J. Math. Computer Sci., 39 (2025), 418–436

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Asymptotic and oscillatory characteristics of solutions of neutral differential equations

Fahd Masood[a], Belal Batiha[b,*], Omar Bazighifan[c,d,*]

[a]Department of Mathematics, Faculty of Education and Science, University of Saba Region, Marib, Yemen.

[b]Mathematics Department, Faculty of Science and Information Technology, Jadara University, Irbid 21110, Jordan.

[c]Department of Mathematics, Faculty of Education, Seiyun University, Hadhramout, Yemen.

[d]Jadara Research Center, Jadara University, Irbid 21110, Jordan.

Abstract

The paper investigates third-order linear neutral differential equations in the non-canonical case, aiming to simplify the complexity of such equations by transforming them into the canonical form. This transformation reduces the number of potential cases for positive solutions and their derivatives from four in the non-canonical case to two in the canonical case, significantly facilitating the derivation of results. Using an iterative method, we establish conditions that exclude the existence of positive solutions fulfilling the equation. Furthermore, by employing a comparison approach with first-order equations, we derive additional conditions that exclude the existence of Kneser-type solutions that satisfy the equation. By combining these conditions, we derive new oscillation criteria that guarantee the oscillation of all solutions satisfying the studied equation. Our findings extend and generalize existing results in the literature. We provide three illustrative examples to demonstrate our results' validity and significance.

Keywords: Oscillatory, nonoscillatory, neutral differential equations, third-order, noncanonical case.

2020 MSC: 34C10, 34K11.

1. Introduction

In this study, we explore the oscillatory properties of a third-order linear neutral differential equation, expressed as

$$\left(\kappa_2(s)\left(\kappa_1(s)\left(\varkappa(s)+p(s)\varkappa(\mathfrak{r}(s))\right)'\right)'\right)'+\mathfrak{q}(s)\varkappa(\mathfrak{y}(s))=0,\ s\geqslant s_0. \tag{1.1}$$

Our analysis is based on the following assumptions.

(A$_1$) $\mathfrak{r},\mathfrak{y}\in C^1([s_0,\infty),\mathbb{R})$, $\mathfrak{y}(s)\leqslant s$, $\mathfrak{y}'(s)>0$, $\left(\mathfrak{y}^{-1}(s)\right)'\geqslant \mathfrak{y}_0>0$, $\mathfrak{r}'(s)\geqslant \mathfrak{r}_0>0$, $\mathfrak{r}(s)\leqslant s$, $\mathfrak{r}\circ\mathfrak{y}=\mathfrak{y}\circ\mathfrak{r}$, and $\lim_{s\to\infty}\mathfrak{r}(s)=\lim_{s\to\infty}\mathfrak{y}(s)=\infty$;

(A$_2$) $p,\mathfrak{q}\in C([s_0,\infty),(0,\infty))$, $0\leqslant p(s)\leqslant p_0<\infty$, and $\mathfrak{q}(s)$ does not vanish identically;

*Corresponding author

Email addresses: b.bateha@jadara.edu.jo (Belal Batiha), o.bazighifan@gmail.com (Omar Bazighifan)

doi: 10.22436/jmcs.039.04.02

Received: 2024-10-26 *Revised:* 2024-11-12 *Accepted:* 2025-02-28

(A_3) $\kappa_1 \in C^2([s_0,\infty),(0,\infty))$, $\kappa_2 \in C^1([s_0,\infty),(0,\infty))$, and (1.1) is in noncanonical case, that is

$$\int_{s_0}^{\infty} \frac{1}{\kappa_1(\nu)} d\nu < \infty \ \text{ and } \ \int_{s_0}^{\infty} \frac{1}{\kappa_2(\nu)} d\nu < \infty.$$

A function $\varkappa \in C^3([S_\varkappa,\infty),\mathbb{R})$, $S_\varkappa \geqslant s_0$, is said to be a solution of (1.1) which has the property $\kappa_2(\kappa_1 y')' \in C^1[S_\varkappa,\infty)$, and it satisfies the equation (1.1) for all $s \in [S_\varkappa,\infty)$. We consider only those solutions $\varkappa$ of (1.1), which exist on some half-line $[S_\varkappa,\infty)$ and satisfy the condition

$$\sup\{|\varkappa(s)| : s \geqslant S\} > 0, \text{ for all } S \geqslant S_\varkappa.$$

For sake of simplicity, we define the operators

$$L_0 y = y,\ \ L_1 y = \kappa_1 y',\ \ L_2 y = \kappa_2(\kappa_1 y')',\ \text{ and }\ L_3 y = \left(\kappa_2(\kappa_1 y')'\right)'.$$

A solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if all of its solutions are oscillatory.

Differential equations (DEs) play a crucial role in modeling dynamic systems across various scientific and engineering fields. They describe the relationship between a function and its derivatives, providing insight into how systems evolve over time. The significance of DEs lies in their ability to represent a wide range of phenomena, from the motion of particles in physics to population dynamics in biology. As foundational tools in mathematics, they enable researchers to analyze and predict the behavior of complex systems, making them indispensable in both theoretical and applied contexts. With advancements in computational methods and numerical analysis, the study of DEs continues to evolve, allowing for more accurate simulations and a deeper understanding of real-world applications (see [16, 23, 33]).

Neutral differential equations (NDEs) are a specific category of DEs that incorporate both derivative terms and delayed arguments of the dependent variable. This unique structure allows them to model systems where the current state is influenced not only by immediate conditions but also by past states. The presence of delays complicates the analysis, as it introduces additional dynamics that can lead to phenomena such as oscillations and stability issues. Research in NDEs has garnered attention due to their relevance in various fields, including control theory, ecology, and engineering, where time delays are often inherent in system behaviors. Their study offers valuable insights into the stability and oscillatory characteristics of systems, enhancing our understanding of time-dependent processes (see [1, 3, 7, 9, 10, 15, 20, 26, 30, 32]).

Oscillatory theorems provide a framework for understanding the oscillatory behavior of solutions to DEs. These theorems establish criteria under which solutions exhibit oscillations, which are critical for analyzing the stability and long-term behavior of dynamical systems. By identifying conditions that lead to oscillations, researchers can predict system responses and design effective control strategies in engineering applications. The development of oscillatory theorems has significantly advanced the field of DEs, allowing for a deeper exploration of the intricate interplay between system parameters and solution behavior. This area of study continues to be pivotal in both pure mathematics and applied sciences, providing essential tools for analyzing the complexities of oscillatory phenomena (see [2, 8, 22, 39]).

In recent years, the study of third-order NDEs has gained significant attention, especially regarding their oscillatory and asymptotic properties. Dzurina [17] established foundational results on the asymptotic behavior of third-order DDEs, which were further developed by Parhi and Padhi [35], who provided critical insights into the long-term behavior of solutions. Baculikova and Dzurina [5] made significant contributions by analyzing the oscillation of third-order NDEs. They also examined the asymptotic properties of specific classes of third-order nonlinear NDEs [6]. Candan [12] expanded this research by investigating the asymptotic properties of third-order nonlinear NDEs. Chatzarakis et al. [14] broadened the scope of existing findings by studying the oscillatory and asymptotic properties of third-order quasilinear DDEs.

Similarly, Li et al. [27] presented general criteria for the oscillation of third-order NDEs, while Elabbasy et al. [19] developed oscillation criteria for third-order DDEs. Moaaz et al. [29] explored the asymptotic behavior of third-order nonlinear mixed-type NDEs. Most recently, Masood et al. [28, 31] introduced new criteria for assessing the asymptotic and oscillatory behavior of solutions to a class of third-order FDEs, enhancing the understanding of Kneser-type solutions.

Below, we review some notable results from recent research that have contributed to the progress of the study of third-order differential equations.

Bohner et al. [11] made significant advancements by extending Hanan's Kneser-type oscillation criterion, which was initially designed for ODEs expressed as

$$\varkappa'''(s) + \mathfrak{q}(s)\varkappa(s) = 0,$$

to encompass DDEs of the form

$$\varkappa'''(s) + \mathfrak{q}(s)\varkappa(\mathfrak{y}(s)) = 0.$$

This extension was proven to maintain sharpness when applied to the delay Euler DEs

$$\varkappa'''(s) + \frac{\mathfrak{q}_0}{s^3}\varkappa(\lambda s) = 0,\ \lambda \in (0,1),\ s \geqslant 1.$$

Jadlovská et al. [24] established effective oscillation criteria for third-order delay differential equations of the form

$$\left(\kappa_2(s)\left(\kappa_1(s)\varkappa'(s)\right)'\right)' + \mathfrak{q}(s)\varkappa(\mathfrak{y}(s)) = 0,$$

under the canonical case, characterized by

$$\int_{s_0}^{\infty}\frac{1}{\kappa_1(\nu)}d\nu = \infty \quad \text{and} \quad \int_{s_0}^{\infty}\frac{1}{\kappa_2(\nu)}d\nu = \infty.$$

Their results ensure that any nonoscillatory solution asymptotically approaches zero and is shown to be sharp when applied to Euler-type delay differential equations. Further contributions were made by Chatzarakis et al. [13] who introduced new criteria aimed at evaluating the oscillatory behavior of third-order NDEs, specifically in

$$\left(\varkappa(s) + p(s)\varkappa(\mathfrak{r}(s))\right)''' + \mathfrak{q}(s)\varkappa^{\alpha}(\mathfrak{y}(s)) = 0.$$

They developed rigorous conditions demonstrating the nonexistence of Kneser-type solutions in these equations, pushing the boundaries of oscillation theory for NDEs. In a separate study, Saker [38] utilized Riccati transformation techniques to establish conditions that guarantee oscillation for every solution of the NDEs represented by

$$\left(\kappa_2(s)\left(\kappa_1(s)\left(\varkappa(s) + p(s)\varkappa(\mathfrak{r}(s))\right)'\right)'\right)' + \mathfrak{q}(s)f\left(\varkappa(s-\mathfrak{y})\right) = 0.$$

Grace et al. [21] also made strides in the non-canonical context by developing criteria for oscillation in third-order delay differential equations, notably those of the form

$$\left(\kappa_2(s)\left(\kappa_1(s)\varkappa'(s)\right)'\right)' + \mathfrak{q}(s)\varkappa(\mathfrak{y}(s)) = 0. \tag{1.2}$$

Baculıkova [4] extended these criteria by employing an appropriate substitution to transform (1.2) into a canonical form, which facilitated the introduction of new oscillation criteria and offered a deeper insight into oscillatory behavior in non-canonical equations. Nithyakala et al. [34] investigated the nonexistence of Kneser-type solutions for third-order noncanonical NDEs (1.1) using Myshkis-type criteria. Complementing this work, Purushothaman et al. [37] focused on the existence and bounds of Kneser-type solutions in noncanonical third-order NDEs (1.1) with $\mathfrak{y}(s) = s$.

