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A Refined form of the generalized Jensen inequality through an arbitrary function

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Abstract

This article is about a refined form of the generalized Jensen inequality in Riemann sense, which is obtained through a positive integrable function. Then manipulation of certain functions with suitable substitutions in the proposed result enables to establish new refined inequalities for the well-known Hölder, and celebrated Hermite–Hadamard inequalities along with power, and quasi-arithmetic means. Further the proposed result also gives refined estimates to Csiszár divergence and its particular variants.

Keywords: Jensen inequality, Hermite–Hadamard inequality, Hölder inequality, Csiszár divergence.

2020 MSC: 26A33, 26A51, 26D15.

1. Introduction

Mathematical inequalities and convexity are closely related concepts that help us to describe and set limits on behaviors across various fields. Convexity is a basic property of functions or shapes that curve upward smoothly, without any dips. If, we connect any two points on the graph of a convex function through a line, the line will always lie above or on the graph itself. This aspect allows us to establish inequalities because convex functions follow certain natural laws, for example Jensen's inequality, which relates the value of a function at an average point to the average of the values of corresponding function. By interpreting convexity, one can set useful boundaries and find the best solutions in different fields like chemistry, computer science, information theory, economics, optimization, and machine learning.

Over a century ago, Jensen introduced an inequality for convex functions that remains one of the most important and widely applied inequalities in mathematical literature, known as Jensen's inequality. It is significant because it serves as the foundation for deriving many well-known inequalities, including Hölder's, Young's, the arithmetic-geometric, Ky-Fan's, Levinson's, Beckenbach–Dresher's, and Minkowski's inequalities. This inequality has been instrumental in solving numerous problems across

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various fields, including engineering [8], information theory [4, 5, 22], mathematical statistics [15], computer science [17], financial economics [6], and mathematical biology [14]. Extensive research also explores its refinements [12], converse results [7], generalizations [19], extensions [24], and bounds for its gap [2, 3]. The following theorem presents a generalized form of this inequality in Riemann sense [13].

Theorem 1.1. *Let $f : I \to \mathbb{R}$ be a function that exhibits convexity. Let g and h be real-valued functions that can be integrated over the interval $[u, v]$. Then, for $g(x) > 0$ and $h(x) \in I$ for all $x \in [u, v]$ with $\int_u^v g(x)dx = G$, the aforesaid inequality follows*

$$f\left(\frac{1}{G}\int_u^v g(x)h(x)dx\right) \leqslant \frac{1}{G}\int_u^v g(x)\,(f \circ h)\,(x)dx.$$

A recent work about this inequality can be followed as, the paper [23] proposes new delay-dependent stability criteria for static neural networks with time-varying delays using an improved Lyapunov–Krasovskii functional, the celebrated Jensen inequality, and parameter-dependent linear matrix inequality. Some enhanced conditions are achieved via augmented vectors, and the achieved results are validated through numerical examples. The article [25] corrects a previous error regarding Jensen's inequality for the asymmetric Choquet integral, proving its validity. It also extends the result by establishing Jensen's inequality for the generalized asymmetric Choquet integral. The next article [11] focuses on finding new inequalities, such as midpoint, the Hermite–Hadamard, and trapezoid types by using fractional extended Riemann–Liouville integrals. It uses well-known mathematical tools like Jensen–Mercer, power-mean, and Hölder inequalities to prove the results. The paper also shows how these new results relate to earlier research and includes examples with graphs to explain and confirm the findings. Further the article [1] introduces improved versions of this inequality, using advanced concepts like 4-convex functions and Green functions. These improvements are developed for both sums (discrete forms) and integrals which make this inequality, its related versions, and the Jensen–Steffensen inequality, more precise. The article also enhances their reverse forms, leading to better ways to calculate key mathematical quantities such as different kinds of averages (power, geometric, and quasi-arithmetic means), the Hermite–Hadamard gap (which measures approximation errors), and the Hölder inequality. Additionally, these improved inequalities are used to find more accurate results for comparing data distributions (divergences) and for a concept in data analysis known as Zipf-Mandelbrot entropy. The article [18] discusses conditions that a real-valued function must meet to be considered preinvex. It also explains some features of preinvex functions and introduces new forms of Jensen's integral inequality for these functions. Several examples are included to understand the concepts. Furthermore, the article [12] explains a way to improve the integral Jensen inequality for finite signed measures using a concept called integral majorization inequalities. The method is unique for the conditions discussed and gives new ideas for working with measures. It also introduces specific improvements, including links to Jensen–Steffensen's inequality, and presents a new, extended version of this inequality. These results make it easier to apply and understand such inequalities. The contents presented in [9] enhance the Jensen inequality by using the concepts of majorization and convexity. The improved inequality leads to more precise bounds for power means and quasi-arithmetic means. The study also uses these refinements to derive bounds for divergences, Shannon entropy, and distance measures in the context of probability distributions. These bounds offer useful understandings for practical applications. This paper [10] expands on a method called cyclic refinement of the discrete Jensen inequality and applies it to integrals. It introduces a new way to improve the integral version of the inequality, which also includes some recent improvements as specific examples. These refined results are then applied to well-known inequalities, like the Fejér and Hermite–Hadamard inequalities, as well as to quasi-arithmetic means and f-divergences, showing their practical value. Similarly, the paper [20] presents a new way to improve the integral Jensen inequality using specific functions. It applies this improvement to various types of means, leading to new versions of the Hermite–Hadamard and Hölder inequalities. The paper also shows how this refined inequality can be used in information theory, such as in calculating entropy and divergences. At the end, a more general version of the refinement is introduced, making it applicable to a wider range of functions. This approach highlights the broader usefulness of the refined Jensen inequality in different areas of mathematics. Finally, the article [21] enhances the classical Jensen inequality by applying finite real sequences. This improvement leads to new inequalities for various means and also provides inequalities for Csiszár-divergence, as well as Shannon and Zipf–Mandelbrot entropies. The refinement is further expanded to include several finite real sequences, increasing its general applicability.

