividing the right-hand side of the previous expression (2.5) by $\lambda!$ and then comparing the coefficients of the same powers of ζ on both sides, we can derive assertion (2.4). □

Theorem 2.4. *Moreover, for the two-variable Δ_h Laguerre-Appell polynomials ${}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)$, the following explicit relation holds:*

$$ {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v) = \sum_{k=0}^{\lambda}\binom{\lambda}{k}\gamma_{k,h}\ {}_{\mathbb{L}}\mathbb{A}_{\lambda-k}^{[h]}(u,v). \tag{2.6}$$

Proof. Expand the generating relation (2.1) using expressions (1.2) and (2.1), with $\gamma(\zeta)=1$, in the specified manner:

$$\gamma(\zeta)(1+h\zeta)^{\frac{v}{h}}(1+h\zeta)^{\frac{D_u^{-1}}{h}} = \sum_{k=0}^{\infty}\gamma_{k,h}\frac{\zeta^k}{k!}\sum_{\lambda=0}^{\infty} {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)\frac{\zeta^{\lambda}}{\lambda!},$$

which can be reformulated as

$$\sum_{\lambda=0}^{\infty} {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)\frac{\zeta^{\lambda}}{\lambda!} = \sum_{\lambda=0}^{\infty}\sum_{k=0}^{\infty}\gamma_{k,h}\ {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)\frac{\zeta^{\lambda+k}}{\lambda!\ k!}.$$

By substituting $\lambda \to \lambda - k$ into the right-hand side of the previous expression, we get:

$$\sum_{\lambda=0}^{\infty} {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)\frac{\zeta^{\lambda}}{\lambda!} = \sum_{\lambda=0}^{\infty}\sum_{k=0}^{\lambda}\gamma_{k,h}\ {}_{\mathbb{L}}\mathbb{A}_{\lambda-k}^{[h]}(u,v)\frac{\zeta^{\lambda}}{(\lambda-k)!\ k!}. \tag{2.7}$$

By multiplying and dividing the right-hand side of expression (2.7) by $\lambda!$ and then comparing the coefficients of matching powers of ζ on both sides, we can derive assertion (2.6). □

3. Summation formulae

The summation equations, sometimes known as sigma notation, for specific two-variable polynomials, are derived in this section. These formulas are necessary for methodically computing sums containing such polynomials and for comprehending intricate interactions between variables. They make it easier to evaluate complex expressions in a variety of mathematical domains, assisting mathematicians in identifying symmetry and patterns in polynomial structures. Combinatorics, probability theory, and mathematical physics all advance as a result, and computer methods for accurate problem-solving are improved. Summation equations are essential tools that make it easier to represent and compute term series.

We now demonstrate these summation formulae by proving the following results.

Theorem 3.1. *For $\lambda \geqslant 0$, the following relation holds:*

$$ {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v) = \sum_{\kappa=0}^{\lambda}\binom{\lambda}{\kappa}\left(-\frac{v}{h}\right)_{\kappa}(-h)^{\kappa}\,{}_{\mathbb{L}}\mathbb{A}_{\lambda-\kappa}^{[h]}(u,0). \tag{3.1}$$

Proof. In view of expression (2.1), it follows that

$$\sum_{\lambda=0}^{\infty} {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)\frac{\zeta^{\lambda}}{\lambda!} = \gamma(\zeta)(1+h\zeta)^{\frac{v}{h}}(1+h\zeta)^{\frac{D_u^{-1}}{h}} = \sum_{\lambda=0}^{\infty} {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,0)\frac{\zeta^{\lambda}}{\lambda!}\sum_{\kappa=0}^{\infty}\left(-\frac{v}{h}\right)_{\kappa}(-h)^{\kappa}\frac{\zeta^{\kappa}}{\kappa!}$$

$$= \sum_{\lambda=0}^{\infty}\left(\sum_{\kappa=0}^{\lambda}\binom{\lambda}{\kappa}\left(-\frac{\nu}{h}\right)_{\kappa}(-h)^{\kappa}{}_{\mathbb{L}}\mathbb{A}_{\lambda-\kappa}^{[h]}(u,0)\right)\frac{\zeta^{\lambda}}{\lambda!}.$$

By comparing the coefficients of ζ, we obtain the relation given in equation (3.1). □

Theorem 3.2. *For $\lambda \geqslant 0$, the following relation holds:*

$$ {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,\nu+1) = \sum_{\kappa=0}^{\lambda}\binom{\lambda}{\kappa}\left(-\frac{1}{h}\right)_{\kappa}(-h)^{\kappa}{}_{\mathbb{L}}\mathbb{A}_{\lambda-\kappa}^{[h]}(u,\nu). \tag{3.2}$$