This research aims to build upon the foundational studies by deriving new oscillation criteria for a specific class of third-order NDEs. By transforming the investigated equations from the noncanonical form into a canonical form and extending the results of previous studies, particularly those of [4, 34, 37], this work introduces significant advancements in differential equation oscillation theory. Employing the iterative method as presented in [24] and the comparison method with first-order equations, the results demonstrate consistency with existing findings while offering novel contributions. These contributions highlight the originality of this study in addressing unresolved aspects of oscillation theory for third-order NDEs.

2. Auxiliary results

In this section, we introduce a series of lemmas and assumptions that will be essential for the analysis in this paper and to streamline the mathematical processes. To facilitate clarity and brevity, we also define the following notations:

$$
\begin{aligned}
y(s) &:= \varkappa(s) + p(s)\varkappa(\mathfrak{r}(s)),\\
\pi_1(s) &:= \int_s^\infty \frac{1}{\kappa_1(\nu)}d\nu,\ \pi_2(s) := \int_s^\infty \frac{1}{\kappa_2(\nu)}d\nu,\ \pi_{12}(s) := \int_s^\infty \frac{\pi_2(\nu)}{\kappa_1(\nu)}d\nu,\ \pi_*(s) := \int_s^\infty \frac{\pi_1(\nu)}{\kappa_2(\nu)}d\nu,\\
a_1(s) &:= \frac{\kappa_1(s)\pi_{12}^2(s)}{\pi_*(s)},\ a_2(s) := \frac{\kappa_2(s)\pi_*^2(s)}{\pi_{12}(s)},\\
\mu_1(s) &:= \int_{s_0}^s \frac{1}{a_1(\nu)}d\nu,\ \mu_2(s) := \int_{s_0}^s \frac{1}{a_2(\nu)}d\nu,\ \mu_{12}(s) := \int_{s_0}^s \frac{\mu_2(\nu)}{a_1(\nu)}d\nu,\\
\mu_1(\sigma,\rho) &:= \int_\rho^\sigma \frac{1}{a_1(\nu)}d\nu,\ \mu_2(\sigma,\rho) := \int_\rho^\sigma \frac{1}{a_2(\nu)}d\nu,\ \mu_{12}(\sigma,\rho) := \int_\rho^\sigma \frac{\mu_2(\nu,\rho)}{a_1(\nu)}d\nu,\\
g(s) &:= \pi_*(s)q(s),\ g_1(s) := g(s)(1-p(\eta(s)))\pi_{12}(\eta(s)),\\
\widetilde{g}(s) &:= \min\{g(s), g(\mathfrak{r}(s))\},\ g_2(s) := \widetilde{g}(s)\pi_{12}(\eta(s)),\\
\lambda_* &:= \liminf_{s\to\infty} \frac{\mu_{12}(s)}{\mu_{12}(\eta(s))},\ \gamma_* := \liminf_{s\to\infty} a_2(s)\mu_{12}(\eta(s))\mu_2(s)g_1(s),
\end{aligned}
$$

and

$$
k_* := \liminf_{s\to\infty} \frac{\mu_2^{\gamma_*}(s)\int_{s_0}^s \frac{\mu_2^{1-\gamma_*}(\nu)}{a_1(\nu)}d\nu}{\mu_{12}(s)},\ \text{for } \kappa\ \gamma_* \in (0,1).
$$

Lemma 2.1 ([25]). *Let $w \in \mathbf{C}^m([s_0,\infty),(0,\infty))$, $w^{(i)}(s) > 0$ for $i = 1, 2, \ldots, m$, and $w^{(m+1)}(s) \leqslant 0$, eventually. Then, eventually,*

$$
\frac{w(s)}{w'(s)} \geqslant \frac{\epsilon}{m}s,
$$

for every $\epsilon \in (0,1)$.

Lemma 2.2 ([40]). *Let α be a ratio of two odd positive integers, $A > 0$ and B are constants. Then*

$$
Bw - Aw^{(\alpha+1)/\alpha} \leqslant \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^\alpha},\ A > 0.
$$

Lemma 2.3 ([18]). *Assume that $\varkappa(s)$ is an eventually positive solution of equation (1.1). Then there exists $s_1 \geqslant s_0$ such that the associated function y can be classified into one of the following cases:*

Case (C_0): $y(s) > 0,\ L_1y(s) < 0,\ L_2y(s) > 0,\ L_3y(s) < 0$;
Case (C_1): $y(s) > 0,\ L_1y(s) < 0,\ L_2y(s) < 0,\ L_3y(s) < 0$;
Case (C_3): $y(s) > 0,\ L_1y(s) > 0,\ L_2y(s) > 0,\ L_3y(s) < 0$;

Case (C_4): $y(s) > 0,\ L_1 y(s) > 0,\ L_2 y(s) < 0,\ L_3 y(s) < 0,$

for $s \geqslant s_1$.

To establish conditions for the oscillation of equation (1.1), we must eliminate all four of these cases. Moreover, for the nonexistence of Kneser-type solutions, Cases (C_0) and (C_1) must be ruled out. However, when equation (1.1) is transformed into its canonical form, the number of classes for non-oscillatory solutions is reduced from four to two, significantly simplifying the analysis.

We begin by transforming the original equation into an equivalent canonical form, as established in [34]. The transformed equation is expressed as follows:

$$\left(a_2(s)\left(a_1(s)\left(\frac{y(s)}{\pi_{12}(s)}\right)'\right)'\right)' + \pi_*(s)\, q(s) \varkappa(\eta(s)) = 0, \tag{2.1}$$

with

$$\int_{s_0}^{\infty} \frac{1}{a_1(\nu)} d\nu = \infty \quad \text{and} \quad \int_{s_0}^{\infty} \frac{1}{a_2(\nu)} d\nu = \infty.$$

By introducing the substitution $w(s) = \frac{y(s)}{\pi_{12}(s)}$, and defining $g(s) := \pi_*(s)\, q(s)$, we can simplify the equation (2.1) to:

$$\left(a_2(s)\left(a_1(s)\, w'(s)\right)'\right)' + g(s)\varkappa(\eta(s)) = 0. \tag{2.2}$$

This transformation indicates that the solution of the noncanonical equation is equivalent to the solution of the canonical form. Next, we define

$$D_0 w = w,\ D_1 w = a_1 w',\ D_2 w = a_2\left(\left(a_1 w'\right)'\right),\ \text{and } D_3 w = \left(a_2\left(\left(a_1 w'\right)'\right)\right)'.$$

We can now conclude that the noncanonical NDE (1.1) has an eventually positive solution if and only if the canonical NDE (2.2) exhibits the same behavior. This simplifies the examination to two distinct classes of eventually positive solutions.

Lemma 2.4 ([24])**.** *Assume that* $\varkappa(s)$ *is an eventually positive solution of equation* (2.2)*. Then there exists* $s_1 \geqslant s_0$ *such that* w *is one of the following cases:*

Case (N_0): $w(s) > 0,\ D_1 w(s) < 0,\ D_2 w(s) > 0,\ D_3 w(s) < 0$;
Case (N_2): $w(s) > 0,\ D_1 w(s) > 0,\ D_2 w(s) > 0,\ D_3 w(s) < 0$.

Definition 2.5. We denote the sets N_0 and N_2 as the collections of eventually positive solutions satisfying the conditions of Cases (N_0) and (N_2), respectively.

Definition 2.6. A solution $\varkappa$ for which the associated function $w \in N_0$ is referred to as a Kneser-type solution.

3. Investigating the nonexistence of N_2-type solutions

In this section, we present a series of lemmas to analyze the asymptotic properties of solutions within the class N_2. To derive these results effectively, we first outline a set of fundamental assumptions, which are based on and extend the framework presented in [24], as follows.