2. Main Results

The following notations will be used throughout the paper: $A^c = [u,v]\backslash A$ and $W_A = \int_A w(x)dx$, where A is a proper subinterval of $[u,v]$, and $w : [u,v] \to \mathbb{R}$ is a positive function.

The main result for the generalized Jensen inequality in the Riemann sense is presented in the following theorem.

Theorem 2.1. *Let* $f : I \to \mathbb{R}$ *be a convex function. Let* $w, g, h : [u,v] \to \mathbb{R}$ *be some integrable functions such that* $w(x), g(x) > 0$ *and* $h(x) \in I$ *for all* $x \in [u,v]$ *with* $\int_u^v g(x)dx = G$ *and* $\int_u^v w(x)dx = W$*. Also, assume that* K, L, M *are some proper subintervals of* $[u,v]$*. Then*

$$\begin{aligned}
f\left(\frac{1}{G}\int_u^v g(x)h(x)dx\right) \leqslant& \frac{1}{GW}\Bigg\{W_L G_K f\left(\frac{\int_K g(x)h(x)dx}{G_K}\right) + W_L G_{K^c} f\left(\frac{\int_{K^c} g(x)h(x)dx}{G_{K^c}}\right) \\
&+ W_{L^c} G_M f\left(\frac{\int_M g(x)h(x)dx}{G_M}\right) + W_{L^c} G_{M^c} f\left(\frac{\int_{M^c} g(x)h(x)dx}{G_{M^c}}\right)\Bigg\} \qquad (2.1)\\
\leqslant& \frac{1}{G}\int_u^v g(x)\,(f\circ h)\,(x)dx.
\end{aligned}$$

Proof. We start by noting that:

$$\begin{aligned}
f\left(\frac{1}{G}\int_u^v g(x)h(x)dx\right) &= f\left(\int_u^v w(x)dx \frac{1}{GW}\int_u^v g(x)h(x)dx\right) \\
&= f\left(\frac{1}{GW}\left(\int_L w(x)dx + \int_{L^c} w(x)dx\right)\int_u^v g(x)h(x)dx\right) \\
&= f\left(\frac{1}{GW}\left(\int_L w(x)dx\int_u^v g(x)h(x)dx + \int_{L^c} w(x)dx\int_u^v g(x)h(x)dx\right)\right) \\
&= f\Bigg(\frac{1}{GW}\Bigg(\int_L w(x)dx\left(\int_K g(x)h(x)dx + \int_{K^c} g(x)h(x)dx\right) \\
&\quad + \int_{L^c} w(x)dx\left(\int_M g(x)h(x)dx + \int_{M^c} g(x)h(x)dx\right)\Bigg)\Bigg) \\
&= f\Bigg(\frac{1}{GW}\Bigg(\int_L w(x)dx\int_K g(x)h(x)dx + \int_L w(x)dx\int_{K^c} g(x)h(x)dx \qquad (2.2)\\
&\quad + \int_{L^c} w(x)dx\int_M g(x)h(x)dx + \int_{L^c} w(x)dx\int_{M^c} g(x)h(x)dx\Bigg)\Bigg) \\
&= f\Bigg(\frac{1}{GW}\Bigg(\int_L w(x)dx\int_K g(x)dx\frac{\int_K g(x)h(x)dx}{\int_K g(x)dx} \\
&\quad + \int_L w(x)dx\int_{K^c} g(x)dx\frac{\int_{K^c} g(x)h(x)dx}{\int_{K^c} g(x)dx} \\
&\quad + \int_{L^c} w(x)dx\int_M g(x)dx\frac{\int_M g(x)h(x)dx}{\int_M g(x)dx} \\
&\quad + \int_{L^c} w(x)dx\int_{M^c} g(x)dx\frac{\int_{M^c} g(x)h(x)dx}{\int_{M^c} g(x)dx}\Bigg)\Bigg).
\end{aligned}$$