Proof. In view of expression (2.1), it follows that

$$\begin{aligned}
&\sum_{\lambda=0}^{\infty}{}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,\nu+1)\frac{\zeta^{\lambda}}{\lambda!} - \sum_{\lambda=0}^{\infty}{}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,\nu)\frac{\zeta^{\lambda}}{\lambda!}\\
&\quad= \gamma(\zeta)(1+h\zeta)^{\frac{\nu}{h}}(1+h\zeta^{2})^{\frac{D_u^{-1}}{h}}\left((1+h\zeta)^{\frac{1}{h}}-1\right)\\
&\quad= \sum_{\lambda=0}^{\infty}{}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,\nu)\frac{\zeta^{\lambda}}{\lambda!}\left(\sum_{\kappa=0}^{\infty}\left(-\frac{1}{h}\right)_{\kappa}(-h)^{\kappa}\frac{\zeta^{\kappa}}{\kappa!}-1\right)\\
&\quad= \sum_{\lambda=0}^{\infty}\left(\sum_{\kappa=0}^{\lambda}\binom{\lambda}{\kappa}\left(-\frac{1}{h}\right)_{\kappa}(-h)^{\kappa}{}_{\mathbb{L}}\mathbb{A}_{\lambda-\kappa}^{[h]}(u,\nu)\right)\frac{\zeta^{\lambda}}{\lambda!} - \sum_{\lambda=0}^{\infty}{}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,\nu)\frac{\zeta^{\lambda}}{\lambda!}.
\end{aligned}$$

By comparing the coefficients of ζ, we obtain the relation given in equation (3.2). □

Theorem 3.3. *For $\lambda \geqslant 0$, the following relation holds:*

$$ {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,\nu) = \sum_{k=0}^{\lambda}\sum_{j=0}^{\lambda-k}\left(-\frac{\nu}{h}\right)_{\lambda-j-k}(-h)^{\lambda-j-k}\left(-\frac{u}{h}\right)_{j}(-1)^{j}A_{k,h}\frac{\lambda!}{(\lambda-j-k)!(j!)^{2}k!}. \tag{3.3}$$

Proof. Using (2.1), we have

$$\begin{aligned}
\sum_{\lambda=0}^{\infty}{}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,\nu)\frac{\zeta^{\lambda}}{\lambda!} &= \gamma(\zeta)(1+h\zeta)^{\frac{\nu}{h}}(1+h\zeta)^{\frac{D_u^{-1}}{h}}\\
&= \gamma(\zeta)\sum_{n=0}^{\infty}\left(-\frac{\nu}{h}\right)_{\lambda}(-h)^{\lambda}\frac{\zeta^{\lambda}}{\lambda!}\sum_{j=0}^{\infty}\left(-\frac{u}{h}\right)_{j}(-1)^{j}(-h)^{j}\frac{\zeta^{j}}{j!j!}\\
&= \sum_{k=0}^{\infty}A_{k,h}\frac{\zeta^{k}}{k!}\sum_{\lambda=0}^{\infty}\sum_{j=0}^{\lambda}\left(-\frac{\nu}{h}\right)_{\lambda-j}(-h)^{\lambda-j}\left(-\frac{u}{h}\right)_{j}(-1)^{j}\frac{\zeta^{\lambda}}{(\lambda-j)!(j!)^{2}}\\
&= \sum_{\lambda=0}^{\infty}\sum_{k=0}^{\lambda}\sum_{j=0}^{\lambda-k}\left(-\frac{\nu}{h}\right)_{\lambda-2j-k}(-h)^{\lambda-j-k}\left(-\frac{u}{h}\right)_{j}(-1)^{j}A_{k,h}\frac{\zeta^{\lambda}}{(\lambda-j-k)!(j!)^{2}k!}.
\end{aligned}$$

By comparing the coefficients of ζ, we obtain the relation given in equation (3.3). □

We now explore the relationship between the Stirling numbers of the first kind and the two-variable Δ_h Laguerre-Appell polynomials,

$$\frac{[\log(1+\zeta)]^{\kappa}}{\kappa!} = \sum_{i=k}^{\infty}S_{1}(i,k)\frac{\zeta^{i}}{i!},\ |\,\zeta\,|< 1.$$

Based on the definition provided above, we have

$$(\nu)_i = \sum_{\kappa=0}^{i} (-1)^{i-\kappa} S_1(i,\kappa)\nu^{\kappa}. \tag{3.4}$$

Theorem 3.4. *For $\lambda \geqslant 0$, the following relation holds:*

$$_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,\nu) = \sum_{\kappa=0}^{\lambda} \binom{\lambda}{\kappa} {_{\mathbb{L}}\mathbb{A}_{\lambda-\kappa}^{[h]}}(u,0) \sum_{j=0}^{\kappa} \nu^j S_1(\kappa,j) h^{\kappa-j}.$$

Proof. From (2.1), we have

$$\begin{aligned}
\sum_{\lambda=0}^{\infty} {_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}}(u,\nu)\frac{\zeta^{\lambda}}{\lambda!} &= e^{\frac{\nu}{h}\log(1+h\zeta)}\gamma(\zeta)(1+h\zeta)^{\frac{D_u^{-1}}{h}} \\
&= \gamma(\zeta)(1+h\zeta)^{\frac{D_u^{-1}}{h}} \sum_{j=0}^{\infty}\left(\frac{\nu}{h}\right)^j \frac{[\log(1+h\zeta)]^j}{j!} \\
&= \sum_{\lambda=0}^{\infty} {_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}}(u,0)\frac{\zeta^{\lambda}}{\lambda!} \sum_{\kappa=0}^{\infty}\sum_{j=0}^{\kappa}\left(\frac{\nu}{h}\right)^j S_1(\kappa,j)h^{\kappa}\frac{\zeta^{\kappa}}{\kappa!} \\
&= \sum_{\lambda=0}^{\infty}\left(\sum_{\kappa=0}^{\lambda}\binom{\lambda}{\kappa} {_{\mathbb{L}}\mathbb{A}_{\lambda-\kappa}^{[h]}}(u,0)\sum_{j=0}^{\kappa}\left(\frac{\nu}{h}\right)^j S_1(\kappa,j)h^{\kappa}\right)\frac{\zeta^{\lambda}}{\lambda!}.
\end{aligned}$$

