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Approximating fixed points of nonexpansive-type mappings in Hadamard spaces

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Abstract

Certain essential properties of a generalized nonexpansive mapping, showcasing an inclusion relation to known mappings within the framework of Hadamard spaces, are investigated in this work. Additionally, a viscosity-type algorithm is proposed for approximating fixed points of such mappings. The strong convergence of the sequences generated by this algorithm is demonstrated under appropriate assumptions. As an application of the findings, the existence of a solution to the variational inequality problem involving the mapping is established. Furthermore, an iterative scheme is derived, exhibiting strong convergence towards solving the variational inequality problem. To demonstrate the implementation of the proposed algorithm, a numerical example is provided in a non-Hilbert CAT(0) space.

Keywords: Convergence analysis, fixed point, Hadamard space, inverse strongly monotone, nonexpansive mapping, variational inequality, viscosity iteration.

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1. Introduction

Let $(\mathcal{H}, d)$ be a metric space, and let $T : \mathcal{H} \to \mathcal{H}$ be a mapping. A point $q \in \mathcal{H}$ is said to be a *fixed point* of T if $Tq = q$. The mapping T is considered *nonexpansive* if it satisfies the inequality

$$d(Tu, Tw) \leqslant d(u, w), \quad \forall\, u, w \in \mathcal{H}.$$

Nonexpansive mappings and their generalizations find numerous applications in real-world phenomena. They often arise as transition operators for initial-value inclusions of the form

$$0 \in \frac{du}{dt} + T(t)u,$$

(see, for example, Bruck [13]). The study of nonexpansive mappings is regarded as a generalization of the renowned contraction principle *à la* Stefan Banach. For instance, a contraction mapping possesses a unique fixed point and the Picard's iterate converges to such a unique fixed point, regardless of the initial

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starting point. In the same setting, a (generalized) nonexpansive mapping may not possess a fixed point. Even when it does, the Picard's scheme may not converge to that point. This issue has sparked great attentions to the problem of finding fixed points of a nonexpansive mapping and its generalizations. For more details, one could refer to [7, 15, 33, 51] and the references therein.

One of the pioneering prominent scheme for approximating fixed points of (generalized) nonexpansive mappings is the Krasnosel'skiĭ-Mann (KM) iteration, which is typically defined in a linear space $\mathcal{H}$ as follows:

$$u_1 \in \mathcal{H}, \quad u_{n+1} = (1-\beta_n)u_n + \beta_n T u_n, \ \ n \geqslant 1. \tag{1.1}$$

Here, $\{\beta_n\}$ is a sequence in $[0, 1]$ that satisfies certain conditions, and $T : \mathcal{H} \to \mathcal{H}$ is the underlying mapping (see, for example, [12, 25, 34] and the references therein). Although the KM scheme described in (1.1) provides a unifying foundation for several algorithms, it only yields weak convergence under specific assumptions (see [23] for an example where it does not converge strongly). Therefore, it becomes necessary to impose additional conditions on the mapping itself (such as continuity) or on the space (such as compactness), or to introduce modifications to the KM scheme to achieve strong convergence.

Let $\mathcal{H}$ be a linear space, and let $T : \mathcal{H} \to \mathcal{H}$ be a mapping. One approach to achieving strong convergence, without imposing strong assumptions on the underlying mapping, is the viscosity scheme, which can be attributed to Moudafi [37]. This method iteratively defines $\{u_n\}$ as follows:

$$u_1 \in \mathcal{H}, \quad u_{n+1} = \frac{\delta_n}{1+\delta_n} f(u_n) + \frac{1}{1+\delta_n} T u_n, \ \ n \geqslant 1.$$

Here, $\{\delta_n\}$ is a sequence of positive real numbers converging to zero, and $f : \mathcal{H} \to \mathcal{H}$ is a contraction mapping. Another approach to achieve strong convergence is the scheme proposed by Yao et al. [50], which is a modification of the KM scheme. The scheme defines $\{u_n\}$ as follows:

$$\begin{cases} w_n = (1-\alpha_n)u_n, \ \ u_0 \in \mathcal{H} \\ u_{n+1} = (1-\beta_n)w_n + \beta_n T w_n, \ \ n \geqslant 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

Recently, Berinde [8] introduced the class of enriched nonexpansive mappings as a superclass of the class of nonexpansive mappings. A mapping T from a normed linear space $(\mathcal{H}, \|\cdot\|)$ to itself is considered an *enriched nonexpansive* (or α-enriched nonexpansive) mapping if there exists $\alpha \in [0, +\infty)$ such that the inequality

$$\|\alpha(u-w) + Tu - Tw\| \leqslant (\alpha+1)\|u-w\|, \ \forall u, w \in \mathcal{H}, \tag{1.3}$$

holds. Later, Berinde [9] presented a strong convergence theorem for the scheme in (1.2) to approximate fixed points of enriched nonexpansive mappings in the context of real Hilbert spaces. We state the main result of Berinde [9] as follows.

Theorem 1.1. *Let $\mathcal{H}$ be a real Hilbert space and let $T : \mathcal{H} \to \mathcal{H}$ be an enriched nonexpansive mapping with a nonempty fixed point set. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0, 1]$ that satisfied the following conditions:* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\beta_n \in [a, b] \subset (0, 1)$. *Then, the sequences $\{u_n\}$ and $\{w_n\}$ generated by*

$$\begin{cases} w_n = (1-\alpha_n)u_n, \\ u_{n+1} = (1-\sigma\beta_n)w_n + \sigma\beta_n T w_n, \ \ n \geqslant 1, \end{cases} \tag{1.4}$$

converge strongly to a fixed point of T, where $u_1 \in \mathcal{H}$ and $\sigma \in (0, 1]$ is a constant.

Note that, the scheme in (1.4) can be rewritten as

$$\begin{cases} w_n = (1-\alpha_n)u_n, \\ u_{n+1} = (1-\beta_n)w_n + (1-\sigma)\beta_n w_n + \sigma\beta_n T w_n, \ \ n \geqslant 1. \end{cases} \tag{1.5}$$

Moreover, it is natural to seek a scheme that converges strongly to a fixed point of an enriched nonexpansive mapping in a more general setting than Hilbert spaces. For further details regarding the concept of enriching techniques and nonexpansive mappings, refer to [10, 39–41, 45–47].

One of the generalizations of Hilbert spaces where fixed point theory plays a significant role is Hadamard spaces. Fixed point theory in Hadamard spaces finds applications in various areas such as gradient flow, diffusion tensor imaging, Calabi flow in Kahler geometry, and many other areas of optimization. Additionally, by utilizing appropriate metrics, Hadamard spaces allow the transformation of certain non-smooth and non-convex constrained optimization problems into smooth and convex unconstrained problems (see, for example, [4, 22, 27, 28, 42, 44] and the references therein). Fixed point approximation methods for nonexpansive mappings in Hadamard spaces can be used to investigate chemotaxis systems; see, e.g., [26].

Fixed point theory is a well-known approach to handling nonlinear convex analysis and optimizations. In the context of Hadamard spaces, Kirk [30, 31] established that a nonexpansive mapping defined on a closed bounded convex subset of a Hadamard space always has a nonempty closed and bounded fixed point set. Since then, several scholars have considered Hadamard spaces as a suitable framework for studying fixed points of certain (generalized) nonexpansive mappings (see, for example, [18, 29, 36, 38] and the references therein).

In [43], the authors introduced and analyzed the class of enriched nonexpansive mappings in the setting of Hadamard spaces, which serves as a superclass of the class of nonexpansive mappings. They established certain properties such as demiclosedness-type, existence and uniqueness of fixed points for such mappings. Additionally, they employed a Krasnosel'skiĭ-type scheme to approximate the fixed points.

Definition 1.2 ([43, Definition 4.3])**.** Let $(\mathcal{H}, d)$ be a Hadamard space and D be a nonempty subset of $\mathcal{H}$. A mapping $T : D \to \mathcal{H}$ is called an *enriched nonexpansive* (or *α-enriched nonexpansive*) if there exists a positive real number α such that

$$d(Tu, Tw)^2 + \alpha^2 d(u, w)^2 + 2\alpha\langle \overrightarrow{uw}, \overrightarrow{TuTw}\rangle \leqslant (\alpha + 1)^2 d(u, w)^2, \tag{1.6}$$

for every points $u, w \in D$, where

$$\langle \overrightarrow{uw}, \overrightarrow{TuTw}\rangle := \frac{1}{2}\left(d^2(u, Tw) + d^2(w, Tu) - d^2(u, Tu) - d^2(w, Tw)\right)$$

denotes the quasilinearization on $\mathcal{H}$ (see [6]).

Note that in an inner-product space $\mathcal{H}$, (1.6) reduces to (1.3) with $D = \mathcal{H}$. Moreover, as an example of a non-Hilbert Hadamard space, consider $\mathbb{R}^2$ equipped with the radial metric ρ defined by $\rho(u, w) = \|u - w\|_2$ if u and w lie on the same Euclidean line passing through $(0, 0)$, and $\rho(u, w) = \|u\|_2 + \|w\|_2$ otherwise. The following is an example of an enriched nonexpansive mapping with respect to the radial metric.

