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General decay synchronization of T-S fuzzy complex-valued BAM neural networks with mixed time delays

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Abstract

This study investigated the general decay synchronization of T-S fuzzy Binary association memory (BAM) neural networks with discrete and distributed time delays over a complex domain. First, a novel and simple nonlinear feedback controller is introduced. Then, based on a suitable Lyapunov-Krasovskii functional and non-separation method of complex-valued systems, sufficient conditions for the concerned derived-response systems to achieve general decay synchronization are obtained through inequality techniques. Finally, the feasibility of the results was verified through numerical simulations. It is worth noting that general decay synchronization provides a more general convergence rate for synchronization errors approaching zero, and the usual logarithmic synchronization, exponential synchronization, and pronominal synchronization can be special cases of general decay synchronization considered in our work when $\rho(t)$ in the designed controller has specific functions.

Keywords: T-S fuzzy, general decay synchronization, complex-valued BAM neural network, mixed time delay.

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1. Introduction

Binary association memory neural networks (BAMNNs), first proposed by Kosok in 1987, consist of two layers of neurons with no interconnection between neurons in the same layer and full interconnection between neurons in different layers [10]. As important neural network models, BAMNNs can store any pair of simulation patterns using unsupervised learning, which is crucial for many applications in pattern recognition and automatic control engineering [1, 4, 5, 11, 27, 37].

It is well known that time delays are inevitable when implementing neural networks (NNs), which can lead to undesirable dynamic behaviors such as oscillations and instability. Time delays are generally categorized into discrete and distributed types during research. Discrete-time delays are easier to identify, making the stability and synchronization analysis of NNs with discrete time delays a popular research topic. However, NNs are inherently spatial in nature, with varying axon lengths along different parallel pathways, a factor that cannot be fully captured by discrete time delays alone. As a result, researchers have started to incorporate distributed time delays into NN models [18, 22–26].

On the other hand, fuzzy logic control, as a simple and effective method for controlling many complex nonlinear systems or even nonanalytic systems, has received much attention from researchers in the field

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of control [8, 14, 17, 34, 36]. For example, [8] discussed a two-wheeled robot's evolutionary fuzzy control and navigation for collaborative object handling in an unknown environment. In [17], a fuzzy-logic control algorithm was used to stabilize an Rössler chaotic dynamic system. In [34], adaptive event-triggered output feedback fuzzy control for nonlinear network systems with packet loss and actuator faults was investigated. In [14], the quantitative moving average strategy for the crude oil futures market was studied based on fuzzy logic rules and a genetic algorithm. [36] investigated discontinuous fuzzy competitive neural network timing synchronisation based on quantitative controls.

Among the various fuzzy methods, T-S fuzzy modeling is one of the most commonly used methods for analyzing and designing fuzzy systems. The T-S fuzzy modeling technique can effectively represent complex systems using fuzzy rules and logic [21]. Many topics, such as the stability and performance analysis of T-S fuzzy models, controller design, and filter design, have attracted extensive attention [12, 31, 32]. Therefore, it is important to consider the synchronization and stability of fuzzy systems. Thus far, the polynomial and exponential stability and synchronization of BAMNNs with T-S fuzzy logic have been obtained. For example, the stability of Cohen-Grossberg BAMNNs with discrete and distributed time-varying time delay T-S fuzzy logic was discussed in [3]. In [13], the stability of T-S fuzzy uncertain BAMNNs with leaky terms and time delays was discussed. The authors in [30] proposed global Lagrange stability for BAMNNs with time-varying delays T-S fuzzy Cohen-Grossberg BAMNNs.

Estimating the rate of synchronization convergence has been a concern for many researchers. However, in many cases, the synchronization convergence rate cannot be obtained using common synchronization methods. For example, consider the differential equation $\dot{z}(t) = -\frac{1}{2z^3}$, $t \geqslant 0$. Although the differential equation is asymptotically stable, it is impossible to obtain a convergence rate. Therefore, scholars have proposed a new rate of convergence, general decay (GD) convergence, and many valuable research results have been obtained [7, 9, 15, 16, 19, 29, 33]. For example, in [33], the GD stability of stochastic time-delay differential equations with Markov switching was studied. In [16], the GD synchronization of BAMNNs with a time delay based on a nonlinear feedback control was studied. In [19] the GD rate stability was presented for neutral-type stochastic functional mixed differential equations with Lévy noise. [15] discussed the GD stability of the backward Euler-Maruyama method for nonlinear stochastic integro-differential equations. The GD synchronization of discrete and distributed time-delay discontinuous fuzzy NNs based on nonlinear feedback control was studied in [9]. GD anti-synchronization and H_∞ anti-synchronization of coupled reaction-diffusion memristive CV-NNs with and without time delays was considered via using severation method of complex-valued systems in [7]. In fact, GD synchronization is more general and encompasses cases of logarithmic synchronization (LS), polynomial synchronization (PS), and exponential synchronization (ES), which can be obtained as shown in Figure 1.

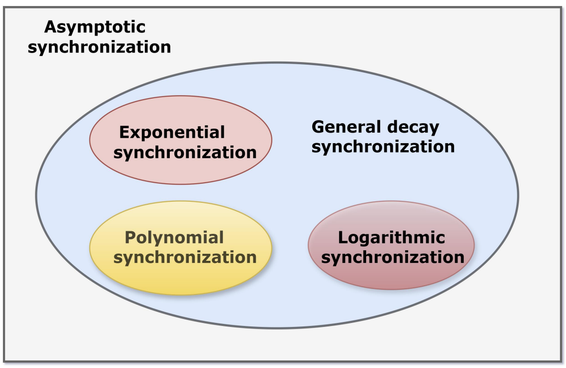

Figure 1: The synchronization relation diagram.

It is worth noting that most of the above results are obtained in the case of real-valued NNs (RV-NNs). In general, a complex-valued neural network (CV-NN) has more complex and superior properties than an RV-NN, and it can solve the XOR problem, symmetry detection problems, etc., which are difficult to solve using an RV-NN. Several studies have been conducted on the synchronization and stability of complex-valued BAMNNs (CV-BAMNNs). For example, the global stability of CV-BAMNNs with time delay was analyzed in [28]. In [20], the existence, uniqueness, and global asymptotic stability of Cohen-Grossberg CV-BAMNNs with time delays were studied. However, in the above studies, a separation method was used to deal with CV-NNs, which increases the system's dimensionality and complicates its theoretical analysis, making it inconvenient for practical applications. Therefore, a non-separation method was proposed [6]. Fixed- and finite-time synchronization problems of a class of CV-NNs were studied by introducing a complex-valued sign function and two discontinuous control strategies. However, there have been few studies on the GD synchronization of CV-NNs using non-separation methods, especially for T-S fuzzy CV-BAMNNs with mixed time delays.

Inspired by the above analysis, this study investigated the GD synchronization of T-S fuzzy CV-BAMNNs with mixed time delays. The main contributions of this study are as follows. 1) The GD synchronization of T-S fuzzy BAMNNs with mixed-delays over complex domains is analyzed for the first time. 2) The novel Lyapunov-Krasovskii functional is constructed to ensure the GD synchronization between drive-response BAMNNs. 3) The controller designed in this study removed excess terms of the controller used in [7, 9, 29], thus can reduce the control costs. 4) The GD synchronization is relatively general, and its convergence rate can be effectively estimated when the synchronous system reaches zero. PS, ES, and LS are special cases of GD synchronization. Therefore, the results of this study have a wide range of applications.

The rest of the article is structured as follows. Section 2 provides the system model and some preliminary knowledge. In Section 3, the sufficient conditions for determining GD synchronization of CV-NNs are given. Section 4 carries on the numerical simulations through Matlab software.

Notations. In this article, $\mathscr{P} = \{1,\dots,p\}$, $\mathscr{Q} = \{1,\dots,q\}$. $\mathcal{R}^m$ represents m-dimensional real vector space, and $\mathbb{C}^m$ represents m-dimensional complex vector space. For any $\chi \in \mathbb{C}$, $\bar{\chi}$ is the conjugate of χ, $\mathbf{Re}(\chi)$ and $\mathbf{Im}(\chi)$ are the real and imaginary parts of χ, respectively. $|\chi|_1 = |\mathbf{Re}(\chi)| + |\mathbf{Im}(\chi)|$ and $|\chi|_2 = \sqrt{\chi\bar{\chi}}$. For any $\mathscr{X} = (x_1,\dots x_m)^T \in \mathbb{C}^m$, $\|\mathscr{X}\| = (\sum_{k=1}^{p} |x_k|_2^2)^{(1/2)}$. Let $\tau = \max\{\tau_{\jmath\ell}, \sigma_{\jmath\ell}\}$ and $\pi = \max\{\pi_{\ell\jmath}, \rho_{\ell\jmath}\}$, we assume $\mathscr{H} = \mathcal{C}([-\tau,0],\mathbb{C}^m)$ be the Banach spaces of all continuous functions on $[-\tau,0]$ and $\mathscr{M} = \mathcal{C}([-\pi,0],\mathbb{C}^n)$ be the Banach spaces of all continuous functions on $[-\pi,0]$, where the norm is defined as $\|\nu\|_2 = \sup_{-\sigma\leqslant s\leqslant 0} (\sum_{k=1}^{p} |\nu_k(s)|_2^2)^{1/2}$.

2. Preliminaries

The BAMNNs model with mixed time delays can be given as follows:

$$\begin{cases} \dot{x}_\jmath(t) = -c_\jmath x_\jmath(t) + \sum\limits_{\ell=1}^{q} a_{\jmath\ell} f_\ell(y_\ell(t)) + \sum\limits_{\ell=1}^{q} b_{\jmath\ell} f_\ell(y_\ell(t-\tau_{\jmath\ell}(t))) + \sum\limits_{\ell=1}^{q} d_{\jmath\ell} \int_{t-\sigma_{\jmath\ell}(t)}^{t} f_\ell(y_\ell(s))ds + J_\jmath, \\ \dot{y}_\ell(t) = -\tilde{c}_\ell y_\ell(t) + \sum\limits_{\jmath=1}^{p} \tilde{a}_{\ell\jmath} g_\jmath(x_\jmath(t)) + \sum\limits_{\jmath=1}^{p} \tilde{b}_{\ell\jmath} g_\jmath(x_\jmath(t-\pi_{\ell\jmath}(t))) + \sum\limits_{\jmath=1}^{p} \tilde{d}_{\ell\jmath} \int_{t-\rho_{\ell\jmath}(t)}^{t} g_\jmath(x_\jmath(s))ds + \tilde{J}_\ell, \end{cases} \tag{2.1}$$

where $\jmath \in \mathscr{P}$, $\ell \in \mathscr{Q}$, $p \geqslant 2$ and $q \geqslant 2$ correspond to the number of neurons in the neural field X_U and Y_U, respectively. $x_\jmath(t) \in \mathbb{C}$ and $y_\ell(t) \in \mathbb{C}$ are the state variables of the $\jmath$-th neuron from the nerve field X_U and the ℓ-th neuron from the nerve field Y_U, respectively. $c_\jmath \in \mathbb{C}$ and $\tilde{c}_\ell \in \mathbb{C}$ represent self-inhibition of neurons; $a_{\jmath\ell} \in \mathbb{C}$, $\tilde{a}_{\ell\jmath} \in \mathbb{C}$, $b_{\jmath\ell} \in \mathbb{C}$, $\tilde{b}_{\ell\jmath} \in \mathbb{C}$, $d_{\jmath\ell} \in \mathbb{C}$, and $\tilde{d}_{\ell\jmath} \in \mathbb{C}$ represent to the connection strengths, the time-varying delay connection strengths and distributed time delay connection strengths, respectively; $f_\ell(y_\ell(t))$ and $g_\jmath(x_\jmath(t))$ are the neuron feedback functions; $\tau_{\jmath\ell}(t)$ and $\pi_{\ell\jmath}(t)$ represent discrete delays and satisfy $0 \leqslant \tau_{\jmath\ell}(t) \leqslant \tau_{\jmath\ell}$, $0 \leqslant \pi_{\ell\jmath}(t) \leqslant \pi_{\ell\jmath}$; $\sigma_{\jmath\ell}(t)$ and $\rho_{\ell\jmath}(t)$ represent distributed delays and satisfy $0 \leqslant \sigma_{\jmath\ell}(t) \leqslant \sigma_{\jmath\ell}$, $0 \leqslant \rho_{\ell\jmath}(t) \leqslant \rho_{\ell\jmath}$, where $\tau_{\jmath\ell}$, $\pi_{\ell\jmath}$, $\sigma_{\jmath\ell}$, and $\rho_{\ell\jmath}$ are the upper bounds for the respective delays; $J_\jmath$ and $\tilde{J}_\ell$ represent external inputs. It is important to note that the variable delays, denoted by $\tau_{\jmath\ell}(t)$, $\pi_{\ell\jmath}(t)$, $\sigma_{\jmath\ell}(t)$, and $\rho_{\ell\jmath}(t)$, as well as the functions $f_\ell(y_\ell(t))$ and $g_\jmath(x_\jmath(t))$ are assumed to be continuous. The initial functions of system (2.1) are

$$\begin{cases} x_j(s) = \phi_j(s), & s \in [-\max\{\tau_{j\ell}, \sigma_{j\ell}\}, 0], \\ y_\ell(s) = \tilde{\phi}_\ell(s), & s \in [-\max\{\pi_{\ell j}, \rho_{\ell j}\}, 0], \end{cases}$$

where $\phi_j(s) = (\phi_1(s), \dots, \phi_p(s))^T \in \mathscr{H}$, $\tilde{\phi}_\ell(s) = (\tilde{\phi}_1(s), \dots, \tilde{\phi}_q(s))^T \in \mathscr{M}$.

Based on the T-S fuzzy model method, consider the following T-S fuzzy CV-BAMNN model with mixed time delays.

Plant Rule k: If $\varphi_1(t)$ and $\psi_1(t)$ are $M_1^k, \dots,$ and $\varphi_m(t)$ and $\psi_m(t)$ are M_m^k, then

$$\begin{cases} \dot{x}_j(t) = -c_j^k x_j(t) + \sum\limits_{\ell=1}^{q} a_{j\ell}^k f_\ell(y_\ell(t)) + \sum\limits_{\ell=1}^{q} b_{j\ell}^k f_\ell(y_\ell(t-\tau_{j\ell}(t))) + \sum\limits_{\ell=1}^{q} d_{j\ell}^k \int_{t-\sigma_{j\ell}(t)}^{t} f_\ell(y_\ell(s))ds + J_j^k, \\ \dot{y}_\ell(t) = -\tilde{c}_\ell^k y_\ell(t) + \sum\limits_{j=1}^{p} \tilde{a}_{\ell j}^k g_j(x_j(t)) + \sum\limits_{j=1}^{p} \tilde{b}_{\ell j}^k g_j(x_j(t-\pi_{\ell j}(t))) + \sum\limits_{j=1}^{p} \tilde{d}_{\ell j}^k \int_{t-\rho_{\ell j}(t)}^{t} g_j(x_j(s))ds + \tilde{J}_\ell^k, \end{cases}$$

where M_l^k $(k = 1, 2, \dots, n,\ l = 1, 2, \dots, m)$ is the fuzzy set, $(\varphi_1(t), \varphi_2(t), \dots, \varphi_m(t), \psi_1(t), \psi_2(t), \dots, \psi_m(t))^T$ is the vector of premise variables, $x_j(t), y_\ell(t)$ are state variables, and n is the number of If-Then rules.