By comparing the coefficients of ζ, we obtain the result. □

Theorem 3.5. *For $\lambda \geqslant 0$, the following relation holds:*

$$_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,0) = \sum_{\kappa=0}^{\lambda} \binom{\lambda}{\kappa} {_{\mathbb{L}}\mathbb{A}_{\lambda-\kappa}^{[h]}}(u,\nu) \sum_{j=0}^{\kappa} \left(-\frac{\nu}{h}\right)^j S_1(\kappa,j) h^{\kappa}.$$

Proof. From (2.1), we have

$$\begin{aligned}
\gamma(\zeta)(1+h\zeta)^{\frac{D_u^{-1}}{h}} &= e^{-\frac{\nu}{h}\log(1+h\zeta)} \sum_{\lambda=0}^{\infty} {_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}}(u,\nu)\frac{\zeta^{\lambda}}{\lambda!} \\
&= \sum_{\lambda=0}^{\infty} {_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}}(u,\nu)\frac{\zeta^{\lambda}}{\lambda!} \sum_{j=0}^{\infty}\left(-\frac{\nu}{h}\right)^j \frac{[\log(1+h\zeta)]^j}{j!} \\
&= \sum_{\lambda=0}^{\infty} {_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}}(u,\nu)\frac{\zeta^{\lambda}}{\lambda!} \sum_{\kappa=0}^{\infty}\sum_{j=0}^{\kappa}\left(-\frac{\nu}{h}\right)^j S_1(\kappa,j)h^{\kappa}\frac{\zeta^{\kappa}}{\kappa!} \\
&= \sum_{\lambda=0}^{\infty}\left(\sum_{\kappa=0}^{\lambda}\binom{\lambda}{\kappa} {_{\mathbb{L}}\mathbb{A}_{\lambda-\kappa}^{[h]}}(u,\nu)\sum_{j=0}^{\kappa}\left(-\frac{\nu}{h}\right)^j S_1(\kappa,j)h^{\kappa}\right)\frac{\zeta^{\lambda}}{\lambda!}.
\end{aligned}$$

By comparing the coefficients of ζ, we obtain the result. □

Theorem 3.6. *For $\lambda \geqslant 0$, the following relation holds:*

$$_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,\nu) = \sum_{l=0}^{\lambda}\sum_{\kappa=0}^{\lambda-l} \frac{\lambda!}{(\lambda-\kappa-l)!(\kappa+l)!} h^{\kappa} {_{\mathbb{L}}\mathbb{A}_{\lambda-\kappa-l}^{[h]}}(u,0) S_1(\kappa+l,l)\nu^{l}.$$

Proof. From (2.1), we have

$$\sum_{\lambda=0}^{\infty} {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)\frac{\zeta^{\lambda}}{\lambda!} = \gamma(\zeta)(1+h\zeta)^{\frac{v}{h}}(1+h\zeta)^{\frac{D_u^{-1}}{h}} = \sum_{\lambda=0}^{\infty} {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,0)\frac{\zeta^{\lambda}}{\lambda!}\sum_{\kappa=0}^{\infty}\left(-\frac{v}{h}\right)_{\kappa}(-h)^{\kappa}\frac{\zeta^{\kappa}}{\kappa!}$$
$$= \sum_{\lambda=0}^{\infty}\left(\sum_{\kappa=0}^{\lambda}\binom{\lambda}{\kappa}\left(-\frac{v}{h}\right)_{\kappa}(-h)^{\kappa}{}_{\mathbb{L}}\mathbb{A}_{\lambda-\kappa}^{[h]}(u,0)\right)\frac{\zeta^{\lambda}}{\lambda!}.$$

By comparing the coefficients of ζ, we obtain

$$ {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v) = \sum_{\kappa=0}^{\lambda}\binom{\lambda}{\kappa}\left(-\frac{v}{h}\right)_{\kappa}(-h)^{\kappa}{}_{\mathbb{L}}\mathbb{A}_{\lambda-\kappa}^{[h]}(u,0).$$

