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Existence and Stability of solution to a fractional mathematical model of the brain metabolite variations in the circadian rhythm containing Caputo derivative

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Abstract

This study intends to investigate whether a solution to the model of brain metabolite variations in the circadian rhythm exists. The model is a fractional-order important model with regard to applications. The existence of a unique solution to the fractional brain model (FBM) with Caputo derivative is demonstrated, and the general form of the model is provided. A stability analysis is performed on the solution. In four different cases, the numerical solution to FBM is given using the proposed numerical method.

Keywords: Brain metabolites, fractional mathematical model, circadian rhythm, existence and uniqueness, fractional derivatives, Caputo derivative, proposed numerical method.

2020 MSC: 34A12, 34A34, 45G05, 92C50.

1. Introduction

Circadian rhythms are intrinsic biological processes that oscillate approximately every 24 hours, regulating various physiological functions, including sleep-wake cycles, hormone release, cognitive performance, and metabolism. These rhythms are vital for maintaining overall health and well-being, as they influence not only daily behaviors but also the underlying biochemical processes in the brain. Recent studies have revealed that variations in brain metabolites—substances produced during metabolic processes in the brain—play a crucial role in cognitive functioning and emotional regulation. Moreover, fluctuations in these metabolites can significantly impact conditions such as sleep disorders, depression, and neurodegenerative diseases [13].

Understanding the dynamics of brain metabolite variations in relation to circadian rhythms is critical for several reasons. First, it provides insights into the mechanisms by which these metabolites affect cognitive and emotional health. Second, disruptions in circadian rhythms are often linked to various psychiatric and neurological disorders, underscoring the need to explore how metabolite variations correlate with circadian cycles [1, 5, 10, 12, 18]. Despite the acknowledged importance of brain metabolites,

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previous research has primarily focused on their roles in static conditions, leaving a gap in the literature regarding their fluctuations over time and their implications for circadian rhythms.

This research aims to address this gap by investigating a fractional-order model of brain metabolite variations within circadian cycles. By employing a Proposed Numerical Method (PNM), as introduced by El-Sayed and Ziada [8], we will analyze these variations more comprehensively, incorporating the complexities introduced by fractional derivatives. This approach allows for a more nuanced understanding of the temporal dynamics of brain metabolism [9, 14–17, 20].

In recent years, the application of fractional calculus has gained significant attention in various fields, including mathematical modeling of biological systems. Notably, a new fractional model and optimal control of tumor-immune surveillance has been proposed, utilizing a non-singular derivative operator [2]. Additionally, stability analysis and system properties of Nipah virus transmission have been explored through a fractional calculus lens [3]. Furthermore, the concept of fractional treatment has been applied to an accelerated mass-spring system, demonstrating the versatility of fractional models in different contexts [6]. These developments provide a broader context for the findings of this study.

The objectives of this study are threefold:

1. To demonstrate the existence of a unique solution to the fractional brain model (FBM) with Caputo derivative.
2. To conduct a stability analysis of the solution, ensuring its reliability in various scenarios.
3. To provide numerical solutions for different cases that reflect real-world biological variations.

This research will be continued as follows: Section 2 is a presentation of the general form of FBM and its main definitions with properties. Section 3 is for proving the existence of the solution. Section 4, for proving the uniqueness of the solution. Section 5, for discussing the stability of the solution. Section 6, for giving the numerical solution of the FBM in four different cases.

2. Brain Model

The brain metabolism variations in the circadian models are discussed for integer orders in [13]. He takes the integer order brain model as the ODEs:

$$\acute{\chi} + \frac{k\chi}{\acute{k}+\chi} = a\sin(bt+c),\ \chi(0) = \chi_0, \quad a,b,c,k,\acute{k} > 0. \tag{2.1}$$

Here, we will deal with the FBM as

$${}^{C}\check{D}^{\beta}\chi + \frac{k\chi}{\acute{k}+\chi} = a\sin(bt+c),\ \chi(0) = \chi_0, \quad a,b,c,k,\acute{k} > 0, \tag{2.2}$$

where ${}^{C}\check{D}^{\beta}(.)$ denotes the CD.