Example 1.3. Consider $\mathbb{R}^2$ with the radial metric ρ, $a \geqslant 1$ and $D = A_a \cup B_a \cup C_a$, where

$$A_a := \{0\} \times \left[\frac{2}{3}a, a\right], \quad B_a := \left[\frac{2}{3}a, a\right] \times \{0\}, \quad \text{and} \quad C_a := \left\{(p, q) : p^2 + q^2 = 1, p > 0, q > 0\right\}.$$

Let T be the mapping defined by $Tu = (a, 0)$ if $u \in A_a$, $Tu = (0, a)$ if $u \in B_a$ and $Tu = u$ if $u \in C_a$. Then T is 1-enriched nonexpansive mapping but not nonexpansive. Indeed, for $u = \left(\frac{2a}{3}, 0\right)$ and $w = \left(0, \frac{2a}{3}\right)$, we have

$$\rho(Tu, Tw) = \rho((0, a), (a, 0)) = 2a > \frac{4a}{3} = \rho(u, w).$$

We now show that T is 1-enriched nonexpansive mapping in four cases.

Case 1: Both u and w are in either A_a or B_a or C_a. Then $\rho(Tu, Tw) \leqslant \rho(u, w)$.
Case 2: Let $u = (0, y)$ in A_a and $w = (x, 0)$ in B_a. Then

$$\begin{aligned}\rho^2(Tu, Tw) + \rho^2(u, w) + 2\langle \overrightarrow{uw}, \overrightarrow{TuTw}\rangle &= 4a^2 + (x+y)^2 + \left[(a-y)^2 + (a-x)^2 - (a+y)^2 - (a+x)^2\right] \\ &= 4a^2 + (x+y)^2 - 4a(x+y) \\ &\leqslant 4a^2 + (x+y)^2 - 16\frac{a^2}{3} \leqslant (x+y)^2 = \rho(u, w) \leqslant 2^2\rho(u, w).\end{aligned}$$

Case 3: Let $u = (0, y)$ in A_a and $w = (p, q)$ in C_a. Then

$$\begin{aligned}&\rho^2(Tu, Tw) + \rho^2(u, w) + 2\langle \overrightarrow{uw}, \overrightarrow{TuTw}\rangle \\ &\quad= (a + \sqrt{p^2+q^2})^2 + (y + \sqrt{p^2+q^2})^2 + \left[(y + \sqrt{p^2+q^2})^2 + (a + \sqrt{p^2+q^2})^2 - (a+y)^2\right] \\ &\quad= 8a^2 + (y+a)^2 \leqslant \frac{25}{3}a^2 + (y+a)^2 = 3\left(\frac{2}{3}a + a\right)^2 + (y+a)^2 \leqslant 4(y+a)^2 = 2^2\rho(u, w).\end{aligned}$$

Case 4: Let $u = (x, 0)$ in B_a and $w = (p, q)$ in C_a. This follows similar lines as in Case 3 with x in places of y.

The purpose of this paper is to establish essential properties of the class of mappings introduced in [43]. We demonstrate that this class, which falls under the category of Lipschitz mappings, encompasses strictly pseudocontractive and inverse-strongly monotone mappings. To approximate the fixed points of these mappings, we propose a viscosity-type scheme and prove the strong convergence of the sequences generated by this scheme towards the fixed point of the studied mapping. Additionally, we provide applications of our findings to solutions of variational inequality problems. Our results extend and complement recent findings in the literature.

Next, we present a collection of known results on geodesic spaces that will be utilized in obtaining our main results.

2. Preliminaries

Recall that a metric space $(\mathcal{H}, d)$ is said to be a *geodesic space* if for every two points $u, w \in \mathcal{H}$, there exists a mapping $\tau : [0, 1] \subset \mathbb{R} \to \mathcal{H}$ satisfying the following:

(i) $\tau(0) = u$;
(ii) $\tau(1) = w$;
(iii) $d(\tau(a), \tau(b)) = |a - b| d(u, w)$ for every $a, b \in [0, 1]$.

The image of τ is called a *geodesic segment* connecting u and w. For $u, w \in \mathcal{H}$ having unique geodesic segment and for any $\ell \in [0, 1]$, there exists a unique point z on the segment connecting u and w, denoted by $(1-\ell)u \oplus \ell w$ with the following conditions:

$$d(u, z) = \ell d(u, w) \quad \text{and} \quad d(z, w) = (1-\ell) d(u, w). \tag{2.1}$$

A collection of three elements (say u, v, w) and three geodesic segments connecting each pair is called a *geodesic triangle* denoted by $\triangle(u, v, w)$. A *comparison triangle* of $\triangle(u, v, w)$ is a triangle $\overline{\triangle}(\bar{u}, \bar{v}, \bar{w}) \in \mathbb{R}^2$ such that

$$\|\bar{u} - \bar{v}\|_2 = d(u, v), \quad \|\bar{u} - \bar{w}\|_2 = d(u, w), \quad \text{and} \quad \|\bar{v} - \bar{w}\|_2 = d(v, w).$$

A geodesic space $(\mathcal{H}, d)$ is called a *CAT(0) space* if every geodesic triangle is slimmer than its comparison triangle in the sense that

$$d(x, y) \leqslant \|\bar{x} - \bar{y}\|_2 \tag{2.2}$$

holds for all points x, y in the triangle with corresponding comparison points $\bar{x}, \bar{y}$ (where a point $\bar{w}$ on the segment connecting $\bar{x}$ to $\bar{y}$ is called a *comparison point* of a point w on the segment connecting x to y if $\|\bar{x}-\bar{w}\|_2 = d(x, w)$). A complete CAT(0) space is called a *Hadamard space*.

Using (2.1) and (2.2), we have the following inequalities (see also [21]) for $u_1, u_2, u_3 \in \mathcal{H}$ and $t \in [0,1]$:

$$d((1-t)u_1 \oplus tu_2, u_3) \leqslant (1-t)d(u_1, u_3) + td(u_2, u_3), \tag{2.3}$$

$$d((1-t)u_1 \oplus tu_2, u_3)^2 \leqslant (1-t)d(u_1, u_3)^2 + td(u_2, u_3)^2 - t(1-t)d(u_1, u_2)^2. \tag{2.4}$$

The inequality (2.4) is a generalisation of CN-inequality of Bruhat and Tits [14] stated as follows. Letting $u, w \in \mathcal{H}$, then

$$d\left(\frac{1}{2}u \oplus \frac{1}{2}w, y\right)^2 \leqslant \frac{1}{2}d(w, y)^2 + \frac{1}{2}d^2(w, y) - \frac{1}{4}d^2(u, w), \tag{2.5}$$

for every $y \in \mathcal{H}$. For further details on the setting of CAT(κ) spaces, see, for example, [11, 33] and the references therein.

The *asymptotic center* of a bounded sequence $\{u_n\}$ in a metric space $(\mathcal{H}, d)$ is the set

$$A(\{u_n\}) := \left\{w \in \mathcal{H} : \limsup_{n\to\infty} d(w, u_n) = \inf_{z\in\mathcal{H}} \limsup_{n\to\infty} d(z, u_n)\right\}.$$

It follows from [20, Proposition 7] that, in a Hadamard space, $A(\{u_n\})$ has exactly one element. Furthermore, the sequence $\{u_n\}$ is said to Δ-*converge* to a point u in $\mathcal{H}$ if $\{u\}$ is the unique asymptotic center for every subsequences of $\{u_n\}$. When $d(u_n, u) \to 0$, we say that the sequence $\{u_n\}$ *converges strongly* to u.

In the sequel, we take $(\mathcal{H}, d)$ to be a Hadamard space. We use $u_n \to u$ (resp. $u_n \xrightarrow{\Delta} u$) to mean $\{u_n\}$ converges strongly (resp. Δ-converges) to u. Also, $T : \mathcal{H} \to \mathcal{H}$ is said to have *demiclosedness-type property* if

$$\left.\begin{array}{r} u_n \xrightarrow{\Delta} u \\ d(u_n, Tu_n) \to 0 \end{array}\right\} \implies u = Tu.$$

We now state some facts that will be useful for the convergence analysis of our scheme.