Applying the Jensen's inequality to (2.2) gives:

$$
\begin{aligned}
f\left(\frac{1}{G}\int_u^v g(x)h(x)dx\right) &\leqslant \frac{1}{GW}\Bigg(\int_L w(x)dx\int_K g(x)dx f\left(\frac{\int_K g(x)h(x)dx}{\int_K g(x)dx}\right)\\
&\quad+\int_L w(x)dx\int_{K^c} g(x)dx f\left(\frac{\int_{K^c} g(x)h(x)dx}{\int_{K^c} g(x)dx}\right)\\
&\quad+\int_{L^c} w(x)dx\int_M g(x)dx f\left(\frac{\int_M g(x)h(x)dx}{\int_M g(x)dx}\right)\\
&\quad+\int_{L^c} w(x)dx\int_{M^c} g(x)dx f\left(\frac{\int_{M^c} g(x)h(x)dx}{\int_{M^c} g(x)dx}\right)\Bigg)\\
&\leqslant \frac{1}{GW}\Bigg(\int_L w(x)dx\int_K g(x)dx\frac{1}{\int_K g(x)dx}\int_K g(x)f(h(x))dx\\
&\quad+\int_L w(x)dx\int_{K^c} g(x)dx\frac{1}{\int_{K^c} g(x)dx}\int_{K^c} g(x)f(h(x))dx\\
&\quad+\int_{L^c} w(x)dx\int_M g(x)dx\frac{1}{\int_M g(x)dx}\int_M g(x)f(h(x))dx\\
&\quad+\int_{L^c} w(x)dx\int_{M^c} g(x)dx\frac{1}{\int_{M^c} g(x)dx}\int_{M^c} g(x)f(h(x))dx\Bigg)\\
&= \frac{1}{GW}\Bigg(\int_L w(x)dx\left(\int_K g(x)f(h(x))dx+\int_{K^c} g(x)f(h(x))dx\right)\\
&\quad+\int_{L^c} w(x)dx\left(\int_M g(x)f(h(x))dx+\int_{M^c} g(x)f(h(x))dx\right)\Bigg)\\
&= \frac{1}{GW}\left(\int_L w(x)dx\int_u^v g(x)f(h(x))dx+\int_{L^c} w(x)dx\int_u^v g(x)f(h(x))dx\right)\\
&= \frac{1}{GW}\left(\int_u^v g(x)f(h(x))dx\left(\int_L w(x)dx+\int_{L^c} w(x)dx\right)\right)\\
&= \frac{1}{GW}\left(\int_u^v g(x)f(h(x))dx\int_u^v w(x)dx\right)\\
&= \frac{1}{G}\left(\int_u^v g(x)\,(f\circ h)\,(x)dx\right).
\end{aligned} \tag{2.3}
$$

Thus we get the required refinement (2.1) of Jensen inequality, the refined term can be seen in (2.3). □

The next corollary delivers a more refined form of the Hölder inequality:

Corollary 2.2. *Let* p,q *are real number greater than 1 with* $\frac{1}{p}+\frac{1}{q}=1$. *Also, let* $w,g_1,g_2:[u,v]\to\mathbb{R}$ *be positive functions such that* $g_1^p(x), g_2^q(x), g_1(x).g_2(x)$ *are integrable functions and* $W=\int_u^v w(x)dx$. *The following inequality holds by further assuming that* K,L,M *are some proper subintervals of* $[u,v]$.