2.1. Brain model in general form

The general form of the FBM will be

$${}^{C}\check{D}^{\beta}\chi(\tau) - F(\tau,\chi(\tau)) = g(\tau), \quad \tau \in (0,T], \beta \in (0,1), \tag{2.3}$$

$$\chi(0) = \chi_0, \tag{2.4}$$

where if we take $\left[F(\tau,\chi(\tau)) = \frac{k\chi}{\acute{k}+\chi}, g(\tau) = a\sin(bt+c)\right]$, it will give our model.

2.2. The main definitions and properties

We take our problems under the following hypothesis:

(i) $F : J \times R \to R$ is continuous and measurable in τ for any $\chi \in R$.

(ii) There exists a bounded measurable function $h(\tau) \in L_1(J)$ and $\exists$ a positive constant m s.t. $|F(\tau,\chi)| \leqslant h(\tau) + m|\chi|$.

(iii) $g, h : J \times J$, where $|g(\tau)| < G, |h(\tau)| < H$.

(iv) Define a mapping $\Phi : \check{E} \to \check{E}$, such that $\check{E}$ is a Banach space of continuous functions on J, and $\|\chi\| = \sup_{\tau\in J} |\chi(\tau)|$.

Definition 2.1. The definition of the CD of order υ is [16]

$$ {}_0^C D_t^{\upsilon} f(t) = \frac{1}{\Gamma(n-\upsilon)} \int_0^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\upsilon-n+1}}, \quad n-1 < \upsilon < n, \tag{2.5} $$

and its corresponding FI is [16]

$$ {}^C I^{\upsilon} F(t) = \frac{1}{\Gamma(\upsilon)} \int_0^t F(\tau)(t-\tau)^{\upsilon-1} d\tau, \quad 0 < \upsilon < 1. \tag{2.6} $$

Moreover,

$$ \left({}^C I^{\upsilon}\right)\left({}^C D^{\upsilon}\right) F(t) = F(t) - F(a). \tag{2.7} $$

This definition is considered one of the classical FDs and is one of the most well-known and famous FDs [16].

Justifications for Choosing the Caputo Derivative:

1. Initial Conditions Compatibility:
   Advantage: The Caputo derivative allows for the specification of initial conditions in the same manner as integer-order differential equations. This is beneficial for modeling real-world problems where initial values are critical, such as in brain metabolite dynamics.
2. Physical Interpretability:
   Advantage: The Caputo derivative has a clear physical interpretation, making it suitable for applications in biological systems. It reflects the memory effect inherent in fractional calculus, which is relevant in modeling biological processes that depend on past states.
3. Flexibility in Modeling:
   Advantage: The Caputo derivative can effectively capture the behavior of systems exhibiting nonlocality and hereditary properties. This is particularly useful in brain metabolism studies, where past metabolite levels influence current dynamics.
4. Established Framework:
   Advantage: The Caputo derivative has a well-established theoretical framework and numerous applications in various fields, including physics and engineering. Its extensive use in literature provides a solid foundation for validating models and results.
5. Numerical Methods Compatibility:
   Advantage: Many numerical methods have been specifically developed for the Caputo derivative, facilitating efficient computational solutions. This compatibility enhances the robustness of the proposed numerical method in the study.

From the above we can conclude that choosing the Caputo derivative for modeling brain metabolite variations within circadian rhythms provides significant advantages in terms of initial conditions, physical interpretability, modeling flexibility, and numerical solution compatibility. These factors strengthen the validity of the research and its potential contributions to understanding complex biological systems.

3. Existence of the solution

Operating with ${}^{C}I^{\beta}$ to both sides of the FDE (2.3)-(2.4), we get the following fractional integral equation (FIE),

$$\chi(\tau) = \chi_0 + {}^{C}I^{\beta}\left[F(\tau,\chi(\tau)) + g(\tau)\right], \tag{3.1}$$

where $\tau \in J = (0,T], \tau \in R^{+}, F(\tau,\chi)$ is continuous function satisfies Lipschitz condition

$$|F(\tau,\chi) - F(\tau,z)| \leqslant c_1 |\chi - z|, \tag{3.2}$$

where c_1 is the Lipschitz constant.