Applying the equality given in (3.4) to the previous expression, we obtain

$$ {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v) = \left(\sum_{\kappa=0}^{\lambda}\binom{\lambda}{\kappa}(-h)^{\kappa}{}_{\mathbb{L}}\mathbb{A}_{\lambda-\kappa}^{[h]}(u,0)\right)\left(\sum_{l=0}^{\kappa}(-1)^{\kappa-l}S_1(\kappa,l)(-h)^{-l}v^{l}\right)$$
$$= \sum_{l=0}^{\lambda}\sum_{\kappa=l}^{\lambda}\frac{\lambda!}{(\lambda-\kappa)!\kappa!}(-h)^{\kappa-l}{}_{\mathbb{L}}\mathbb{A}_{\lambda-\kappa}^{[h]}(u,0)(-1)^{\kappa-l}S_1(\kappa,l)v^{l}$$
$$= \sum_{l=0}^{\lambda}\sum_{\kappa=0}^{\lambda-l}\frac{\lambda!}{(\lambda-\kappa-l)!(\kappa+l)!}(-h)^{\kappa}{}_{\mathbb{L}}\mathbb{A}_{\lambda-\kappa-l}^{[h]}(u,0)(-1)^{\kappa}S_1(\kappa+l,l)v^{l}.$$

This concludes the proof of the theorem. □

Theorem 3.7. *For $\lambda \geqslant 0$, the following relation holds*

$$ {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v+w) = \sum_{l=0}^{\lambda}\sum_{\kappa=0}^{\lambda-l}\frac{\lambda!}{(\lambda-\kappa-l)!(\kappa+l)!}h^{\kappa}{}_{\mathbb{L}}\mathbb{A}_{\lambda-\kappa-l}^{[h]}(u,v)S_1(\kappa+l,l)w^{l}. \tag{3.5}$$

Proof. Taking $v+w=s$ instead of v in (2.1), we have

$$\sum_{\lambda=0}^{\infty} {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v+s)\frac{\zeta^{\lambda}}{\lambda!} = \gamma(\zeta(1+h\zeta)^{\frac{v+s}{h}}(1+h\zeta)^{\frac{D_u^{-1}}{h}} = \left(\sum_{\lambda=0}^{\infty} {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)\frac{\zeta^{\lambda}}{\lambda!}\right)\left(\sum_{\kappa=0}^{\infty}\left(-\frac{s}{h}\right)_{\kappa}(-h)^{\kappa}\frac{\zeta^{\kappa}}{\kappa!}\right).$$

Applying the Cauchy rule and equating the coefficients of ζ on both sides of the resulting equation, we obtain:

$$ {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v+s) = \sum_{\kappa=0}^{\lambda}\binom{\lambda}{\kappa}\left(-\frac{s}{h}\right)_{\kappa}(-h)^{\kappa}{}_{\mathbb{L}}\mathbb{A}_{\lambda-\kappa}^{[h]}(u,v).$$

Then using (3.4) for $\left(-\frac{s}{h}\right)_{\kappa}$, we have obtain (3.5). □

4. Monomiality principle and determinant form

A fundamental idea in polynomial theory, the monomiality principle offers a framework for comprehending and working with polynomial expressions. Any polynomial can be uniquely represented as a linear combination of monomials, which are straightforward algebraic words that elevate a single variable to a non-negative integer power, according to this principle. The structure of polynomials is simplified by this monomial representation, which also facilitates their analysis and manipulation in a variety of mathematical contexts. Mathematicians can identify crucial characteristics like degree, leading coefficient, and roots by breaking down complicated polynomial expressions into their monomial constituents.

This allows for the development of sophisticated mathematical methods and algorithms as well as greater understanding of polynomial behavior.

In many engineering and scientific fields, the monomiality concept is essential in real-world applications. For example, in computer mathematics, the accuracy and efficiency of algorithms for numerical integration, approximation, and polynomial interpolation depend on monomial bases. Polynomials are used to describe complicated systems in domains such as image analysis, control theory, and signal processing, and the monomiality principle provides an understandable and straightforward framework. Furthermore, the principle's wide relevance and significance in theoretical and practical problem-solving across scientific and technical domains are highlighted by the fact that polynomials in physics define physical laws and occurrences.

Research on hybrid special polynomials has further investigated the idea of monomiality. Dattoli [12, 13] developed the concept of poweroids, which Steffenson first proposed in 1941 [28]. In this case, the multiplicative and derivative operators $\hat{\mathfrak{J}}$ and $\hat{\mathfrak{K}}$ are crucial for a polynomial set $g_k(u_1)_{k\in\mathbb{N}}$. These operators meet the requirements of the following expressions:

$$g_{k+1}(u_1) = \hat{\mathfrak{J}}\{g_k(u_1)\} \quad \text{and} \quad k\ g_{k-1}(u_1) = \hat{\mathfrak{K}}\{g_k(u_1)\}.$$

Therefore, applying multiplicative and differential operations to the polynomial set $\{g_k(u_1)\}_{m\in\mathbb{N}}$ yields a quasi-monomial domain. It is essential to use the following formula for this quasi-monomial:

$$[\hat{\mathfrak{K}}, \hat{\mathfrak{J}}] = \hat{\mathfrak{K}}\hat{\mathfrak{J}} - \hat{\mathfrak{J}}\hat{\mathfrak{K}} = \hat{1}.$$

As a result, this reveals a Weyl group structure. If the set $\{g_k(u_1)\}_{k\in\mathbb{N}}$ is quasi-monomial, the roles and applications of the operators $\hat{\mathfrak{J}}$ and $\hat{\mathfrak{K}}$ can be leveraged to understand the importance of the given set. Thus, the following axioms hold true.