Lemma 2.1 ([43, Lemma 4.7])**.** *Every enriched nonexpansive mapping $T : \mathcal{H} \to \mathcal{H}$ has demiclosedness-type property.*

Lemma 2.2 ([19, Proposition 2.1])**.** *The asymptotic centre of any bounded sequence in $\mathcal{H}$ is contained in $\mathcal{H}$.*

Lemma 2.3 ([32, Proposition 3.6])**.** *Every bounded sequence in $\mathcal{H}$ has a Δ-convergent subsequence.*

Lemma 2.4 ([1, Theorem 2.6])**.** *A bounded sequence $\{w_n\}$ in $\mathcal{H}$ is said to Δ-converges to a point w in $\mathcal{H}$ if and only if $\limsup_{n\to\infty}\langle \overrightarrow{w_n w}, \overrightarrow{zw}\rangle \leqslant 0$ for all z in $\mathcal{H}$.*

Lemma 2.5 ([49, Lemma 2.5])**.** *Let $\{\theta_n\}$ be a sequence in $[0, +\infty) \subset \mathbb{R}$ with*

$$\theta_{n+1} \leqslant (1-\sigma_n)\theta_n + \sigma_n\phi_n + \gamma_n, \quad n \geqslant 1,$$

where $\{\sigma_n\}$, $\{\phi_n\}$, and $\{\gamma_n\}$ satisfy the following conditions:

$$\{\sigma_n\} \subset [0,1], \quad \sum_{n=1}^{\infty} \sigma_n = \infty, \quad \limsup_{n\to\infty} \phi_n \leqslant 0, \quad \{\gamma_n\} \subset [0, \infty), \quad \textit{and} \quad \sum_{n=1}^{\infty} \gamma_n < \infty.$$

Then $\lim_{n\to\infty} \theta_n = 0$.

Let D be a nonempty closed convex subset of $\mathcal{H}$. It has been established (see, e.g., [11, pp. 176]) that for any $x \in \mathcal{H}$, there exists a unique point $u \in D$ such that $d(x, u) = \min\{d(x, w) : w \in D\}$ and the mapping $P_D : \mathcal{H} \to D$ defined by $P_D x = u$ is called the *metric projection*. Moreover, it follows from Proposition 2.4 of [11] that P_D is nonexpansive. We have the following important result for the projection mapping.

Lemma 2.6 ([17, Theorem 2.2])**.** *Let $x \in \mathcal{H}$ and $u \in D$. Then $u = P_D x$ if and only if $\langle \overrightarrow{ux}, \overrightarrow{yu}\rangle \geqslant 0$, $\forall\, y \in D$.*

3. Main results

3.1. Generalized nonexpansive mappings

Let D be a nonempty subset of $\mathcal{H}$. As in [2, 24], a mapping $T : D \to \mathcal{H}$ is called *strictly pseudocontractive* if there exists a real number, say $\kappa \in [0, 1)$ such that

$$d^2(Tu, Tw) \leqslant d^2(u, w) + 4\kappa d^2\left(\frac{1}{2}u \oplus \frac{1}{2}Tw, \frac{1}{2}w \oplus \frac{1}{2}Tu\right), \quad \forall\, u, w \in D. \tag{3.1}$$

To indicate the constant κ involved, we say that T is κ-strictly pseudocontractive mapping.

Proposition 3.1. *Every κ-strictly pseudocontractive mapping is $\frac{\kappa}{1-\kappa}$-enriched nonexpansive mapping.*

Proof. Assume that $T : D \to \mathcal{H}$ is κ-strictly pseudocontractive and let $u, w \in D$. Then, by (3.1) and (2.4), we have

$$\begin{aligned}
d^2(Tu, Tw) &\leqslant d^2(u, w) + 4\kappa d^2\left(\frac{1}{2}u \oplus \frac{1}{2}Tw, \frac{1}{2}w \oplus \frac{1}{2}Tu\right)\\
&\leqslant d^2(u, w) + 4\kappa\left[\frac{1}{2}d^2\left(u, \frac{1}{2}w \oplus \frac{1}{2}Tu\right) + \frac{1}{2}d^2\left(Tw, \frac{1}{2}w \oplus \frac{1}{2}Tu\right) - \frac{1}{4}d^2\left(u, Tw\right)\right]\\
&\leqslant d^2(u, w) + 2\kappa\left[\frac{1}{2}d^2(u, w) + \frac{1}{2}d^2(u, Tu) - \frac{1}{4}d^2(w, Tu)\right]\\
&\quad + 2\kappa\left[\frac{1}{2}d^2(Tw, w) + \frac{1}{2}d^2(Tw, Tu) - \frac{1}{4}d^2(w, Tu)\right] - \kappa d^2(u, Tw)\\
&= (1 + k)d^2(u, w) + \kappa d^2(Tu, Tw) - 2\kappa\langle\overrightarrow{uw}, \overrightarrow{TuTw}\rangle.
\end{aligned}$$

Consequently, we have

$$d^2(Tu, Tw) + 2\frac{\kappa}{1-\kappa}\langle\overrightarrow{uw}, \overrightarrow{TuTw}\rangle \leqslant \frac{1+\kappa}{1-\kappa}d^2(u, w),$$

which holds if and only if

$$d^2(Tu, Tw) + \left(\frac{\kappa}{1-\kappa}\right)^2 d(u, w) + 2\frac{\kappa}{1-\kappa}\langle\overrightarrow{uw}, \overrightarrow{TuTw}\rangle \leqslant \left(\frac{\kappa}{1-\kappa} + 1\right)^2 d^2(u, w).$$

Hence T is $\frac{\kappa}{1-\kappa}$-enriched nonexpansive mapping. □

Let $T : D \to \mathcal{H}$ be a mapping. Define a function $\phi_T : D \times D \to \mathbb{R}$ by

$$\phi_T(u, w) = d^2(u, w) + d^2(Tu, Tw) - 2\langle\overrightarrow{uw}, \overrightarrow{TuTw}\rangle \quad (u, w \in D).$$

In [2], the notation '$I - T$' is said to be *inverse-strongly monotone* if there exists a number $\alpha > 0$ such that

$$d^2(u, w) - \langle\overrightarrow{uw}, \overrightarrow{TuTw}\rangle \geqslant \alpha\phi_T(u, w), \quad \forall\, u, w \in D. \tag{3.2}$$

To indicate the parameter α involved in (3.2), we also refer $I - T$ as *α-inverse-strongly monotone*. Note that in an inner-product setting, (3.2) reduces to $\alpha\|Au - Aw\|^2 \leqslant \langle Au - Aw, u - w\rangle$, where $A \equiv I - T$.

Proposition 3.2. *Let $T : D \to \mathcal{H}$ be a mapping such that $I - T$ is α-inverse-strongly monotone. Then for any $\beta \in (0, \alpha]$, $I - T$ is β-inverse-strongly monotone.*

Proof. We observe that $\phi_T(D \times D) \subset [0, +\infty)$. Indeed, by Cauchy Schwartz inequality (that is, $\langle \overrightarrow{xy}, \overrightarrow{vz} \rangle \leqslant d(x, y)d(v, z)$ for all $x, y, u, w \in D$), we have that

$$\begin{aligned}\phi_T(u, w) &= d^2(u, w) + d^2(Tu, Tw) - 2\langle \overrightarrow{uw}, \overrightarrow{TuTw} \rangle \\ &\geqslant d^2(u, w) + d^2(Tu, Tw) - 2d(u, w)d(Tu, Tw) = \big[d(u, w) - d(Tu, Tw)\big]^2 \geqslant 0.\end{aligned}$$

Consequently, we get

$$d^2(u, w) - 2\langle \overrightarrow{uw}, \overrightarrow{TuTw} \rangle \geqslant \alpha\phi_T(u, w) \geqslant \beta\phi_T(u, w), \ \forall\, u, w \in D$$

as desired. □

Proposition 3.3. *Let* $T : D \to \mathcal{H}$ *be a mapping such that* $I - T$ *is* α*-inverse-strongly monotone. Then, there exists a positive real number, say* α^* *such that* T *is* α^**-enriched nonexpansive mapping.*

Proof. Let $u, w \in D$ and let $\alpha^o = \min\left\{\frac{1}{2}, \alpha\right\}$. Then by Proposition 3.2, we have that $I - T$ is α^o-inverse-strongly monotone. Thus,

$$d^2(u, w) - \langle \overrightarrow{uw}, \overrightarrow{TuTw} \rangle \geqslant \alpha^o\phi_T(u, w)$$

if and only if

$$\alpha^o d^2(u, w) + \alpha^o d^2(Tu, Tw) + (1 - 2\alpha^o)\langle \overrightarrow{uw}, \overrightarrow{TuTw} \rangle \leqslant d^2(u, w)$$

if and only if

$$d^2(u, w) + d^2(Tu, Tw) + 2\left(\frac{1}{2\alpha^o} - 1\right)\langle \overrightarrow{uw}, \overrightarrow{TuTw} \rangle \leqslant \frac{1}{\alpha^o}d^2(u, w)$$

if and only if

$$d^2(Tu, Tw) + \left(\frac{1}{2\alpha^o} - 1\right)^2 d^2(u, w) + 2\left(\frac{1}{2\alpha^o} - 1\right)\langle \overrightarrow{uw}, \overrightarrow{TuTw} \rangle \leqslant \left(\frac{1}{2\alpha^o}\right)^2 d^2(u, w)$$