Let $\eta_k(\theta(t))$ be the normalized membership function of inferential fuzzy set $N_k(\theta(t))$. The derived system is as follows:

$$\begin{cases} \dot{x}_j(t) = \sum\limits_{k=1}^{m} \eta_k(\theta(t)) \Big\{ -c_j^k x_j(t) + \sum\limits_{\ell=1}^{q} a_{j\ell}^k f_\ell(y_\ell(t)) \\ \qquad + \sum\limits_{\ell=1}^{q} b_{j\ell}^k f_\ell(y_\ell(t-\tau_{j\ell}(t))) + \sum\limits_{\ell=1}^{q} d_{j\ell}^k \int_{t-\sigma_{j\ell}(t)}^{t} f_\ell(y_\ell(s))ds + J_j^k \Big\}, \\ \dot{y}_\ell(t) = \sum\limits_{k=1}^{m} \eta_k(\theta(t)) \Big\{ -\tilde{c}_\ell^k y_\ell(t) + \sum\limits_{j=1}^{p} \tilde{a}_{\ell j}^k g_j(x_j(t)) \\ \qquad + \sum\limits_{j=1}^{p} \tilde{b}_{\ell j}^k g_j(x_j(t-\pi_{\ell j}(t))) + \sum\limits_{j=1}^{p} \tilde{d}_{\ell j}^k \int_{t-\rho_{\ell j}(t)}^{t} g_j(x_j(s))ds + \tilde{J}_\ell^k \Big\}, \end{cases} \tag{2.2}$$

where

$$\eta_k(\theta(t)) = \frac{N_k(\theta(t))}{\sum\limits_{k=1}^{n} N_k(\theta(t))}, \quad N_k(\theta(t)) = \prod_{h=1}^{m} M_l^k(\theta(t)),$$

and $M_l^k(z_l(t))$ is the rank of the membership function of $z_l(t)$ in M_l^k. The basic properties of the fuzzy weighting function are

$$\eta_k(\theta(t)) \geqslant 0,\ k = 1, 2, \dots, n,\ \sum_{k=1}^{n} \eta_k(\theta(t)) = 1,\ \text{for any } \theta(t).$$

System (2.2) can be regarded as the driving system. The response system is introduced as follows:

$$\begin{cases} \dot{v}_j(t) = \sum\limits_{k=1}^{m} \eta_k(\theta(t)) \Big\{ -c_j^k v_j(t) + \sum\limits_{\ell=1}^{q} a_{j\ell}^k f_\ell(z_\ell(t)) + \sum\limits_{\ell=1}^{q} b_{j\ell}^k f_\ell(z_\ell(t-\tau_{j\ell}(t))) \\ \qquad + \sum\limits_{\ell=1}^{q} d_{j\ell}^k \int_{t-\sigma_{j\ell}(t)}^{t} f_\ell(z_\ell(s))ds + J_j^k + u_j^k(t) \Big\}, \\ \dot{z}_\ell(t) = \sum\limits_{k=1}^{m} \eta_k(\theta(t)) \Big\{ -\tilde{c}_\ell^k z_\ell(t) + \sum\limits_{j=1}^{p} \tilde{a}_{\ell j}^k g_j(v_j(t)) + \sum\limits_{j=1}^{p} \tilde{b}_{\ell j}^k g_j(v_j(t-\pi_{\ell j}(t))) \\ \qquad + \sum\limits_{j=1}^{p} \tilde{d}_{\ell j}^k \int_{t-\rho_{\ell j}(t)}^{t} g_j(v_j(s))ds + \tilde{J}_\ell^k + \tilde{u}_\ell^k(t) \Big\}. \end{cases} \tag{2.3}$$

where $u_j^k(t)$ and $\tilde{u}_\ell^k(t)$ are controllers. Let $\alpha_j(t) = v_j(t) - x_j(t)$, $\beta_\ell(t) = z_\ell(t) - y_\ell(t)$, then the corresponding error system is

$$\begin{cases} \dot{\alpha}_j(t) = \sum_{k=1}^{m} \eta_k(\theta(t)) \Big\{ -c_j^k \alpha_j(t) + \sum_{\ell=1}^{q} a_{j\ell}^k \tilde{f}_\ell(\beta_\ell(t)) \\ \qquad + \sum_{\ell=1}^{q} b_{j\ell}^k \tilde{f}_\ell(\beta_\ell(t - \tau_{j\ell}(t))) + \sum_{\ell=1}^{q} d_{j\ell}^k \int_{t-\sigma_{j\ell}(t)}^{t} \tilde{f}_\ell(\beta_\ell(s))ds + u_j^k(t) \Big\}, \\ \dot{\beta}_\ell(t) = \sum_{k=1}^{m} \eta_k(\theta(t)) \Big\{ -\tilde{c}_\ell^k \beta_\ell(t) + \sum_{j=1}^{p} \tilde{a}_{\ell j}^k \tilde{g}_j(\alpha_j(t)) \\ \qquad + \sum_{j=1}^{p} \tilde{b}_{\ell j}^k \tilde{g}_j(\alpha_j(t - \pi_{\ell j}(t))) + \sum_{j=1}^{p} \tilde{d}_{\ell j}^k \int_{t-\rho_{\ell j}(t)}^{t} \tilde{g}_j(\alpha_j(s))ds + \tilde{u}_\ell^k(t) \Big\}, \end{cases} \tag{2.4}$$

where $\tilde{f}_\ell(\beta_\ell(\cdot)) = f_\ell(z_\ell(\cdot)) - f_\ell(y_\ell(\cdot))$ and $\tilde{g}_j(\alpha_j(\cdot)) = g_j(v_j(\cdot)) - g_j(x_j(\cdot))$.

To facilitate the discussion of the synchronization between the systems (2.2) and (2.3), the following are preliminary preparations for this study.

Assumption 2.1. *$\tau_{j\ell}(t)$, $\pi_{\ell j}(t)$, $\rho_{\ell j}(t)$, and $\sigma_{j\ell}(t)$ are differentiable, and there are real numbers $0 < \tau_{j\ell}^* < 1$, $0 < \pi_{\ell j}^* < 1$, $0 < \rho_{\ell j}^* < \frac{1}{2}$, $0 < \sigma_{j\ell}^* < \frac{1}{2}$ such that*

$$\dot{\tau}_{j\ell}(t) \leqslant \tau_{j\ell}^*, \quad \dot{\pi}_{\ell j}(t) \leqslant \pi_{\ell j}^*, \quad \dot{\rho}_{\ell j}(t) \leqslant \rho_{\ell j}^*, \quad \dot{\sigma}(t)_{j\ell} \leqslant \sigma_{j\ell}^*.$$

Assumption 2.2. *There exist positive real numbers L_ℓ^f and L_j^g such that the activation functions $f_\ell(\cdot)$ and $g_j(\cdot)$ satisfy the following inequalities*

$$|f_\ell(\varepsilon_2) - f_\ell(\varepsilon_1)| \leqslant L_\ell^f |\varepsilon_2 - \varepsilon_1|, \qquad |g_j(\varepsilon_2) - g_j(\varepsilon_1)| \leqslant L_j^g |\varepsilon_2 - \varepsilon_1|.$$

Assume that the activation functions $f_\ell(\cdot)$ and $g_j(\cdot)$ in system (2.1) are sufficiently smooth, i.e., they satisfy Assumption 2.2, and that the time-delay functions $\tau_{j\ell}(t)$, $\pi_{\ell j}(t)$, $\sigma_{j\ell}(t)$, and $\rho_{\ell j}(t)$ satisfy the differentiability and boundedness conditions in Assumption 2.1. Under these conditions, the solution to system (2.1) exists and is unique in a local time interval.

The definition of GD synchronization for functions of Ψ-type is given below.

Definition 2.3 ([16])**.** The function Ψ: $\mathcal{R}^+ \to [1, +\infty)$ is called a Ψ-type function if it satisfies the following criteria:

(1) $\dot{\Psi}(\cdot)$ exists and it is non-decreasing function;
(2) $\Psi(0) = 1$ and $\Psi(+\infty) = +\infty$;
(3) $\tilde{\Psi}(t) = \dot{\Psi}(t)/\Psi(t)$ is non-decreasing and $\Psi^* = \sup_{t \geqslant 0} \tilde{\Psi}(t) < +\infty$;
(4) for any t, $s \geqslant 0$, $\Psi(t + s) \leqslant \Psi(t)\Psi(s)$.

Definition 2.4 ([16])**.** If there is a constant $\varepsilon > 0$, such that

$$\limsup_{t \to +\infty} \frac{\log(\|\alpha(t)\| + \|\beta(t)\|)}{\log \Psi(t)} \leqslant -\varepsilon,$$

then the error system (2.4) said to be GD stable, where $\alpha(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_p(t))^T$, $\beta(t) = (\beta_1(t), \beta_2(t), \ldots, \beta_q(t))^T$ and $\varepsilon > 0$ is the convergence rate when the synchronization error approaches zero.

Assumption 2.5 ([16])**.** *For the function $\Psi(t)$, $\tilde{\Psi}(t)$ given in Definition 2.3, there is a positive function $\rho(t) \in \mathcal{C}(\mathcal{R}, \mathcal{R}^+)$ and a real number δ, such that*

$$\tilde{\Psi}(t) \leqslant 1, \quad \sup_{t \in [0, +\infty)} \int_0^t \Psi^\delta(s)\rho(s)ds < +\infty, \textit{ for any } \ t \geqslant 0.$$

Lemma 2.6 ([16]). *Under Assumption 2.5, suppose that the continuous functions* $f(t)$ *and* $g(t)$ *satisfy differential equations* $\dot{\alpha}(t) = \mathscr{F}(t, f(t), g(t))$ *and* $\dot{\beta}(t) = \mathscr{G}(t, f(t), g(t))$*, where* $\mathscr{F}(t, f(t), g(t))$ *and* $\mathscr{G}(t, f(t), g(t))$ *are locally bounded. If there is a differential functional* $\mathcal{V}(t, \alpha(t), \beta(t)) : \mathcal{R}^+ \times \mathcal{R}^m \times \mathcal{R}^m \to \mathcal{R}^+$*, and normal numbers* λ_1*,* λ_2 *for any* $(t, \alpha(t), \beta(t)) \in \mathcal{R}^+ \times \mathcal{R}^m \times \mathcal{R}^m$ *such that*

$$\lambda_1(\|\alpha(t)\| + \|\beta(t)\|)^2 \leqslant \mathcal{V}(t, \alpha(t), \beta(t)), \quad \frac{d\mathcal{V}(t, \alpha(t), \beta(t))}{dt} \leqslant -\theta\mathcal{V}(t, \alpha(t), \beta(t)) + \lambda_2\rho(t),$$

where $\theta > 0$ *and* $\rho(t)$ *are defined in Assumption 2.5, then* $\alpha(t)$ *and* $\beta(t)$ *will achieve GD stability in the sense of Definition 2.4, with convergence rate* $\theta/2$.

Lemma 2.7 ([6]). *The following is true for any* $\gamma \in \mathbb{C}$:

$$\chi + \bar{\chi} = 2\boldsymbol{Re}(\chi) \leqslant 2|\chi|_2.$$

3. Main results

In this section, we will derive some sufficient criteria for GD synchronization for master-slave systems (2.2) and (2.3). First, design a set of T-S fuzzy controller consisting of fuzzy implications, which is given as follows.

Plant Rule k: If $\varphi_1(t)$ is $M_1^k, \ldots,$ and $\varphi_m(t)$ is M_m^k, then

$$\begin{cases} u_j^k(t) = -\mu_j^k \frac{\|\alpha(t)\|^2\alpha_j(t)}{\|\alpha(t)\|^2+\rho(t)}, \\ \tilde{u}_\ell^k(t) = -\lambda_\ell^k \frac{\|\beta(t)\|^2\beta_\ell(t)}{\|\beta(t)\|^2+\rho(t)}, \end{cases} \tag{3.1}$$

where μ_j^k and λ_ℓ^k are positive control gains satisfying:

$$\begin{aligned} \Delta_j^k &\triangleq \mathbf{Re}(-c_j^k) - \mu_j^k + \sum_{\ell=1}^{q}\Bigg(\frac{|a_{j\ell}^k|_2 L_\ell^f}{2} + \frac{|a_{\ell j}^k|_2 L_j^f}{2} + \frac{|b_{j\ell}^k|_2 L_\ell^f}{2} \\ &\quad + \frac{|d_{j\ell}^k|_2 L_\ell^f}{2} + \frac{\Gamma_{j\ell}^k}{1-\pi_{j\ell}^*} + \Gamma_{\ell j}^k \pi_{\ell j} + 2\Pi_{\ell j}^k \rho_{\ell j} + \Pi_{\ell j}^k \frac{1}{2}\rho_{\ell j}^2\Bigg) < 0, \\ \Lambda_\ell^k &\triangleq \mathbf{Re}(-\tilde{c}_\ell^k) - \lambda_\ell^k + \sum_{j=1}^{p}\Bigg(\frac{|\tilde{a}_{\ell j}^k|_2 L_j^g}{2} + \frac{|\tilde{a}_{j\ell}^k|_2 L_\ell^g}{2} + \frac{|\tilde{b}_{\ell j}^k|_2 L_j^g}{2} \\ &\quad + \frac{|\tilde{d}_{\ell j}^k|_2 L_j^g}{2} + \frac{\tilde{\Gamma}_{\ell j}^k}{1-\tau_{\ell j}^*} + \tilde{\Gamma}_{j\ell}^k \tau_{j\ell} + 2\tilde{\Pi}_{j\ell}^k \sigma_{j\ell} + \tilde{\Pi}_{j\ell}^k \frac{1}{2}\sigma_{j\ell}^2\Bigg) < 0, \end{aligned} \tag{3.2}$$

where $\Gamma_{\ell j}^k = \frac{|\tilde{b}_{\ell j}^k|_2 L_j^g}{2}$, $\Pi_{\ell j}^k = \frac{|\tilde{d}_{\ell j}^k|_2 L_j^g}{2}$, $\tilde{\Gamma}_{j\ell}^k = \frac{|b_{j\ell}^k|_2 L_\ell^g}{2}$, $\tilde{\Pi}_{j\ell}^k = \frac{|d_{j\ell}^k|_2 L_\ell^g}{2}$. Based on the above T-S fuzzy rules, the controller in response system (2.3) can be given as

$$\begin{cases} u_j^k(t) = \sum\limits_{k=1}^{m} \eta_k(\theta(t))\Big\{-\mu_j^k \frac{\|\alpha(t)\|^2\alpha_j(t)}{\|\alpha(t)\|^2+\rho(t)}\Big\}, \\ \tilde{u}_\ell^k(t) = \sum\limits_{k=1}^{m} \eta_k(\theta(t))\Big\{-\lambda_\ell^k \frac{\|\beta(t)\|^2\beta_\ell(t)}{\|\beta(t)\|^2+\rho(t)}\Big\}. \end{cases} \tag{3.3}$$