(i) $g_k(u_1)$ gives differential equation

$$\hat{\mathfrak{J}}\hat{\mathfrak{K}}\{g_k(u_1)\} = k\ g_k(u_1), \tag{4.1}$$

provided $\hat{\mathfrak{J}}$ and $\hat{\mathfrak{K}}$ exhibits differential traits.

(ii) The expression

$$g_k(u_1) = \hat{\mathfrak{J}}^k\ \{1\}, \tag{4.2}$$

gives the explicit form, with $g_0(u_1) = 1$.

(iii) Further, the expression

$$e^{w\hat{\mathfrak{J}}}\{1\} = \sum_{k=0}^{\infty} g_k(u_1)\frac{w^k}{k!}, \quad |w| < \infty,$$

illustrates the behavior of the generating expression and is derived using identity (4.2).

These methods, which come from quantum mechanics, mathematical physics, and classical optics, are still very applicable and helpful in contemporary study. In these domains, they offer dependable and useful tools for investigating intricate phenomena. The efficiency of these techniques demonstrates how important they are to expanding our comprehension of complex systems.

Considering the significance of these methods, we concentrate on verifying the notion of monomiality in particular for the Δ_h Laguerre-Appell polynomials, especially represented as $\mathbb{LA}n^{[h]}(u,v)$. For the analysis and interpretation of a wide range of phenomena, these polynomials provide an essential mathematical foundation. By verifying the concept of monomiality in this situation, we want to provide light on the basic characteristics that control how these polynomials behave and are used.

Our validation efforts are presented in this section. As important mathematical structures, these results highlight the validity and usefulness of the Δ_h Laguerre-Appell polynomials. The usefulness of the monomiality notion is confirmed using a comprehensive analysis and validation, which strengthens the dependability of these mathematical tools for both theoretical investigation and real-world applications.

Theorem 4.1. *The Δ_h Laguerre-based Appell polynomials ${}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)$ satisfy the following multiplicative and differential operators:*

$$\hat{M_{\mathbb{L}\mathbb{A}}} = \left(\frac{v}{1+{}_v\Delta_h} + \frac{2\, D_u^{-1}\, {}_v\Delta_h}{h+{}_v\Delta_h{}^2} + \frac{\gamma'(\frac{{}_v\Delta_h}{h})}{\gamma(\frac{{}_v\Delta_h}{h})}\right) \tag{4.3}$$

and

$$\hat{D_{\mathbb{L}\mathbb{A}}} = \frac{{}_v\Delta_h}{h}. \tag{4.4}$$

Proof. Given expression (1.1), by differentiating expression (2.1) with respect to v, we get

$$\begin{aligned}{}_v\Delta_h\left\{\gamma(\zeta)(1+h\zeta)^{\frac{v}{h}}(1+h\zeta)^{\frac{D_u^{-1}}{h}}\right\} &= (1+h\zeta)^{\frac{v+h}{h}}(1+h\zeta^2)^{\frac{D_u^{-1}}{h}} - \gamma(\zeta)(1+h\zeta)^{\frac{v}{h}}(1+h\zeta)^{\frac{D_u^{-1}}{h}}\\ &= (1+h\zeta-1)\gamma(\zeta)(1+h\zeta)^{\frac{v}{h}}(1+h\zeta)^{\frac{D_u^{-1}}{h}} = h\zeta\, \gamma(\zeta)(1+h\zeta)^{\frac{v}{h}}(1+h\zeta)^{\frac{D_u^{-1}}{h}},\end{aligned}$$

thus, we have

$$\frac{{}_v\Delta_h}{h}\left[\gamma(\zeta)(1+h\zeta)^{\frac{v}{h}}(1+h\zeta)^{\frac{D_u^{-1}}{h}}\right] = \zeta\left[\gamma(\zeta)(1+h\zeta)^{\frac{v}{h}}(1+h\zeta)^{\frac{D_u^{-1}}{h}}\right],$$

which gives the identity

$$\frac{{}_v\Delta_h}{h}\left[{}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)\right] = \zeta\left[{}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)\right]. \tag{4.5}$$

Differentiating expression (2.1) with respect to ζ yields:

$$\frac{\partial}{\partial\zeta}\left\{\gamma(\zeta)(1+h\zeta)^{\frac{v}{h}}(1+h\zeta^2)^{\frac{D_u^{-1}}{h}}\right\} = \frac{\partial}{\partial\zeta}\left\{\sum_{\lambda=0}^{\infty} {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)\frac{\zeta^{\lambda}}{\lambda!}\right\},$$

$$\left(\frac{v}{1+h\zeta} + 2\frac{D_u^{-1}\zeta}{1+h\zeta} + \frac{\gamma'(\zeta)}{\gamma(\zeta)}\right)\left\{\sum_{\lambda=0}^{\infty} {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)\frac{\zeta^{\lambda}}{\lambda!}\right\} = \sum_{\lambda=0}^{\infty}\lambda\, {}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)\frac{\zeta^{\lambda-1}}{\lambda!}. \tag{4.6}$$

By applying the identity given in expression (4.5) and substituting $n \to \lambda+1$ into the right-hand side of expression (4.6), we can establish assertion (4.3). Additionally, considering the identity from expression (4.5), we have

$$\frac{{}_v\Delta_h}{h}\left[{}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)\right] = \left[\lambda\, {}_{\mathbb{L}}\mathbb{A}_{\lambda-1}^{[h]}(u,v)\right],$$

This yields the expression for the derivative operator given in (4.4). □

We then derive the differential equation for the Δ_h Laguerre-based Appell polynomials ${}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)$ by proving the following result.