if and only if

$$d^2(Tu, Tw) + \alpha^{*2} d^2(u, w) + 2\alpha^*\langle \overrightarrow{uw}, \overrightarrow{TuTw} \rangle \leqslant (\alpha^* + 1)^2\, d^2(u, w),$$

where $\alpha^* = \dfrac{1}{2\alpha^o} - 1 \geqslant 0$ as desired. □

Proposition 3.4. *Let* $T : D \to \mathcal{H}$ *be* α*-enriched nonexpansive mapping. Then*

$$d(Tu, Tw) \leqslant (2\alpha + 1)d(u, w), \quad \forall\, u, w \in D.$$

Proof. Let $u, w \in D$. From Cauchy Schwartz inequality and the hypothesis that T is α-enriched nonexpansive mapping, we have

$$\begin{aligned}\big[d(Tu, Tw) - \alpha d(u, w)\big]^2 &= d^2(Tu, Tw) + \alpha^2 d^2(u, w) - 2\alpha d(u, w)d(Tu, Tw) \\ &\leqslant d^2(Tu, Tw) + \alpha^2 d^2(u, w) - 2\alpha\langle \overrightarrow{uw}, \overrightarrow{TwTu} \rangle \\ &= d^2(Tu, Tw) + \alpha^2 d^2(u, w) + 2\alpha\langle \overrightarrow{uw}, \overrightarrow{TuTw} \rangle \leqslant (\alpha + 1)^2 d^2(u, w).\end{aligned}$$

Consequently, we have

$$\big|d(Tu, Tw) - \alpha d(u, w)\big| \leqslant (\alpha + 1)d(u, w).$$

Thus $d(Tu, Tw) \leqslant (2\alpha + 1)d(u, w)$ as desired. □

Proposition 3.5. *Let* D *be a closed convex subset of* $\mathcal{H}$*. Suppose that* $T : D \to \mathcal{H}$ *is* α*-enriched nonexpansive mapping with* $F(T) \neq \emptyset$*. Then* $F(P_D T) = F(T)$*.*

Proof. Clearly, $F(T) \subset F(P_D T)$. For the converse, let $u = P_D Tu$. Then for $p \in F(T)$, we obtain from Lemma 2.6, that

$$d^2(u, Tu) \leqslant d^2(u, Tu) + 2\langle \overrightarrow{TuP_DTu}, \overrightarrow{P_DTup} \rangle \leqslant d^2(u, Tu) + 2\langle \overrightarrow{Tuu}, \overrightarrow{up} \rangle = d^2(p, Tu) - d^2(p, u). \tag{3.3}$$

Since T is α-enriched nonexpansive mapping, we have that

$$d^2(p, Tu) + \alpha^2 d^2(p, u) + 2\alpha\langle \overrightarrow{up}, \overrightarrow{Tup} \rangle \leqslant (\alpha + 1)^2 d^2(p, u).$$

This holds if and only if

$$(\alpha + 1)d^2(p, Tu) \leqslant (\alpha + 1)d^2(p, u) + \alpha d^2(u, Tu). \tag{3.4}$$

Thus, we obtain from (3.3) and (3.4) that

$$d^2(u, Tu) \leqslant d^2(p, Tu) - d^2(p, u) \leqslant \frac{\alpha}{\alpha + 1} d^2(u, Tu).$$

Hence

$$\frac{1}{\alpha + 1} d^2(u, Tu) \leqslant 0$$

and consequently, we have that $u \in F(T)$. □

3.2. *Viscosity-type scheme for approximating fixed points*

Let $T : \mathcal{H} \to \mathcal{H}$ be a mapping. Recall that, as in [16], for $u, v, w \in \mathcal{H}$ and $t_1, t_2, t_3 \in [0, 1]$ such that $t_1 + t_2 + t_3 = 1$, the oplus ($\oplus$) term for a convex combination of three points can take the following order form:

$$t_1 u \oplus t_2 v \oplus t_3 w := t_1 u \oplus (1 - t_1)\left(\frac{t_2}{1 - t_1} v \oplus \frac{t_3}{1 - t_1} w\right). \tag{3.5}$$

This paper concerns the following scheme

$$\begin{cases} w_n = (1 - \alpha_n)u_n \oplus \alpha_n f(u_n), \ u_1 \in \mathcal{H}, \\ u_{n+1} = (1 - \sigma\beta_n)w_n \oplus \sigma\beta_n T w_n, \quad n \geqslant 1, \end{cases} \tag{3.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$, $\sigma \in (0, 1]$ is a constant, and $f : \mathcal{H} \to \mathcal{H}$ is a contraction mapping. Note that, in the particular case where $\mathcal{H}$ is a Hilbert space and $f \equiv 0$, the scheme in (3.6) reduces to the scheme in (1.5). In addition to that, if $\alpha_n \equiv 0$, then (3.6) reduces to the KM scheme in (1.1).

Lemma 3.6. *Let* $T : \mathcal{H} \to \mathcal{H}$ *be an enriched nonexpansive mapping with* $F(T) \neq \emptyset$*. Suppose* $\{u_n\}$ *and* $\{w_n\}$ *are sequences from* (3.6)*. Then for any* $p \in F(T)$*,*

(P1) $d(u_{n+1}, p) \leqslant d(w_n, p)$, $\forall\, n \geqslant 1$;
(P2) *the sequences* $\{u_n\}$ *and* $\{w_n\}$ *are bounded;*
(P3) *if* $\beta_n \in [a, b] \subset (0, 1)$*, then there exists some positive number* M *such that*

$$d^2(w_n, Tw_n) \leqslant M\big(d^2(w_n, p) - d^2(u_{n+1}, p)\big), \ \forall\, n \geqslant 1.$$

Proof. Let $u, w \in \mathcal{H}$. By (2.4), we have

$$\begin{aligned}
&d^2\big((1-\sigma)u \oplus \sigma Tu, (1-\sigma)w \oplus \sigma Tw\big)\\
&\quad\leqslant (1-\sigma)d^2\big(u,(1-\sigma)w \oplus \sigma Tw\big) + \sigma d^2\big(Tu,(1-\sigma)w\oplus \sigma Tw\big) - \sigma(1-\sigma)d^2(u,Tu)\\
&\quad\leqslant (1-\sigma)\Big[(1-\sigma)d^2(u,w) + \sigma d^2(u,Tw) - \sigma(1-\sigma)d^2(w,Tw)\Big]\\
&\qquad + \sigma\Big[(1-\sigma)d^2(Tu,w) + \sigma d^2(Tu,Tw) - \sigma(1-\sigma)d^2(w,Tw)\Big] - \sigma(1-\sigma)d^2(u,Tu)\\
&\quad= (1-\sigma)^2d^2(u,w) + \sigma^2d^2(Tu,Tw) + \sigma(1-\sigma)\Big[d^2(u,Tw) + d^2(w,Tu) - d^2(u,Tu) - d^2(w,Tw)\Big]\\
&\quad= (1-\sigma)^2d^2(u,w) + \sigma d^2(Tu,Tw) + 2\sigma(1-\sigma)\langle\overrightarrow{uw},\overrightarrow{TuTw}\rangle.
\end{aligned}$$

Let $\sigma = \dfrac{1}{1+\alpha}$. Then, we have from the immediate inequality that

$$d^2\big((1-\sigma)u\oplus\sigma Tu,(1-\sigma)w\oplus\sigma Tw\big) \leqslant \frac{1}{(\alpha+1)^2}\Big[d^2(Tu,Tw)+\alpha^2d^2(u,w)+2\alpha\langle\overrightarrow{uw},\overrightarrow{TuTw}\rangle\Big].$$

Consequently, as T is α-enriched nonexpansive mapping, it follows from Definition 1.2 that

$$d\big((1-\sigma)u\oplus\sigma Tu,(1-\sigma)w\oplus\sigma Tw\big) \leqslant d(u,w). \tag{3.7}$$

Let $p \in F(T)$. Then by (3.6), (3.5), and (2.3), we have

$$\begin{aligned}
d^2(u_{n+1},p) &= d^2\left((1-\beta_n)w_n \oplus (1-\sigma)\beta_n w_n \oplus \sigma\beta_n Tw_n, p\right)\\
&= d^2\left((1-\beta_n)w_n \oplus \beta_n\left((1-\sigma)w_n\oplus\sigma Tw_n\right),p\right) \qquad (3.8)\\
&\leqslant (1-\beta_n)d^2(w_n,p) + \beta_n d^2\left((1-\sigma)w_n\oplus\sigma Tw_n,p\right) - \beta_n(1-\beta_n)d^2\left(w_n,(1-\sigma)w_n\oplus\sigma Tw_n\right)\\
&= (1-\beta_n)d^2(w_n,p) + \beta_n d^2\left((1-\sigma)w_n\oplus\sigma Tw_n,p\right) - \sigma^2\beta_n(1-\beta_n)d^2(w_n,Tw_n).
\end{aligned}$$