$$
\begin{aligned}
\int_u^v g_1(x)g_2(x)dx &\leqslant \Bigg\{\frac{1}{W\int_u^v g_2^q(x)dx}\Bigg(W_L\int_K g_2^q(x)dx\left(\frac{\int_K g_1(x)g_2(x)dx}{\int_K g_2^q(x)dx}\right)^p\\
&\quad+W_L\int_{K^c} g_2^q(x)dx\left(\frac{\int_{K^c} g_1(x)g_2(x)dx}{\int_{K^c} g_2^q(x)dx}\right)^p+W_{L^c}\int_M g_2^q(x)dx\left(\frac{\int_M g_1(x)g_2(x)dx}{\int_M g_2^q(x)dx}\right)^p\\
&\quad+W_{L^c}\int_{M^c} g_2^q(x)dx\left(\frac{\int_{M^c} g_1(x)g_2(x)dx}{\int_{M^c} g_2^q(x)dx}\right)^p\Bigg)\Bigg\}^{\frac{1}{p}}\int_u^v g_2^q(x)dx\\
&\leqslant \left(\int_u^v g_1^p(x)dx\right)^{\frac{1}{p}}\left(\int_u^v g_2^q(x)dx\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.4}
$$

Proof. Employing the convex function $f(x) = x^p$, $x > 0$, $p > 1$ in (2.1) and substituting $g(x)$ by $g_2^q(x)$ and $h(x)$ by $g_1(x)g_2^{\frac{-q}{p}}(x)$, we get the following inequality

$$\begin{aligned}
\left(\frac{\int_u^v g_1(x)g_2(x)dx}{\int_u^v g_2^q(x)dx}\right)^p \leqslant \frac{1}{W\int_u^v g_2^q(x)dx}\Bigg\{ & W_L \int_K g_2^q(x)dx \left(\frac{\int_K g_1(x)g_2(x)dx}{\int_K g_2^q(x)dx}\right)^p \\
& + W_L \int_{K^c} g_2^q(x)dx \left(\frac{\int_{K^c} g_1(x)g_2(x)dx}{\int_{K^c} g_2^q(x)dx}\right)^p \\
& + W_{L^c} \int_M g_2^q(x)dx \left(\frac{\int_M g_1(x)g_2(x)dx}{\int_M g_2^q(x)dx}\right)^p \\
& + W_{L^c} \int_{M^c} g_2^q(x)dx \left(\frac{\int_{M^c} g_1(x)g_2(x)dx}{\int_{M^c} g_2^q(x)dx}\right)^p \Bigg\} \\
\leqslant \frac{\int_u^v g_1^p(x)dx}{\int_u^v g_2^q(x)dx}.
\end{aligned}$$

Multiplying the above inequality by $\left(\int_u^v g_2^q(x)dx\right)^p$ and then taking power $\frac{1}{p}$, we obtain the required inequalities in (2.4). □

The following corollary provides another refined inequality of the Hölder inequality:

Corollary 2.3. *Let $p, q \in \mathbb{R}$. Also, assume that $w, g_1, g_2 : [u, v] \to \mathbb{R}$ are positive functions such that $g_1^p(x)$, $g_2^q(x)$, and $g_1.g_2(x)$ are integrable functions and $W = \int_u^v w(x)dx$. Further assuming that K, L, M are some proper subintervals of $[u, v]$, then the following inequality holds*

(i). *The following inequality holds for $p > 1$, and $q = \frac{p}{p-1}$.*

$$\begin{aligned}
\int_u^v g_1(x)g_2(x)dx \leqslant \frac{1}{W}\Bigg\{ & W_L \left(\left(\int_K g_1^p(x)dx\right)^{\frac{1}{p}}\left(\int_K g_2^q(x)dx\right)^{\frac{1}{q}} + \left(\int_{K^c} g_1^p(x)dx\right)^{\frac{1}{p}}\left(\int_{K^c} g_2^q(x)dx\right)^{\frac{1}{q}}\right) \\
& + W_{L^c}\left(\left(\int_M g_1^p(x)dx\right)^{\frac{1}{p}}\left(\int_M g_2^q(x)dx\right)^{\frac{1}{q}} + \left(\int_{M^c} g_1^p(x)dx\right)^{\frac{1}{p}}\left(\int_{M^c} g_2^q(x)dx\right)^{\frac{1}{q}}\right)\Bigg\} \\
\leqslant \left(\int_u^v g_1^p(x)dx\right)^{\frac{1}{p}}\left(\int_u^v g_2^q(x)dx\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.5}$$