Theorem 4.2. *The Δ_h Laguerre-based Appell polynomials ${}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)$ satisfy the differential equation:*

$$\left(\frac{v}{1+{}_v\Delta_h} + \frac{2\, D_u^{-1}\, {}_v\Delta_h}{h+{}_v\Delta_h{}^2} + \frac{\gamma'(\frac{{}_v\Delta_h}{h})}{\gamma(\frac{{}_v\Delta_h}{h})} - \frac{\lambda h}{{}_v\Delta_h}\right){}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v) = 0. \tag{4.7}$$

Proof. By substituting expressions (4.3) and (4.4) into expression (4.1), we establish the result given in (4.7). □

Next, we present the determinant representation of the Δ_h Laguerre-based Appell polynomials ${}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)$ by proving the following theorem.

Theorem 4.3. *The Δ_h Laguerre-based Appell polynomials ${}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)$ are expressed through the following determinant:*

$$
{}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)=\frac{(-1)^{\lambda}}{(\gamma_{0,h})^{\lambda+1}}\begin{vmatrix}
1 & \mathbb{L}_1^{[h]}(u,v) & \mathbb{L}_2^{[h]}(u,v) & \cdots & \mathbb{L}_{\lambda-1}^{[h]}(u,v) & \mathbb{L}_{\lambda}^{[h]}(u,v)\\
\gamma_{0,h} & \gamma_{1,h} & \gamma_{2,h} & \cdots & \gamma_{\lambda-1,h} & \gamma_{\lambda,h}\\
0 & \gamma_{0,h} & \binom{2}{1}\gamma_{1,h} & \cdots & \binom{\lambda-1}{1}\gamma_{\lambda-2,h} & \binom{\lambda}{1}\gamma_{\lambda-1,h}\\
0 & 0 & \gamma_{0,h} & \cdots & \binom{\lambda-1}{2}\gamma_{\lambda-3,h} & \binom{\lambda}{2}\gamma_{\lambda-2,h}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & \gamma_{0,h} & \binom{\lambda}{\lambda-1}\gamma_{1,h}
\end{vmatrix},
$$

where $\gamma_{\lambda,h}$, $\lambda=0,1,\dots$ are the coefficients of Maclaurins series of $\frac{1}{\gamma(\zeta)}$.

Proof. Multiplying both sides of equation (2.1) by $\frac{1}{\gamma(\zeta)}=\sum_{\lambda=0}^{\infty}\gamma_{\lambda,h}\frac{\zeta^{\lambda}}{\lambda!}$, we find

$$
\sum_{\lambda=0}^{\infty}\mathbb{L}_{\lambda}^{[h]}(u,v)\frac{\zeta^{\lambda}}{\lambda!}=\sum_{\lambda=0}^{\infty}\sum_{\kappa=0}^{\infty}\gamma_{\kappa,h}\frac{\zeta^{\kappa}}{\kappa!}\,{}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)\frac{\zeta^{\lambda}}{\lambda!},
$$

which on using the Cauchy product rule becomes

$$
\mathbb{L}_{\lambda}^{[h]}(u,v)=\sum_{\kappa=0}^{\lambda}\binom{\lambda}{\kappa}\gamma_{\kappa,h}\,{}_{\mathbb{L}}\mathbb{A}_{\lambda-\kappa}^{[h]}(u,v).
$$

This equality results in a system of m equations with the unknowns ${}_{\mathbb{L}}\mathbb{A}_{\lambda}^{[h]}(u,v)$, where $\lambda=0,1,2,\dots$. Solving this system using Cramer's rule, and noting that the denominator is the determinant of a lower triangular matrix with determinant $(\gamma_{0,h})^{\lambda+1}$, we can achieve the desired result by transposing the numerator matrix and replacing the i-th row with the $(i+1)$-th row for $i=1,2,\dots,\lambda-1$. □

5. Examples

There are many different types of polynomials in the Appell polynomial family, and each one is defined by selecting a certain function$\gamma(\zeta)$. The names, generating functions, and related constants of each polynomial in this family are distinct. We illustrate the generating function for these polynomials below. The generating function for the Bernoulli polynomials $\Delta_h\ \beta_{\lambda}^{[h]}(v)$ is specifically provided by:

$$
\frac{\log(1+h\zeta)^{\frac{1}{h}}}{(1+h\zeta)^{\frac{1}{h}}-1}(1+h\zeta)^{\frac{v}{h}}=\sum_{\lambda=0}^{\infty}\beta_{\lambda}^{[h]}(v)\frac{\zeta^{\lambda}}{\lambda!},\quad |\,\zeta\,|<2\pi.
$$

The generating expression for Δ_h-Euler polynomials $E_{\lambda}^{[h]}(v)$ is given by

$$
\frac{2}{(1+h\zeta)^{\frac{1}{h}}+1}(1+h\zeta)^{\frac{v}{h}}=\sum_{\lambda=0}^{\infty}E_{\lambda}^{[h]}(v)\frac{\zeta^{\lambda}}{\lambda!},\quad |\,\zeta\,|<\pi.
$$