Theorem 3.1 ([19]). *Let the hypothesis* $(i)-(iv)$ *be satisfied, then there exists at least one solution* $\chi \in C(J)$ *of the problem* (2.3)-(2.4).

Proof. Define the set $Q_{\check{r}} = \{\chi \in C(J) : |\chi(\tau)| < \check{r}\}$, where $\check{r} = \frac{\chi_0 + \frac{[H+G]T^{\beta}}{\Gamma(\beta+1)}}{1-mT^{\beta}}$. Let $\chi \in Q_{\check{r}}$, then we get

$$\begin{aligned}
|\mathcal{F}\chi(\tau)| &= \left|\chi_0 + {}^{C}I^{\beta}\left[F(\tau,\chi(\tau)) + g(\tau)\right]\right| \\
&\leqslant |\chi_0| + \left|{}^{C}I^{\beta}\left[F(\tau,\chi(\tau)) + g(\tau)\right]\right| \\
&\leqslant |\chi_0| + \frac{1}{\Gamma(\beta)}\left|\int_0^{\tau}\left|(\tau-\xi)^{\beta-1}\right| F(\tau,\chi(\tau)) + g(\tau)\right| d\xi \\
&\leqslant |\chi_0| + \frac{1}{\Gamma(\beta)}\int_0^{\tau}\left|(\tau-\xi)^{\beta-1}\right| \left[|F(\tau,\chi(\tau))| + |g(\tau)|\right] d\xi \\
&\leqslant |\chi_0| + \frac{1}{\Gamma(\beta)}\int_0^{\tau}\left|(\tau-\xi)^{\beta-1}\right| \left[|h(\tau)| + m|\chi(\xi)| + G\right] d\xi \\
&\leqslant |\chi_0| + \frac{[H+m\check{r}+G]}{\Gamma(\beta)}\int_0^{\tau}\left|(\tau-\xi)^{\beta-1}\right| d\xi \\
&\leqslant |\chi_0| + [H+m\check{r}+G]\frac{T^{\beta}}{\beta\Gamma(\beta)} \\
&\leqslant \chi_0 + \frac{[H+m\check{r}+G]T^{\beta}}{\Gamma(\beta+1)} = \check{r},
\end{aligned}$$

and $\{\mathcal{F}\chi(\tau)\} \subset Q_{\check{r}}$.

Now let $\chi \in Q_{\check{r}}$. Let $\tau_2, \tau_1 \in J$ s.t. $\tau_2 > \tau_1 > 0$ and $|\tau_2 - \tau_1| < \delta$, then

$$\begin{aligned}
|\mathcal{F}\chi(\tau_2) - \mathcal{F}\chi(\tau_1)| &= {}^{C}I^{\beta}\left[F(\tau_2,\chi(\tau_2)) + g(\tau_2)\right] - {}^{C}I^{\beta}\left[F(\tau_1,\chi(\tau_1)) + g(\tau_1)\right] \\
&= {}^{C}I^{\beta}\left|\left[F(\tau_2,\chi(\tau_2)) + g(\tau_2)\right] - \left[F(\tau_1,\chi(\tau_1)) + g(\tau_1)\right]\right| \\
&= {}^{C}I^{\beta}\left|\left[F(\tau_2,\chi(\tau_2)) - F(\tau_1,\chi(\tau_1))\right] + \left[g(\tau_2) - g(\tau_1)\right]\right| \\
&\leqslant {}^{C}I^{\beta}\left[|F(\tau_2,\chi(\tau_2)) - F(\tau_1,\chi(\tau_1))| + |g(\tau_2) - g(\tau_1)|\right] \\
&\leqslant {}^{C}I^{\beta}\left[\eta\check{r}|\tau_2 - \tau_1| + G|\tau_2 - \tau_1|\right] \\
&\leqslant [\eta\check{r}+G]\ |\tau_2 - \tau_1|\,{}^{C}I^{\beta}(1) \\
&\leqslant [\eta\check{r}+G]\ |\tau_2 - \tau_1|\frac{1}{\Gamma(\beta)}\int_0^{T}(\tau-\xi)^{\beta-1}\,d\xi \\
&\leqslant \frac{[\eta\check{r}+G]\ \delta T^{\beta}}{\Gamma(\beta+1)},
\end{aligned}$$