Theorem 3.1. *Suppose that the Assumptions 2.1, 2.2, and 2.5 hold true, and the control gains* μ_j^k *and* λ_ℓ^k *satisfy inequalities (3.2), then the drive-response CV-BAMNNs (2.2) and (2.3) will realize GD synchronization via controller (3.3).*

Proof. Consider the following Lyapunov-Krasovskii functional $\mathcal{V}(t) = \mathcal{V}_1(t) + \mathcal{V}_2(t)$, where

$$\begin{aligned}
\mathcal{V}_1(t) &= \frac{1}{2}\sum_{\jmath=1}^{p} \alpha_\jmath(t)\overline{\alpha_\jmath(t)} + \sum_{\jmath=1}^{p}\sum_{\ell=1}^{q} \frac{\Gamma_{\jmath\ell}^{k}}{1-\pi_{\jmath\ell}^{*}} \int_{t-\pi_{\jmath\ell}(t)}^{t} \alpha_\ell(s)\overline{\alpha_\ell(s)}ds \\
&\quad + \sum_{\jmath=1}^{p}\sum_{\ell=1}^{q} \Gamma_{\jmath\ell}^{k} \int_{-\pi_{\jmath\ell}}^{0}\int_{t+s}^{t} \alpha_\ell(s)\overline{\alpha_\ell(s)}dsd\sigma + \sum_{\jmath=1}^{p}\sum_{\ell=1}^{q} 2\Pi_{\jmath\ell}^{k} \int_{-\rho_{\jmath\ell}(t)}^{0}\int_{t+s}^{t} \alpha_\ell(\sigma)\overline{\alpha_\ell(\sigma)}d\sigma ds \\
&\quad + \sum_{\jmath=1}^{p}\sum_{\ell=1}^{q} \Pi_{\jmath\ell}^{k} \int_{-\rho_{\jmath\ell}}^{0}\int_{\epsilon}^{0}\int_{t+s}^{t} \alpha_\ell(\sigma)\overline{\alpha_\ell(\sigma)}d\sigma dsd\epsilon, \\
\mathcal{V}_2(t) &= \frac{1}{2}\sum_{\ell=1}^{q} \beta_\ell(t)\overline{\beta_\ell(t)} + \sum_{\jmath=1}^{p}\sum_{\ell=1}^{q} \frac{\tilde{\Gamma}_{\ell\jmath}^{k}}{1-\tau_{\ell\jmath}^{*}} \int_{t-\tau_{\ell\jmath}(t)}^{t} \beta_\jmath(s)\overline{\beta_\jmath(s)}ds \\
&\quad + \sum_{\jmath=1}^{p}\sum_{\ell=1}^{q} \tilde{\Gamma}_{\ell\jmath}^{k} \int_{-\tau_{\ell\jmath}}^{0}\int_{t+s}^{t} \beta_\jmath(s)\overline{\beta_\jmath(s)}dsd\sigma + \sum_{\jmath=1}^{p}\sum_{\ell=1}^{q} 2\tilde{\Pi}_{\ell\jmath}^{k} \int_{-\sigma_{\ell\jmath}(t)}^{0}\int_{t+s}^{t} \beta_\jmath(\sigma)\overline{\beta_\jmath(\sigma)}d\sigma ds \\
&\quad + \sum_{\jmath=1}^{p}\sum_{\ell=1}^{q} \tilde{\Pi}_{\ell\jmath}^{k} \int_{-\sigma_{\ell\jmath}}^{0}\int_{\epsilon}^{0}\int_{t+s}^{t} \beta_\jmath(\sigma)\overline{\beta_\jmath(\sigma)}d\sigma dsd\epsilon,
\end{aligned} \tag{3.4}$$

where each term is non-negative, and the functional is zero only when $\alpha_\jmath(t) = 0$ and $\beta_\ell(t) = 0$. Therefore, functionals (3.4) are positive definite, and consequently, $\mathcal{V}(t)$ is also positive definite. Let

$$\mathcal{V}_{13}(t) = \Pi_{\jmath\ell}^{k} \int_{-\rho_{\jmath\ell}}^{0}\int_{\epsilon}^{0}\int_{t+s}^{t} \alpha_\ell(\sigma)\overline{\alpha_\ell(\sigma)}d\sigma dsd\epsilon, \qquad \mathcal{V}_{23}(t) = \tilde{\Pi}_{\ell\jmath}^{k} \int_{-\sigma_{\ell\jmath}}^{0}\int_{\epsilon}^{0}\int_{t+s}^{t} \beta_\jmath(\sigma)\overline{\beta_\jmath(\sigma)}d\sigma dsd\epsilon.$$

By using Leibniz's formula we obtain

$$\frac{d}{dt}\int_{t+s}^{t} \alpha_\ell(\sigma)\overline{\alpha_\ell(\sigma)}d\sigma = \alpha_\ell(t)\overline{\alpha_\ell(t)} - \alpha_\ell(t+s)\overline{\alpha_\ell(t+s)},$$

thus

$$\begin{aligned}
\dot{\mathcal{V}}_{13}(t) &= \frac{d}{dt}\Pi_{\jmath\ell}^{k} \int_{-\rho_{\jmath\ell}}^{0}\int_{\epsilon}^{0}\int_{t+s}^{t} \alpha_\ell(\sigma)\overline{\alpha_\ell(\sigma)}d\sigma dsd\epsilon \\
&= \Pi_{\jmath\ell}^{k} \int_{-\rho_{\jmath\ell}}^{0}\int_{\epsilon}^{0} \left(\alpha_\ell(t)\overline{\alpha_\ell(t)} - \alpha_\ell(t+s)\overline{\alpha_\ell(t+s)}\right) dsd\epsilon \\
&= \Pi_{\jmath\ell}^{k} \left(\int_{-\rho_{\jmath\ell}}^{0}\int_{\epsilon}^{0} \alpha_\ell(t)\overline{\alpha_\ell(t)}dsd\epsilon - \int_{-\rho_{\jmath\ell}}^{0}\int_{\epsilon}^{0} \alpha_\ell(t+s)\overline{\alpha_\ell(t+s)}dsd\epsilon\right) \\
&= \Pi_{\jmath\ell}^{k} \left(\int_{-\rho_{\jmath\ell}}^{0} \alpha_\ell(t)\overline{\alpha_\ell(t)}(-\epsilon)d\epsilon - \int_{-\rho_{\jmath\ell}}^{0}\int_{-\rho_{\jmath\ell}}^{s} \alpha_\ell(t+s)\overline{\alpha_\ell(t+s)}d\epsilon ds\right) \\
&= \Pi_{\jmath\ell}^{k} \left(\frac{1}{2}\rho_{\jmath\ell}^{2}\alpha_\ell(t)\overline{\alpha_\ell(t)} - \int_{-\rho_{\jmath\ell}}^{0} (s+\rho_{\jmath\ell})\alpha_\ell(t+s)\overline{\alpha_\ell(t+s)}ds\right) \\
&= \Pi_{\jmath\ell}^{k} \left(\frac{1}{2}\rho_{\jmath\ell}^{2}\alpha_\ell(t)\overline{\alpha_\ell(t)} - \int_{-\rho_{\jmath\ell}}^{0}\int_{t+s}^{t} \alpha_\ell(s)\overline{\alpha_\ell(s)}dsd\sigma\right).
\end{aligned} \tag{3.5}$$

Similarly

$$\dot{\mathcal{V}}_{23}(t) = \frac{d}{dt}\tilde{\Pi}_{\ell\jmath}^{k} \int_{-\sigma_{\ell\jmath}}^{0}\int_{\epsilon}^{0}\int_{t+s}^{t} \beta_\jmath(\sigma)\overline{\beta_\jmath(\sigma)}d\sigma dsd\epsilon = \tilde{\Pi}_{\ell\jmath}^{k}\left(\frac{1}{2}\sigma_{\ell\jmath}^{2}\beta_\jmath(t)\overline{\beta_\jmath(t)} - \int_{-\sigma_{\ell\jmath}}^{0}\int_{t+s}^{t} \beta_\jmath(s)\overline{\beta_\jmath(s)}dsd\sigma\right). \tag{3.6}$$

Then, it is not hard to see that there are positive scalars $\Theta > 1$ and $\tilde{\Theta} > 1$ such that

$$\begin{aligned}
\frac{1}{2}\sum_{\jmath=1}^{p}\alpha_\jmath(t)\overline{\alpha_\jmath(t)} \leqslant \mathcal{V}_1(t) \leqslant \Theta\sum_{\jmath=1}^{p}\alpha_\jmath(t)\overline{\alpha_\jmath(t)} + \frac{\Theta}{\Delta^k}\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\Big(\Gamma_{\jmath\ell}^k\int_{t-\pi_{\jmath\ell}}^{t}\alpha_\jmath(s)\overline{\alpha_\jmath(s)}ds \\
+ \Pi_{\jmath\ell}^k\int_{-\rho_{\jmath\ell}}^{0}\int_{t+s}^{t}\alpha_\jmath(s)\overline{\alpha_\jmath(s)}dsd\sigma\Big), \\
\frac{1}{2}\sum_{\ell=1}^{q}\beta_\ell(t)\overline{\beta_\ell(t)} \leqslant \mathcal{V}_2(t) \leqslant \tilde{\Theta}\sum_{\ell=1}^{q}\beta_\ell(t)\overline{\beta_\ell(t)} + \frac{\tilde{\Theta}}{\Lambda^k}\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\Big(\tilde{\Gamma}_{\ell\jmath}^k\int_{t-\tau_{\ell\jmath}}^{t}\beta_\ell(s)\overline{\beta_\ell(s)}ds \\
+ \tilde{\Pi}_{\ell\jmath}^k\int_{-\sigma_{\ell\jmath}}^{0}\int_{t+s}^{t}\beta_\ell(s)\overline{\beta_\ell(s)}dsd\sigma\Big),
\end{aligned} \tag{3.7}$$

where $\Delta^k = \min_{\jmath\in\mathscr{P}}\{\Delta_\jmath^k\}$ and $\Lambda^k = \min_{\ell\in\mathscr{Q}}\{\Lambda_\ell^k\}$. Compute the derivative of $\mathcal{V}_\jmath(t)$ ($\jmath = 1, 2$) with respect to time t and combining (3.5) and (3.6), we have

$$\begin{aligned}
\dot{\mathcal{V}}_1(t) &= \sum_{\jmath=1}^{p}\Big\{\frac{1}{2}\Big[\sum_{k=1}^{m}\eta_k(\theta(t))\Big(-c_\jmath^k\alpha_\jmath(t) + \sum_{\ell=1}^{q}a_{\jmath\ell}^k\tilde{f}_\ell(\beta_\ell(t)) + \sum_{\ell=1}^{q}b_{\jmath\ell}^k\tilde{f}_\ell(\beta_\ell(t-\tau_{\jmath\ell}(t))) \\
&\quad + \sum_{\ell=1}^{q}d_{\jmath\ell}^k\int_{t-\sigma_{\jmath\ell}(t)}^{t}\tilde{f}_\ell(\beta_\ell(s))ds + u_\jmath^k(t)\Big)\overline{\alpha_\jmath(t)} + \sum_{k=1}^{m}\eta_k(\theta(t))\Big(-\overline{c_\jmath^k}\overline{\alpha_\jmath(t)} + \sum_{\ell=1}^{q}\overline{a_{\jmath\ell}^k\tilde{f}_\ell(\beta_\ell(t))} \\
&\quad + \sum_{\ell=1}^{q}\overline{b_{\jmath\ell}^k\tilde{f}_\ell(\beta_\ell(t-\tau_{\jmath\ell}(t)))} + \sum_{\ell=1}^{q}\overline{d_{\jmath\ell}^k\int_{t-\sigma_{\jmath\ell}(t)}^{t}\tilde{f}_\ell(\beta_\ell(s))ds} + \overline{u_\jmath^k(t)}\Big)\alpha_\jmath(t)\Big] \\
&\quad + \sum_{\ell=1}^{q}\frac{\Gamma_{\jmath\ell}^k}{1-\pi_{\jmath\ell}^*}\Big(\alpha_\ell(t)\overline{\alpha_\ell(t)} - (1-\dot{\pi}_{\jmath\ell}(t))\alpha_\ell(t-\pi_{\jmath\ell}(t))\overline{\alpha_\ell(t-\pi_{\jmath\ell}(t))}\Big) \\
&\quad + \sum_{\ell=1}^{q}\Gamma_{\jmath\ell}^k\Big(\pi_{\jmath\ell}\alpha_\ell(t)\overline{\alpha_\ell(t)} - \int_{t-\pi_{\jmath\ell}}^{t}\alpha_\ell(s)\overline{\alpha_\ell(s)}ds\Big) + \sum_{\ell=1}^{q}2\Pi_{\jmath\ell}^k\Big(\rho_{\jmath\ell}(t)\alpha_\ell(t)\overline{\alpha_\ell(t)} \\
&\quad - (1-\dot{\rho}_{\jmath\ell}(t))\int_{t-\rho_{\jmath\ell}(t)}^{t}\alpha_\ell(s)\overline{\alpha_\ell(s)}ds\Big) + \sum_{\ell=1}^{q}\Pi_{\jmath\ell}^k\Big(\frac{1}{2}\rho_{\jmath\ell}^2\alpha_\ell(t)\overline{\alpha_\ell(t)} - \int_{-\rho_{\jmath\ell}}^{0}\int_{t+s}^{t}\alpha_\ell(s)\overline{\alpha_\ell(s)}dsd\sigma\Big)\Big\} \\
&= \sum_{k=1}^{m}\eta_k(\theta(t))\Big\{\sum_{\jmath=1}^{p}\mathbf{Re}(-c_\jmath^k)\alpha_\jmath(t)\overline{\alpha_\jmath(t)} + \frac{1}{2}\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\Big(\alpha_\jmath(t)\overline{a_{\jmath\ell}^k\tilde{f}_\ell(\beta_\ell(t))} + \overline{\alpha_\jmath(t)}a_{\jmath\ell}^k\tilde{f}_\ell(\beta_\ell(t))\Big) \\
&\quad + \frac{1}{2}\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\Big(\alpha_\jmath(t)\overline{b_{\jmath\ell}^k\tilde{f}_\ell(\beta_\ell(t-\tau_{\jmath\ell}(t)))} + \overline{\alpha_\jmath(t)}b_{\jmath\ell}^k\tilde{f}_\ell(\beta_\ell(t-\tau_{\jmath\ell}(t)))\Big) \\
&\quad + \frac{1}{2}\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\Big(\alpha_\jmath(t)\overline{d_{\jmath\ell}^k\int_{t-\sigma_{\jmath\ell}(t)}^{t}\tilde{f}_\ell(\beta_\ell(s))ds} + \overline{\alpha_\jmath(t)}d_{\jmath\ell}^k\int_{t-\sigma_{\jmath\ell}(t)}^{t}\tilde{f}_\ell(\beta_\ell(s))ds\Big) \\
&\quad + \frac{1}{2}\sum_{\jmath=1}^{p}\Big(-\mu_\jmath^k\frac{\|\alpha(t)\|^2\alpha_\jmath(t)}{\|\alpha(t)\|^2+\rho(t)}\overline{\alpha_\jmath(t)} - \mu_\jmath^k\frac{\|\alpha(t)\|^2\overline{\alpha_\jmath(t)}}{\|\alpha(t)\|^2+\rho(t)}\alpha_\jmath(t)\Big) \\
&\quad + \sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\frac{\Gamma_{\jmath\ell}^k}{1-\pi_{\jmath\ell}^*}\Big(\alpha_\ell(t)\overline{\alpha_\ell(t)} - (1-\dot{\pi}_{\jmath\ell}(t))\alpha_\ell(t-\pi_{\jmath\ell}(t))\overline{\alpha_\ell(t-\pi_{\jmath\ell}(t))}\Big) \\
&\quad + \sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\Gamma_{\jmath\ell}^k\Big(\pi_{\jmath\ell}\alpha_\ell(t)\overline{\alpha_\ell(t)} - \int_{t-\pi_{\jmath\ell}}^{t}\alpha_\ell(s)\overline{\alpha_\ell(s)}ds\Big)
\end{aligned}$$