The generating expression for Δ_h-Genocchi polynomials $G_{\lambda}^{[h]}(v)$ is given by

$$
\frac{2\log(1+h\zeta)^{\frac{1}{h}}}{(1+h\zeta)^{\frac{1}{h}}+1}(1+h\zeta)^{\frac{v}{h}}=\sum_{\lambda=0}^{\infty}G_{\lambda}^{[h]}(v)\frac{\zeta^{\lambda}}{\lambda!},\quad |\,\zeta\,|<\pi.
$$

For $h \to 0$, these polynomials reduce to the $g_\lambda(v)$, $E_\lambda(v)$, and $G_\lambda(v)$ polynomials [7]. Numerous applications in number theory, combinatorics, and numerical analysis can be found for the Δ_h Bernoulli, Euler, and Genocchi polynomials and numbers. In real-world situations, these numbers and polynomials are used to solve issues and obtain mathematical formulas. For instance, Bernoulli numbers are essential in many mathematical formulas, such as sums of powers of natural numbers, trigonometric and hyperbolic functions, and Taylor series expansions. They make significant correlations and patterns among numbers visible.

In graph theory, automata theory, and the counting of particular sequences, which facilitates the analysis of discrete structures, Euler numbers are employed extensively in Taylor series expansions and are linked to trigonometric and hyperbolic secant functions.

In automata theory and graph theory, genochchi numbers are helpful, especially when counting ascending sequences from top to bottom. This aids in the understanding of discrete structures and entails looking at how sequence elements are arranged.

Hence, the Δ_h polynomials, Bernoulli, Euler, and Genocchi numbers, are essential in many areas of mathematics because they allow for the investigation of connections, the creation of formulas, and the examination of patterns and structures.

By appropriately choosing the function $\gamma(t)$ in equation (2.1), we can derive generating functions for the Δ_h Laguerre-based Bernoulli ${}_{\mathbb{L}}{}^{[h]}_{\lambda}(u,v)$, Euler ${}_{\mathbb{L}}\mathbb{E}^{[h]}_{\lambda}(u,v)$, and Genocchi ${}_{\mathbb{L}}\mathbb{G}^{[h]}_{\lambda}(u,v)$ polynomials:

$$\begin{aligned}
\frac{\log(1+h\zeta)}{h(1+h\zeta)^{\frac{1}{h}}-h}(1+h\zeta)^{\frac{v}{h}}(1+h\zeta)^{\frac{D_u^{-1}}{h}} &= {}_{\mathbb{L}}\mathbb{B}^{[h]}_{\lambda}(u,v)\frac{\zeta^\lambda}{\lambda!},\\
\frac{2}{(1+h\zeta)^{\frac{1}{h}}+1}(1+h\zeta)^{\frac{v}{h}}(1+h\zeta)^{\frac{D_u^{-1}}{h}} &= {}_{\mathbb{L}}\mathbb{E}^{[h]}_{\lambda}(u,v)\frac{\zeta^\lambda}{\lambda!},\\
\frac{2\log(1+h\zeta)}{h(1+h\zeta)^{\frac{1}{h}}+h}(1+h\zeta)^{\frac{v}{h}}(1+h\zeta)^{\frac{D_u^{-1}}{h}} &= {}_{\mathbb{L}}\mathbb{G}^{[h]}_{\lambda}(u,v)\frac{\zeta^\lambda}{\lambda!}.
\end{aligned}$$

Further, in view of expression (2.6), the polynomials ${}_{S}\mathbb{B}^{[h]}_{\lambda}(u,v)$, Δ_h, ${}_{\mathbb{L}}\mathbb{E}^{[h]}_{\lambda}(u,v)$, and Δ_h, ${}_{\mathbb{L}}\mathbb{G}^{[h]}_{\lambda}(u,v)$ satisfy the following explicit form:

$$\begin{aligned}
{}_{\mathbb{L}}\mathbb{B}^{[h]}_{\lambda}(u,v) &= \sum_{k=0}^{\lambda}\binom{\lambda}{k}\ {}_{k,h}\ {}_{\mathbb{L}}\mathbb{A}^{[h]}_{\lambda-k}(u,v),\\
{}_{\mathbb{L}}\mathbb{E}^{[h]}_{\lambda}(u,v) &= \sum_{k=0}^{\lambda}\binom{\lambda}{k}\ \mathbb{E}_{k,h}\ {}_{\mathbb{L}}\mathbb{A}^{[h]}_{\lambda-k}(u,v)\\
{}_{\mathbb{L}}\mathbb{G}^{[h]}_{\lambda}(u,v) &= \sum_{k=0}^{\lambda}\binom{\lambda}{k}\ \mathbb{G}_{k,h}\ {}_{\mathbb{L}}\mathbb{A}^{[h]}_{\lambda-k}(u,v).
\end{aligned}$$