where $\eta = \sup\{|F(\tau_2,\chi(\tau_2)) - F(\tau_1,\chi(\tau_1))|\}$. Then $\{\mathcal{F}\chi(\tau)\}$ is equi-continuous on J and by Arzela-Ascoli Theorem [4] $\{\mathcal{F}\chi(\tau)\}$ is relatively compact. Then $\mathcal{F}$ is compact [7].

Let $\{\chi_m(\tau)\} \subset \mathbb{Q}_{\check{r}}$ s.t. $\chi_\kappa \to \chi_0$, then

$$\begin{aligned}
\mathcal{F}\chi_\kappa(\tau) &= \chi_0 + {}^{C}I^{\beta}\left[F(\tau,\chi_\kappa(\tau)) + g(\tau)\right] \\
\lim_{\kappa\to\infty}\mathcal{F}\chi_\kappa(\tau) &= \chi_0 + \lim_{\kappa\to\infty}{}^{C}I^{\beta}\left[F(\tau,\chi_\kappa(\tau)) + g(\tau)\right] \\
\lim_{\kappa\to\infty}\mathcal{F}\chi_\kappa(\tau) &= \chi_0 + {}^{C}I^{\beta}\left[\lim_{\kappa\to\infty}F(\tau,\chi_\kappa(\tau)) + g(\tau)\right] \\
\lim_{\kappa\to\infty}\mathcal{F}\chi_\kappa(\tau) &= \chi_0 + {}^{C}I^{\beta}\left[F\left(\tau,\lim_{\kappa\to\infty}\chi_\kappa(\tau)\right) + g(\tau)\right] \\
\lim_{\kappa\to\infty}\mathcal{F}\chi_\kappa(\tau) &= \chi_0 + {}^{C}I^{\beta}\left[F\left(\tau,\lim_{\kappa\to\infty}\chi_0(\tau)\right) + g(\tau)\right] \\
&= \mathcal{F}\chi_0(\tau).
\end{aligned}$$

Then $\mathcal{F}$ is continuous.

By Schauder's fixed point Theorem [11], $\exists$ at least one solution $\chi \in \mathbb{Q}_{\check{r}} \subset \mathbb{C}(J)$ to FIE (3.1). Consequently, $\exists$ at least one solution $\chi \in \mathbb{C}(J)$ to the FDE (2.3)-(2.4). □

4. Uniqueness of the solution

Theorem 4.1 ([19]). *Let $F(\tau,\chi)$ satisfy the Lipschitz condition (3.2). If $T^{\beta} < \frac{\Gamma(\beta+1)}{c_1}$, then the FIE (3.1) has a unique solution $\chi \in \mathbb{C}(J)$.*

Proof. From (3.1), the mapping Φ will be defined as

$$\Phi\chi = \chi_0 + {}^{C}I^{\beta}\left[F(\tau,\chi(\tau)) + g(\tau)\right].$$

Let $\chi, z \in \check{E}$, then

$$\Phi\chi - \Phi z = {}^{C}I^{\beta}F(\tau,\chi(\tau)) - {}^{C}I^{\beta}F(\tau,z(\tau)),$$

$$\begin{aligned}
\|\Phi\chi - \Phi z\| &\leqslant \sup_{\tau\in J}{}^{C}I^{\beta}\left|F(\tau,\chi(\tau)) - F(\tau,z(\tau))\right| \\
&\leqslant c_1\|\chi - z\|\frac{1}{\Gamma(\beta)}\int_0^{\tau}(\tau-\xi)^{\beta-1}\,d\xi \\
&\leqslant \frac{c_1 T^{\beta}}{\Gamma(\beta+1)}\|\chi - z\| \\
&\leqslant N_1\|\chi - z\|.
\end{aligned}$$