(ii). *The following inequality holds for $0 < p < 1$, and $q = \frac{p}{p-1}$.*

$$\begin{aligned}
\int_u^v g_1(x)g_2(x)dx \geqslant \frac{1}{W}\Bigg\{ & W_L \left(\left(\int_K g_1^p(x)dx\right)^{\frac{1}{p}}\left(\int_K g_2^q(x)dx\right)^{\frac{1}{q}} + \left(\int_{K^c} g_1^p(x)dx\right)^{\frac{1}{p}}\left(\int_{K^c} g_2^q(x)dx\right)^{\frac{1}{q}}\right) \\
& + W_{L^c}\left(\left(\int_M g_1^p(x)dx\right)^{\frac{1}{p}}\left(\int_M g_2^q(x)dx\right)^{\frac{1}{q}} + \left(\int_{M^c} g_1^p(x)dx\right)^{\frac{1}{p}}\left(\int_{M^c} g_2^q(x)dx\right)^{\frac{1}{q}}\right)\Bigg\} \\
\geqslant \left(\int_u^v g_1^p(x)dx\right)^{\frac{1}{p}}\left(\int_u^v g_2^q(x)dx\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.6}$$

(iii). *The inequality in (2.6) again holds for $p < 0$, and $q = \frac{p}{p-1}$.*

Proof.

(i). Using the concave function $f(x) = x^{\frac{1}{p}}$, $x > 0$ in (2.1) and substituting $g(x)$ by $g_2^q(x)$ and $h(x)$ by $g_1^p(x)g_2^{-q}(x)$, we get the desired result (2.5).

(ii). In this case the function $f(x) = x^{\frac{1}{p}}$ becomes convex, thus using (2.1) for this function and substituting $g(x)$ by $g_2^q(x)$ and $h(x)$ by $g_1^p(x)g_2^{-q}(x)$, we get the result in (2.6).

(iii). Given the specified conditions on p and q, it is evident that $0 < q < 1$, implying that this scenario represents a reflection of Case ii. Thus, substituting p, q, g_1, g_2 with q, p, g_2, g_1. After looking into the result, we see the same result as given in (2.6). □

The next corollary offers a more precise form of the Hermite-Hadamard inequality:

Corollary 2.4. *Consider $f : [u, v] \to \mathbb{R}$ as a convex function, and let $w : [u, v] \to \mathbb{R}$ denote a positive, integrable function satisfying $W = \int_u^v w(x)dx$. The following inequality holds by further assuming that $[a_1, b_1], [a_2, b_2]$, and L are some proper subintervals of $[u, v]$.*

$$
\begin{aligned}
f\left(\frac{u+v}{2}\right) &\leqslant \frac{1}{(v-u)W}\Bigg[W_L(b_1-a_1)f\left(\frac{a_1+b_1}{2}\right)+W_L(a_1-u+v-b_1)f\left(\frac{a_1^2-u^2+v^2-b_1^2}{2(a_1-u+v-b_1)}\right) \\
&\quad + W_{L^c}(b_2-a_2)f\left(\frac{a_2+b_2}{2}\right)+W_{L^c}(a_2-u+v-b_2)f\left(\frac{a_2^2-u^2+v^2-b_2^2}{2(a_2-u+v-b_2)}\right)\Bigg] \\
&\leqslant \frac{1}{v-u}\int_u^v f(x)dx.
\end{aligned} \tag{2.7}
$$

Proof. Substituting $g(x) = 1, h(x) = x, K = [a_1, b_1], M = [a_2, b_2]$ in (2.1), we obtain (2.7) □

Definition 2.5. The following formula defines the power mean of order $\lambda \in \mathbb{R}$ for two positive integrable functions g and h over the interval $[u, v]$:

$$
\mathcal{M}_\lambda(\mathbf{g}, \mathbf{h}) = \begin{cases} \left(\frac{1}{\int_u^v g(x)dx}\int_u^v g(x)h^\lambda(x)dx\right)^{\frac{1}{\lambda}}, & \lambda \neq 0, \\ \exp\left(\frac{\int_u^v g(x)\log h(x)dx}{\int_u^v g(x)dx}\right), & \lambda = 0. \end{cases}
$$

Also, if J is a proper subinterval of $[u, v]$, then

$$
\mathcal{M}_\lambda(\mathbf{g}, \mathbf{h}, J) = \begin{cases} \left(\frac{1}{\int_J g(x)dx}\int_J g(x)h^\lambda(x)dx\right)^{\frac{1}{\lambda}}, & \lambda \neq 0, \\ \exp\left(\frac{\int_J g(x)\log h(x)dx}{\int_J g(x)dx}\right), & \lambda = 0. \end{cases}
$$

The following corollary provides an estimate for power mean:

Corollary 2.6. *Assuming that g and h are two positive integrable functions defined on $[u, v]$, further with the condition that K, L, M are some proper subintervals of $[u, v]$, then for s, t such that $t \geqslant s$ the following inequalities for power mean exist*

$$
\begin{aligned}
\mathcal{M}_s(\mathbf{g}, \mathbf{h}) &\leqslant \Bigg[\frac{1}{GW}\Big(W_L G_K \mathcal{M}_s^t(\mathbf{g}, \mathbf{h}, K) + W_L G_{K^c}\mathcal{M}_s^t(\mathbf{g}, \mathbf{h}, K^c) \\
&\quad + W_{L^c}G_M\mathcal{M}_s^t(\mathbf{g}, \mathbf{h}, M) + W_{L^c}G_{M^c}\mathcal{M}_s^t(\mathbf{g}, \mathbf{h}, M^c)\Big)\Bigg]^{\frac{1}{t}} \\
&\leqslant \mathcal{M}_t(\mathbf{g}, \mathbf{h}), \quad t \neq 0.
\end{aligned} \tag{2.8}
$$

$$
\begin{aligned}
\mathcal{M}_s(\mathbf{g}, \mathbf{h}) &\leqslant \exp\Bigg[\frac{1}{GW}\Big(W_L G_K \log\mathcal{M}_s(\mathbf{g}, \mathbf{h}, K) + W_L G_{K^c}\log\mathcal{M}_s(\mathbf{g}, \mathbf{h}, K^c) \\
&\quad + W_{L^c}G_M\log\mathcal{M}_s(\mathbf{g}, \mathbf{h}, M) + W_{L^c}G_{M^c}\log\mathcal{M}_s(\mathbf{g}, \mathbf{h}, M^c)\Big)\Bigg] \\
&\leqslant \mathcal{M}_t(\mathbf{g}, \mathbf{h}), \quad t = 0.
\end{aligned} \tag{2.9}
$$

$$\begin{aligned}\mathcal{M}_s(\mathbf{g},\mathbf{h}) \leqslant &\Big[\frac{1}{GW}\Big(W_L G_K \mathcal{M}_t^s(\mathbf{g},\mathbf{h},K) + W_L G_{K^c}\mathcal{M}_t^s(\mathbf{g},\mathbf{h},K^c)\\ &+ W_{L^c} G_M \mathcal{M}_t^s(\mathbf{g},\mathbf{h},M) + W_{L^c} G_{M^c}\mathcal{M}_t^s(\mathbf{g},\mathbf{h},M^c)\Big)\Big]^{\frac{1}{s}}\\ &\leqslant \mathcal{M}_t(\mathbf{g},\mathbf{h}), \quad s \neq 0.\end{aligned} \tag{2.10}$$

$$\begin{aligned}\mathcal{M}_s(\mathbf{g},\mathbf{h}) \leqslant \exp &\Big[\frac{1}{GW}\Big(W_L G_K \log \mathcal{M}_t(\mathbf{g},\mathbf{h},K) + W_L G_{K^c}\log \mathcal{M}_t(\mathbf{g},\mathbf{h},K^c)\\ &+ W_{L^c} G_M \log \mathcal{M}_t(\mathbf{g},\mathbf{h},M) + W_{L^c} G_{M^c}\log \mathcal{M}_t(\mathbf{g},\mathbf{h},M^c)\Big)\Big]\\ &\leqslant \mathcal{M}_t(\mathbf{g},\mathbf{h}), \quad s = 0.\end{aligned} \tag{2.11}$$

Proof. Assume that $f(x) = x^{\frac{t}{s}}$ for the given real numbers s and t with the condition that $t \geqslant s$. Also, we have $f''(x) = \frac{t}{s}\left(\frac{t}{s}-1\right)x^{\frac{t}{s}-2}$. Here we have exactly three cases, first $t > 0, s < 0$, second $t > 0, s > 0$, and third $t < 0$, $s < 0$. In the first and second cases, it is straightforward to verify that the function f is convex, hence, we can apply (2.1) to this function with $h \to h^s$ and finally taking the power $\frac{1}{t}$, we get (2.8). In the third case the assumed function becomes concave, hence this time using the main result provided in inequality (2.1) for concave function, letting $h \to h^s$, and finally taking the power $\frac{1}{t}$ for $t < 0$, we obtain the same inequality (2.8).

Adopting the same procedure for the function $f(x) = x^{\frac{s}{t}}$, $x > 0$, letting $h \to h^t$ and taking power as $\frac{1}{s}$, we obtain (2.10) using (2.1).