All results presented depend on the positivity of γ_*. Considering $\gamma \in (0, \gamma_*)$ and $\lambda \in (1, \lambda_*)$, where ($\lambda_* > 1$) or $\lambda = \lambda_*$, when $\lambda_* = 1$, there exists a sufficiently large $s_1 \geqslant s_0$ that satisfies the following conditions

$$\frac{\mu_{12}(s)}{\mu_{12}(\eta(s))} \geqslant \lambda, \tag{3.1}$$

$$a_2(s)\,\mu_{12}(\eta(s))\mu_2(s)g_1(s) \geqslant \gamma, \tag{3.2}$$

$$\frac{\mu_2^{\gamma}(s)\int_{s_0}^{s}\frac{\mu_2^{1-\gamma}(\nu)}{a_1(\nu)}d\nu}{\mu_{12}(s)} \geqslant k. \tag{3.3}$$

To define the sequence $\{\gamma_n\}_{n=0}$ under the conditions $\gamma_* \in (0,1)$, $k_* \in [1,\infty)$ and $\lambda_* \in [1,\infty)$ the following sequence $\{\gamma_n\}_{n=0}$ starts with $\gamma_0 = \gamma_*$. For $n \in \mathbb{N}$ the sequence is defined recursively as

$$\gamma_n = \frac{\gamma_0 k_{n-1}\lambda_*^{1-1/k_{n-1}}}{1-\gamma_{n-1}},$$

where k_n is defined as

$$k_n = \liminf_{s\to\infty}\frac{\mu_2^{\gamma_n}(s)\int_{s_0}^{s}\frac{\mu_2^{1-\gamma_n}(\nu)}{a_1(\nu)}d\nu}{\mu_{12}(s)},\ n \in \mathbb{N}_0.$$

The existence of γ_{n+1} is guaranteed if $\gamma_i < 1$ and $k_i \in [1,\infty)$ for all $i = 0,1,\dots,n$. As a consequence, we find that

$$\frac{\gamma_1}{\gamma_0} = \frac{k_0\lambda^{1-1/k_0}}{1-\gamma_0} > 1.$$

Additionally, we express $k_1 \geqslant k_0$. By induction on n, it can be shown that

$$\frac{\gamma_{n+1}}{\gamma_n} \geqslant \ell_n > 1,$$

where

$$\ell_0 = \frac{k_0\lambda_*^{1-1/k_{n-1}}}{1-\gamma_0},$$

and

$$\ell_n = \frac{k_n\lambda_*^{1/k_{n-1}-1/k_n}(1-\gamma_{n-1})}{k_{n-1}(1-\gamma_n)}. \tag{3.4}$$

Thus, we conclude that $k_n \geqslant k_{n-1}$ for all n.

Lemma 3.1. *Assume that $\varkappa \in N_2$, then equation (2.2) implies*

$$D_3 w(s) + g_1(s)\,w(\eta(s)) \leqslant 0. \tag{3.5}$$

Proof. Consider

$$y(s) = \varkappa(s) + p(s)\,\varkappa(\mathfrak{r}(s)),$$

which leads to $y(s) \geqslant \varkappa(s)$ and

$$\varkappa(s) = y(s) - p(s)\,\varkappa(\,(s)) \geqslant y(s) - p(s)\,y(\mathfrak{r}(s)).$$

Since $y' > 0$, it follows that

$$\varkappa(s) \geqslant (1-p(s))\,y(s).$$

Substituting this into equation (2.2), we get

$$D_3 w(s) = -g(s)\,\varkappa(\eta(s)) \leqslant -g(s)(1-p(\eta(s)))\,y(\eta(s)).$$

Therefore

$$D_3 w(s) \leqslant -g(s)(1-p(\eta(s)))\,y(\eta(s)).$$

Since $w(s) = \frac{y(s)}{\pi_{12}(s)}$, we find

$$D_3 w(s) \leqslant -g(s)(1-p(\eta(s)))\,\pi_{12}(\eta(s))\,w(\eta(s)) = -g_1(s)\,w(\eta(s)).$$

Thus, the proof is complete. □

Lemma 3.2. *Assume that* $\gamma_* > 0$ *and* $\varkappa \in N_2$. *Then the following holds.*

($B_{1,1}$): $\lim_{s\to\infty} D_2 w(s) = \lim_{s\to\infty} D_1 w(s)/\mu_2(s) = \lim_{s\to\infty} w(s)/\mu_{12}(s)$;

($B_{1,2}$): $D_1 w/\mu_2$ *is decreasing and* $D_1 w \geqslant \mu_2 D_2 w$;

($B_{1,3}$): w/μ_{12} *is decreasing and* $w > (\mu_{12}/\mu_2) D_1 w$,

for s *sufficiently large*

Proof. Let $\varkappa \in N_2$ and choose $s_1 \geqslant s_0$ such that $\varkappa(\eta(s)) > 0$ and γ satisfies (3.2) for $s \geqslant s_1$.

($B_{1,1}$): Since $D_2 w$ is a positive decreasing function, obviously

$$\lim_{s\to\infty} D_2 w = l \geqslant 0.$$

If $l > 0$, then $D_2 w \geqslant l > 0$. Thus, for any $\varepsilon \in (0,1)$, we get

$$w(s) \geqslant l \int_{s_1}^{s} \frac{1}{a_1(u)} \int_{s_1}^{u} \frac{1}{a_2(v)} dv du \geqslant \tilde{l}\mu_{12}(s), \ \tilde{l} = \varepsilon l. \tag{3.6}$$

From (3.5) and (3.6), we find

$$-D_3 w(s) \geqslant \tilde{l} g_1(s) \mu_{12}(\eta(s)).$$

Integrating from s_1 to s, we have

$$D_2 w(s_1) \geqslant \tilde{l} \int_{s_1}^{s} g_1(v) \mu_{12}(\eta(v)) dv \geqslant \gamma \tilde{l} \int_{s_1}^{s} \frac{1}{a_2(v)\mu_2(v)} dv = \gamma \tilde{l} \ln \frac{\mu_2(s)}{\mu_2(s_1)} \to \infty \text{ as } s \to \infty,$$

which leads to a contradiction. Therefore, it follows that $l = 0$. By applying Hôpital's rule, we can conclude that condition ($B_{1,1}$) is satisfied.

($B_{1,2}$): Using the fact that $D_2 w$ is positive and decreasing, we see that

$$\begin{aligned} D_1 w(s) &= D_1 w(s_1) + \int_{s_1}^{s} \frac{D_2 w(v)}{a_2(v)} dv \geqslant D_1 w(s_1) + D_2 w(s) \int_{s_1}^{s} \frac{1}{a_2(v)} dv \\ &= D_1 w(s_1) + D_2 w(s) \int_{s_1}^{s} \frac{1}{a_2(v)} dv - D_2 w(s) \int_{s_0}^{s_1} \frac{1}{a_2(v)} dv \geqslant \mu_2(s) D_2 w(s). \end{aligned}$$

In view of ($B_{1,1}$), there is a $s_2 > s_1$ such that

$$D_1 w(s_1) - D_2 w(s) \int_{s_0}^{s_1} \frac{1}{a_2(v)} dv > 0, \ s \geqslant s_2.$$

Thus,

$$D_1 w(s) > \mu_2(s) D_2 w(s), \ s \geqslant s_2,$$

and consequently

$$\left(\frac{D_1 w}{\mu_2}\right)'(s) = \frac{\mu_2(s) D_2 w(s) - D_1 w(s)}{a_2(s) \mu_2^2(s)} < 0, \ s \geqslant s_2.$$

($B_{1,3}$): Since $(D_1 w/\mu_2)' < 0$ and tending to zero, then

$$\begin{aligned} w(s) &= w(s_2) + \int_{s_2}^{s} \frac{D_1 w(v)}{\mu_2(v)} \frac{\mu_2(v)}{a_1(v)} dv \geqslant w(s_2) + \frac{D_1 w(s)}{\mu_2(s)} \int_{s_2}^{s} \frac{\mu_2(v)}{a_1(v)} dv \\ &\geqslant w(s_2) + \frac{D_1 w(s)}{\mu_2(s)} \mu_{12}(s) + \frac{D_1 w(s)}{\mu_2(s)} \int_{s_0}^{s_2} \frac{\mu_2(v)}{a_1(v)} dv > \frac{D_1 w(s)}{\mu_2(s)} \mu_{12}(s), \end{aligned}$$

for $s \geqslant s_3$ for some $s_3 > s_2$. Therefore

$$\left(\frac{w}{\mu_{12}}\right)'(s) = \frac{D_1 w(s) \mu_{12}(s) - w(s) \mu_2(s)}{a_1(s) \mu_{12}^2(s)} < 0, \ s \geqslant s_2.$$

□

The following lemma provides further properties of solutions that are part of the class N_2.

Lemma 3.3. *Assume that* $\gamma_* > 0$ *and* $\varkappa \in N_2$. *Then for* $\gamma \in (0, \gamma_*)$ *and* s *sufficiently large:*

($B_{2,1}$): $D_1 w / \mu_2^{1-\gamma_*}$ *is decreasing, and* $(1-\gamma_*) D_1 w > \mu_2 L_2 w$;

($B_{2,2}$): $\lim_{s\to\infty} D_1 w(s) / \mu_2^{1-\gamma_*}(s) = 0$;

($B_{2,3}$): $w/\mu_{12}^{1/k}$ *is decreasing and* $w > k(\mu_{12}/\mu_2) D_1 w$.

Proof. Let $\varkappa \in N_2$ and choose $s_1 \geqslant s_0$ such that $y(\eta(s)) > 0$ and parts ($B_{1,1}$)-($B_{1,3}$) in Lemma 3.2 hold for $s \geqslant s_1 \geqslant s_0$. Additionally, choose a fixed but sufficiently large $\gamma \in (\gamma_*/(1+\gamma_*), \gamma_*)$ and $k \leqslant k_*$ that fulfills conditions (3.2) and (3.3) for $s \geqslant s_1$. Since

$$\frac{\gamma}{1-\gamma} > \gamma_*,$$

there are constants $c_1 \in (0,1)$ and $c_2 > 0$ such that

$$\frac{c_1 \gamma}{1-\gamma} > \gamma_* + c_2. \tag{3.7}$$

($B_{2,1}$): Define

$$\omega(s) = D_1 w(s) - \mu_2(s) D_2 w(s) > 0. \tag{3.8}$$

By taking the derivative of ω and applying (3.5) and (3.2), we can observe that

$$\omega'(s) = (D_1 w(s) - \mu_2(s) D_2 w(s))' = -\mu_2(s) D_3 w(s) \geqslant \mu_2(s) g_1(s) w(\eta(s)) \geqslant \frac{\gamma}{a_2(s)} \frac{w(\eta(s))}{\mu_{12}(\eta(s))}. \tag{3.9}$$

By virtue of ($B_{1,3}$), we have

$$\omega'(s) \geqslant \frac{\gamma}{a_2(s)} \frac{w(s)}{\mu_{12}(s)} \geqslant \frac{\gamma}{a_2(s)} \frac{D_1 w(s)}{\mu_2(s)}.$$