$$+\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}2\Pi_{\jmath\ell}^{k}\Big(\rho_{\jmath\ell}\alpha_\ell(t)\overline{\alpha_\ell(t)}-(1-\dot{\rho}_{\jmath\ell}(t))\int_{t-\rho_{\jmath\ell}(t)}^{t}\alpha_\ell(s)\overline{\alpha_\ell(s)}ds\Big)$$
$$+\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\Pi_{\jmath\ell}^{k}\Big(\frac{1}{2}\rho_{\jmath\ell}^{2}\alpha_\ell(t)\overline{\alpha_\ell(t)}-\int_{-\rho_{\jmath\ell}}^{0}\int_{t+s}^{t}\alpha_\ell(s)\overline{\alpha_\ell(s)}dsd\sigma\Big)\Big\}$$

and

$$\begin{aligned}
\dot{\mathcal{V}}_2(t)=&\sum_{\ell=1}^{p}\Big\{\frac{1}{2}\Big[\sum_{k=1}^{m}\eta_k(\theta(t))\Big(-\tilde{c}_\ell^k\beta_\ell(t)+\sum_{\jmath=1}^{p}\tilde{a}_{\ell\jmath}^k\tilde{g}_\jmath(\alpha_\jmath(t))+\sum_{\jmath=1}^{p}\tilde{b}_{\ell\jmath}^k\tilde{g}_\jmath(\alpha_\jmath(t-\pi_{\ell\jmath}(t)))\\
&+\sum_{\jmath=1}^{p}\tilde{d}_{\ell\jmath}^k\int_{t-\rho_{\ell\jmath}(t)}^{t}\tilde{g}_\jmath(\alpha_\jmath(s))ds+\tilde{u}_\ell^k(t)\Big)\overline{\beta_\ell(t)}+\sum_{k=1}^{m}\eta_k(\theta(t))\Big(-\overline{\tilde{c}_\ell^k\beta_\ell(t)}\\
&+\sum_{\jmath=1}^{p}\overline{\tilde{a}_{\ell\jmath}^k\tilde{g}_\jmath(\alpha_\jmath(t))}+\sum_{\jmath=1}^{p}\overline{\tilde{b}_{\ell\jmath}^k\tilde{g}_\jmath(\alpha_\jmath(t-\pi_{\ell\jmath}(t)))}+\sum_{\jmath=1}^{p}\overline{\tilde{d}_{\ell\jmath}^k\int_{t-\rho_{\ell\jmath}(t)}^{t}\tilde{g}_\jmath(\alpha_\jmath(s))ds}\\
&+\overline{\tilde{u}_\ell^k(t)}\Big)\beta_\ell(t)\Big]+\sum_{\jmath=1}^{p}\frac{\tilde{\Gamma}_{\ell\jmath}^k}{1-\tau_{\ell\jmath}^*}\Big(\beta_\jmath(t)\overline{\beta_\jmath(t)}-(1-\dot{\tau}_{\ell\jmath}(t))\beta_\jmath(t-\tau_{\ell\jmath}(t))\overline{\beta_\jmath(t-\tau_{\ell\jmath}(t))}\Big)\\
&+\sum_{\jmath=1}^{p}\tilde{\Gamma}_{\ell\jmath}^k\Big(\tau_{\ell\jmath}\beta_\jmath(t)\overline{\beta_\jmath(t)}-\int_{t-\tau_{\ell\jmath}}^{t}\beta_\jmath(s)\overline{\beta_\jmath(s)}ds\Big)+\sum_{\jmath=1}^{p}2\tilde{\Pi}_{\ell\jmath}^k\Big(\sigma_{\ell\jmath}(t)\beta_\jmath(t)\overline{\beta_\jmath(t)}\\
&-(1-\dot{\sigma}_{\ell\jmath}(t))\int_{t-\sigma_{\ell\jmath}(t)}^{t}\beta_\jmath(s)\overline{\beta_\jmath(s)}ds\Big)+\sum_{\jmath=1}^{p}\tilde{\Pi}_{\ell\jmath}^k\Big(\frac{1}{2}\sigma_{\ell\jmath}^2\beta_\jmath(t)\overline{\beta_\jmath(t)}-\int_{-\sigma_{\ell\jmath}}^{0}\int_{t+s}^{t}\beta_\jmath(s)\overline{\beta_\jmath(s)}dsd\sigma\Big)\Big\}\\
=&\sum_{k=1}^{m}\eta_k(\theta(t))\Big\{\sum_{\ell=1}^{q}\mathbf{Re}(-\tilde{c}_\ell^k)\beta_\ell(t)\overline{\beta_\ell(t)}+\frac{1}{2}\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\Big(\beta_\ell(t)\overline{\tilde{a}_{\ell\jmath}^k\tilde{g}_\jmath(\alpha_\jmath(t))}+\overline{\beta_\ell(t)}\tilde{a}_{\ell\jmath}^k\tilde{g}_\jmath(\alpha_\jmath(t))\Big)\\
&+\frac{1}{2}\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\Big(\beta_\ell(t)\overline{\tilde{b}_{\ell\jmath}^k\tilde{g}_\jmath(\alpha_\jmath(t-\pi_{\ell\jmath}(t)))}+\overline{\beta_\ell(t)}\tilde{b}_{\ell\jmath}^k\tilde{g}_\jmath(\alpha_\jmath(t-\pi_{\ell\jmath}(t)))\Big)\\
&+\frac{1}{2}\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\Big(\beta_\ell(t)\overline{\tilde{d}_{\ell\jmath}^k\int_{t-\rho_{\ell\jmath}(t)}^{t}\tilde{g}_\jmath(\alpha_\jmath(s))ds}+\overline{\beta_\ell(t)}\tilde{d}_{\ell\jmath}^k\int_{t-\rho_{\ell\jmath}(t)}^{t}\tilde{g}_\jmath(\alpha_\jmath(s))ds\Big)\\
&+\frac{1}{2}\sum_{\ell=1}^{q}\Big(-\lambda_\ell^k\frac{\|\beta(t)\|^2\beta_\ell(t)}{\|\beta(t)\|^2+\rho(t)}\overline{\beta_\ell(t)}-\lambda_\ell^k\frac{\|\beta(t)\|^2\overline{\beta_\ell(t)}}{\|\beta(t)\|^2+\rho(t)}\beta_\ell(t)\Big)\\
&+\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\frac{\tilde{\Gamma}_{\ell\jmath}^k}{1-\tau_{\ell\jmath}^*}\Big(\beta_\jmath(t)\overline{\beta_\jmath(t)}-(1-\dot{\tau}_{\ell\jmath}(t))\beta_\jmath(t-\tau_{\ell\jmath}(t))\overline{\beta_\jmath(t-\tau_{\ell\jmath}(t))}\Big)\\
&+\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\tilde{\Gamma}_{\ell\jmath}^k\Big(\tau_{\ell\jmath}\beta_\jmath(t)\overline{\beta_\jmath(t)}-\int_{t-\tau_{\ell\jmath}}^{t}\beta_\jmath(s)\overline{\beta_\jmath(s)}ds\Big)\\
&+\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}2\tilde{\Pi}_{\ell\jmath}^k\Big(\sigma_{\ell\jmath}\beta_\jmath(t)\overline{\beta_\jmath(t)}-(1-\dot{\sigma}_{\ell\jmath}(t))\int_{t-\sigma_{\ell\jmath}(t)}^{t}\beta_\jmath(s)\overline{\beta_\jmath(s)}ds\Big)\\
&+\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\tilde{\Pi}_{\ell\jmath}^k\Big(\frac{1}{2}\sigma_{\ell\jmath}^2\beta_\jmath(t)\overline{\beta_\jmath(t)}-\int_{-\sigma_{\ell\jmath}}^{0}\int_{t+s}^{t}\beta_\jmath(s)\overline{\beta_\jmath(s)}dsd\sigma\Big)\Big\}.
\end{aligned}$$

According to Assumption 2.2, Lemma 2.7, and inequality $2\Xi\Theta \leqslant \Xi^2 + \Theta^2$ for any $\Xi,\ \Theta > 0$, one has

$$
\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{p}\sum_{\ell=1}^{q}\left(\alpha_j(t)\overline{a_{j\ell}^{k}\tilde{f}_\ell(\beta_\ell(t))} + \overline{\alpha_j(t)}a_{j\ell}^{k}\tilde{f}_\ell(\beta_\ell(t))\right)\\
&\leqslant \sum_{j=1}^{p}\sum_{\ell=1}^{q}|\overline{\alpha_j(t)}a_{j\ell}^{k}\tilde{f}_\ell(\beta_\ell(t))|_2 \leqslant \sum_{j=1}^{p}\sum_{\ell=1}^{q}|a_{j\ell}^{k}|_2 L_\ell^f|\alpha_j(t)|_2|\alpha_\ell(t)|_2 \leqslant \sum_{j=1}^{p}\sum_{\ell=1}^{q}|a_{j\ell}^{k}|_2\frac{L_\ell^f}{2}\left(|\alpha_j(t)|_2^2 + \alpha_\ell(t)|_2^2\right).
\end{aligned} \tag{3.8}
$$

Similarly

$$
\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{p}\sum_{\ell=1}^{q}\left(\alpha_j(t)\overline{b_{j\ell}\tilde{f}_\ell(\beta_\ell(t-\tau_{j\ell}(t)))} + \overline{\alpha_j(t)}b_{j\ell}^{k}\tilde{f}_\ell(\beta_\ell(t-\tau_{j\ell}(t)))\right)\\
&\quad\leqslant \sum_{j=1}^{p}\sum_{\ell=1}^{q}|b_{j\ell}^{k}|_2\frac{L_\ell^f}{2}\left(|\alpha_j(t)|_2^2 + |\beta_\ell(t-\tau_{j\ell}(t))|_2^2\right),\\
&\frac{1}{2}\sum_{j=1}^{p}\sum_{\ell=1}^{q}\left(\alpha_j(t)\overline{d_{j\ell}^{k}\int_{t-\sigma_{j\ell}(t)}^{t}\tilde{f}_\ell(\beta_\ell(s))ds} + \overline{\alpha_j(t)}d_{j\ell}^{k}\int_{t-\sigma_{j\ell}(t)}^{t}\tilde{f}_\ell(\beta_\ell(s))ds\right)\\
&\quad\leqslant \sum_{j=1}^{p}\sum_{\ell=1}^{q}|d_{j\ell}^{k}|_2\frac{L_\ell^f}{2}\left(|\alpha_j(t)|_2^2 + \int_{t-\sigma_{j\ell}(t)}^{t}|\beta_\ell(s)|_2^2ds\right),\\
&\frac{1}{2}\sum_{j=1}^{p}\sum_{\ell=1}^{q}\left(\beta_\ell(t)\overline{\tilde{a}_{\ell j}^{k}\tilde{g}_j(\alpha_j(t))} + \overline{\beta_\ell(t)}\tilde{a}_{\ell j}^{k}\tilde{g}_j(\alpha_j(t))\right) \leqslant \sum_{j=1}^{p}\sum_{\ell=1}^{q}|\tilde{a}_{\ell j}^{k}|_2\frac{L_j^g}{2}\left(|\beta_\ell(t)|_2^2 + \beta_j(t)|_2^2\right),\\
&\frac{1}{2}\sum_{j=1}^{p}\sum_{\ell=1}^{q}\left(\beta_\ell(t)\overline{\tilde{b}_{\ell j}^{k}\tilde{g}_j(\alpha_j(t-\pi_{\ell j}(t)))} + \overline{\beta_\ell(t)}\tilde{b}_{\ell j}^{k}\tilde{g}_j(\alpha_j(t-\pi_{\ell j}(t)))\right)\\
&\quad\leqslant \sum_{j=1}^{p}\sum_{\ell=1}^{q}|\tilde{b}_{\ell j}^{k}|_2\frac{L_j^g}{2}\left(|\beta_\ell(t)|_2^2 + |\alpha_j(t-\pi_{\ell j}(t))|_2^2\right),\\
&\frac{1}{2}\sum_{j=1}^{p}\sum_{\ell=1}^{q}\left(\beta_\ell(t)\overline{\tilde{d}_{\ell j}^{k}\int_{t-\rho_{\ell j}(t)}^{t}\tilde{g}_j(\alpha_j(s))ds} + \overline{\beta_\ell(t)}\tilde{d}_{\ell j}^{k}\int_{t-\rho_{\ell j}(t)}^{t}\tilde{g}_j(\alpha_j(s))ds\right)\\
&\quad\leqslant \sum_{j=1}^{p}\sum_{\ell=1}^{q}|\tilde{d}_{\ell j}^{k}|_2\frac{L_j^g}{2}\left(|\beta_\ell(t)|_2^2 + \int_{t-\rho_{\ell j}(t)}^{t}|\alpha_j(s)|_2^2ds\right).
\end{aligned} \tag{3.9}
$$