Under the condition $0 < N_1 < 1$, the mapping Φ is contraction, and hence for $T^{\beta} < \frac{\Gamma(\beta+1)}{c_1}$ there exists a unique solution $\chi \in \mathbb{C}(J)$. □

Corollary 4.2. *Consider the NDE:*

$$\frac{d\chi(\tau)}{d\tau} - F(\tau,\chi(\tau)) = g(\tau),\ \chi(0) = \chi_0. \tag{4.1}$$

Operating with $I(.) = \int_0^{\tau}(.)\,d\tau$ to both sides of equation (4.1), we get

$$\chi(\tau) = \chi_0 + \int_0^{\tau}\left[F(\xi,\chi(\xi)) + g(\xi)\right]d\xi = \chi_0 + I\left[F(\tau,\chi(\tau)) + g(\tau)\right]. \tag{4.2}$$

In problem (2.3), let $\beta \to 1$, then the NIE (4.2) which is equivalent to the NDE (4.1) has a unique solution $\chi \in \mathbb{C}(J)$, if $T < \frac{1}{c_1}$.

5. Stability

Theorem 5.1. *Let the hypothesis* (i)-(iv) *be satisfied and* $\exists$ *a unique solution of the FDE* (2.3)-(2.4)*, then its solution is uniformly stable.*

Proof. Let $\tilde{\chi}$ is also a solution to the FDE (2.3)-(2.4)

$$\begin{aligned}
|\chi(\tau)-\tilde{\chi}(\tau)| &= \left|\left[\chi_0 + {}^{C}I^{\beta}\left[F(\tau,\chi(\tau))\right]\right]-\left[\tilde{\chi}_0 + {}^{C}I^{\beta}\left[F(\tau,\chi_s(\tau))\right]\right]\right| \\
&\leqslant |\chi_0-\tilde{\chi}_0| + {}^{C}I^{\beta}\left|F(\tau,\chi(\tau))-F(\tau,\tilde{\chi}(\tau))\right| \\
&\leqslant |\chi_0-\tilde{\chi}_0| + c_1\,{}^{C}I^{\beta}\left|\chi(\tau)-\tilde{\chi}(\tau)\right| \\
&\leqslant |\chi_0-\tilde{\chi}_0| + c_1\left|\chi(\tau)-\tilde{\chi}(\tau)\right|\frac{1}{\Gamma(\beta)}\int_0^{\tau}(\tau-\text{ş})^{\beta-1}\,d\text{ş} \\
&\leqslant |\chi_0-\tilde{\chi}_0| + \left|\chi(\tau)-\tilde{\chi}(\tau)\right|\frac{c_1T^{\beta}}{\beta\Gamma(\beta)} \\
&\leqslant |\chi_0-\tilde{\chi}_0| + \frac{c_1T^{\beta}}{\Gamma(\beta+1)}\left|\chi(\tau)-\tilde{\chi}(\tau)\right| \\
&\leqslant |\chi_0-\tilde{\chi}_0| + \frac{c_1T^{\beta}}{\Gamma(\beta+1)}\left|\chi(\tau)-\tilde{\chi}(\tau)\right|,
\end{aligned}$$

$$\begin{aligned}
\left|\chi(\tau)-\tilde{\chi}(\tau)\right| - \frac{c_1T^{\beta}}{\Gamma(\beta+1)}\left|\chi(\tau)-\tilde{\chi}(\tau)\right| &\leqslant |\chi_0-\tilde{\chi}_0|, \\
\left[1-\frac{c_1T^{\beta}}{\Gamma(\beta+1)}\right]\left|\chi(\tau)-\tilde{\chi}(\tau)\right| &\leqslant |\chi_0-\tilde{\chi}_0|, \\
\left|\chi(\tau)-\tilde{\chi}(\tau)\right| &\leqslant \left[1-\frac{c_1T^{\beta}}{\Gamma(\beta+1)}\right]^{-1}|\chi_0-\tilde{\chi}_0|.
\end{aligned}$$

If $\left[1-\frac{c_1T^{\beta}}{\Gamma(\beta+1)}\right]^{-1} < \delta(\varepsilon)$, then $|\chi(\tau)-\tilde{\chi}(\tau)| < \varepsilon$, which implies that the solution of the FDE (2.3)-(2.4) is uniformly stable. □

6. Proposed numerical method (El-Sayed method)

This method was introduced for the first time by El-Sayed and Ziada in [8] and they used it to overcome the shortage in the numerical method given by Podlubny in ([16], chapter 8). The numerical method in [16] has a disadvantage; it solves only FDE with homogenous initial conditions (zeros initial conditions). The El-Sayed method (EM) was created to overcome this disadvantage.