Furthermore, in view of expressions (4.8), the polynomials ${}_{\mathbb{L}}\mathbb{B}^{[h]}_{\lambda}(u,v)$, Δ_h, ${}_{\mathbb{L}}\mathbb{E}^{[h]}_{\lambda}(u,v)$, and Δ_h , ${}_{\mathbb{L}}\mathbb{G}^{[h]}_{\lambda}(u,v)$ satisfy the following determinant representations:

$$
{}_{\mathbb{L}}\mathbb{B}^{[h]}_{\lambda}(u,v) = \frac{(-1)^\lambda}{(\gamma_{0,h})^{\lambda+1}}
\begin{vmatrix}
1 & \mathbb{B}^{[h]}_{1}(u,v) & \mathbb{B}^{[h]}_{2}(u,v) & \cdots & \mathbb{B}^{[h]}_{\lambda-1}(u,v) & \mathbb{B}^{[h]}_{\lambda}(u,v)\\
\gamma_{0,h} & \gamma_{1,h} & \gamma_{2,h} & \cdots & \gamma_{\lambda-1,h} & \gamma_{\lambda,h}\\
0 & \gamma_{0,h} & \binom{2}{1}\gamma_{1,h} & \cdots & \binom{\lambda-1}{1}\gamma_{\lambda-2,h} & \binom{\lambda}{1}\gamma_{\lambda-1,h}\\
0 & 0 & \gamma_{0,h} & \cdots & \binom{\lambda-1}{2}\gamma_{\lambda-3,h} & \binom{\lambda}{2}\gamma_{\lambda-2,h}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & \gamma_{0,h} & \binom{n}{\lambda-1}\gamma_{1,h}
\end{vmatrix},
$$

$$
{}_{\mathbb{L}}\mathbb{E}_{\lambda}^{[h]}(u,v) = \frac{(-1)^{\lambda}}{(\gamma_{0,h})^{\lambda+1}} \begin{vmatrix} 1 & \mathbb{E}_1^{[h]}(u,v) & \mathbb{E}_2^{[h]}(u,v) & \cdots & \mathbb{E}_{\lambda-1}^{[h]}(u,v) & \mathbb{E}_{\lambda}^{[h]}(u,v) \\ \gamma_{0,h} & \gamma_{1,h} & \gamma_{2,h} & \cdots & \gamma_{\lambda-1,h} & \gamma_{\lambda,h} \\ 0 & \gamma_{0,h} & \binom{2}{1}\gamma_{1,h} & \cdots & \binom{\lambda-1}{1}\gamma_{\lambda-2,h} & \binom{\lambda}{1}\gamma_{\lambda-1,h} \\ 0 & 0 & \gamma_{0,h} & \cdots & \binom{\lambda-1}{2}\gamma_{\lambda-3,h} & \binom{\lambda}{2}\gamma_{\lambda-2,h} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_{0,h} & \binom{\lambda}{\lambda-1}\gamma_{1,h} \end{vmatrix},
$$

$$
{}_{\mathbb{L}}\mathbb{G}_{\lambda}^{[h]}(u,v) = \frac{(-1)^{\lambda}}{(\gamma_{0,h})^{\lambda+1}} \begin{vmatrix} 1 & \mathbb{G}_1^{[h]}(u,v) & \mathbb{G}_2^{[h]}(u,v) & \cdots & \mathbb{G}_{\lambda-1}^{[h]}(u,v) & \mathbb{G}_{\lambda}^{[h]}(u,v) \\ \gamma_{0,h} & \gamma_{1,h} & \gamma_{2,h} & \cdots & \gamma_{\lambda-1,h} & \gamma_{\lambda,h} \\ 0 & \gamma_{0,h} & \binom{2}{1}\gamma_{1,h} & \cdots & \binom{\lambda-1}{1}\gamma_{\lambda-2,h} & \binom{\lambda}{1}\gamma_{\lambda-1,h} \\ 0 & 0 & \gamma_{0,h} & \cdots & \binom{\lambda-1}{2}\gamma_{\lambda-3,h} & \binom{\lambda}{2}\gamma_{\lambda-2,h} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_{0,h} & \binom{\lambda}{\lambda-1}\gamma_{1,h} \end{vmatrix}.
$$

6. Conclusion

An important development in polynomial theory, namely in quantum mechanics and entropy modeling in quantum systems, is the presentation and analysis of the Δ_h Laguerre-Appell polynomials. Through the application of particular operational criteria and the monomiality principle, the new class of polynomials we introduced offers new insights on underexplored mathematical fields.

The study provides closed formulas and a thorough explanation of the basic characteristics of Δ_h Laguerre polynomials, which not only improves our comprehension of these polynomials but also creates connections with well-known polynomial families, thereby contributing to the more comprehensive mathematical framework.

Numerous interesting avenues could be investigated in future studies. Deeper understanding of Δ_h Laguerre-Appell polynomials' behavior and uses may be possible with additional examination of their algebraic and structural characteristics. Further exploration of its application in other mathematical physics and quantum mechanics contexts may also yield new avenues for study and useful applications.

Future research, particularly in fields such as information theory, statistical mechanics, and computer science, may concentrate on bridging the gap between mathematical theory and practical applications, given the interdisciplinary nature of this study. It may be possible to fully realize the potential of Δ_h hybrid polynomials and their contributions to numerous domains by cooperation amongst different scientific disciplines.

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