Introducing the above inequalities (3.8)-(3.9) into the derivative of $\mathcal{V}(t)$, we get

$$
\begin{aligned}
\dot{\mathcal{V}}(t) \leqslant &\sum_{k=1}^{m}\eta_k(\theta(t))\Bigg\{\sum_{j=1}^{p}\mathbf{Re}(-c_j^k)|\alpha_j(t)|_2^2 + \sum_{j=1}^{p}\sum_{\ell=1}^{q}|a_{j\ell}^{k}|_2\frac{L_\ell^f}{2}\left(|\alpha_j(t)|_2^2 + \alpha_\ell(t)|_2^2\right)\\
&+\sum_{j=1}^{p}\sum_{\ell=1}^{q}|b_{j\ell}^{k}|_2\frac{L_\ell^f}{2}\left(|\alpha_j(t)|_2^2 + |\beta_\ell(t-\tau_{j\ell}(t))|_2^2\right) + \sum_{j=1}^{p}\sum_{\ell=1}^{q}|d_{j\ell}^{k}|_2\frac{L_\ell^f}{2}\Bigg(|\alpha_j(t)|_2^2\\
&+\int_{t-\sigma_{j\ell}(t)}^{t}|\beta_\ell(s)|_2^2ds\Bigg) - \sum_{j=1}^{p}\mu_j^k\frac{\|\alpha(t)\|^2|\alpha_j(t)|_2^2}{\|\alpha(t)\|^2+\rho(t)} + \sum_{j=1}^{p}\sum_{\ell=1}^{q}\frac{\Gamma_{j\ell}^{k}}{1-\pi_{j\ell}^{*}}\Bigg(|\alpha_\ell(t)|_2^2\\
&-(1-\dot{\pi}_{j\ell}(t))|\alpha_\ell(t-\pi_{j\ell}(t))|_2^2\Bigg) + \sum_{j=1}^{p}\sum_{\ell=1}^{q}\Gamma_{j\ell}^{k}\left(\pi_{j\ell}|\alpha_\ell(t)|_2^2 - \int_{t-\pi_{j\ell}}^{t}|\alpha_\ell(s)|_2^2ds\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{p}\sum_{\ell=1}^{q}2\Pi_{j\ell}^{k}\left(\rho_{j\ell}|\alpha_\ell(t)|_2^2-(1-\dot\rho_{j\ell}(t))\int_{t-\rho_{j\ell}(t)}^{t}|\alpha_\ell(s)|_2^2ds\right)\\
&+\sum_{j=1}^{p}\sum_{\ell=1}^{q}\Pi_{j\ell}^{k}\left(\frac{1}{2}\rho_{j\ell}^2|\alpha_\ell(t)|_2^2-\int_{-\rho_{j\ell}}^{0}\int_{t+s}^{t}|\alpha_\ell(s)|_2^2dsd\sigma\right)+\sum_{\ell=1}^{q}\mathbf{Re}(-\tilde c_\ell^k)|\beta_\ell(t)|_2^2\\
&+\sum_{j=1}^{p}\sum_{\ell=1}^{q}|\tilde a_{\ell j}^k|_2\frac{L_j^g}{2}\left(|\beta_\ell(t)|_2^2+\beta_j(t)|_2^2\right)+\sum_{j=1}^{p}\sum_{\ell=1}^{q}|\tilde b_{\ell j}^k|_2\frac{L_j^g}{2}\left(|\beta_\ell(t)|_2^2+|\alpha_j(t-\pi_{\ell j}(t))|_2^2\right)\\
&+\sum_{j=1}^{p}\sum_{\ell=1}^{q}|\tilde d_{\ell j}^k|_2\frac{L_j^g}{2}\left(|\beta_\ell(t)|_2^2+\int_{t-\rho_{\ell j}(t)}^{t}|\alpha_j(s)|_2^2ds\right)-\sum_{\ell=1}^{q}\lambda_\ell^k\frac{\|\beta(t)\|^2|\beta_\ell(t)|_2^2}{\|\beta(t)\|^2+\rho(t)}\\
&+\sum_{j=1}^{p}\sum_{\ell=1}^{q}\frac{\tilde\Gamma_{\ell j}^k}{1-\tau_{\ell j}^*}\left(|\beta_j(t)|_2^2-(1-\dot\tau_{\ell j}(t))|\beta_j(t-\tau_{\ell j}(t))|_2^2\right)\\
&+\sum_{j=1}^{p}\sum_{\ell=1}^{q}\tilde\Gamma_{\ell j}^k\left(\tau_{\ell j}|\beta_j(t)|_2^2-\int_{t-\tau_{\ell j}}^{t}|\beta_j(s)|_2^2ds\right)\\
&+\sum_{j=1}^{p}\sum_{\ell=1}^{q}2\tilde\Pi_{\ell j}^k\left(\sigma_{\ell j}|\beta_j(t)|_2^2-(1-\dot\sigma_{\ell j}(t))\int_{t-\sigma_{\ell j}(t)}^{t}|\beta_j(s)|_2^2ds\right)\\
&+\sum_{j=1}^{p}\sum_{\ell=1}^{q}\tilde\Pi_{\ell j}^k\left(\frac{1}{2}\sigma_{\ell j}^2|\beta_j(t)|_2^2-\int_{-\sigma_{\ell j}}^{0}\int_{t+s}^{t}|\beta_j(s)|_2^2dsd\sigma\right)\Bigg\}\\
\leqslant&\sum_{k=1}^{m}\eta_k(\theta(t))\Bigg\{\sum_{j=1}^{p}\Bigg(\mathbf{Re}(-c_j)^k-\mu_j^k+\sum_{\ell=1}^{q}|a_{j\ell}^k|_2\frac{L_\ell^f}{2}+\sum_{\ell=1}^{q}|a_{\ell j}^k|_2\frac{L_j^f}{2}+\sum_{\ell=1}^{q}|b_{j\ell}^k|_2\frac{L_\ell^f}{2}\\
&+\sum_{\ell=1}^{q}|d_{j\ell}^k|_2\frac{L_\ell^f}{2}+\sum_{\ell=1}^{q}\frac{\Gamma_{j\ell}^k}{1-\pi_{j\ell}^*}+\sum_{\ell=1}^{q}\Gamma_{\ell j}^k\pi_{\ell j}+\sum_{\ell=1}^{q}2\Pi_{\ell j}^k\rho_{\ell j}+\sum_{\ell=1}^{q}\Pi_{\ell j}^k\frac{1}{2}\rho_{\ell j}^2\Bigg)|\alpha_j(t)|_2^2\\
&+\sum_{\ell=1}^{q}\Bigg(\mathbf{Re}(-\tilde c_\ell^k)-\lambda_\ell^k+\sum_{j=1}^{p}|\tilde a_{\ell j}^k|_2\frac{L_j^g}{2}+\sum_{j=1}^{p}|\tilde a_{j\ell}^k|_2\frac{L_\ell^g}{2}+\sum_{j=1}^{p}|\tilde b_{\ell j}^k|_2\frac{L_j^g}{2}\\
&+\sum_{j=1}^{p}|\tilde d_{\ell j}^k|_2\frac{L_j^g}{2}+\sum_{j=1}^{p}\frac{\tilde\Gamma_{\ell j}^k}{1-\tau_{\ell j}^*}+\sum_{j=1}^{p}\tilde\Gamma_{j\ell}^k\tau_{j\ell}+\sum_{j=1}^{p}2\tilde\Pi_{j\ell}^k\sigma_{j\ell}+\sum_{j=1}^{p}\tilde\Pi_{j\ell}^k\frac{1}{2}\sigma_{j\ell}^2\Bigg)|\beta_\ell(t)|_2^2\\
&+\sum_{j=1}^{p}\sum_{\ell=1}^{q}|b_{j\ell}^k|_2\frac{L_\ell^f}{2}|\beta_\ell(t-\tau_{j\ell}(t))|_2^2+\sum_{j=1}^{p}\sum_{\ell=1}^{q}|d_{j\ell}^k|_2\frac{L_\ell^f}{2}\int_{t-\sigma_{j\ell}(t)}^{t}|\beta_\ell(s)|_2^2ds\\
&-\sum_{j=1}^{p}\sum_{\ell=1}^{q}\Gamma_{j\ell}^k|\alpha_\ell(t-\pi_{j\ell}(t))|_2^2-\sum_{j=1}^{p}\sum_{\ell=1}^{q}\Gamma_{j\ell}^k\int_{t-\pi_{j\ell}}^{t}|\alpha_\ell(s)|_2^2ds\\
&-\sum_{j=1}^{p}\sum_{\ell=1}^{q}\Pi_{j\ell}^k\int_{t-\rho_{j\ell}(t)}^{t}|\alpha_\ell(s)|_2^2ds-\sum_{j=1}^{p}\sum_{\ell=1}^{q}\Pi_{j\ell}^k\int_{-\rho_{j\ell}}^{0}\int_{t+s}^{t}|\alpha_\ell(s)|_2^2dsd\sigma\\
&+\sum_{j=1}^{p}\sum_{\ell=1}^{q}|\tilde b_{\ell j}^k|_2\frac{L_j^g}{2}|\alpha_j(t-\pi_{\ell j}(t))|_2^2+\sum_{j=1}^{p}\sum_{\ell=1}^{q}|\tilde d_{\ell j}^k|_2\frac{L_j^g}{2}\int_{t-\rho_{\ell j}(t)}^{t}|\alpha_j(s)|_2^2ds\\
&-\sum_{j=1}^{p}\sum_{\ell=1}^{q}\tilde\Gamma_{\ell j}^k|\beta_j(t-\tau_{\ell j}(t))|_2^2-\sum_{j=1}^{p}\sum_{\ell=1}^{q}\tilde\Gamma_{\ell j}^k\int_{t-\tau_{\ell j}}^{t}|\beta_j(s)|_2^2ds
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\tilde{\Pi}_{\ell\jmath}^{k}\int_{t-\sigma_{\ell\jmath}(t)}^{t}|\beta_\jmath(s)|_2^2ds-\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\tilde{\Pi}_{\ell\jmath}^{k}\int_{-\sigma_{\ell\jmath}}^{0}\int_{t+s}^{t}|\beta_\jmath(s)|_2^2dsd\sigma\\
&+\sum_{\jmath=1}^{p}\mu_\jmath^k|\alpha_\jmath(t)|_2^2-\sum_{\jmath=1}^{p}\mu_\jmath^k\frac{\|\alpha(t)\|^2|\alpha_\jmath(t)|_2^2}{\|\alpha(t)\|^2+\rho(t)}+\sum_{\jmath=1}^{p}\lambda_\ell^k|\beta_\ell(t)|_2^2-\sum_{\ell=1}^{q}\lambda_\ell^k\frac{\|\beta(t)\|^2|\beta_\ell(t)|_2^2}{\|\beta(t)\|^2+\rho(t)}\Bigg\}\\
\leqslant&\sum_{k=1}^{m}\eta_k(\theta(t))\Bigg\{\sum_{\jmath=1}^{p}\Bigg[\mathbf{Re}(-c_\jmath)^k-\mu_\jmath^k+\sum_{\ell=1}^{q}\Bigg(\frac{|a_{\jmath\ell}^k|_2L_\ell^f}{2}+\frac{|a_{\ell\jmath}^k|_2L_\jmath^f}{2}+\frac{|b_{\jmath\ell}^k|_2L_\ell^f}{2}\\
&+\frac{|d_{\jmath\ell}^k|_2L_\ell^f}{2}+\frac{\Gamma_{\jmath\ell}^k}{1-\pi_{\jmath\ell}^*}+\Gamma_{\ell\jmath}^k\pi_{\ell\jmath}+2\Pi_{\ell\jmath}^k\rho_{\ell\jmath}+\Pi_{\ell\jmath}^k\frac{1}{2}\rho_{\ell\jmath}^2\Bigg)\Bigg]|\alpha_\jmath(t)|_2^2\\
&+\sum_{\ell=1}^{q}\Bigg[\mathbf{Re}(-\tilde{c}_\ell^k)-\lambda_\ell^k+\sum_{\jmath=1}^{p}\Bigg(\frac{|\tilde{a}_{\ell\jmath}^k|_2L_\jmath^g}{2}+\frac{|\tilde{a}_{\jmath\ell}^k|_2L_\ell^g}{2}+\frac{|\tilde{b}_{\ell\jmath}^k|_2L_\jmath^g}{2}+\frac{|\tilde{d}_{\ell\jmath}^k|_2L_\jmath^g}{2}+\frac{\tilde{\Gamma}_{\ell\jmath}^k}{1-\tau_{\ell\jmath}^*}\\
&+\tilde{\Gamma}_{\jmath\ell}^k\tau_{\jmath\ell}+2\tilde{\Pi}_{\jmath\ell}^k\sigma_{\jmath\ell}+\tilde{\Pi}_{\jmath\ell}^k\frac{1}{2}\sigma_{\jmath\ell}^2\Bigg)\Bigg]|\beta_\ell(t)|_2^2-\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\Gamma_{\jmath\ell}^k\int_{t-\pi_{\jmath\ell}}^{t}|\alpha_\ell(s)|_2^2ds\\
&-\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\Pi_{\jmath\ell}^k\int_{-\rho_{\jmath\ell}}^{0}\int_{t+s}^{t}|\alpha_\ell(s)|_2^2dsd\sigma-\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\tilde{\Gamma}_{\ell\jmath}^k\int_{t-\tau_{\ell\jmath}}^{t}|\beta_\jmath(s)|_2^2ds\\
&-\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\tilde{\Pi}_{\ell\jmath}^k\int_{-\sigma_{\ell\jmath}}^{0}\int_{t+s}^{t}|\beta_\jmath(s)|_2^2dsd\sigma+\max_{\jmath\in\mathscr{P}}\{\mu_\jmath^k\}\frac{\|\alpha(t)\|^2}{\|\alpha(t)\|^2+\rho(t)}+\max_{\ell\in\mathscr{Q}}\{\lambda_\ell^k\}\frac{\|\beta(t)\|^2}{\|\beta(t)\|^2+\rho(t)}\Bigg\}.
\end{aligned}
$$

By letting $\mu=\max_{\jmath\in\mathscr{P}}\{\mu_\jmath^k\}>0$ and $\lambda=\max_{\ell\in\mathscr{Q}}\{\lambda_\ell^k\}>0$ and employing the inequality $0\leqslant\omega_1\omega_2/(\omega_1+\omega_2)\leqslant\omega_1$ for any $\omega_1>0,\omega_2>0$, one has

$$
\begin{aligned}
\dot{\mathcal{V}}(t)\leqslant&\sum_{k=1}^{m}\eta_k(\theta(t))\Bigg\{-\sum_{\jmath=1}^{p}\Delta_\jmath^k|\alpha_\jmath(t)|_2^2-\sum_{\ell=1}^{q}\Lambda_\ell^k|\beta_\ell(t)|_2^2+\mu\rho(t)+\lambda\rho(t)-\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\Gamma_{\jmath\ell}^k\int_{t-\pi_{\jmath\ell}}^{t}|\alpha_\ell(s)|_2^2ds\\
&-\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\Pi_{\jmath\ell}^k\int_{-\rho_{\jmath\ell}}^{0}\int_{t+s}^{t}|\alpha_\ell(s)|_2^2dsd\sigma-\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\tilde{\Gamma}_{\ell\jmath}^k\int_{t-\tau_{\ell\jmath}}^{t}|\beta_\jmath(s)|_2^2ds\\
&-\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\tilde{\Pi}_{\ell\jmath}^k\int_{-\sigma_{\ell\jmath}}^{0}\int_{t+s}^{t}|\beta_\jmath(s)|_2^2dsd\sigma\Bigg\}.
\end{aligned}
\tag{3.10}
$$