6.1. *Steps for the solution:*

Step 1: *Using substitution to transform the initial conditions to be homogenous,* this substitution is

$$\chi = \sum_{j=0}^{n-1} c_j\frac{\tau^j}{j!} + X. \tag{6.1}$$

Step 2: *Obtain the solution algorithm* using the following relations:

$$D^{\alpha_i}X = h^{-\alpha_i}\sum_{j=0}^{m} w_j^{(\alpha_i)}X_{m-j}, \quad w_j^{(\alpha_i)} = (-1)^j\frac{\Gamma(\alpha_i+1)}{\Gamma(\alpha_i+1-j)},$$

$$\tau_m = mh \ \ (m = 0, 1, 2, \ldots), \ \ X(\tau) = X_m(\tau_m), \tag{6.2}$$

to obtain the solution algorithm, then get the values of X_m.

Step 3: *Return to the original variables.* From relation (6.1) we get

$$\chi_m = \sum_{j=0}^{n-1} c_j \frac{(\tau_m)^j}{j!} + X_m. \tag{6.3}$$

7. Examples and applications

Example 7.1. Consider the FBM:

$$\begin{aligned} {}^{C}\check{D}^{\beta}\chi + \frac{0.007\chi}{10.5+\chi} &= 0.5\sin(0.265\tau + 1.4), \\ \chi(0) &= 12.75. \end{aligned} \tag{7.1}$$

The solution algorithm of FBM (7.1) using EM will be

$$X_m = \frac{1}{h^{-\beta}}\left(0.5\sin(0.265\tau_m + 1.4) - \frac{0.007(X_{m-1} - \chi_0)}{10.5 + (X_{m-1} - \chi_0)} - h^{-\beta}\sum_{j=1}^{m} w_j^{(\beta)} X_{m-j}\right), \quad m = 1, 2, 3, \cdots. \tag{7.2}$$

Figures 1-6 show EM solutions at different values of β ($\beta = 1, 0.9, 0.8, 0.7, 0.6$, and 0.5), respectively.

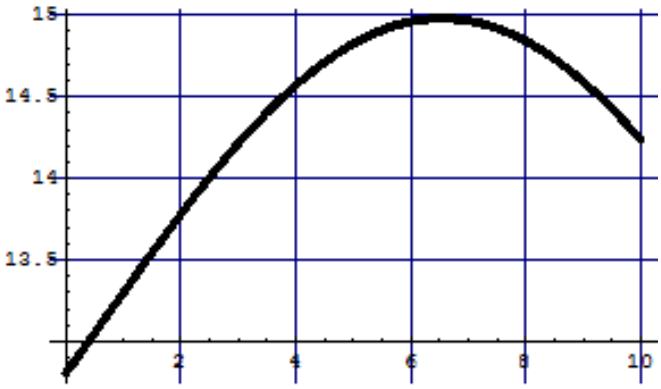

Figure 1: EM Solution ($\beta = 1$).

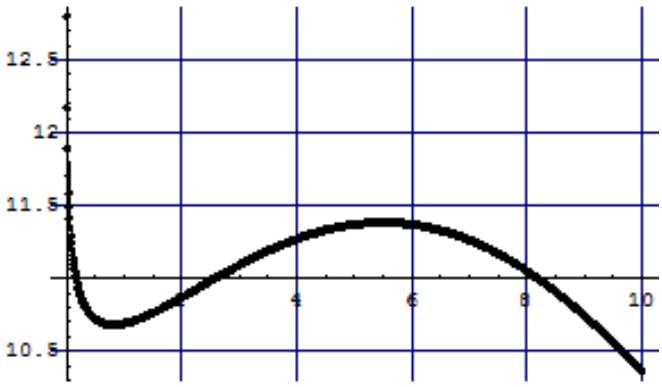

Figure 2: EM Solution ($\beta = 0.9$).