By integrating this inequality from s_2 to s and using the fact that $(D_1 w/\mu_2)' < 0$ and tends to zero, there exists $s_3 \geqslant s_2$ such that

$$\begin{aligned} \omega(s) &\geqslant \omega(s_2) + \gamma \int_{s_2}^{s} \frac{D_1 y(\nu)}{a_2(\nu)\mu_2(\nu)} d\nu \\ &\geqslant w(s_2) + \gamma \frac{D_1 w(s)}{\mu_2(s)} \int_{s_2}^{s} \frac{1}{a_2(\nu)} d\nu \\ &= y(s_2) + \gamma \frac{D_1 w(s)}{\mu_2(s)} \mu_2(s) - \gamma \frac{D_1 w(s)}{\mu_2(s)} \int_{s_0}^{s_2} \frac{1}{a_2(\nu)} d\nu > \gamma D_1 w(s),\ s \geqslant s_3. \end{aligned} \tag{3.10}$$

Then $(1-\gamma) D_1 w(s) > \mu_2(s) D_2 w(s)$, and

$$\left(\frac{D_1 w(s)}{\mu_2^{1-\gamma}(s)}\right)' = \frac{D_2 w(s) \mu_2(s) - (1-\gamma) D_1 w(s)}{a_2(s) \mu_2^{1+\gamma}(s)} < 0. \tag{3.11}$$

It follows directly from (3.11) and the fact that $D_1 w$ is increasing that $\gamma < 1$. Using this in (3.10) and taking (3.7) into account, we find that there is $s_4 > s_3$ such that

$$\begin{aligned} y(s) &\geqslant y(s_2) + \gamma \int_{s_3}^{s} \frac{D_1 w(\nu)}{a_2(\nu)\mu_2(\nu)} d\nu \\ &\geqslant Y(s_2) + \gamma \frac{D_1 w(s)}{\mu_2^{1-\gamma}(s)} \int_{s_3}^{s} \frac{1}{a_2(\nu)\mu_2^{\gamma}(\nu)} d\nu \end{aligned}$$

$$\geqslant \frac{\gamma}{1-\gamma}\frac{D_1 w(s)}{\mu_2^{1-\gamma}(s)}\left(\mu_2^{1-\gamma}(s)-\mu_2^{1-\gamma}(s_3)\right) \geqslant \frac{c_1\gamma}{1-\gamma} D_1 w(s) \geqslant (\gamma_* + c_2) D_1 w(s), \ s \geqslant s_4,$$

which implies

$$(1-\gamma_*) D_1 w(s) > (1-\gamma_* - c_2) D_1 w(s) > D_2 w(s) \mu_2(s),$$

and

$$\left(\frac{D_1 w(s)}{\mu_2^{1-\gamma_*-c_2}(s)}\right)' < 0. \tag{3.12}$$

Therefore, the conclusion is reached directly.

($B_{2,2}$): It is clear that (3.12) also implies that $D_1 w/\mu_2^{1-\gamma_*} \to 0$ as $s \to \infty$, since otherwise

$$\frac{D_1 y(s)}{\mu_2^{1-\gamma_*-c_2}(s)} = \frac{D_1 y(s)}{\mu_2^{1-\gamma_*}(s)}\mu_2^{c_2}(s) \to \infty \text{ as } s \to \infty,$$

which is a contradiction.

($B_{2,3}$): Using that by ($B_{2,1}$) and ($B_{2,2}$), $\left(D_1 w/\kappa_2^{1-\gamma_*}\right)' < 0$ and tending to zero, we deduce that

$$\begin{aligned} w(s) &= w(s_4) + \int_{s_4}^{s} \frac{D_1 w(\nu)}{\mu_2^{1-\gamma_*}(\nu)}\frac{\mu_2^{1-\gamma_*}(\nu)}{a_1(\nu)} d\nu \\ &\geqslant w(s_4) + \frac{D_1 w(s)}{\mu_2^{1-\gamma_*}(s)} \int_{s_4}^{s} \frac{\mu_2^{1-\gamma_*}(\nu)}{a_1(\nu)} d\nu \\ &= w(s_4) + \frac{D_1 w(s)}{\mu_2^{1-\gamma_*}(s)} \int_{s_0}^{s} \frac{\mu_2^{1-\gamma_*}(\nu)}{a_1(\nu)} d\nu - \frac{D_1 w(s)}{\mu_2^{1-\gamma_*}(s)} \int_{s_0}^{s_4} \frac{\mu_2^{1-\gamma_*}(\nu)}{a_1(\nu)} d\nu \\ &> \frac{D_1 w(s)}{\mu_2^{1-\gamma_*}(s)} \int_{s_0}^{s} \frac{\mu_2^{1-\gamma_*}(\nu)}{a_1(\nu)} d\nu \geqslant k\frac{\mu_{12}(s)}{\mu_2(s)} D_1 w(s), \ s \geqslant s_5 \geqslant s_4. \end{aligned}$$

Therefore

$$\left(\frac{w(s)}{\mu_{12}^{1/k}(s)}\right)' = \frac{k\mu_{12}(s) D_1 w(s) - \mu_2(s) w(s)}{k a_1(s) \mu_2^{1/k+1}(s)} < 0, \ s \geqslant s_5.$$

Thus, the proof is complete. □

Corollary 3.4. *Assume that*

$$\gamma_* \geqslant 1. \tag{3.13}$$

Then $N_2 = \varnothing$.

Proof. This follows from $(1-\gamma_*) D_1 w(s) > \mu_2(s) D_2 w(s)$, and the fact that $D_2 w$ is positive. □

Corollary 3.5. *Suppose that*

$$\gamma_* > 0 \ \textit{and} \ \lambda_* = \infty. \tag{3.14}$$

Then $N_2 = \varnothing$.

Proof. Let $\varkappa \in N_2$ and choose $s_1 \geqslant s_0$ such that $w(\eta(s)) > 0$ and parts ($B_{2,1}$)-($B_{2,3}$) in Lemma 3.2 hold for $s \geqslant s_1 \geqslant s_0$. Additionally, choose a fixed but sufficiently large $\lambda \leqslant \lambda_*$, $\gamma \leqslant \gamma_*$, and $k \leqslant k_*$ that fulfills conditions (3.1), (3.2), and (3.3) for $s \geqslant s_1$. Using (3.9) and the decreasing of $w/\mu_{12}^{1/k}$, we have

$$y'(s) \geqslant \gamma \frac{w(\eta(s))}{\mu_{12}^{1/k}(\eta(s))} \frac{1}{a_2(s)\mu_{12}^{1-1/k}(\eta(s))}$$

$$\geqslant \gamma \frac{w(s)}{\mu_{12}^{1/k}(s)} \frac{1}{a_2(s)\, \mu_{12}^{1-1/k}(\eta(s))}$$
$$\geqslant \gamma \frac{\mu_{12}^{1-1/k}(s)}{a_2(s)\, \mu_{12}^{1-1/k}(\eta(s))} \frac{w(s)}{\mu_{12}(s)} \geqslant \gamma k \lambda^{1-1/k}(s) \frac{D_1 w(s)}{a_2(s)\, \mu_2(s)}.$$

By integrating this inequality from s_2 to s and using that $(D_1 w/\mu_2)' < 0$ and tending to zero, we get

$$y(s) \geqslant k\gamma\lambda^{1-1/k} D_1 w(s), \; s \geqslant s_2.$$

Thus

$$\left(1 - k\gamma\lambda^{1-1/k}\right) D_1 w(s) \geqslant \mu_2(s)\, D_2 w(s).$$

Since λ can take on any large value, we can choose $\lambda > (1/k\gamma)^{k/(k-1)}$, which contradicts the positivity of $D_2 w$. This concludes the proof of the Corollary. □

Example 3.6. Consider the third-order NDE given by

$$\left(e^s \left(e^s \left(\varkappa(s) + p_0 \varkappa(\mathfrak{r}_0 s)\right)'(s)\right)'\right)' + \mathfrak{q}_0 \varkappa(\eta_0 s) = 0, \tag{3.15}$$

where $0 \leqslant p_0 < 1$. By comparing this equation with equation (1.1), we can observe that

$$\kappa_1(s) = e^s, \; \kappa_2(s) = e^s, \; \eta(s) = \eta_0 s, \; \mathfrak{r}(s) = \mathfrak{r}_0 s, \; \mathfrak{q}(s) = \mathfrak{q}_0.$$

As a result, we obtain:

$$\pi_1(s) = e^{-s}, \; \pi_2(s) = e^{-s}, \; \pi_{12}(s) = \frac{1}{2} e^{-2s}, \; \pi_*(s) = \frac{1}{2} e^{-2s},$$

and

$$a_1(s) = \frac{1}{2} e^{-s} \text{ and } a_2(s) = \frac{1}{2} e^{-s}.$$

Now, we can transform (3.15) into canonical form

$$\left(e^{-s}\left(e^{-s} w'(s)\right)'\right)' + 2\mathfrak{q}_0 \varkappa(\eta_0 s) = 0.$$

Moreover,

$$\mu_1(s) = e^s, \; \mu_2(s) = e^s, \; \mu_{12}(s) = \frac{1}{2} e^{2s}.$$

Condition (3.14) yields

$$\gamma_* = e^{-s} \frac{1}{2} e^{2\eta_0 s} e^s 2\mathfrak{q}_0 (1 - p_0) \frac{1}{2} e^{-2\eta_0 s} > 1,$$

which holds under the condition

$$\mathfrak{q}_0 > \frac{2}{1 - p_0}.$$

Condition (3.14) leads to

$$\lambda_* = \liminf_{s \to \infty} e^{2(1-\eta_0)s} = \infty.$$

Corollary 3.7. *Suppose that* $\gamma_* > 0$ *and* $k_* = \infty$. *Then* $N_2 = \varnothing$.