The inequalities in (2.9) and (2.11) can be deduced by taking $t \to 0$ and $s \to 0$ in (2.8) and (2.10) respectively. □

Definition 2.7. For g as a positive and h as any integrable function defined on $[u, v]$, and ψ as a strictly monotone continuous function whose domain is contained within the image of h, the quasi-arithmetic mean is defined as

$$\mathcal{M}_\psi(\mathbf{g},\mathbf{h}) = \psi^{-1}\left(\frac{1}{\int_u^v g(x)dx}\int_u^v g(x)\psi(h(x))dx\right).$$

Also, if J is a proper subinterval of $[u, v]$, then

$$\mathcal{M}_\psi(\mathbf{g},\mathbf{h},J) = \psi^{-1}\left(\frac{1}{\int_J g(x)dx}\int_J g(x)\psi(h(x))dx\right).$$

The following corollary provides an inequality for quasi-arithmetic mean:

Corollary 2.8. *Let $g, h : [u, v] \to \mathbb{R}$ be integrable functions, where it is required that $g(x) > 0$, for each $x \in [u, v]$. Assume that ψ is a strictly monotone continuous function whose domain belong to the image of the function h. In the case where $p \circ \psi^{-1}$ is convex, the following inequality holds true*

$$\begin{aligned}p\left(\mathcal{M}_\psi(\mathbf{g},\mathbf{h})\right) \leqslant \frac{1}{GW}&\Big[W_L G_K p\left(\mathcal{M}_\psi(\mathbf{g},\mathbf{h},K)\right) + W_L G_{K^c} p\left(\mathcal{M}_\psi(\mathbf{g},\mathbf{h},K^c)\right)\\ &+ W_{L^c} G_M p\left(\mathcal{M}_\psi(\mathbf{g},\mathbf{h},M)\right) + W_{L^c} G_{M^c} p\left(\mathcal{M}_\psi(\mathbf{g},\mathbf{h},M^c)\right)\\ &\leqslant \frac{1}{G}\int_u^v g(x)p(h(x))dx.\end{aligned} \tag{2.12}$$

Proof. Using (2.1) for $h \to \psi \circ h$ and $f \to p \circ \psi^{-1}$, we obtain the above inequality. □

3. Applications in Information Theory

Information Theory helps us to understand how information is stored, shared, and compressed. This notion was introduced by Claude Shannon in 1948 to explain how messages travel through noisy channels and how uncertainty can be measured using entropy. Key ideas include Shannon's entropy, which measures mutual information, and unpredictability, which shows how much two different things can be shared in common. Most important concept is Csiszár divergence, which builds on Kullback-Leibler divergence to compare different probability distributions as well as it can yield entropies directly by manipulating certain substitutions [13]. These ideas are widely used in communication systems, data compression, error correction, and cyber security. They also play a big role in artificial intelligence, neuroscience, and language processing. The following definition is from [16]:

Definition 3.1. Let $[\alpha,\beta]$ be a real-number interval, and let $f:[\alpha,\beta]\to\mathbb{R}$ be a function. Also let $s:[u,v]\to[\alpha,\beta]$, $t:[u,v]\to(0,\infty)$ be two densities such that $\frac{s(x)}{t(x)}\in[\alpha,\beta]$ for all $x\in[u,v]$, the Csiszár divergence is then given by

$$D_c(s,t)=\int_u^v t(x)f\left(\frac{s(x)}{t(x)}\right)dx.$$

Theorem 3.2. *Let $f:[\alpha,\beta]\to\mathbb{R}$ be a function. Assuming a positive function $w:[u,v]\to\mathbb{R}$ and considering that $W=\int_u^v w(x)dx$. Further, assume that $s:[u,v]\to[\alpha,\beta]$ and $t:[u,v]\to(0,\infty)$ be two functions and $\frac{s(x)}{t(x)}\in[\alpha,\beta]$ for all $x\in[u,v]$. Also, assuming that K,L,M are some proper subintervals of $[u,v]$ then*

$$\begin{aligned}\int_u^v t(x)dxf\left(\frac{\int_u^v s(x)dx}{\int_u^v t(x)dx}\right)\leqslant&\frac{1}{W}\Bigg\{W_L\int_K t(x)dxf\left(\frac{\int_K s(x)dx}{\int_K t(x)dx}\right)+W_L\int_{K^c} t(x)dxf\left(\frac{\int_{K^c} s(x)dx}{\int_{K^c} t(x)dx}\right)\\&+W_{L^c}\int_M t(x)dxf\left(\frac{\int_M s(x)dx}{\int_M t(x)dx}\right)+W_{L^c}\int_{M^c} t(x)dxf\left(\frac{\int_{M^c} s(x)dx}{\int_{M^c} t(x)dx}\right)\Bigg\}\\\leqslant&D_c(s,t).\end{aligned}\tag{3.1}$$