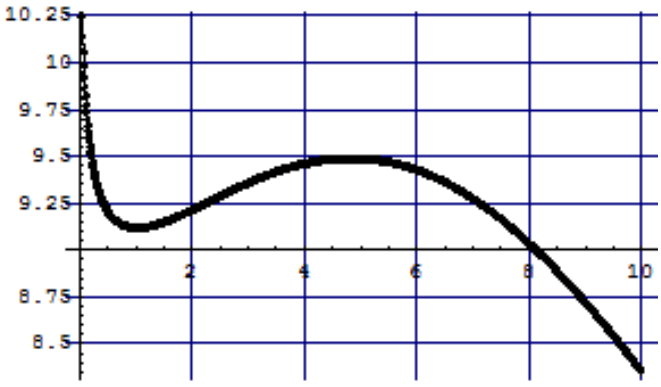

Figure 3: EM Solution ($\beta = 0.8$).

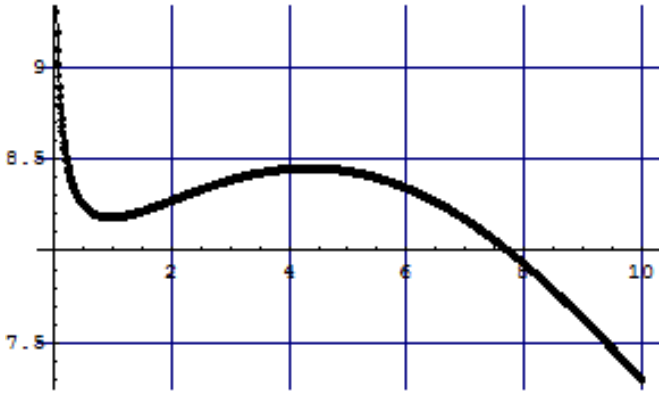

Figure 4: EM Solution ($\beta = 0.7$).

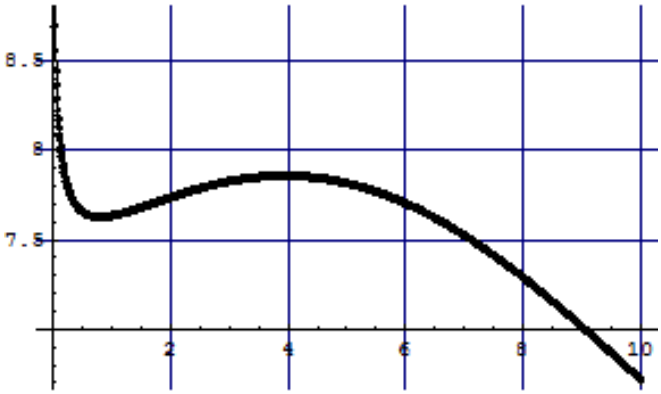

Figure 5: EM Solution ($\beta = 0.6$).

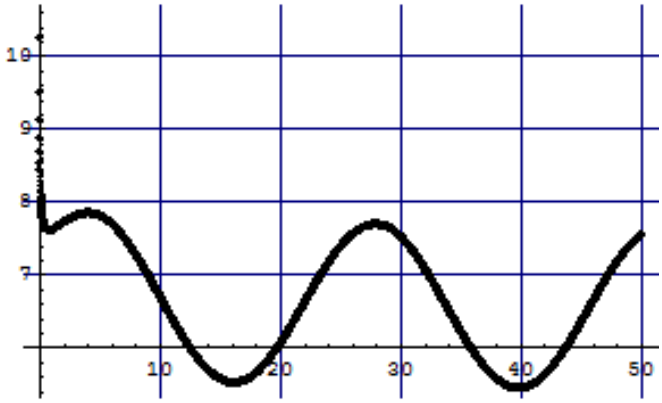

Figure 6: EM Solution ($\beta = 0.5$).