Choosing a small enough constant Ξ such that $\Xi\Theta<\Delta^k$ and $\Xi\tilde{\Theta}<\Lambda^k$, then we get by combining the inequalities (3.7) and (3.10),

$$
\begin{aligned}
\dot{\mathcal{V}}(t)+\Xi\mathcal{V}(t)\leqslant&\sum_{k=1}^{m}\eta_k(\theta(t))\Bigg\{-\sum_{\jmath=1}^{p}\Delta_\jmath^k|\alpha_\jmath(t)|_2^2-\sum_{\ell=1}^{q}\Lambda_\ell^k|\beta_\ell(t)|_2^2+\mu\rho(t)+\lambda\rho(t)\\
&-\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\Gamma_{\jmath\ell}^k\int_{t-\pi_{\jmath\ell}}^{t}|\alpha_\ell(s)|_2^2ds-\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\Pi_{\jmath\ell}^k\int_{-\rho_{\jmath\ell}}^{0}\int_{t+s}^{t}|\alpha_\ell(s)|_2^2dsd\sigma\\
&-\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\tilde{\Gamma}_{\ell\jmath}^k\int_{t-\tau_{\ell\jmath}}^{t}|\beta_\jmath(s)|_2^2ds-\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\tilde{\Pi}_{\ell\jmath}^k\int_{-\sigma_{\ell\jmath}}^{0}\int_{t+s}^{t}|\beta_\jmath(s)|_2^2dsd\sigma\Bigg\}\\
&+\Xi\Bigg[\Theta\sum_{\jmath=1}^{p}|\alpha_\jmath(t)|_2^2+\frac{\Theta}{\Delta^k}\sum_{\jmath=1}^{p}\sum_{\ell=1}^{q}\Bigg(\Gamma_{\jmath\ell}^k\int_{t-\pi_{\jmath\ell}}^{t}|\alpha_\jmath(s)|_2^2ds+\Pi_{\jmath\ell}^k\int_{-\rho_{\jmath\ell}}^{0}\int_{t+s}^{t}|\alpha_\jmath(s)|_2^2dsd\sigma\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\tilde{\Theta}\sum_{\ell=1}^{q}|\beta_\ell(t)|_2^2+\frac{\tilde{\Theta}}{\Lambda^k}\sum_{j=1}^{p}\sum_{\ell=1}^{q}\left(\tilde{\Gamma}_{\ell j}^k\int_{t-\tau_{\ell j}}^{t}|\beta_\ell(s)|_2^2ds+\tilde{\Pi}_{\ell j}^k\int_{-\sigma_{\ell j}}^{0}\int_{t+s}^{t}|\beta_\ell(s)|_2^2dsd\sigma\right)\Bigg]\\
\leqslant&\sum_{k=1}^{m}\eta_k(\theta(t))\Bigg\{(\Xi\Theta-\Delta^k)\sum_{j=1}^{p}|\alpha_j(t)|_2^2+(\Xi\tilde{\Theta}-\Lambda^k)\sum_{\ell=1}^{q}|\beta_\ell(t)|_2^2+(\frac{\Xi\Theta}{\Delta^k}-1)\\
&\times\sum_{j=1}^{p}\sum_{\ell=1}^{q}\left(\Gamma_{j\ell}^k\int_{t-\pi_{j\ell}}^{t}|\alpha_j(s)|_2^2ds+\Pi_{j\ell}^k\int_{-\rho_{j\ell}}^{0}\int_{t+s}^{t}|\alpha_j(s)|_2^2dsd\sigma\right)+(\frac{\Xi\tilde{\Theta}}{\Lambda^k}-1)\\
&\times\sum_{j=1}^{p}\sum_{\ell=1}^{q}\left(\tilde{\Gamma}_{\ell j}^k\int_{t-\tau_{\ell j}}^{t}|\beta_\ell(s)|_2^2ds+\tilde{\Pi}_{\ell j}^k\int_{-\sigma_{\ell j}}^{0}\int_{t+s}^{t}|\beta_\ell(s)|_2^2dsd\sigma\right)+(\mu+\lambda)\rho(t)\Bigg\}\\
\leqslant&\sum_{k=1}^{m}\eta_k(\theta(t))(\mu+\lambda)\rho(t)=(\mu+\lambda)\rho(t).
\end{aligned}
$$

Thus, according to Lemma 2.6, drive-response CV-BAMNNs (2.2) and (2.3) will realize GD synchronization under nonlinear feedback controller (3.3), and the rate at which $\alpha_j(t)$ and $\beta_\ell(t)$ converge to zero is Ξ. The proof is completed. □

If there is no distributed delay in system (2.2), then it will degenerate into the following form:

$$
\begin{cases}
\dot{x}_j(t)=\sum\limits_{k=1}^{m}\eta_k(\theta(t))\Big\{-c_j^kx_j(t)+\sum\limits_{\ell=1}^{q}a_{j\ell}^kf_\ell(y_\ell(t))+\sum\limits_{\ell=1}^{q}b_{j\ell}^kf_\ell(y_\ell(t-\tau_{j\ell}(t)))+J_j^k\Big\},\\
\dot{y}_\ell(t)=\sum\limits_{k=1}^{m}\eta_k(\theta(t))\Big\{-\tilde{c}_\ell^ky_\ell(t)+\sum\limits_{j=1}^{p}\tilde{a}_{\ell j}^kg_j(x_j(t))+\sum\limits_{j=1}^{p}\tilde{b}_{\ell j}^kg_j(x_j(t-\pi_{\ell j}(t)))+\tilde{J}_\ell^k\Big\}.
\end{cases}\tag{3.11}
$$

System (3.11) can be regarded as the driving system, and the corresponding response system can be introduced as follows:

$$
\begin{cases}
\dot{v}_j(t)=\sum\limits_{k=1}^{m}\eta_k(\theta(t))\Big\{-c_j^kv_j(t)+\sum\limits_{\ell=1}^{q}a_{j\ell}^kf_\ell(z_\ell(t))+\sum\limits_{\ell=1}^{q}b_{j\ell}^kf_\ell(z_\ell(t-\tau_{j\ell}(t)))+J_j^k+u_j^k(t)\Big\},\\
\dot{z}_\ell(t)=\sum\limits_{k=1}^{m}\eta_k(\theta(t))\Big\{-\tilde{c}_\ell^kz_\ell(t)+\sum\limits_{j=1}^{p}\tilde{a}_{\ell j}^kg_j(v_j(t))+\sum\limits_{j=1}^{p}\tilde{b}_{\ell j}^kg_j(v_j(t-\pi_{\ell j}(t)))+\tilde{J}_\ell^k+\tilde{u}_\ell^k(t)\Big\}.
\end{cases}\tag{3.12}
$$

In this case, the following corollary can be drawn from Theorem 3.1.

Corollary 3.2. *Suppose that the Assumptions 2.1, 2.2, and 2.5 hold true, and the control gains μ_j^k and λ_ℓ^k satisfy the inequality (3.13), then systems (3.11) and (3.12) will realize GD synchronization via controller (3.3),*

$$
\begin{aligned}
\Delta_j^k&\triangleq \boldsymbol{Re}(-c_j^k)-\mu_j^k+\sum_{\ell=1}^{q}\left(\frac{|a_{j\ell}^k|_2L_\ell^f}{2}+\frac{|a_{\ell j}^k|_2L_j^f}{2}+\frac{|b_{j\ell}^k|_2L_\ell^f}{2}+\frac{\Gamma_{j\ell}^k}{1-\pi_{j\ell}^*}+\Gamma_{\ell j}^k\pi_{\ell j}\right)<0,\\
\Lambda_\ell^k&\triangleq \boldsymbol{Re}(-\tilde{c}_\ell^k)-\lambda_\ell^k+\sum_{j=1}^{p}\left(\frac{|\tilde{a}_{\ell j}^k|_2L_j^g}{2}+\frac{|\tilde{a}_{j\ell}^k|_2L_\ell^g}{2}+\frac{|\tilde{b}_{\ell j}^k|_2L_j^g}{2}+\frac{\tilde{\Gamma}_{\ell j}^k}{1-\tau_{\ell j}^*}+\tilde{\Gamma}_{j\ell}^k\tau_{j\ell}\right)<0.
\end{aligned}\tag{3.13}
$$

If there is no delay in drive CV-BAMNNs model (2.2) and (2.3), then they will degenerate into the following forms, respectively,

$$
\begin{cases}
\dot{x}_j(t)=\sum\limits_{k=1}^{m}\eta_k(\theta(t))\Big\{-c_j^kx_j(t)+\sum\limits_{\ell=1}^{q}a_{j\ell}^kf_\ell(y_\ell(t))+J_j^k\Big\},\\
\dot{y}_\ell(t)=\sum\limits_{k=1}^{m}\eta_k(\theta(t))\Big\{-\tilde{c}_\ell^ky_\ell(t)+\sum\limits_{j=1}^{p}\tilde{a}_{\ell j}^kg_j(x_j(t))+\tilde{J}_\ell^k\Big\},
\end{cases}\tag{3.14}
$$

$$\begin{cases} \dot{v}_j(t) = \sum\limits_{k=1}^{m} \eta_k(\theta(t))\Big\{ -c_j^k v_j(t) + \sum\limits_{\ell=1}^{q} a_{j\ell}^k f_\ell(z_\ell(t)) + J_j^k + u_j^k(t) \Big\}, \\ \dot{z}_\ell(t) = \sum\limits_{k=1}^{m} \eta_k(\theta(t))\Big\{ -\tilde{c}_\ell^k z_\ell(t) + \sum\limits_{j=1}^{p} \tilde{a}_{\ell j}^k g_j(v_j(t)) + \tilde{J}_\ell^k + \tilde{u}_\ell^k(t) \Big\}. \end{cases} \tag{3.15}$$

Corollary 3.3. *Suppose that the Assumptions 2.1, 2.2, and 2.5 hold true, and the control gains μ_j^k and λ_ℓ^k satisfy the following inequality (3.16), then systems (3.14) and (3.15) will realize GD synchronization via controller (3.3),*

$$\begin{aligned} \Delta_j^k &\triangleq \boldsymbol{Re}(-c_j^k) - \mu_j^k + \sum_{\ell=1}^{q} \left(\frac{|a_{j\ell}^k|_2 L_\ell^f}{2} + \frac{|a_{\ell j}^k|_2 L_j^f}{2} \right) < 0, \\ \Lambda_\ell^k &\triangleq \boldsymbol{Re}(-\tilde{c}_\ell^k) - \lambda_\ell^k + \sum_{j=1}^{p} \left(\frac{|\tilde{a}_{j\ell}^k|_2 L_\ell^g}{2} + \frac{|\tilde{a}_{\ell j}^k|_2 L_j^g}{2} \right) < 0. \end{aligned} \tag{3.16}$$

Remark 3.4. In recent years, the asymptotic and exponential synchronization issues of various CV-NNs have been widely studied [20]. However, only a few scholars have studied the GD synchronization of CV-NNs. In this paper, we have studied the GD synchronization of T-S fuzzy CV-BAMNNs by designing a new variable-transformed Lyapunov-Krasovskii functional. As mentioned in before, since GD synchronization provides a convenient way to estimate the convergence rate of system synchronization that other methods cannot well estimate the convergence rate of synchronization, the results of this paper have more efficient in terms of estimating the rate at which the synchronization error tends to zero.

Remark 3.5. When $m = 1$, the system (2.2) is reduced to a general CV-BAMNN with mixed time delays. In recent years, considerable research has been conducted on the synchronization and stability of real-valued BAMNNs including GD synchronization [3, 13, 15, 16]. However, there are few studies on the GD synchronization of CV-BAMNNs, not alone GD synchronization of T-S fuzzy BAMNNs in complex domain. Since the T-S fuzzy modeling technique can effectively represent complex systems via fuzzy rules and logic [21], the considered research topic have advantages in term of model complexity and practical applicability.

Remark 3.6. In [7, 9, 29], the GD synchronization fuzzy NNs with mixed-delays was achieved through using a type of complex controller containing two additional linear terms. Because of the some shortcomings used techniques in [7, 9, 29], it was necessary to design a complicated controller to cancel out the time delay term. In this paper, by deigning novel Lyapunov-Krasovskii functional we removed the extra term in the controller of [7, 9, 29] while insuring the GD synchronization of the considered T-S fuzzy CV-BAMNNs at the same time, thus it can effectively reduce the control costs.

Remark 3.7. In recent years, many researchers have studied the synchronization of various fuzzy complex-valued neural networks [2, 35], and have introduced new fuzzy rules in the complex domain to incorporate fuzziness. However, the fuzzy definitions they introduced have some issues, as comparing the magnitudes of complex numbers is not straightforward. To address this, some studies compare the maximum of the real part of the complex numbers, but this method is not entirely valid. Although they have defined some new fuzzy rules in the complex domain, they have not provided practical application cases to demonstrate the effectiveness and rationality of these rules. In contrast, this paper employs the T-S fuzzy rules, which completely avoid the issue of comparing the magnitudes of complex numbers, as T-S fuzzy rules do not involve such comparisons at all.

Remark 3.8. How to effectively analyze the stability of the system is an interesting and important issue. Previous studies used separation methods to separate the real and imaginary parts and discuss the stability conditions for each part [20, 28]. While this approach is intuitive, it increases the computational complexity. To reduce complexity, this paper adopts a non-separation method, constructing the product of a complex number and its conjugate, and using a related Lyapunov function to study stability. However, it is worth exploring whether there are better methods for directly analyzing the stability of complex-valued systems. Although the non-separation method is better than the separation method in terms of reducing complexity, there may still be more efficient and accurate methods for studying the stability and synchronization of the system, which warrants further exploration.

Remark 3.9. In classical Lyapunov stability theory, the stability of a system requires the constructed Lyapunov function to be positive definite, and its time derivative to be semi-negative definite; to achieve asymptotic stability, the derivative needs to be strictly negative definite. However, in the context of GD stability, these conditions are relaxed, we do not require the derivative to be strictly semi-negative definite, but instead only need to satisfy $\dot{\mathcal{V}}(t) \leqslant -\theta\mathcal{V}(t) + \lambda\rho(t)$. When $\rho(t) = 0$, the condition reduces to the classical Lyapunov stability condition. When $\rho(t) > 0$, the condition allows the derivative to be positive at certain moments, as long as the overall trend satisfies the decay condition. This relaxation makes GD stability more general than classical Lyapunov stability, and applicable to more complex systems with time-varying parameters or external disturbances.

4. Numerical simulations

In this section, two numerical examples are given to verify the feasibility of the above theoretical results.