Proof. The proof can be derived using the same steps outlined in Corollary 3.5. Since k can be chosen to be arbitrarily large, we omit the details. □

We will demonstrate how the outcomes of Lemma 3.3 can be improvied through iterative methods.

Lemma 3.8. *Suppose that $\delta_* > 0$ and $\varkappa \in N_2$. Then, for any $n \in \mathbb{N}_0$ and s sufficiently large,*

$(B_{n,1})$: $D_1 w/\mu_2^{1-\gamma_n}$ *is decreasing and* $(1-\gamma_n) D_1 w > \mu_2 D_2 w$;

$(B_{n,2})$: $\lim_{s\to\infty} D_1 w(s)/\mu_n^{1-\gamma_n}(s) = 0$;

$(B_{n,3})$: $w/\mu_{12}^{1/\varepsilon_n k_n}$ *is decreasing and* $w > \varepsilon_n k_n (\mu_{12}/\mu_2) D_1 w$ *for any* $\varepsilon_n \in (0,1)$.

Proof. Let $\varkappa \in N_2$ with $y(\eta(s)) > 0$ and parts $(B_{1,1})$-$(B_{1,3})$ in Lemma 3.2 hold for $s \geqslant s_1 \geqslant s_0$. Additionally, choose a fixed but sufficiently large $\gamma \leqslant \gamma_*$, and $k \leqslant k_*$ that fulfills conditions (3.2) and (3.3) for $s \geqslant s_1$. We will use mathematical induction on n. For the base case where $n = 0$, the result is established by Lemma 3.3 with $\varepsilon_0 = k/k_*$. Now, let us assume that the conditions $(B_{n,1})$-$(B_{n,3})$ are satisfied for all $n \geqslant 1$ for $s \geqslant s_n \geqslant s_1$. Our goal is to demonstrate that these conditions also hold for $n+1$.

$(B_{n,n+1})$: Applying $(B_{n,3})$ in (3.9), we get

$$\begin{aligned} y'(s) &\geqslant \gamma \frac{w(\eta(s))}{\mu_{12}^{1/\varepsilon_n k_n}(\eta(s))} \frac{1}{a_2(s)\,\mu_{12}^{1-1/\varepsilon_n k_n}(\eta(s))} \\ &\geqslant \gamma \frac{w(s)}{\mu_{12}^{1/\varepsilon_n k_n}(s)} \frac{1}{a_2(s)\,\mu_{12}^{1-1/\varepsilon_n k_n}(\eta(s))} \\ &\geqslant \gamma \frac{\mu_{12}^{1-1/\varepsilon_n k_n}(s)}{a_2(s)\,\mu_{12}^{1-1/\varepsilon_n k_n}(\eta(s))} \frac{w(s)}{\mu_{12}(s)} \geqslant \gamma \varepsilon_n k_n \lambda^{1-1/\varepsilon_n k_n}(s) \frac{D_1 w(s)}{a_2(s)\,\mu_2(s)}. \end{aligned}$$

Integrating this inequality from s_n to s and using $(B_{n,1})$ and $(B_{n,2})$, we have

$$\begin{aligned} y(s) &\geqslant y(s_n) + \varepsilon_n k_n \gamma \lambda^{1-1/\varepsilon_n k_n} \int_{s_n}^{s} \frac{D_1 w(\nu)}{a_2(\nu)\,\mu_2(\nu)} d\nu \\ &\geqslant y(s_n) + \varepsilon_n k_n \gamma \lambda^{1-1/\varepsilon_n k_n} \frac{D_1 w(s)}{\mu_2^{1-\gamma_n}(s)} \int_{s_n}^{s} \frac{1}{a_2(\nu)\,\mu_2^{\gamma_n}(\nu)} d\nu \\ &\geqslant y(s_n) + \frac{\varepsilon_n k_n \gamma \lambda^{1-1/\varepsilon_n k_n}}{1-\gamma_n} \frac{D_1 w(s)}{\mu_2^{1-\gamma_n}(s)} \left[\mu_2^{1-\gamma_n}(s) - \mu_2^{1-\gamma_n}(s_n)\right] \\ &> \frac{\varepsilon_n k_n \gamma \lambda^{1-1/\varepsilon_n k_n}}{1-\gamma_n} D_1 w(s) = m\gamma_{n+1} L_1 y(s), \end{aligned} \tag{3.16}$$

where

$$m = \frac{\gamma}{\gamma_*} \varepsilon_n \frac{\lambda^{1-1/\varepsilon_n k_n}}{\lambda_*^{1-1/k_n}} \in (0,1),$$

and

$$\lim_{\substack{\lambda\to\lambda_* \\ \varepsilon_n \to 1 \\ \gamma\to\gamma_*}} m = 1.$$

Choose m such that

$$m > \frac{1}{1-\gamma_n+\gamma_{n+1}} = \frac{1}{1+\gamma_n(\ell_n - 1)},$$

where ℓ_n satisfies (3.4). Then

$$\frac{m\gamma_{n+1}}{1-\mu\gamma_{n+1}} > \frac{\gamma_{n+1}}{(1+\gamma_n(\ell_n-1))\left(1-\frac{\ell_n\gamma_n}{1+\gamma_n(\ell_n-1)}\right)} = \frac{\gamma_{n+1}}{1-\gamma_n},$$

and there exist two constants $c_1 \in (0,1)$ and $c_2 > 0$ such that

$$c_1 \frac{m(1-\gamma_n)\gamma_{n+1}}{1-\mu\gamma_{n+1}} > \gamma_{n+1} + c_2.$$

According to the Definition (3.8) of y, we deduce that

$$(1-\mu\gamma_{n+1})\,D_1w\,(s) = \mu_2\,(s)\,D_2w\,(s)\,,$$

and

$$\left(\frac{D_1w\,(s)}{\mu_2^{1-\mu\gamma_{n+1}}\,(s)}\right)' < 0,\ s \geqslant s_n'.$$

By applying the monotonicity established in (3.16), we can observe that there exists a sufficiently large $s_n'' \geqslant s_n'$ such that

$$\begin{aligned}
y\,(s) &\geqslant y\,(s_n) + \varepsilon_n k_n \gamma \lambda^{1-1/\varepsilon_n k_n} \int_{s_n}^{s} \frac{D_1w\,(\nu)}{a_2\,(\nu)\,\mu_2\,(\nu)} d\nu \\
&\geqslant y\,(s_n) + \frac{\varepsilon_n k_n \gamma \lambda^{1-1/\varepsilon_n k_n}}{1-\mu\gamma_{n+1}} \frac{D_1w\,(s)}{\mu_2^{1-\mu\gamma_{n+1}}\,(s)} \left(\mu_2^{1-\mu\gamma_{n+1}}\,(s) - \mu_2^{1-\mu\gamma_{n+1}}\,(s_n)\right) \\
&> \frac{c_1 \varepsilon_n k_n \gamma \lambda^{1-1/\varepsilon_n k_n}}{1-\mu\gamma_{n+1}} D_1w\,(s) = c_1 m \gamma_{n+1} \frac{1-\gamma_n}{1-m\gamma_{n+1}} D_1w\,(s) > (\gamma_{n+1}+c_2)\,D_1w\,(s)\,,\ s \geqslant s_n''.
\end{aligned}$$

Then

$$(1-\gamma_{n+1}-c_2)\,D_1w\,(s) > \mu_2\,(s)\,D_2w\,(s)\,,$$

and

$$\left(\frac{D_1w\,(s)}{\mu_2^{1-\gamma_{n+1}-c_2}\,(s)}\right)' < 0. \tag{3.17}$$

Consequently, we arrive at the conclusion.

$(B_{n+1,2})$: It is clear that (3.17) also indicates that $D_1w/\mu_2^{1-\gamma_{n+1}} \to 0$ as $s \to \infty$. If this were not the case, then

$$\frac{D_1w\,(s)}{\mu_2^{1-\gamma_{n+1}-c_2}\,(s)} = \frac{D_1w\,(s)}{\mu_2^{1-\gamma_{n+1}}\,(s)} \mu_2^{c_2}\,(s) \to \infty \text{ as } s \to \infty,$$

which leads to a contradiction.

$(B_{n+1,3})$: Using that by $(B_{n+1,1})$ and $(B_{n+1,2})$, $\left(D_1w/\mu_2^{1-\gamma_{n+1}}\right)' < 0$ and tending to zero, we obtain for any $\varepsilon_n \in (0,1)$,

$$\begin{aligned}
w\,(s) &= w\,(s_n'') + \int_{s_n''}^{s} \frac{D_1w\,(\nu)}{\mu_2^{1-\gamma_{n+1}}\,(\nu)} \frac{\mu_2^{1-\gamma_{n+1}}\,(\nu)}{a_1\,(\nu)} d\nu \\
&\geqslant w\,(s_n'') + \frac{D_1w\,(s)}{\mu_2^{1-\gamma_{n+1}}\,(s)} \int_{s_n''}^{s} \frac{\mu_2^{1-\gamma_{n+1}}\,(\nu)}{a_1\,(\nu)} d\nu \\
&= y\,(s_n'') + \frac{D_1w\,(s)}{\mu_2^{1-\gamma_{n+1}}\,(s)} \int_{s_0}^{s} \frac{\mu_2^{1-\gamma_{n+1}}\,(\nu)}{a_1\,(\nu)} d\nu - \frac{D_1w\,(s)}{\mu_2^{1-\gamma_{n+1}}\,(s)} \int_{s_0}^{s_n''} \frac{\mu_2^{1-\gamma_{n+1}}\,(\nu)}{a_1\,(\nu)} d\nu \\
&> \frac{D_1w\,(s)}{\mu_2^{1-\gamma_{n+1}}\,(s)} \int_{s_0}^{s} \frac{\mu_2^{1-\gamma_{n+1}}\,(\nu)}{a_1\,(\nu)} d\nu \geqslant \varepsilon_{n+1} k_{n+1} \frac{\mu_{12}\,(s)}{\mu_2\,(s)} D_1w\,(s)\,,\ s \geqslant s_{n+1} \geqslant s_n'',
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{w}{\mu_{12}^{1/\varepsilon_{n+1}k_{n+1}}}\right)'(s) &= \frac{\varepsilon_{n+1} k_{n+1} \mu_{12}^{1/\varepsilon_{n+1}k_{n+1}}\,(s)\,D_1w\,(s) - \mu_{12}^{1/\varepsilon_{n+1}k_{n+1}-1}\,(s)\,\mu_2\,(s)\,w\,(s)}{\varepsilon_{n+1} k_{n+1} a_1\,(s)\,\mu_{12}^{2/\varepsilon_{n+1}k_{n+1}}\,(s)} \\
&= \frac{\varepsilon_{n+1} k_{n+1} \mu_{12}\,(s)\,D_1w\,(s) - \mu_2\,(s)\,w\,(s)}{\varepsilon_{n+1} k_{n+1} a_1\,(s)\,\mu_{12}^{1/\varepsilon_{n+1}k_{n+1}+1}\,(s)} < 0.
\end{aligned}$$