This and (3.7) imply

$$\begin{aligned}
d^2(u_{n+1},p) &\leqslant (1-\beta_n)d^2(w_n,p) + \beta_n d^2\left((1-\sigma)w_n\oplus\sigma Tw_n,(1-\sigma)p\oplus\sigma p\right) - \sigma^2\beta_n(1-\beta_n)d^2(w_n,Tw_n)\\
&\leqslant d^2(w_n,p) - \sigma^2\beta_n(1-\beta_n)d^2(w_n,Tw_n) \leqslant d^2(w_n,p).
\end{aligned} \tag{3.9}$$

Consequently, (P1) holds. It follows from (2.3) that, for $p \in F(T)$ and f a contraction mapping with constant $k \in [0,1)$, we have

$$\begin{aligned}
d(w_n,p) &= d\left((1-\alpha_n)u_n\oplus\alpha_n f(u_n),p\right)\\
&\leqslant (1-\alpha_n)d(u_n,p) + \alpha_n d(f(u_n),p)\\
&\leqslant (1-\alpha_n)d(u_n,p) + \alpha_n d(f(u_n),f(p)) + \alpha_n d(f(p),p)\\
&\leqslant (1-\alpha_n)d(u_n,p) + \alpha_n kd(u_n,p) + \alpha_n d(f(p),p)\\
&\leqslant (1-\alpha_n(1-k))d(u_n,p) + \alpha_n(1-k)\frac{d(f(p),p)}{1-k} \leqslant \max\left\{d(u_n,p),\frac{d(f(p),p)}{1-k}\right\}.
\end{aligned}$$

This and (P1) imply that

$$\begin{aligned}
d^2(u_{n+1},p) &\leqslant \max\left\{d(u_n,p),\frac{d(f(p),p)}{1-k}\right\}\\
&\leqslant \max\left\{d(u_{n-1},p),\frac{d(f(p),p)}{1-k}\right\}
\end{aligned}$$

$$\vdots$$
$$\leqslant \max\left\{d\left(u_1,p\right), \frac{d\left(f(p),p\right)}{1-k}\right\}.$$

Hence, $\{u_n\}$ is bounded and consequently (P2) holds. The last part of the proof follows from (3.9). Indeed,

$$d^2\left(u_{n+1},p\right) \leqslant d^2\left(w_n,p\right) - \sigma^2\beta_n(1-\beta_n)d^2\left(w_n,Tw_n\right)$$

implies that

$$\sigma^2\beta_n(1-\beta_n)d^2\left(w_n,Tw_n\right) \leqslant d^2\left(w_n,p\right) - d^2\left(u_{n+1},p\right),$$

which yields

$$d^2\left(w_n,Tw_n\right) \leqslant M\left(d^2\left(w_n,p\right) - d^2\left(u_{n+1},p\right)\right)$$

for some $M = \dfrac{1}{a(1-b)\sigma^2}$. □

Theorem 3.7. *Let $T : \mathcal{H} \to \mathcal{H}$ be an enriched nonexpansive mapping with nonempty fixed point set. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0,1]$ that satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\{\alpha_n\}$ *is nonincreasing and* $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $\beta_n \in [a,b] \subset (0,1)$.

For $u_1 \in \mathcal{H}$, the sequences $\{u_n\}$ and $\{w_n\}$ generated by (3.6) converge strongly to the same fixed point of T.

Proof. Let $p \in F(T)$. By Lemma 3.6 (P2), $\left\{d(u_n,p)\right\}$ is bounded. It follows from Lemma 3.6 (P1), (3.6), and (2.3) that

$$d(u_{n+1},p) \leqslant d\left(w_n,p\right) \leqslant (1-\alpha_n)d(u_n,p) + \alpha_n d\left(f(w_n),p\right) \leqslant d(u_n,p) + \alpha_n d\left(f(w_n),p\right).$$

This and the assumption (C1) imply that

$$\liminf_{n\to\infty} d\left(u_n,p\right) = \liminf_{n\to\infty} d\left(w_n,p\right), \quad \limsup_{n\to\infty} d\left(u_n,p\right) = \limsup_{n\to\infty} d\left(w_n,p\right). \tag{3.10}$$

We now give the rest of the proof in two cases.

Case1: The sequence $\left\{d(u_n,p)\right\}$ is nonincreasing.

By Lemma 3.6 (P2), the sequence $\left\{d(u_n,p)\right\}$ is bounded and therefore it is convergent. Also, by (3.10), we get

$$\lim_{n\to\infty} d\left(u_n,p\right) = \lim_{n\to\infty} d\left(w_n,p\right).$$

We now have

$$d^2\left(w_n,p\right) - d^2\left(u_{n+1},p\right) \to 0, \text{ as } n \to \infty.$$

This and Lemma 3.6 (P3) imply that

$$d\left(w_n,Tw_n\right) \to 0, \text{ as } n \to \infty. \tag{3.11}$$

Now by (3.6), (2.1), and (C1), we have

$$d(w_n,u_n) = \alpha_n d\left(u_n,f(u_n)\right) \to 0 \text{ as } n \to \infty. \tag{3.12}$$

Using (3.7), we get

$$\begin{aligned} d(u_n, Tu_n) &= \frac{1}{\sigma} d(u_n, (1-\sigma)u_n \oplus \sigma Tu_n) \\ &\leqslant d(u_n, w_n) + d(w_n, (1-\sigma)w_n \oplus \sigma Tw_n) + ((1-\sigma)w_n \oplus \sigma Tw_n, (1-\sigma)u_n \oplus \sigma Tu_n) \\ &\leqslant d(u_n, w_n) + \sigma d(w_n, Tw_n) + d(w_n, u_n) \leqslant 2d(u_n, w_n) + d(w_n, Tw_n). \end{aligned}$$

Thus, (3.12) and (3.11) imply that

$$d(u_n, Tu_n) \to 0, \text{ as } n \to \infty. \tag{3.13}$$

By Lemma 3.6 (P2), $\{u_n\}$ is bounded. Thus, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ that Δ-converges to some point $u^o \in \mathcal{H}$. By Lemma 2.1, we get that $u^o \in F(T)$. Next we prove that $\{u_n\}$ converges strongly to the point u^o. Indeed, without loss of generality, we may assume that $\{u_{n_j}\}$ is a subsequence of $\{u_n\}$ that Δ-converges to u^o and

$$\limsup_{n\to\infty} \langle \overrightarrow{u_n u^o}, \overrightarrow{f(u^o)u^o} \rangle = \lim_{j\to\infty} \langle \overrightarrow{u_{n_j} u^o}, \overrightarrow{f(u^o)u^o} \rangle.$$

This and Lemma 2.4 imply that

$$\limsup_{n\to\infty} \langle \overrightarrow{u_n u^o}, \overrightarrow{f(u^o)u^o} \rangle = \lim_{j\to\infty} \langle \overrightarrow{u_{n_j} u^o}, \overrightarrow{f(u^o)u^o} \rangle \leqslant 0. \tag{3.14}$$

It follows from quasilinearization, Cauchy Schwartz inequality, and the assumption that f is contraction with $k \in (0,1)$ that

$$\begin{aligned} \langle \overrightarrow{u_n u^o}, \overrightarrow{f(u_n)u^o} \rangle &= \langle \overrightarrow{u_n u^o}, \overrightarrow{f(u_n)f(u^o)} \rangle + \langle \overrightarrow{u_n u^o}, \overrightarrow{f(u^o)u^o} \rangle \\ &\leqslant d(u_n, u^o) d(f(u_n), f(u^o)) + \langle \overrightarrow{u_n u^o}, \overrightarrow{f(u^o)u^o} \rangle \leqslant k d^2(u_n, u^o) + \langle \overrightarrow{u_n u^o}, \overrightarrow{f(u^o)u^o} \rangle. \end{aligned}$$

Thus, Lemma 3.6 (P1) and (2.4) imply that

$$\begin{aligned} d^2(u_{n+1}, u^o) &\leqslant d^2((1-\alpha_n)u_n \oplus \alpha_n f(u_n), u^o) \\ &\leqslant (1-\alpha_n) d^2(u_n, u^o) + \alpha_n d^2(f(u_n), u^o) - \alpha_n(1-\alpha_n) d^2(u_n, f(u_n)) \\ &= (1-\alpha_n)^2 d^2(u_n, u^o) + \alpha_n^2 d^2(f(u_n), u^o) + 2\alpha_n(1-\alpha_n) \langle \overrightarrow{u_n u^o}, \overrightarrow{f(u_n)u^o} \rangle. \\ &\leqslant (1-\alpha_n)^2 d^2(u_n, u^o) + \alpha_n^2 d^2(f(u_n), u^o) + 2\alpha_n(1-\alpha_n) k d^2(u_n, u^o) \\ &\quad + 2\alpha_n(1-\alpha_n) \langle \overrightarrow{f(u^o)u^o}, \overrightarrow{u_n u^o} \rangle. \end{aligned}$$