Example 4.1. Consider the T-S fuzzy CV-BAMNNs (2.2) with the following parameters $\mathfrak{m} = 2$, $\eta_1(\upsilon(t)) = \sin^2(t)$, $\eta_2(\upsilon(t)) = \cos^2(t)$, and

$$C^1 = \begin{pmatrix} 0.82 - 2.29i & 0 \\ 0 & 0.63 - 3.19i \end{pmatrix}, \quad A^1 = \begin{pmatrix} 1.09 + 3.17i & -2.91 + 0.21i \\ -3.05 + 3.57i & 1.09 - 2.54i \end{pmatrix},$$

$$B^1 = \begin{pmatrix} 0.63 - 1.89i & 0.21 + 1.47i \\ 1.47 - 0.42i & -1.26 + 1.47i \end{pmatrix}, \quad D^1 = \begin{pmatrix} -3.15 - 2.31i & 5.25 + 0.42i \\ 1.68 + 2.31i & -2.73 - 2.73i \end{pmatrix},$$

$$\tilde{C}^1 = \begin{pmatrix} 0.95 - 2.25i & 0 \\ 0 & 0.46 - 3.00i \end{pmatrix}, \quad \tilde{A}^1 = \begin{pmatrix} -2.94 + 3.59i & 3.17 - 3.61i \\ -2.60 + 10.82i & 2.42 - 2.86i \end{pmatrix},$$

$$\tilde{B}^1 = \begin{pmatrix} 0.84 - 2.52i & 0.42 + 1.68i \\ 0.63 - 0.84i & -0.84 + 1.89i \end{pmatrix}, \quad \tilde{D}^1 = \begin{pmatrix} -3.15 - 2.73i & 5.88 + 1.05i \\ 1.68 + 2.10i & -2.73 - 2.10i \end{pmatrix},$$

$$C^2 = \begin{pmatrix} 2.64 - 2.07i & 0 \\ 0 & 0.57 - 2.89i \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0.79 + 2.87i & -2.63 + 1.90i \\ -2.76 + 3.23i & 0.99 - 2.30i \end{pmatrix},$$

$$B^2 = \begin{pmatrix} 0.57 - 1.71i & 1.90 + 1.33i \\ 1.33 - 0.38i & -1.14 + 1.33i \end{pmatrix}, \quad D^2 = \begin{pmatrix} -2.85 - 2.09i & 4.75 + 0.38i \\ 1.52 + 2.09i & -2.47 - 2.47i \end{pmatrix},$$

$$\tilde{C}^2 = \begin{pmatrix} 0.86 - 2.03i & 0 \\ 0 & 0.42 - 2.72i \end{pmatrix}, \quad \tilde{A}^2 = \begin{pmatrix} -2.66 + 3.25i & 2.87 - 3.27i \\ -2.36 + 9.79i & 2.19 - 2.58i \end{pmatrix},$$

$$\tilde{B}^2 = \begin{pmatrix} 0.76 - 2.28i & 0.38 + 1.52i \\ 0.57 - 0.76i & -0.76 + 1.71i \end{pmatrix}, \quad \tilde{D}^2 = \begin{pmatrix} -2.85 - 2.47i & 5.32 + 0.95i \\ 1.52 + 1.90i & -2.47 - 1.90i \end{pmatrix},$$

$f(z) = g(z) = 0.8(\tanh(\mathbf{Re}(z)) + \tanh(\mathbf{Im}(z))i)$, $\tau_{\jmath\ell} = e^t/(1+e^t)$, $\sigma_{\jmath\ell} = e^t/2(1+e^t)$, $\pi_{\jmath\ell} = e^t/(1+e^t)$, $\rho_{\jmath\ell} = e^t/2(1+e^t)$, $J_\jmath^k = \tilde{J}_\ell^k = 0$. The Matlab simulation of the system (2.2) with initial values $x_1(s) = -0.4 + 0.8i$, $x_2(s) = 0.9 - 0.7i$, $y_1(s) = 0.6 - 0.3i$ and $y_2(s) = -1.2 + 0.8i$ for $s \in [-1, 0]$ are given in Figures 2-3. It can be seen that the drive system (2.2) with above set of parameters have chaotic attractors.

By easy computation we have $L_\ell^f = 0.5$, $L_\jmath^g = 0.5$, $\tau_{\jmath\ell} = \sigma_{\jmath\ell} = \pi_{\jmath\ell} = \rho_{\jmath\ell} = 1$. Therefore, Assumptions 2.1 and 2.2 are satisfied. Letting $\rho(t) = -4.4\frac{e^t}{1+e^t}$ and $\psi(t) = e^t$ in nonlinear feedback controller (3.3), and choosing $\lambda^1 = 5.2$, $\lambda^2 = 7.3$, $\mu^1 = 1.5$, and $\mu^2 = 0.3$, then all the conditions of Theorem 3.1 are satisfied, thus the GD synchronization between systems (2.2) and (2.3) are guaranteed. The time evolution of the synchronization error of the drive-response systems (2.2) and (2.3) are provided in Figure 4. The initial conditions of systems (2.2) and (2.3) are taken as $x_1(s) = 0.2h - 0.4 + (0.8 + 0.15h)i$, $x_2(s) = 0.2h + 0.9 - (0.7 + 0.15h)i$ and $y_1(s) = 0.2h + 0.6 - (0.3 + 0.15h)i$ and $y_2(s) = 0.2h - 1.2 + (0.8 + 0.15h)i$ for $h = \{-3, -2, -1, 0, 1, 2, 3\}$ and $s \in [-1, 0]$. The synchronization curves between systems (2.2) and (2.3) are given in Figures 5-6.

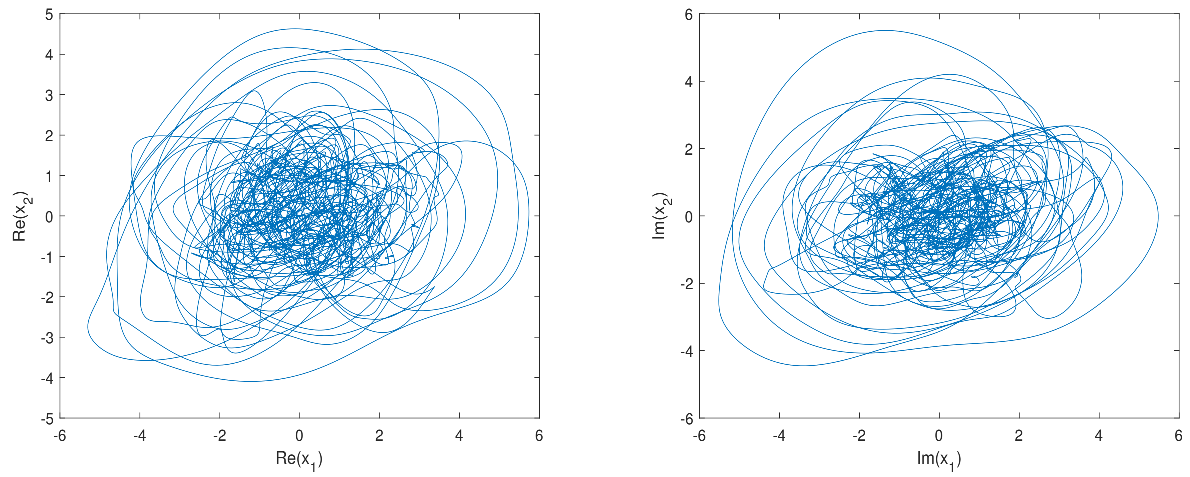

Figure 2: The chaotic attractor of real and imaginary parts of the systems (2.2) and (2.3).

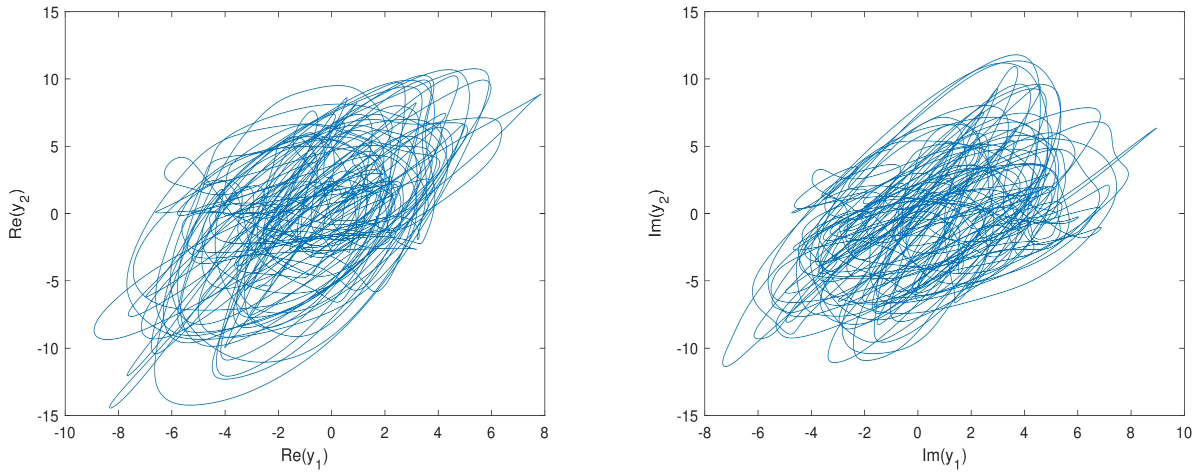

Figure 3: The chaotic attractor of real and imaginary parts of the systems (2.2) and (2.3).

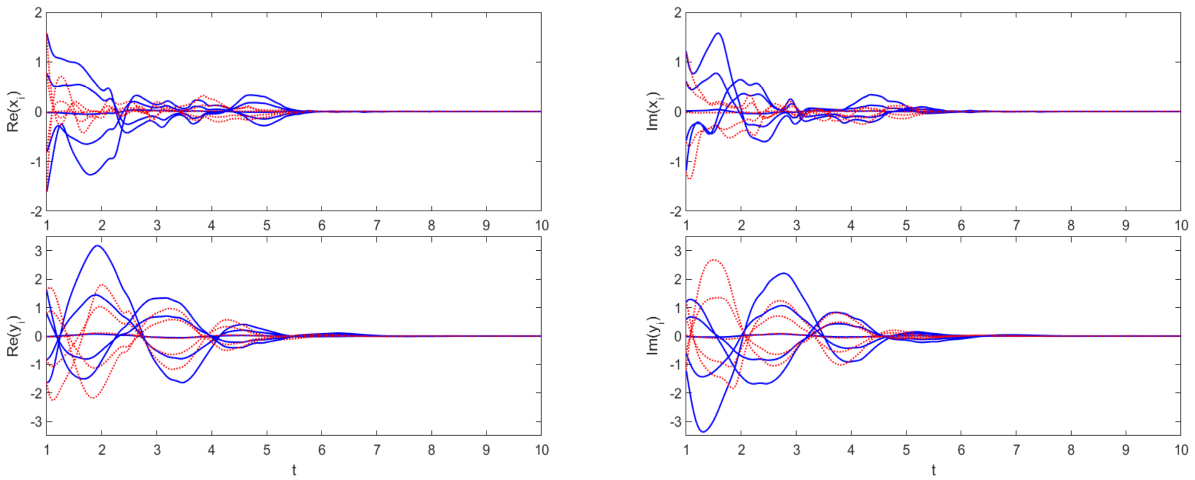

Figure 4: The real and imaginary parts of synchronization error $\alpha_\jmath(t)$ and $\beta_\ell(t)$ estimates of systems (2.2) and (2.3).

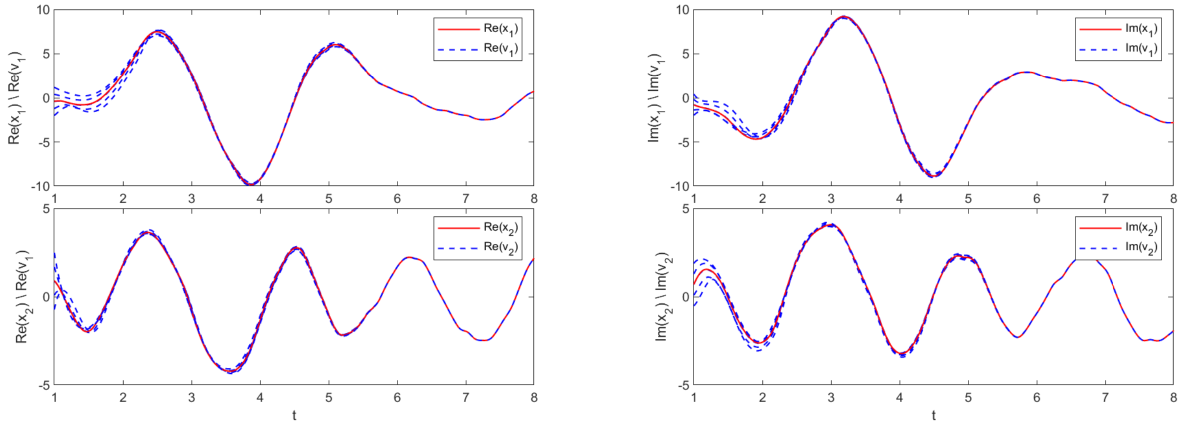

Figure 5: The synchronization curve of the real and imaginary parts of x_1, v_1, x_2, and v_2.

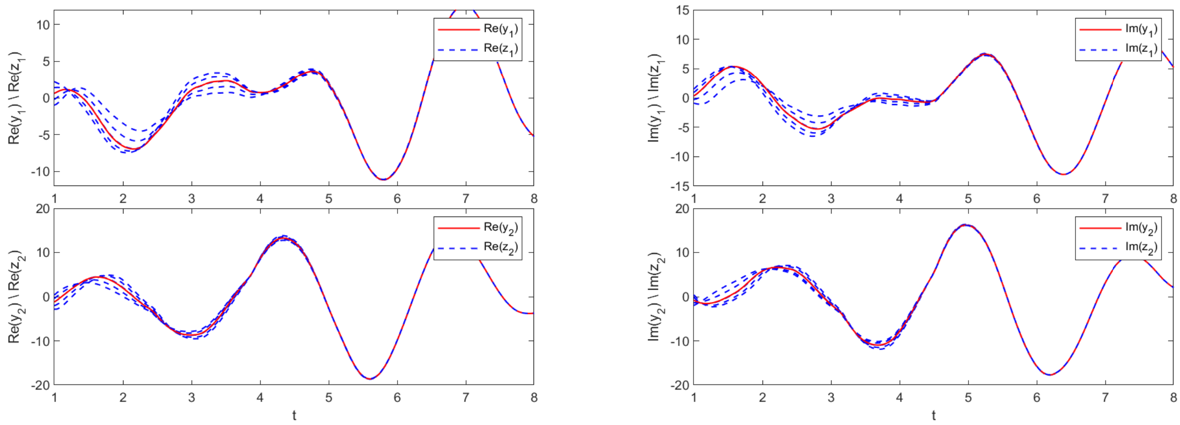

Figure 6: The synchronization curve of the real and imaginary parts of y_1, z_1, y_2, and z_2.