This concludes the proof of the Corollary. □

Corollary 3.9. *Suppose that* $\gamma_i < 1,\ i = 0, 1, 2, \ldots, n-1$, *and* $\gamma_n \geqslant 1$. *Then* $N_2 = \varnothing$.

Proof. This follows directly from $(1-\gamma_n)\, D_1 w(s) > \mu_2(s)\, D_2 w(s)$, and the fact that $D_2 w$ is positive. □

Remark 3.10. By setting $p(t) = 0$ in equation (1.1), the results obtained in this section are entirely identical to those in [24], confirming their validity.

4. Criteria indicating the nonexistence of Kneser solutions

In this section, we outline specific criteria that ensure there are no Kneser solutions fulfilling case (N_0).

Lemma 4.1. *Let* $\zeta(s) \in C([s_0, \infty), (0, \infty))$ *such that* $\mathfrak{y}(s) < \zeta(s)$ *and* $\mathfrak{r}^{-1}(\zeta(s)) < s$. *If the differential equation*

$$\omega'(s) + \frac{\mathfrak{r}_0}{\mathfrak{r}_0 + p_0} g_2(s)\, \mu_{12}(\zeta(s), \mathfrak{y}(s))\, \omega\left(\mathfrak{r}^{-1}(\zeta(s))\right) = 0, \tag{4.1}$$

is oscillatory, then it follows that $N_0 = \varnothing$.

Proof. Let $\varkappa \in N_0$, say $\varkappa(s) > 0$ and $\varkappa(\mathfrak{y}(s)) > 0$ for $s \geqslant s_1 \geqslant s_0$. This implies that

$$(-1)^k w^{(k)}(s) > 0, \text{ for } k = 0, 1, 2, 3.$$

From (2.2), we see that

$$0 \geqslant \frac{p_0}{\mathfrak{r}(s)} D_3 w(\mathfrak{r}(s)) + \pi_*(s)\, p_0 g(\mathfrak{r}(s))\, \varkappa(\mathfrak{y}(\mathfrak{r}(s))) \geqslant \frac{p_0}{\mathfrak{r}_0} D_3 w(\mathfrak{r}(s)) + \pi_*(s)\, p_0 g(\mathfrak{r}(s))\, \varkappa(\mathfrak{y}(\mathfrak{r}(s))). \tag{4.2}$$

Combining (2.2) and (4.2), we obtain

$$\begin{aligned} 0 &\geqslant D_3 w(s) + \frac{p_0}{\mathfrak{r}_0} D_3 w(\mathfrak{r}(s)) + [g(s)\, \varkappa(\mathfrak{y}(s)) + p_0 g(\mathfrak{r}(s))\, \varkappa(\mathfrak{y}(\mathfrak{r}(s)))] \\ &\geqslant D_3 w(s) + \frac{p_0}{\mathfrak{r}_0} D_3 w(\mathfrak{r}(s)) + \widetilde{g}(s)\, [\varkappa(\mathfrak{y}(s)) + p_0 \varkappa(\mathfrak{y}(\mathfrak{r}(s)))]. \end{aligned} \tag{4.3}$$

From definition of y, we have

$$y(\mathfrak{y}(s)) = \varkappa(\mathfrak{y}(s)) + p(\mathfrak{y}(s))\, \varkappa(\mathfrak{r}(\mathfrak{y}(s))) \leqslant \varkappa(\mathfrak{y}(s)) + p_0 \varkappa(\mathfrak{r}(\mathfrak{y}(s))).$$

By using the latter inequality in (4.3), we get

$$0 \geqslant D_3 w(s) + \frac{p_0}{\mathfrak{r}_0} D_3 w(\mathfrak{r}(s)) + \widetilde{g}(s)\, y(\mathfrak{y}(s)).$$

Since $w(s) = y(s)/\pi_{12}(s)$, then

$$0 \geqslant D_3 w(s) + \frac{p_0}{\mathfrak{r}_0} D_3 w(\mathfrak{r}(s)) + \widetilde{g}(s)\, \pi_{12}(\mathfrak{y}(s))\, w(\mathfrak{y}(s)).$$

That is

$$\left(D_2 w(s) + \frac{p_0}{\mathfrak{r}_0} D_2 w(\mathfrak{r}(s))\right)' + g_2(s)\, w(\mathfrak{y}(s)) \leqslant 0. \tag{4.4}$$

On the other hand, it follows from the monotonicity of $D_2 w$ that

$$-D_1 w(\rho) \geqslant D_1 w(\sigma) - D_1 w(\rho) = \int_\rho^\sigma \frac{D_2 w(\nu)}{a_2(\nu)} d\nu \geqslant D_2 w(\sigma) \int_\rho^\sigma \frac{1}{a_2(\nu)} d\nu = D_2 w(\sigma)\, \mu_2(\sigma, \rho). \tag{4.5}$$

Integrating (4.5) from ρ to σ, we have

$$w(\rho) \geqslant -w(\sigma) + w(\rho) = \int_{\rho}^{\sigma} \frac{D_2 w(\nu)\, \mu_2(\sigma,\rho)}{a_1(\nu)} d\nu \geqslant D_2 w(\sigma) \int_{\rho}^{\sigma} \frac{\mu_2(\sigma,\rho)}{a_1(\nu)} d\nu = D_2 w(\sigma)\, \mu_{12}(\sigma,\rho).$$

Then

$$w(\rho) \geqslant D_2 w(\sigma)\, \mu_{12}(\sigma,\rho). \tag{4.6}$$

Thus, we have

$$w(\eta(s)) \geqslant D_2 w(\zeta(s))\, \mu_{12}(\zeta(s), \eta(s)),$$

which, by virtue of (4.4) yields that

$$0 \geqslant \left(D_2 w(s) + \frac{p_0}{\mathfrak{r}_0} D_2 w(\mathfrak{r}(s)) \right)' + g_2(s)\, \mu_{12}(\zeta(s), \eta(s))\, D_2 w(\zeta(s)). \tag{4.7}$$

Now, set

$$\omega(s) = D_2 w(s) + \frac{p_0}{\mathfrak{r}_0} D_2 w(\mathfrak{r}(s)) > 0.$$

Since $D_2 w$ is a non-increasing function, it follows that

$$\omega(s) \leqslant D_2 w(\mathfrak{r}(s)) \left(1 + \frac{p_0}{\mathfrak{r}_0}\right),$$

which can be rewritten as

$$D_2 w(\zeta(s)) \geqslant \frac{\mathfrak{r}_0}{\mathfrak{r}_0 + p_0} \omega\left(\mathfrak{r}^{-1}(\zeta(s))\right). \tag{4.8}$$

Substituting (4.8) into (4.7), we see that ω is a positive solution of the differential inequality

$$\omega'(s) + \frac{\mathfrak{r}_0}{\mathfrak{r}_0 + p_0} g_2(s)\, \mu_{12}(\zeta(s), \eta(s))\, \omega\left(\mathfrak{r}^{-1}(\zeta(s))\right) \leqslant 0.$$

In view of [36, Theorem 1], we have that (4.1), we obtain that ω satisfies the following differential inequality. □

Corollary 4.2. *Let $\zeta(s) \in C([s_0,\infty), (0,\infty))$ such that $\eta(s) < \zeta(s)$ and $\mathfrak{r}^{-1}(\zeta(s)) < s$. If*

$$\liminf_{s\to\infty} \int_{\mathfrak{r}^{-1}(\zeta(s))}^{s} g_2(\nu)\, \mu_{12}(\zeta(\nu), \eta(\nu))\, d\nu > \frac{\mathfrak{r}_0 + p_0}{\mathfrak{r}_0 e}, \tag{4.9}$$

then $N_0 = \varnothing$.

Theorem 4.3. *Let $\delta(s) \in C([s_0,\infty), (0,\infty))$ such that $\delta(s) < s$ and $\eta(s) < \mathfrak{r}(\delta(s))$. If*

$$\limsup_{s\to\infty} \mu_{12}(\mathfrak{r}(\delta(s)), \eta(s)) \int_{\delta(s)}^{s} g_2(\nu)\, d\nu > \frac{\mathfrak{r}_0 + p_0}{\mathfrak{r}_0}, \tag{4.10}$$

then $N_0 = \varnothing$.