Thus

$$\begin{aligned} &d^2(u_{n+1}, u^o) \\ &\leqslant (1-\alpha_n)^2 d^2(u_n, u^o) + \alpha_n^2 d^2(f(u_n), u^o) + 2\alpha_n(1-\alpha_n) k d^2(u_n, u^o) + 2\alpha_n(1-\alpha_n) \langle \overrightarrow{u_n u^o}, \overrightarrow{f(u^o)u^o} \rangle. \\ &\leqslant (1-2\alpha_n(1-k)) d^2(u_n, u^o) + \alpha_n^2 \left[d^2(u_n, u^o) + d^2(f(u_n), u^o) \right] + 2\alpha_n(1-\alpha_n) \langle \overrightarrow{u_n u^o}, \overrightarrow{f(u^o)u^o} \rangle \qquad (3.15) \\ &= (1-2\alpha_n(1-k)) d^2(u_n, u^o) + 2\alpha_n(1-k)\phi_n, \end{aligned}$$

where

$$\phi_n := \frac{\alpha_n}{2(1-k)} \left[d^2(u_n, u^o) + d^2(f(u_n), u^o) \right] + \frac{(1-\alpha_n)}{(1-k)} \langle \overrightarrow{u_n u^o}, \overrightarrow{f(u^o)u^o} \rangle.$$

Thus from (3.15), (3.14), and the conditions (C1) and (C2), we conclude by Lemma 2.5 that $\{u_n\}$ converges strongly to u^o.

Case2: The sequence $\{d(u_n, p)\}$ is not nonincreasing.

Let $\Gamma_n = d^2(u_n, p)$ and $\xi : \mathbb{N} \to \mathbb{N}$ be defined by

$$\xi(n) = \max\{m \in \mathbb{N} : m \leqslant n, \Gamma_m \leqslant \Gamma_{m+1}\}. \tag{3.16}$$

It follows from (3.16) that ξ is nondecreasing sequence such that

$$\xi(n) \to \infty \text{ as } n \to \infty \text{ and } \xi(n) \leqslant \xi(n+1),\ n \geqslant n_o \text{ (for some large enough } n_o).$$

Consequently, we have that

$$\Gamma_{\xi(n)} \leqslant \Gamma_{\xi(n)+1}, \quad \Gamma_n \leqslant \Gamma_{\xi(n)+1}. \tag{3.17}$$

It follows from (P3) that

$$\begin{aligned} d^2\left(w_{\xi(n)}, Tw_{\xi(n)}\right) &\leqslant M\left(d^2\left(w_{\xi(n)}, p\right) - d^2\left(u_{\xi(n)+1}, p\right)\right) \\ &\leqslant M\Big[(1-\alpha_{\xi(n)})d^2\left(u_{\xi(n)}, p\right) + \alpha_{\xi(n)} d^2\left(f(u_{\xi(n)}), p\right) - d^2\left(u_{\xi(n)+1}, p\right)\Big] \\ &\leqslant M\Big(\Gamma_{\xi(n)} - \Gamma_{\xi(n)+1}\Big) + \alpha_{\xi(n)} M d^2\left(f(u_{\xi(n)}), p\right) \leqslant \alpha_{\xi(n)} M d^2\left(f(u_{\xi(n)}), p\right). \end{aligned}$$

This, the boundedness of $\{u_n\}$ and (C1) imply that

$$d\left(w_{\xi(n)}, Tw_{\xi(n)}\right) \to 0 \text{ as } n \to \infty.$$

Following similar argument as in Case 1 (from (3.13) up to (3.14)) with $\{u_{\xi(n)}\}$ in place of $\{u_n\}$, we obtain some $u^* \in F(T)$ with

$$\limsup_{n\to\infty} \langle \overrightarrow{u_{\xi(n)}u^*}, \overrightarrow{f(u^*)u^*} \rangle \leqslant 0. \tag{3.18}$$

Thus, we have by Lemma 3.6 (P1) and (2.4) that

$$d^2(u_{\xi(n)+1}, u^*) \leqslant (1 - 2\alpha_{\xi(n)}(1-k))d^2(u_{\xi(n)}, u^*) + 2\alpha_{\xi(n)}(1-k)\phi_{\xi(n)}, \tag{3.19}$$

where

$$\phi_{\xi(n)} := \frac{\alpha_{\xi(n)}}{2(1-k)}\left[d^2(u_{\xi(n)}, u^*) + d^2(f(u_{\xi(n)}), u^*)\right] + \frac{(1-\alpha_{\xi(n)})}{(1-k)}\langle \overrightarrow{u_{\xi(n)}u^*}, \overrightarrow{f(u^*)u^*} \rangle.$$

Using (3.19), (3.18), and the conditions (C1) and (C2), we conclude by Lemma 2.5 that $\{d\left(u_{\xi(n)}, u^*\right)\}$ converges strongly to 0. Moreover, using (3.17), we have that

$$d\left(u_n, u^*\right) \leqslant d\left(u_{\xi(n)+1}, u^*\right) \to 0, \quad \text{as } \ n \to \infty.$$

Therefore, from Cases 1 and 2, we conclude that $\{u_n\}$ converges strongly to a fixed point of T. Let the limits of $\{u_n\}$ be $\tilde{u}$. Then from (P1), we get

$$d(u_{n+1}, \tilde{u}) \leqslant d(w_n, \tilde{u}) \leqslant (1-\alpha_n)d(u_n, \tilde{u}) + \alpha_n d(f(u_n), \tilde{u}).$$

This and (C1) imply that $\{w_n\}$ converges to $\tilde{u}$, which completes the proof. □

Since every nonexpansive mapping is enriched nonexpansive, we obtain the following corollary.

Corollary 3.8. *Let* $T : \mathcal{H} \to \mathcal{H}$ *be a nonexpansive mapping with a nonempty fixed point set. Suppose that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in the interval* $[0, 1]$ *that satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\{\alpha_n\}$ *is nonincreasing and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $\beta_n \in [a, b] \subset (0, 1)$.

For $u_1 \in \mathcal{H}$, *the sequences* $\{u_n\}$ *and* $\{w_n\}$ *generated by*

$$\begin{cases} w_n = (1-\alpha_n)u_n \oplus \alpha_n f(u_n), \ u_1 \in \mathcal{H}, \\ u_{n+1} = (1-\beta_n)w_n \oplus \beta_n T w_n, \ \ n \geqslant 1, \end{cases}$$

converge strongly to the same fixed point of T *and* $f : \mathcal{H} \to \mathcal{H}$ *is a contraction mapping.*

Theorem 3.7 implies the following corollary which gives the convergence of the viscosity-type scheme in the setting of Hilbert spaces.

Corollary 3.9. *Let* $\mathcal{H}$ *be a Hilbert space and* $T : \mathcal{H} \to \mathcal{H}$ *be an enriched nonexpansive mapping with nonempty fixed point set. Suppose that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in the interval* $[0, 1]$ *that satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\{\alpha_n\}$ *is nonincreasing and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $\beta_n \in [a, b] \subset (0, 1)$.

For $u_1 \in \mathcal{H}$, *the sequences* $\{u_n\}$ *and* $\{w_n\}$ *generated by*

$$\begin{cases} w_n = (1-\alpha_n)u_n + \alpha_n f(u_n), \\ u_{n+1} = (1-\sigma\beta_n)w_n + \sigma\beta_n T w_n, \ \ n \geqslant 1, \end{cases}$$

converge strongly to the same fixed point of T, *where* $\sigma \in (0, 1]$ *is a constant and* $f : \mathcal{H} \to \mathcal{H}$ *is a contraction mapping.*

In the particular case of $f \equiv 0$, Corollary 3.9 yields the next corollary which is the main result (Theorem 2) in [9].

Corollary 3.10. *Let* $\mathcal{H}$ *be a Hilbert space and* $T : \mathcal{H} \to \mathcal{H}$ *be an enriched nonexpansive mapping with nonempty fixed point set. Suppose that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in the interval* $[0, 1]$ *that satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\{\alpha_n\}$ *is nonincreasing and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $\beta_n \in [a, b] \subset (0, 1)$.

For $u_1 \in \mathcal{H}$, *the sequences* $\{u_n\}$ *and* $\{w_n\}$ *generated by*

$$\begin{cases} w_n = (1-\alpha_n)u_n, \\ u_{n+1} = (1-\sigma\beta_n)w_n + \sigma\beta_n T w_n, \ \ n \geqslant 1, \end{cases}$$

converge strongly to the same fixed point of T, *where* $\sigma \in (0, 1]$ *is a constant.*

In the next corollary, we have the main result (Theorem 1 in [50]), which follows from the fact that every nonexpansive mapping is 0-enriched nonexpansive.

Corollary 3.11. *Let* $\mathcal{H}$ *be a Hilbert space and* $T : \mathcal{H} \to \mathcal{H}$ *be a nonexpansive mapping with nonempty fixed point set. Suppose thagt* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in the interval* $[0, 1]$ *that satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\{\alpha_n\}$ *is nonincreasing and* $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $\beta_n \in [a, b] \subset (0, 1)$.