Example 4.2. First choosing two If-Then rules as $\eta_1(\upsilon(t)) = sin^2(t)$, $\eta_2(\upsilon(t)) = cos^2(t)$ and then considering the T-S fuzzy CV-BAMNNs (2.2) with the following set of parameters:

$$C^1 = \begin{pmatrix} 0.82-2.29i & 0 & 0 \\ 0 & 0.63-3.19i & 0 \\ 0 & 0 & 0.63-3.19i \end{pmatrix}, \quad \tilde{C}^1 = \begin{pmatrix} 0.95-2.25i & 0 & 0 \\ 0 & 0.46-3.00i & 0 \\ 0 & 0 & 0.46-3.00i \end{pmatrix},$$

$$A^1 = \begin{pmatrix} 1.09+3.17i & -2.91+0.21i & 2.14+2.96i \\ -3.05+3.57i & 1.09-2.54i & 4.41+1.05i \\ -1.45+1.70i & 0.52-1.21i & 1.89-1.45i \end{pmatrix}, \quad \tilde{A}^1 = \begin{pmatrix} -2.21+4.28i & 2.25-0.44i & 2.73-4.47i \\ -1.45+2.52i & 1.28-2.52i & 4.22-2.35i \\ -2.73+4.41i & 1.13-6.32i & 4.94-2.02i \end{pmatrix},$$

$$B^1 = \begin{pmatrix} -2.94+3.59i & 3.17-3.61i & 2.63+4.22i \\ 1.47-0.42i & -1.26+1.47i & 2.75-1.87i \\ -2.60+10.82i & 2.42-2.86i & 4.10+3.05i \end{pmatrix}, \quad \tilde{B}^1 = \begin{pmatrix} -1.05+2.84i & 4.22-1.18i & 3.44+3.44i \\ -0.42+2.29i & 1.22-4.31i & 4.33-6.74i \\ -6.34+2.25i & 2.16-0.44i & 0.71-3.93i \end{pmatrix},$$

$$D^1 = \begin{pmatrix} -1.41-4.85i & 2.04-6.45i & 2.73+1.07i \\ -1.26+0.63i & 1.09-1.41i & 0.21-1.05i \\ -2.65+0.25i & 2.23-4.22i & 0.50-0.74i \end{pmatrix}, \quad \tilde{D}^1 = \begin{pmatrix} -2.52+3.44i & 2.81-0.48i & 2.81+4.94i \\ -0.74+1.28i & 1.28+0.42i & 2.52-1.68i \\ -0.74+4.62i & 2.31-2.73i & 3.99-1.68i \end{pmatrix},$$

$$C^2 = \begin{pmatrix} 2.64-2.07i & 0 & 0 \\ 0 & 0.48+0.59i & 0 \\ 0 & 0 & 0.57-2.89i \end{pmatrix}, \quad \tilde{C}^2 = \begin{pmatrix} 0.86-2.03i & 0 & 0 \\ 0 & 2.49-0.61i & 0 \\ 0 & 0 & 0.42-2.72i \end{pmatrix},$$

$$A^2 = \begin{pmatrix} -2.03+4.28i & 2.25-0.44i & 2.73-4.47i \\ -1.45+2.52i & 1.28-2.52i & 4.22-2.35i \\ -2.73+4.41i & 1.13-6.32i & 4.94-2.01i \end{pmatrix}, \quad \tilde{A}^2 = \begin{pmatrix} -2.00+3.88i & 2.03-0.40i & 2.47-4.05i \\ -1.31+2.28i & 1.16-2.28i & 3.82-2.13i \\ -2.47+3.99i & 1.03-5.72i & 4.47-1.82i \end{pmatrix},$$

$$B^2 = \begin{pmatrix} -1.05+2.84i & 4.22-1.18i & 3.44+3.44i \\ -0.42+2.29i & 1.22-4.31i & 4.33-6.74i \\ -6.34+2.25i & 2.16-0.44i & 0.71-3.93i \end{pmatrix}, \qquad \tilde{B}^2 = \begin{pmatrix} -0.95+2.57i & 3.82-1.06i & 3.12+3.12i \\ -0.38+2.07i & 1.10-3.90i & 3.91-6.10i \\ -5.74+2.03i & 1.96-0.40i & 0.65-3.55i \end{pmatrix},$$

$$D^2 = \begin{pmatrix} -2.52+3.44i & 2.81-0.48i & 2.81+4.94i \\ -0.74+1.28i & 1.28+0.42i & 2.52-1.68i \\ -0.74+4.62i & 2.31-2.73i & 3.99-1.68i \end{pmatrix}, \qquad \tilde{D}^2 = \begin{pmatrix} 2.28+3.12i & 2.55-0.44i & 2.56+4.47i \\ -0.67+1.16i & 1.16+0.38i & 2.28-1.52i \\ -0.67+4.18i & 2.09-2.47i & 3.61-1.52i \end{pmatrix},$$

$f(z) = g(z) = 0.8(\tanh(\mathbf{Re}(z)) + \tanh(\mathbf{Im}(z))i)$, $\tau_{\jmath\ell} = e^t/(1+e^t)$, $\sigma_{\jmath\ell} = e^t/2(1+e^t)$, $\pi_{\jmath\ell} = e^t/(1+e^t)$, $\rho_{\jmath\ell} = e^t/2(1+e^t)$, $J_\jmath^k = \tilde{J}_\ell^k = 0$.

The Matlab simulation of the system (2.2) under the above set of parameters with initial values $x(s) = [-0.4+0.8i, 0.9-0.7i, 0.3+0.5i]^T$ and $y(s) = [0.6-0.3i, -1.2+0.8i, 0.5-0.2i]^T$ for $s \in [-1, 0]$ is given in Figures 7-8. It can be seen that the drive system (2.2) have chaotic attractors.

By computation, on has $L_\ell^f = 0.5$, $L_\jmath^g = 0.5$, $\tau_{\jmath\ell} = \sigma_{\jmath\ell} = \pi_{\jmath\ell} = \rho_{\jmath\ell} = 1/2$. Therefore, Assumptions 2.1 and 2.2 are satisfied. In (3.3), let $\rho(t) = e^{-8.8t}$, $\psi(t) = e^t$, and choosing $\lambda = 6.8$, $\mu = 8.2$, then all the conditions of Theorem 3.1 are satisfied, and thus the GD synchronization between systems (2.2) and (2.3) are guaranteed. The time evolution of the synchronization error of the drive-response systems (2.2) and (2.3) are provided in Figure 9. The initial conditions of systems (2.2) and (2.3) are taken as $v(s) = [0.2h-0.4+(0.8+0.15h)i, 0.2h-0.7+(0.9+0.15h)i, 0.2h-0.7+(0.9+0.15h)i]^T$, $z(s) = [0.2h+0.8+(0.3+0.15h)i, 0.2h-1.2+(0.8+0.15h)i, 0.2h-1.2+(0.8+0.15h)i]^T$ for $h \in [-6, 6]$. The synchronization curves between systems (2.2) and (2.3) are given in Figures 10-11.

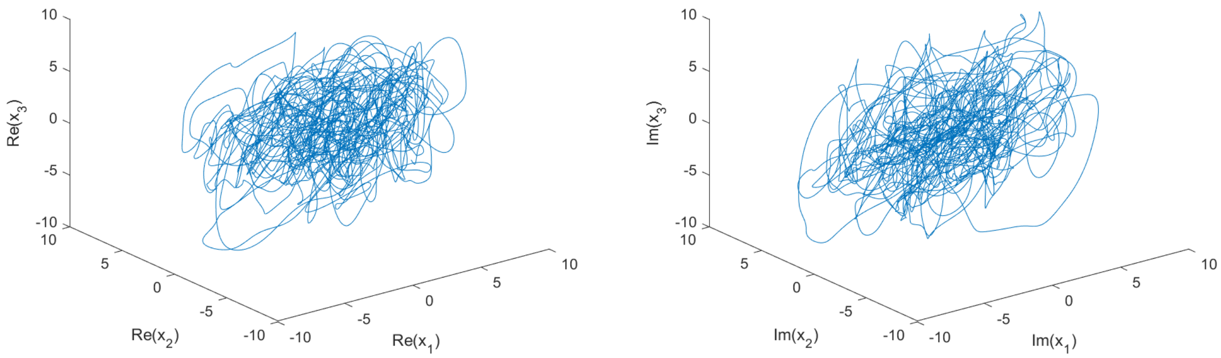

Figure 7: The chaotic attractor of real and imaginary parts of the systems (2.2) and (2.3).

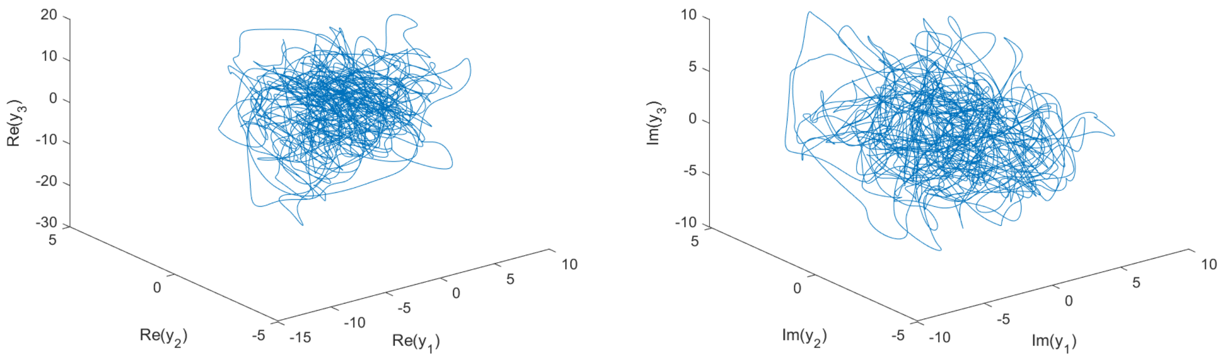

Figure 8: The chaotic attractor of real and imaginary parts of the systems (2.2) and (2.3).

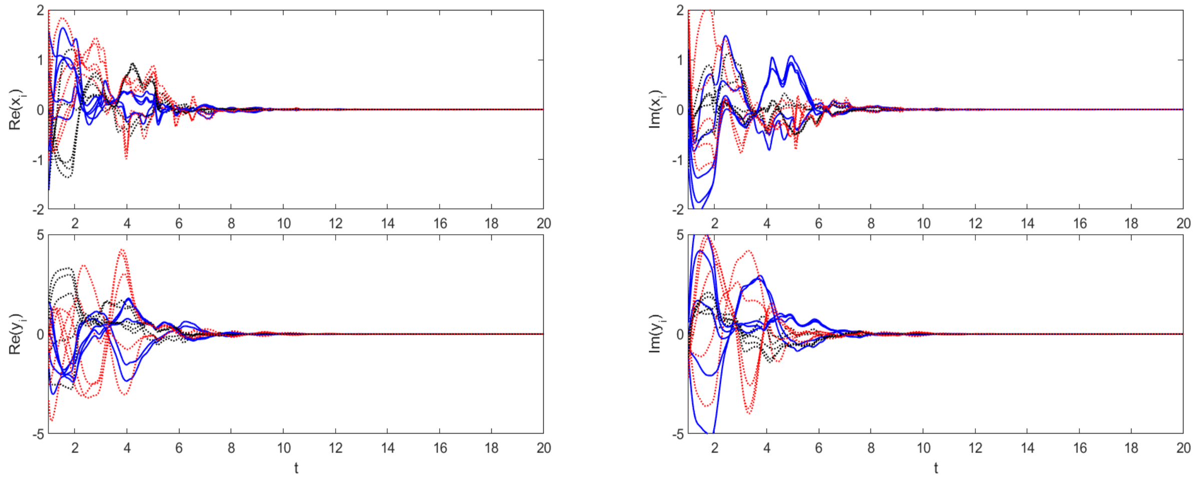

Figure 9: The real and imaginary parts of synchronization error $\alpha_j(t)$ and $\beta_\ell(t)$ estimates of systems (2.2) and (2.3).

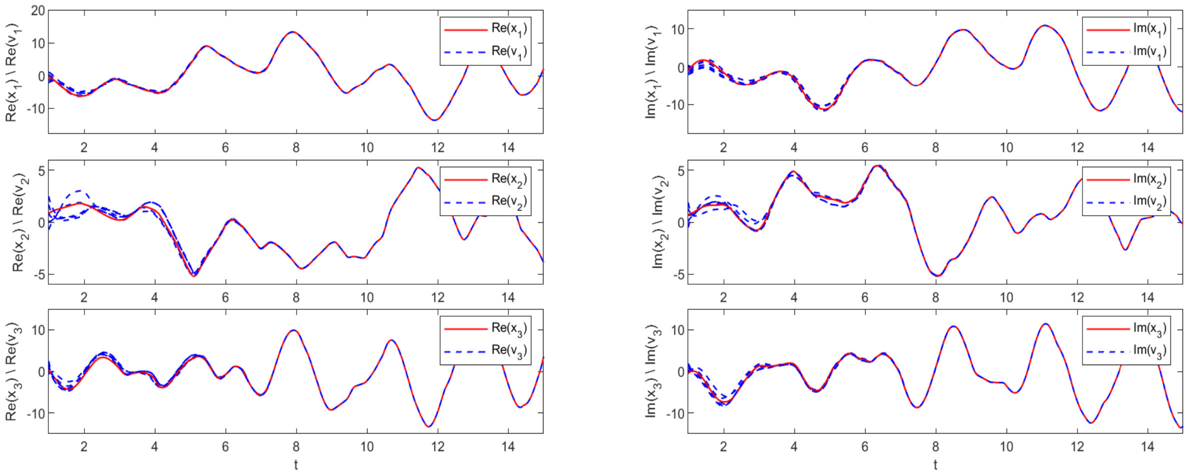

Figure 10: The synchronization curve of the real and imaginary parts of x_1, v_1, x_2, v_2, x_3, and v_3.

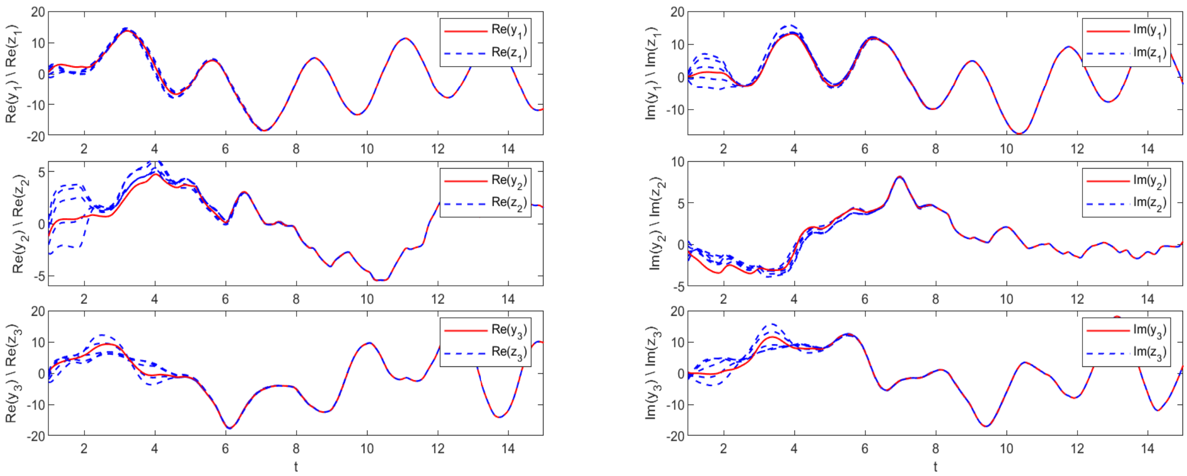

Figure 11: The synchronization curve of the real and imaginary parts of y_1, z_1, y_2, z_2, y_3, and z_3.

5. Conclusion

This paper studied the GD synchronization problem of a class of T-S fuzzy CV-BAMNNs with mixed time delays. Inspired by [6], we use a nonseparation method to discuss the GD synchronization of CV-NNs directly. Also, to dealing with time delay terms, we designed a type of novel Lyapunov-Krasovskii functional to offset the extra term caused by the time delay and reduce the control cost. Finally, the feasibility of the results was verified through numerical simulations. When the number of T-S fuzzy rules is one, the results of this study can be extended to the GD synchronization of CV-BAMNNs with mixed delays. To the best our knowledge, the results in this paper are the first one to consider the GD synchronization of T-S fuzzy CV-BAMNNs by nonseparation method by combining the three main factors such as T-S fuzzy logic, complex-variable and mixed delays.

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