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A NOTE ON COMPACT OPERATORS VIA ORTHOGONALITY

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Abstract

In this paper, we extend the usual notion of orthogonality to Banach spaces. Also, we establish a characterization of compact operators on Banach spaces that admit orthonormal Schauder bases.
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Key Words: Orthogonality, compact operator.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper K is the field of real or complex numbers, E is a Banach space over K and norm denoted by $\|.\|$, and $(x_n) = (x_n)_{n=1}^{N} = (x_n)_{n\in L}$ is a finite or infinite sequence in E, where either N is a positive integer and $L = \{1, 2, ..., N\}$ or $N = \infty$ and $L = \{1, 2, ...\}$. For $J(\neq \emptyset) \subset L$, the closure of the span of the set$\{x_n : n \in J\}$ is denoted by $[x_n : n \in J]$.
The reader is referred to [2] for undefined terms and notation.
The notion of orthogonality goes a long way back in time. Usually this notion is associated with Hilbert spaces or, more generally, inner product spaces. Various extensions have been introduced through the decades. Thus, for instance, x is orthogonal to y in E
(a) In the sense of (G. Birkhoff [1]) if for every $\alpha \in K$

$$\|x + \alpha y\| \geq \|x\|;$$

(b) In the sense of (B. D. Roberts [5]) if for every $\alpha \in K$

$$\|x + \alpha y\| = \|x - \alpha y\|;$$

(c) In the isosceles sense (R. C. James [4]) if

$$\|x + y\| = \|x - y\|;$$

(d) In the Pythagorean sense (R. C. James [4]) if

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2;$$

(e) In the sense of (I. Singer [7]) if

$$\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\| = \|\frac{x}{\|x\|} - \frac{y}{\|y\|}\|.$$

One of the natural and simple properties of orthogonality in a Hilbert space H that one would like to hold true in a Banach space is that x is orthogonal to y in H if and only if

(1.1) $\quad \|x + \lambda_1 y\| = \|x + \lambda_2 y\|, \qquad for\ all\ \lambda_1, \lambda_2 \in K, |\lambda_1| = |\lambda_2|$

Clearly, in any Banach space, Eq. (1.1) is equivalent to

(1.2) $\quad \|\lambda x + \mu y\| = \||\lambda| x + |\mu| y\|, \qquad for\ all\ \lambda, \mu \in K$

Hence, we introduce the following definition:

Definition 1. A finite or infinite sequence $(x_n)_{n\in L}$ in a Banach space E is said to be orthogonal if

(1.6) $\quad \|\sum_{n\in L} a_n x_n\| = \|\sum_{n\in L} |a_n| x_n\|, \qquad for\ each\ \sum_{n\in L} a_n x_n \in E.$

If in addition $\|x_n\| = 1$ for all $n \in L$ then $(x_n)_{n\in L}$ is said to be orthonormal. We write $x \perp y$ if x is orthogonal to y.

It is clear from the definition that $(x_n)_{n\in L}$ is orthogonal in E if and only if $(x_n)_{n\in L}$ is orthogonal in $[x_n : n \in L]$.

Note that Definition 1 is an extension of the usual notion of orthogonality since in a Hilbert space H , $x \perp y$ in our sense if and only if $\langle x, y\rangle = 0$, where $\langle ., .\rangle$ denotes the inner product in H.

Theorem 2. [6] Given a sequence $(x_n)_{n\in L}$ in E, the following are equivalent:

(i) The sequence $(x_n)_{n\in L}$ is orthogonal in E.

(ii) For each pair of sequences $(b_n)_{n\in L}$and $(c_n)_{n\in L}$ in K satisfying $|b_n| = |c_n|$ for all $n \in L$, $\sum_{n\in L} c_n x_n$ converges if and only if $\sum_{n\in L} b_n x_n$ converges and if both converge,

$$\|\sum_{n\in L} b_n x_n\| = \|\sum_{n\in L} c_n x_n\|$$

2. CHARACTERIZATION OF COMPACT OPERATORS

Let $L(F, E)$ denote the set of bounded linear operators from the normed space F into the Banach space E. It is known that if F and E are Hilbert spaces, then $T \in L(F, E)$ is compact, if and only if, T is the limit in $L(F, E)$ of a sequence of finite-rank operators [2]. This gives a convenient and practical characterization of compact operators in Hilbert spaces. We show here that the same characterization still holds true for any Banach space E that admits an orthonormal Schauder basis and any normed space F. More precisely, we have:

Theorem 3. Suppose that $\{e_n\}_{n=1}^{\infty}$ is an orthonormal Schauder basis of the Banach space E and that F is a normed space. For each positive integer k, let P_k be the projection on $[e_n : 1 \leq n \leq k]$ defined by

$$P_k(\sum_{n=1}^{\infty} \alpha_n e_n) = \sum_{n=1}^{k} \alpha_n e_n, \qquad \sum_{n=1}^{\infty} \alpha_n e_n \in E.$$

Then, an operator $T \in L(F, E)$ is compact, if and only if, $P_k \circ T$ converges to T in $L(F, E)$.

proof . The sufficiency part follows from the fact that for every Banach space E and every normed space F, the limit in $L(F, E)$ of a sequence of finite-rank operators is a compact operator [3].

Now, suppose that $T \in L(F, E)$ is compact. For each positive integer k, let $T_k = P_k \circ T$. Note that since $\{e_n\}_{n=1}^{\infty}$is orthonormal, it follows by Theorem 2 that $P_k \in L(E)$ and $\|P_k\| = 1$ for all k. Clearly we have, since $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis of E,

$$\lim_{k\to\infty} P_k(y) = y, \qquad for\ each\ y \in E$$

Let B be the closed unit ball in F. Since T is compact, it follows that $K = cl(T(B))$ is a compact subset of E. We need to show that

$$\lim_{k\to\infty} \sup_{x\in B} \|T_k(x) - T(x)\| = 0.$$

Suppose this is not true. Then there exist $\varepsilon > 0$, a subsequence $\{T_{k_j}\}$, and a sequence $\{x_{k_j}\}$ in B such that

$$(*) \qquad \|T_{k_j}(x_{k_j}) - T(x_{k_j})\| > \varepsilon, \qquad for\ all\ j$$

Since K is compact, there exists a subsequence of $\{x_{k_j}\}$, say $\{x_{k_j}\}$, such that the sequence $\{T(x_{k_j})\}$converges in K to some $y \in K$. Then we have, since $\|P_{k_j}\| = 1$ for all j,

$$\begin{aligned} \|T_{k_j}(x_{k_j}) - T(x_{k_j})\| &\leq \|P_{k_j}(T(x_{k_j})) - P_{k_j}(y)\| + \|T(x_{k_j}) - P_{k_j}(y)\| \\ &\leq \|T(x_{k_j}) - y\| + \|T(x_{k_j}) - P_{k_j}(y)\| \end{aligned}$$

Letting $j \to \infty$, we obtain, since $\{T(x_{k_j})\}$ and $\{P_{k_j}(y)\}$ both converge to y, that

$$\lim_{j\to\infty} \|T_{k_j}(x_{k_j}) - T(x_{k_j})\| = 0,$$

which contradicts $(*)$.
As a corollary we have,
Corollary 4. If E is a Banach space that admits an orthonormal Schauder basis and F is a normed space, then an operator $T \in L(F, E)$ is compact if and only if it is the limit in $L(F, E)$ of a sequence of finite-rank operators.

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