For $u_1 \in \mathcal{H}$, *the sequences* $\{u_n\}$ *and* $\{w_n\}$ *generated by*

$$\begin{cases} w_n = (1-\alpha_n)u_n, \\ u_{n+1} = (1-\beta_n)w_n + \beta_n T w_n, \quad n \geqslant 1, \end{cases}$$

converge strongly to the same fixed point of T.

Remark 3.12. To rule out the assumption that the set of fixed point of the studied mapping is not empty in all the results, consider the mapping to be defined on a bounded closed convex subset of a Hadamard space. This is justified by Theorem 4.8 in [43].

3.3. *Variational inequality problem*

A variational inequality problem (VIP for short) in setting of a Hadamard space $(\mathcal{H}, d)$ is the problem of

$$\text{finding } u \in D \text{ such that } \langle \overrightarrow{uTu}, \overrightarrow{wu} \rangle \geqslant 0, \ \forall\, w \in D, \tag{3.20}$$

where D is a nonempty subset of a $\mathcal{H}$ and $T : D \to \mathcal{H}$ is a mapping. The set of solutions of the variational inequality problem (3.20) is usually denoted by $VI(D, T)$.

Remark 3.13. It is not difficult to deduce that if $\mathcal{H}$ is a Hilbert space, then (3.20) reduces to

$$\text{finding } u \in D \text{ such that } \langle Au, w-u \rangle \geqslant 0, \quad \forall\, w \in D,$$

with $A \equiv I - T$.

Variational inequality problem has vast applications in real-life phenomena and it has been extensively studied by many researchers (see, e.g., [2, 3, 5, 35, 48] and the references therein).

It is our purpose in this section, to prove an existence theorem for VIP involving α-enriched nonexpansive mappings. We also provide a scheme that converges to a member of the set of solutions of the problem provided the set is not empty. We shall need the following facts for our results.

Lemma 3.14 ([2, Lemma 4.1])**.** *Let* D *be a nonempty convex subset of* $\mathcal{H}$ *and* $T : D \to \mathcal{H}$ *be a mapping. Then* $VI(D, T) = VI(D, T_\sigma)$, *where* $\sigma \in (0, 1]$ *and* $T_\sigma w := (1-\sigma)w \oplus Tw$ *for all* $w \in D$.

Lemma 3.15 ([43, Theorem 4.8])**.** *Let* D *be a nonempty bounded closed convex subset of* $\mathcal{H}$ *and* $T : D \to D$ *be an α-enriched nonexpansive. Then the set* $F(T)$ *of fixed points of* T *is a nonempty closed and convex set.*

Theorem 3.16. *Let* D *be a nonempty bounded closed convex subset of* $\mathcal{H}$ *and* $T : D \to \mathcal{H}$ *be an α-enriched nonexpansive mapping. Then* $VI(D, T)$ *is nonempty, closed, and convex.*

Proof. Observe that

$$u = P_D Tu \iff \langle \overrightarrow{uTu}, \overrightarrow{wu} \rangle \geqslant 0, \quad \forall\, w \in D \iff u \in VI(D, T).$$

Hence, by Proposition 3.5, we have $VI(D, T) = F(T)$. Consequently, Lemma 3.15 completes the proof. □

Theorem 3.17. *Let* D *be a nonempty closed convex subset of* $\mathcal{H}$ *and* $T : D \to \mathcal{H}$ *be an α-enriched nonexpansive mapping with* $VI(D, T) \neq \emptyset$. *Suppose that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in the interval* $[0, 1]$ *that satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\{\alpha_n\}$ *is nonincreasing and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $\beta_n \in [a, b] \subset (0, 1)$.

For $u_1 \in \mathcal{H}$, *then the sequences* $\{u_n\}$ *and* $\{w_n\}$ *generated by*

$$\begin{cases} w_n = (1-\alpha_n)u_n \oplus \alpha_n f(u_n), \ u_1 \in \mathcal{H}, \\ u_{n+1} = (1-\beta_n)w_n \oplus \beta_n P_D((1-\sigma)w_n \oplus \sigma T w_n), \ \ n \geqslant 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ *are sequences in* $[0, 1]$, $\sigma \in (0, 1]$ *is a constant and* $f : \mathcal{H} \to \mathcal{H}$ *is a contraction mapping, converge strongly to the same element of* $VI(D, T)$.

Proof. Let $G : D \to D$ be the mapping such that $w \mapsto P_D(T_\sigma w)$. Then for any $x, y \in D$, we obtain from the fact that P_D is nonexpansive mapping and (3.7) that

$$\begin{aligned} d(Gx, Gy) &= d\big(P_D((1-\sigma)x \oplus \sigma Tx), P_D((1-\sigma)y \oplus \sigma Ty)\big) \\ &\leqslant d\big((1-\sigma)x \oplus \sigma Tx, (1-\sigma)y \oplus \sigma Ty\big) \leqslant d(x, y). \end{aligned}$$

This implies that G is nonexpansive. Thus, by Corollary 3.8, we have that the sequences $\{u_n\}$ and $\{w_n\}$ both converge to a fixed point of G. From Lemmas 2.6 and 3.14, we have that

$$F(G) = F(P_D T_\sigma) = VI(D, T_\sigma) = VI(D, T).$$

□

Remark 3.18. To rule out the assumption that $VI(D, T) \neq \emptyset$ in Theorem 3.17, we can consider the set D to be nonempty bounded closed and convex subset. This is justify by Theorem 3.16.

3.4. Numerical example

In this section, we give an example of enriched nonexpansive mapping which is not nonexpansive in a non-Hilbert CAT(0) space and also conduct some numerical experiments to show the implementation of the proposed scheme. All codes are written in Matlab R2021b and run on Acer (11th Gen Intel(R) Core(TM) i5-1135G7 @ 2.40GHz) laptop.

Example 3.19. Let $\mathcal{H} = \mathbb{R}^2$ be endowed with the metric $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d\big((u_1, u_2), (w_1, w_2)\big) = \sqrt{(u_1 - w_1)^2 + (u_1^2 - u_2 - w_1^2 + w_2)^2}. \tag{3.21}$$

The geodesic connecting $w = (w_1, w_2)$ and $z = (z_1, z_2)$ is the mapping $\tau : [0, 1] \to \mathcal{H}$, namely $\tau(t) = (1-t)w \oplus tz$, with the following explicit form:

$$\Big((1-t)w_1 + tz_1, \big((1-t)w_1 + tz_1\big)^2 - (1-t)(w_1^2 - w_2) - t(z_1^2 - z_2)\Big).$$

It is not difficult to see that the CN-inequlity (2.5) is satisfied with d defined in (3.21). Thus $(\mathcal{H}, d)$ is a non-Hilbert CAT(0) space. See, also, [22, Example 5.2]. Let $\eta \geqslant 2$ and $D = \left[\frac{1}{\eta}, \eta\right] \times \mathbb{R}$. The mapping $T : D \to D$ defined by

$$Tu = \left(\frac{1}{u_1}, \frac{1}{u_1^2}\right),$$

where $u = (u_1, u_2)$ is not nonexpansive mapping but enriched nonexpansive mapping. Let $u = (u_1, u_2)$, $w = (w_1, w_2) \in D$. Then we get that

$$d(Tu, Tw)^2 + 2\alpha\langle \overrightarrow{uw}, \overrightarrow{TuTw}\rangle = \left|\frac{1}{u_1} - \frac{1}{w_1}\right|^2 + \alpha\left[\left|u_1 - \frac{1}{u_1}\right|^2 + \left|w_1 - \frac{1}{u_1}\right|^2\right.$$

$$-\left|u_1-\frac{1}{u_1}\right|^2-\left|w_1-\frac{1}{w_1}\right|^2\Bigg]=\left(\frac{1}{(u_1w_1)^2}-\frac{2\alpha}{u_1w_1}\right)|u_1-w_1|^2.$$

Thus,

$$d(Tu,Tw)^2+\alpha^2 d(u,w)^2+2\alpha\langle\overrightarrow{uw},\overrightarrow{TuTw}\rangle\leqslant\left(\alpha-\frac{1}{u_1w_1}\right)^2 d^2(u,w).$$

Now, we easily see that any $\alpha\geqslant\frac{\eta^2-1}{2}$ satisfies the following:

$$\left|\alpha-\frac{1}{u_1w_1}\right|\leqslant\alpha+1,\quad\forall\, u_1,w_1\in D.$$

Hence, the mapping T is an α-enriched nonexpansive mapping with $\alpha\geqslant\frac{\eta^2-1}{2}$. Furthermore, observe that for $u=(1,1)$ and $w=\left(\frac{1}{2},\frac{1}{4}\right)$,

$$d(Tu,Tw)=\left|\frac{1}{1}-\frac{1}{1/2}\right|=1>\frac{1}{2}=\sqrt{\frac{1}{4}+0}=d(u,w).$$

Hence T is not nonexpansive mapping.