Proof. Using the same procedure as in the proof of Theorem 4.1, we obtain

$$\left(D_2 w(s) + \frac{p_0}{\mathfrak{r}_0} D_2 w(\mathfrak{r}(s)) \right)' + g_2(s)\, w(\eta(s)) \leqslant 0.$$

Integrating the previous inequality over $(\eta(s), s)$ and utilizing the property that w is decreasing, we have

$$D_2 w(\delta(s)) + \frac{p_0}{\mathfrak{r}_0} D_2 w(\mathfrak{r}(\delta(s))) \geqslant D_2 w(s) + \frac{p_0}{\mathfrak{r}_0} D_2 w(\ (s)) + \int_{\delta(s)}^{s} g_2(\nu)\, w(\eta(\nu))\, d\nu$$

$$\geqslant w(\mathfrak{y}(s))\int_{\delta(s)}^{s} g_2(\nu)\,d\nu.$$

Since $\mathfrak{r}(\delta(s)) < \mathfrak{r}(s)$ and $D_3 w \leqslant 0$, we get

$$D_2 w(\mathfrak{r}(\delta(s)))\left(1+\frac{p_0}{\mathfrak{r}_0}\right) \geqslant w(\mathfrak{y}(s))\int_{\delta(s)}^{s} g_2(\nu)\,d\nu. \tag{4.11}$$

By utilizing (4.6) with $\sigma = \mathfrak{r}(\delta(s))$ and $\rho = \mathfrak{y}(s)$ into (4.11), we deduce that

$$D_2 w(\mathfrak{r}(\delta(s)))\left(1+\frac{p_0}{\mathfrak{r}_0}\right) \geqslant D_2 w(\mathfrak{r}(\delta(s)))\,\mu_{12}(\mathfrak{r}(\delta(s)),\mathfrak{y}(s))\int_{\delta(s)}^{s} g_2(\nu)\,d\nu.$$

That is

$$\frac{\mathfrak{r}_0+p_0}{\mathfrak{r}_0} \geqslant \mu_{12}(\mathfrak{r}(\delta(s)),\mathfrak{y}(s))\int_{\delta(s)}^{s} g_2(\nu)\,d\nu.$$

Taking the lim sup of both sides of the inequality reveals a contradiction with (4.10). Thus, we can conclude the proof. □

Corollary 4.4. *Letting $\delta(s) = \mathfrak{r}(s)$ in Theorem 4.3, if $\mathfrak{y}(s) < \mathfrak{r}(\mathfrak{r}(s))$, such that*

$$\limsup_{s\to\infty} \mu_{12}(\mathfrak{r}(\mathfrak{r}(s)),\mathfrak{y}(s))\int_{\mathfrak{r}(s)}^{s} g_2(\nu)\,d\nu > \frac{\mathfrak{r}_0+p_0}{\mathfrak{r}_0} \tag{4.12}$$

holds, then $N_0 = \varnothing$.

5. Oscillation theorems

Combining the results from previous two sections, we are prepared to state the main results of this paper.

Theorem 5.1. *Assume that (3.13) and (4.9) are satisfied. Then, it follows that (1.1) exhibits oscillatory behavior.*

Proof. Assume for the sake of contradiction that $\varkappa$ is an eventually positive solution of equation (2.2). According to Lemma 2.4, we can deduce that there are two potential scenarios regarding the behavior of w and its derivatives. By applying Corollaries 3.4 and 4.2, we observe that the conditions (3.13) and (4.9) imply the absence of solutions to equation (2.2) that fulfill the requirements of cases (N_2) and (N_0), respectively. Consequently, this leads us to conclude that our initial assumption must be false, thereby confirming that the solutions of equation (2.2) are indeed oscillatory. Thus, we can conclude the proof. □

Theorem 5.2. *Assume that (3.13) and (4.10) are satisfied. Then, it follows that (1.1) exhibits oscillatory behavior.*

Proof. Assume for the sake of contradiction that $\varkappa$ is an eventually positive solution of equation (2.2). According to Lemma 2.4, we can deduce that there are two potential scenarios regarding the behavior of w and its derivatives. By applying Corollary 3.4 and Theorem 4.3, we observe that the conditions (3.13) and (4.10) imply the absence of solutions to equation (2.2) that fulfill the requirements of cases (N_2) and (N_0), respectively. Consequently, this leads us to conclude that our initial assumption must be false, thereby confirming that the solutions of equation (2.2) are indeed oscillatory. Thus, we can conclude the proof. □

Theorem 5.3. *Assume that all the assumptions of Corollary 3.5 or Corollary 3.7 and all the assumptions of Corollary 4.2 or Theorem 4.3 or Corollary 4.4 are satisfied. Therefore, it follows that (1.1) exhibits oscillatory behavior.*

Proof. Based on Corollary 3.5 and Corollary 3.7, it follows that (N_2) is empty, and (N_0) is empty by Corollary 4.2, Theorem 4.3, and Corollary 4.4. So, all solutions of (1.1) are oscillatory. □

Example 5.4. Consider the neutral differential equations

$$\left(s^2\left(s^2\left(\varkappa(s)+p_0\varkappa(\mathfrak{r}_0 s)\right)'(s)\right)'\right)'+\mathfrak{q}_0 s\varkappa(\mathfrak{y}_0 s)=0, \tag{5.1}$$

with $\mathfrak{r}_0, \mathfrak{y}_0 \in (0,1)$, $0 < p_0 \leqslant 1$, and $\mathfrak{q}_0 > 0$. Comparing this equation with (1.1), we observe that

$$\kappa_1(s)=\kappa_2(s)=s^2,\ \mathfrak{y}(s)=\mathfrak{y}s,\ \mathfrak{r}(s)=\mathfrak{r}_0 s,\ \mathfrak{q}(s)=\mathfrak{q}_0 s.$$

Consequently, we obtain

$$\pi_1(s)=\pi_2(s)=\frac{1}{s},\quad \pi_{12}(s)=\pi_*(s)=\frac{1}{2s^2}.$$

Defining $w(s)=y(s)/\pi_{12}(s)$, the original equation is transformed into its canonical form

$$w'''(s)+\frac{2\mathfrak{q}_0}{s}\varkappa(\mathfrak{y}_0 s)=0.$$

We also derive the following additional relations

$$\mu_1(s)=\mu_2(s)=s,\ \mu_{12}(s)=\frac{s^2}{2},\ \ g(s)=\frac{2\mathfrak{q}_0}{s},\ \ g_1(s)=\frac{\mathfrak{q}_0}{\mathfrak{y}_0^2 s^3}(1-p_0),\ \ \widetilde{g}(s)=\frac{2\mathfrak{q}_0}{s},\ \ g_2(s)=\frac{\mathfrak{q}_0}{\mathfrak{y}_0^2 s^3}.$$

From condition (3.13), we obtain

$$\gamma_*=\frac{\mathfrak{y}_0^2 s^2}{2}s\frac{\mathfrak{q}_0}{s^3}(1-p_0)>1,$$

which is satisfied if

$$\mathfrak{q}_0>\frac{2}{\mathfrak{y}_0^2(1-p_0)}. \tag{5.2}$$

From conditon (4.9), we obtain

$$\liminf_{s\to\infty}\int_{\frac{\zeta_0}{\mathfrak{r}_0}s}^{s}\frac{\mathfrak{q}_0}{\mathfrak{y}_0^2 v^3}\frac{(\zeta_0-\mathfrak{y}_0)^2}{2}v^2 dv=\mathfrak{q}_0\frac{(\zeta_0-\mathfrak{y}_0)^2}{2\mathfrak{y}_0^2}\ln\frac{\mathfrak{r}_0}{\zeta_0},$$

which holds, if

$$\mathfrak{q}_0>\frac{2\mathfrak{y}_0^2}{(\zeta_0-\mathfrak{y}_0)^2\ln\frac{0}{\zeta_0}}\frac{(\mathfrak{r}_0+p_0)}{\mathfrak{r}_0 e}. \tag{5.3}$$

Similarly, condition (4.10) leads to

$$\limsup_{s\to\infty}\frac{(\mathfrak{r}_0\delta_0-\mathfrak{y}_0)^2 s^2}{2}\int_{\delta_0 s}^{s}\frac{\mathfrak{q}_0}{\mathfrak{y}_0^2 v^3}dv=\frac{(\mathfrak{r}_0\delta_0-\mathfrak{y}_0)^2\mathfrak{q}_0(1-\delta_0)\mathfrak{q}_0}{4\delta_0\mathfrak{y}_0^2},$$

which holds if

$$\mathfrak{q}_0>\frac{4\delta_0\mathfrak{y}_0^2(\mathfrak{r}_0+p_0)}{(\mathfrak{r}_0\delta_0-\mathfrak{y}_0)^2(1-\delta_0)\mathfrak{r}_0}. \tag{5.4}$$

Lastly, condition (4.12) is satisfied if

$$\mathfrak{q}_0>\frac{4\mathfrak{y}_0^2(\mathfrak{r}_0+p_0)}{(\mathfrak{r}_0^2-\mathfrak{y}_0)^2(1-\mathfrak{r}_0)}. \tag{5.5}$$

Thus, when conditions (5.2)-(5.5) are fulfilled, all assumptions of Theorems 5.1-5.3 are satisfied, implying that all solutions of the equation (5.1) are oscillatory.

Example 5.5. Consider the simplified form of equation (5.1):

$$\left(s^2\left(s^2\left(\varkappa(s)+\frac{1}{2}\varkappa\left(\frac{2}{3}s\right)\right)'\right)'\right)'+q_0 s\varkappa\left(\frac{1}{4}s\right)=0. \tag{5.6}$$

Here $\eta(s)=\frac{1}{4}s$, $\mathfrak{r}(s)=\frac{2}{3}s$, $q(s)=q_0 s$. This equation is transformed into its canonical form as

$$w'''(s)+\frac{2q_0}{s}\varkappa\left(\frac{1}{4}s\right)=0.$$

Additionally, we have

$$g(s)=\frac{2q_0}{s},\quad g_1(s)=\frac{8q_0}{s^3},\quad \widetilde{g}(s)=\frac{2q_0}{s},\quad g_2(s)=\frac{16q_0}{s^3}.$$

Condition (5.2) is satisfied if $q_0>64$. By choosing $\zeta(s)=\frac{1}{3}s$, we observe that $\eta(s)=\frac{1}{4}s<\frac{1}{3}s=\zeta(s)$ and $\mathfrak{r}^{-1}(\zeta(s))=\frac{1}{2}s<s$. Therefore condition (5.3) holds if $q_0>16.718$. Choosing $\delta(s)=\frac{1}{2}s$ with $\delta(s)<s$ and $\eta(s)=\frac{1}{4}s<\frac{1}{3}s=\mathfrak{r}(\delta(s))$, therefore, condition (5.4) is met if $q_0>63.0$. Finally, condition (5.5) is satisfied if $q_0>15.429$. Thus, for $q_0>64$, all conditions are met, and Theorems 5.1-5.3 guarantee oscillatory solutions when $q_0>64$.