Proof. Putting $h(x)=\frac{s(x)}{t(x)}$, $g(x)=t(x)$, and $[\alpha,\beta]=I$ in (2.1), one can obtain (3.1). □

Definition 3.3. Given a positive density function $t(x)$ on the interval $[u,v]$, the Shannon entropy is calculated as

$$S_e(t)=-\int_u^v t(x)\log(t(x))\,dx.$$

Corollary 3.4. *Assuming a positive function $w:[u,v]\to\mathbb{R}$ and considering that $\int_u^v w(x)dx=W$. Also, $t:[u,v]\to(0,\infty)$ is a density function such that $\frac{1}{t(x)}\in[\alpha,\beta]\subset\mathbb{R}^+$ for all $x\in[u,v]$. Then the following inequality holds by further assuming that K,L,M are some proper subintervals of $[u,v]$.*

$$\begin{aligned}-\log(v-u)\leqslant&\frac{1}{W}\Bigg\{W_L\int_K t(x)dx\log\frac{\int_K t(x)dx}{m(K)}+W_L\int_{K^c} t(x)dx\log\frac{\int_{K^c} t(x)dx}{m(K^c)}\\&+W_{L^c}\int_M t(x)dx\log\frac{\int_M t(x)dx}{m(M)}+W_{L^c}\int_{M^c} t(x)dx\log\frac{\int_{M^c} t(x)dx}{m(M^c)}\Bigg\}\\\leqslant&-S_e(t).\end{aligned}\tag{3.2}$$

Proof. Manipulating $f(x)=-\log x$, $x\in[\alpha,\beta]$, and $s(x)=1$, in (3.1), one get (3.2). □

Definition 3.5. For two positive density functions say $s(x)$ and $t(x)$ defined on $[u,v]$, the Kullback-Leibler divergence is given by

$$D_{KL}^{(s,t)}=\int_u^v s(x)\log\left(\frac{s(x)}{t(x)}\right)dx.$$

Corollary 3.6. *Assuming a positive function* $w : [u, v] \to \mathbb{R}$ *and considering that* $W = \int_u^v w(x)dx$. *Further, assume that* s, t *are some densities defined on* $[u, v]$ *such that* $\frac{s(x)}{t(x)} \in [\alpha, \beta] \subset \mathbb{R}^+$ *for all* $x \in [u, v]$. *Also, assuming that* K, L, M *are some proper subintervals of* $[u, v]$ *then*

$$\begin{aligned} 0 \leqslant & \frac{1}{W}\left\{ W_L \int_K s(x)dx \log \frac{\int_K s(x)dx}{\int_K t(x)dx} + W_L \int_{K^c} s(x)dx \log \frac{\int_{K^c} s(x)dx}{\int_{K^c} t(x)dx} \right. \\ & \left. + W_{L^c} \int_M s(x)dx \log \frac{\int_M s(x)dx}{\int_M t(x)dx} + W_{L^c} \int_{M^c} s(x)dx \log \frac{\int_{M^c} s(x)dx}{\int_{M^c} t(x)dx} \right\} \\ \leqslant & D_{KL}^{(s,t)}. \end{aligned} \tag{3.3}$$

Proof. Manipulating $f(x) = x \log x$, $x \in [\alpha, \beta]$ in (3.1), one get (3.3). □

Remark 3.7. As above, similar inequalities for variational distance, Jeffrey's divergence, Bhattacharyya coefficients, and Hellinger distance can be established.

4. Conclusion

The article concludes by highlighting the significance of refining the generalized Jensen inequality in the Riemann sense through a positive integrable function. This refined inequality framework serves as a foundation for deriving enhanced versions of several classical inequalities. Appropriate function manipulations and substitutions within this framework have enabled to establish new refined forms of the Hölder and Hermite-Hadamard inequalities, along with improvements to power and quasi-arithmetic means, providing greater accuracy in these contexts.

Furthermore, the implications of the refined inequality extend to information theory and related fields by yielding improved estimates for the Csiszár divergence, an important measure in probability and statistics, as well as for its various specific forms. This advancement allows for more precise estimates and tighter bounds in applications where Csiszár divergence and similar measures play a role, enhancing analytical accuracy and potentially impacting broader areas where such inequalities are applied. The study, therefore, not only contributes refined mathematical tools but also broadens the practical applications of these classical inequalities across different disciplines.

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