Observe that, $F(T)=\{(1,1)\}$. Moreover, the sequence $\{u_n\}$ defined iteratively by

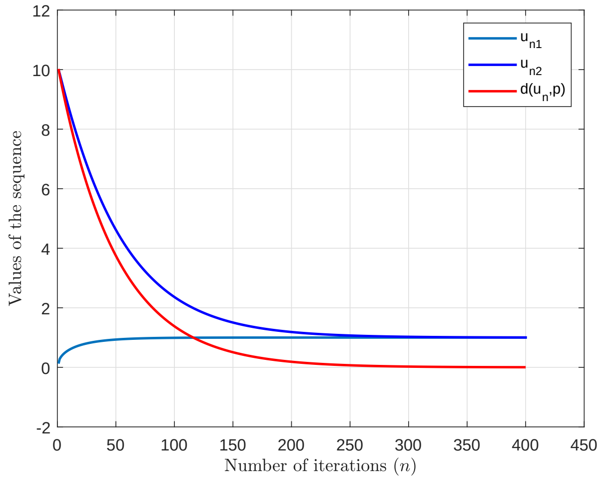
$$\begin{cases}w_n=(1-\alpha_n)u_n\oplus\alpha_n f(u_n),\ u_1\in\mathcal{H},\\ u_{n+1}=\left(1-\frac{2}{\eta_2^2+1}\beta_n\right)w_n\oplus\frac{2}{\eta_2^2+1}\beta_n Tw_n,\ \ n\geqslant 1,\end{cases}$$

converges to the fixed point of T. In the experiment, we take $\eta=10$ and consider f such that $f(u_1,u_2):=\left(\frac{1}{2},\frac{1}{4}u_2\right)$, $\alpha_n=\frac{1}{n+1}$, and $\beta_n=\frac{n}{2n+2}$. The results are shown in Table 1 and Figure 1.

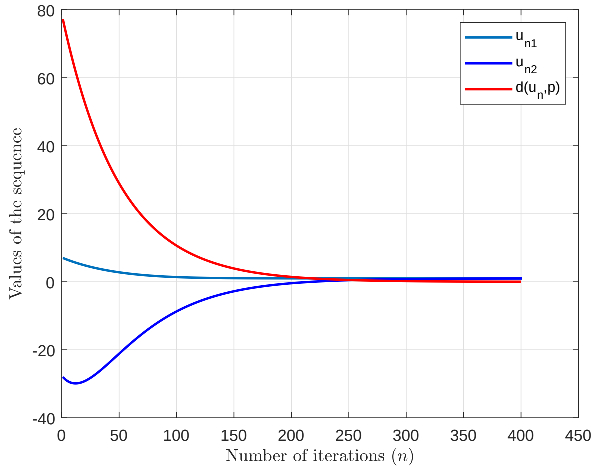
Table 1: Few values of the sequences $\{u_n\}$ and $\{w_n\}$ for Example 3.19.

	$u_1=(1/8,10)$		$u_1=(10,-12)$	
n	u_n	w_n	u_n	w_n
1	(0.125, 10)	(0.3125, 6.2148)	(10, -12)	(5.25, -30.0625)
2	(0.28094, 9.8656)	(0.35396, 7.3885)	(9.804, -13.6645)	(6.7026, -29.4848)
3	(0.34586, 9.7125)	(0.3844, 7.8869)	(9.6118, -15.2208)	(7.3339, -27.9342)
4	(0.39627, 9.5599)	(0.41701, 8.1242)	(9.4236, -16.6738)	(7.6389, -26.9135)
5	(0.43839, 9.4089)	(0.44866, 8.2323)	(9.2391, -18.0284)	(7.7826, -26.382)
⋮	⋮	⋮	⋮	⋮
395	(1, 1.0038)	(0.99874, 1.0012)	(1, 0.95766)	(0.99874, 0.95522)
396	(1, 1.0037)	(0.99874, 1.0012)	(1, 0.9585)	(0.99875, 0.95606)
397	(1, 1.0036)	(0.99874, 1.0011)	(1, 0.95932)	(0.99875, 0.95689)
398	(1, 1.0036)	(0.99875, 1.001)	(1, 0.96013)	(0.99875, 0.9577)
399	(1, 1.0035)	(0.99875, 1.001)	(1, 0.96092)	(0.99876, 0.95849)
400	(1, 1.0034)	(0.99875, 1.0009)	(1, 0.96169)	(0.99876, 0.95927)

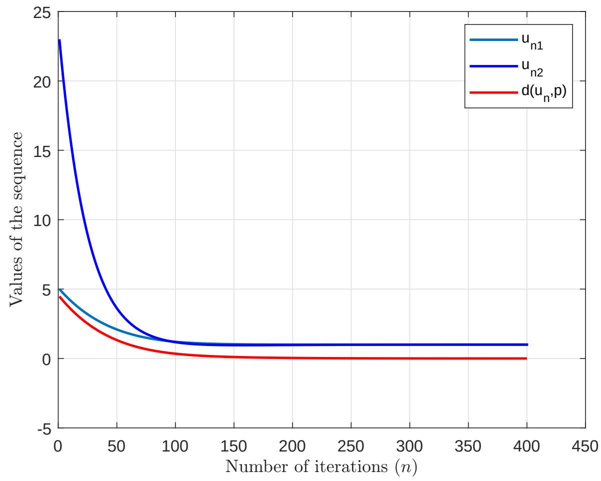
You may note that even though the second terms of both sequences may fluctuate at the beginning, the sequences eventually converge to the limit. The initial fluctuations were as a result of the non-linearity structure of the metric d. Moreover, after some $n_o\geqslant 1$, the difference between u_n and w_n is very negligible. Thus, for the sake of readability, we present the figures for only values of $\{u_n\}$ and $\{d(u_n,p)\}$ up to 400 iterations, where $p=(1,1)$ (the fixed point).



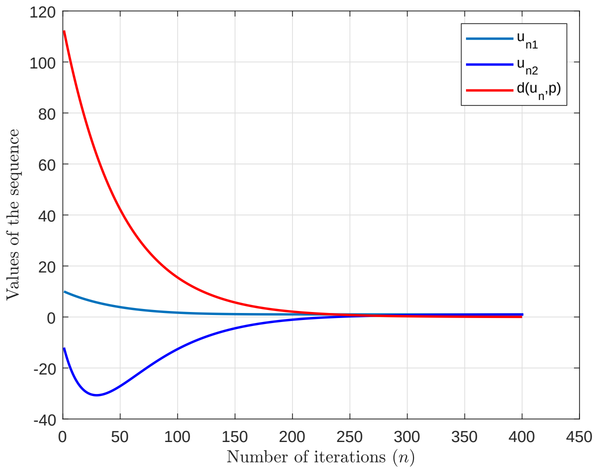
(a) Convergence results for $u_1 = (\frac{1}{8}, 10)$.



(b) Convergence results for $u_1 = (7, -28)$.



(c) Convergence results for $u_1 = (5, 23)$.



(d) Convergence results for $u_1 = (10, -12)$.

Figure 1: Convergence trajectory of $\{u_n\}$ based on Example 3.19.

3.5. Concluding remarks

In this work, we analyzed a generalized class of nonexpansive mappings in the setting of Hadamard spaces. We gave an example of such mappings in a non-Hilbert CAT(0) space and established that this class of mappings is a subclass of the class of Lipchitz mappings with a constant greater or equal to one. Moreover, we have shown that this class of mappings contains the class of inverse-strongly monotone mappings and the class of strictly pseudocontractive mappings which are known with a significant role in optimization problems. We also introduced a viscosity-type scheme for approximating a fixed point of the mapping (provided such a point exists) and obtained strong convergence of the sequences generated therefrom. Furthermore, we gave applications of our results in establishing the existence of solutions of variational inequality problems involving such mappings. In the applications, we gave an iterative scheme that strongly converges to a solution of variational inequality problem assuming existence. Finally, we gave a numerical example in a non-Hilbert setting to show the implementation of the proposed algorithm. Our results extend and complement some recent results in the literature. In particular, Proposition 3.5 is a generalization of Lemma 3.5 of [2] from inverse-strongly monotone to enriched nonexpansive mappings, Theorem 3.7 extends the main result of [9] from Hilbert spaces to Hadamard space with more general iterative procedure and Corollary 3.8 extends the main theorem of [50] from Hilbert spaces to Hadamard spaces with a generalized approximation scheme. Most of the basic properties of the class of nonexpansive mappings (such as the demiclosedness-type property and existence of fixed point if the domain of the self-mapping is nonempty bounded closed and convex) are also obtained for the class of enriched nonexpansive mappings ([43]). In addition to that, we established Propositions 3.1-3.5, which are of special interest. Also, in most geodesic spaces, the class of nonexpansive mappings is closed under convex combination operation and also closed under composition, it is therefore interesting future work to seek such results (or otherwise) for the class of enriched nonexpansive mappings.

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