6. Conclusion

In this study, we have effectively explored third-order linear neutral differential equations in their unconventional form. Our transformative approach of recasting these equations into the classical framework has yielded significant insights, particularly in reducing the complexity associated with the potential cases of positive solutions and their derivatives. By applying an iterative methodology, we successfully established rigorous conditions that eliminate the possibility of Kneser-type solutions, thereby enhancing the clarity of our results. Furthermore, our comparative analysis with first-order equations provided a robust foundation for further excluding the presence of positive solutions. The oscillation criteria derived from our systematic approach not only substantiate the oscillatory behavior of all solutions under examination but also contribute to the broader discourse within the field by extending and generalizing existing findings. We encourage future researchers to build upon our results, potentially applying the same methodologies to third-order quasi-linear neutral differential equations of the form

$$\left(\kappa_2(s)\left(\kappa_1(s)\left(y'(s)\right)^{\alpha}\right)'\right)'+q(s)\varkappa^{\alpha}(\eta(s))=0.$$

Such endeavors could yield additional insights and broaden the applicability of our findings, fostering a deeper understanding of the realm of differential equations.

Acknowledgment

The authors would like to thank the anonymous reviewers for their work and constructive comments that contributed to improving the manuscript.

References

[1] R. P. Agarwal, M. Bohner, T. Li, C. Zhang, *Oscillation of second-order Emden-Fowler neutral delay differential equations*, Ann. Mat. Pura Appl. (4), **193** (2014), 1861–1875. 1

[2] R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation theory for difference and functional differential equations*, Kluwer Academic Publishers, Dordrecht, (2000). 1

[3] B. Almarri, F. Masood, O. Moaaz, A. Muhib, *Amended Criteria for Testing the Asymptotic and Oscillatory Behavior of Solutions of Higher-Order Functional Differential Equations*, Axioms, **11** (2022), 21 pages. 1
[4] B. Baculíková, *Asymptotic properties of noncanonical third order differential equations*, Math. Slovaca, **69** (2019), 1341–1350. 1
[5] B. Baculíková, J. Džurina, *On the asymptotic behavior of a class of third order nonlinear neutral differential equations*, Cent. Eur. J. Math., **8** (2010), 1091–1103. 1
[6] B. Baculíková, J. Džurina, *Oscillation of third-order neutral differential equations*, Math. Comput. Model., **52** (2010), 215–226. 1
[7] B. Baculíková, J. Džurina, *Oscillation theorems for second order neutral differential equations*, Comput. Math. Appl., **61** (2011), 94–99. 1
[8] D. D. Baĭnov, D. P. Mishev, *Oscillation theory for neutral differential equations with delay*, Adam Hilger, Ltd., Bristol, (1991). 1
[9] O. Bazighifan, *An Approach for Studying Asymptotic Properties of Solutions of Neutral Differential Equations*, Symmetry, **12** (2020), 8 pages. 1
[10] O. Bazighifan, *On the oscillation of certain fourth-order differential equations with* p*-Laplacian like operator*, Appl. Math. Comput., **386** (2020), 8 pages. 1
[11] M. Bohner, J. R. Graef, I. Jadlovská, *Asymptotic properties of Kneser solutions to third-order delay differential equations*, J. Appl. Anal. Comput., **12** (2022), 2024–2032. 1
[12] T. Candan, *Asymptotic properties of solutions of third-order nonlinear neutral dynamic equations*, Adv. Difference Equ., **2014** (2014), 10 pages. 1
[13] G. E. Chatzarakis, J. Džurina, I. Jadlovská, *Oscillatory and asymptotic properties of third-order quasilinear delay differential equations*, J. Inequal. Appl., **2019** (2019), 17 pages. 1
[14] G. E. Chatzarakis, S. R. Grace, I. Jadlovská, *Oscillation criteria for third-order delay differential equations*, Adv. Differ. Equ., **2017** (2017), 11 pages. 1
[15] G. E. Chatzarakis, S. R. Grace, I. Jadlovská, T. Li, E. TunÇ, *Oscillation criteria for third-order Emden-Fowler differential equations with unbounded neutral coefficients*, Complexity, **2019** (2019), 7 pages. 1
[16] K. L. Cooke, *Differential-difference equations*, Academic Press, New York-London, (1963). 1
[17] J. Džurina, *Asymptotic properties of third order delay differential equations*, Czechoslov. Math. J., **45** (1995), 443–448. 1
[18] J. Džurina, I. Jadlovská, *Oscillation of third-order differential equations with noncanonical operators*, Appl. Math. Comput., **336** (2018), 336, 394–402. 2.3
[19] E. M. Elabbasy, T. S. Hassan, B. M. Elmatary, *Oscillation Criteria for Third Order Delay Nonlinear Differential Equations*, Electron. J. Qual. Theory Differ. Equ., **2012** (2012), 11 pages. 1
[20] S. R. Grace, J. Džurina, I. Jadlovská, T. Li, *An improved approach for studying oscillation of second-order neutral delay differential equations*, J. Inequal. Appl., **2018** (2018), 13 pages. 1
[21] S. R. Grace, I. Jadlovská, A. Zafer, *On oscillation of third-order noncanonical delay differential equations*, Appl. Math. Comput., **362** (2019), 7 pages. 1
[22] I. Györi, G. Ladas, *Oscillation theory of delay differential equations: with applications*, Oxford University Press, Oxford, (1991). 1
[23] J. K. Hale, *Theory of functional differential equations*, Springer-Verlag, New York-Heidelberg, (1977). 1
[24] I. Jadlovská, G. E. Chatzarakis, J. Džurina, S. R. Grace, *On sharp oscillation criteria for general third-order delay differential equations*, Mathematics, **9** (2021), 18 pages. 1, 2.4, 3, 3.10
[25] I. T. Kiguradze, T. A. Chanturia, *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Kluwer Academic Publishers Group, Dordrecht, (1993). 2.1
[26] T. Li, Y. V. Rogovchenko, *Oscillation criteria for even-order neutral differential equations*, Appl. Math. Lett., **61** (2016), 35–41. 1
[27] T. Li, C. Zhang, G. Xing, *Oscillation of third-order neutral delay differential equations*, Abstract Appl. Anal., **2012** (2012), 11 pages. 1
[28] F. Masood, C. Cesarano, O. Moaaz, S. S. Askar, A. M. Alshamrani, H. El-Metwally, *Kneser-Type Oscillation Criteria for Half-Linear Delay Differential Equations of Third Order*, Symmetry, **15** (2023), 18 pages. 1
[29] O. Moaaz, D. Chalishajar, O. Bazighifan, *Asymptotic Behavior of Solutions of the Third Order Nonlinear Mixed Type Neutral Differential Equations*, Mathematics, **8** (2020), 13 pages. 1
[30] O. Moaaz, F. Masood, C. Cesarano, S. A. M. Alsallami, E. M. Khalil, M. L. Bouazizi, *Neutral Differential Equations of Second-Order: Iterative Monotonic Properties*, Mathematics, **10** (2022), 11 pages. 1
[31] F. Masood, O. Moaaz, G. AlNemer, H. El-Metwally, *More Effective Criteria for Testing the Asymptotic and Oscillatory Behavior of Solutions of a Class of Third-Order Functional Differential Equations*, Axioms, **12** (2023), 22 pages. 1
[32] F. Masood, O. Moaaz, S. S. Santra, U. Fernandez-Gamiz, H. El-Metwally, *On the monotonic properties and oscillatory behavior of solutions of neutral differential equations*, Demonstr. Math., **56** (2023), 23 pages. 1
[33] A. D. Myškis, *On solutions of linear homogeneous differential equations of the first order of stable type with a retarded argument*, Mat. Sbornik N.S., **28(70)** (1951), 641–658. 1
[34] G. Nithyakala, G. E. Chatzarakis, G. Ayyappan, E. Thandapani, *Third-Order Noncanonical Neutral Delay Differential Equations: Nonexistence of Kneser Solutions via Myshkis Type Criteria*, Mathematics, **12** (2024), 10 pages. 1, 2
[35] N. Parhi, S. Padhi, *Asymptotic behaviour of solutions of third order delay-differential equations*, Indian J. Pure Appl.

Math., **33** (2002), 1609–1620. 1

[36] C. Philos, C. *On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays*, Arch. Math. (Basel), **36** (1981), 168–178. 4

[37] G. Purushothaman, K. Suresh, E. Thandapani, E. Tunç, *Existence and bounds for Kneser-type solutions to noncanonical third-order neutral differential equations*, Electron. J. Differ. Equ., **2024** (2024), 13 pages. 1

[38] S. H. Saker, *Oscillation criteria of certain class of third-order nonlinear delay differential equations*, Math. Slovaca, **56** (2006), 433–450. 1

[39] A. Zafer, *Oscillatory and nonoscillatory properties of solutions of functional differential equations and difference equations*, ProQuest LLC, Ann Arbor, MI, (1992). 1

[40] C. Zhang, R. P. Agarwal, M. Bohner, T. Li, *New results for oscillatory behavior of even-order half-linear delay differential equations*, Appl. Math. Lett., **26** (2013), 179–183. 2.2