Figures 1-6: These figures illustrate the EM (Euler-Maruyama) solutions for varying values of the fractional order parameter β (1, 0.9, 0.8, 0.7, 0.6, and 0.5). Each figure demonstrates how the solutions evolve with decreasing fractional order, showcasing the impact of fractional calculus on the dynamics of brain metabolite variations.

Example 7.2. Consider the FBM:

$$\begin{aligned} {}^{C}\check{D}^{\beta}\chi + \frac{0.01\chi}{5+\chi} &= 0.5\sin(0.265\tau - 0.85), \\ \chi(0) &= 18.25. \end{aligned} \tag{7.3}$$

The solution algorithm of EBM (7.3) using EM will be

$$X_m = \frac{1}{h^{-\beta}}\left(0.5\sin(0.265\tau_m - 0.85) - \frac{0.01\,(X_{m-1} - \chi_0)}{5 + (X_{m-1} - \chi_0)} - h^{-\beta}\sum_{j=1}^{m} w_j^{(\beta)} X_{m-j}\right), \quad m = 2, 3, \cdots. \tag{7.4}$$

Figures 7-12 show EM solutions at different values of β ($\beta = 1$, 0.9, 0.8, 0.7, 0.6, and 0.5), respectively.

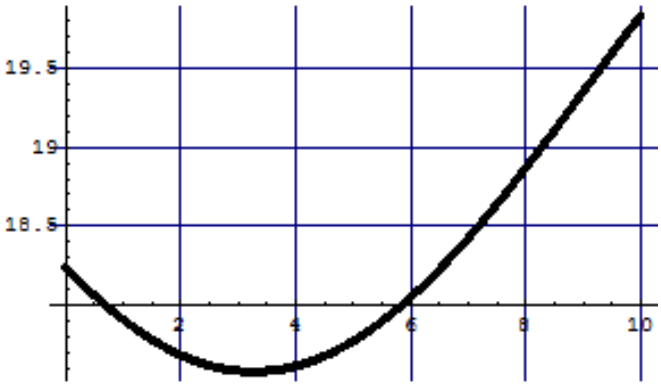

Figure 7: EM Solution ($\beta = 1$).

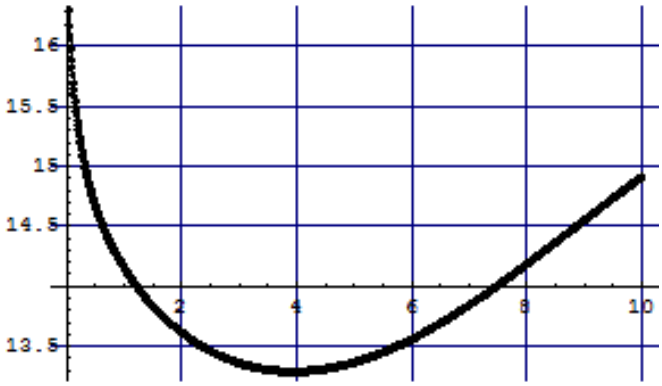

Figure 8: EM Solution ($\beta = 0.9$).

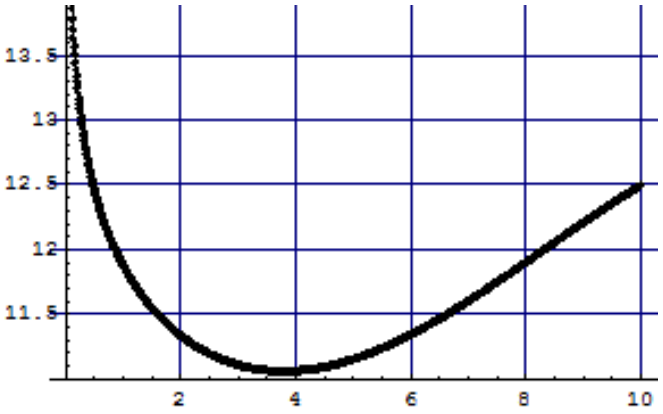

Figure 9: EM Solution ($\beta = 0.8$).

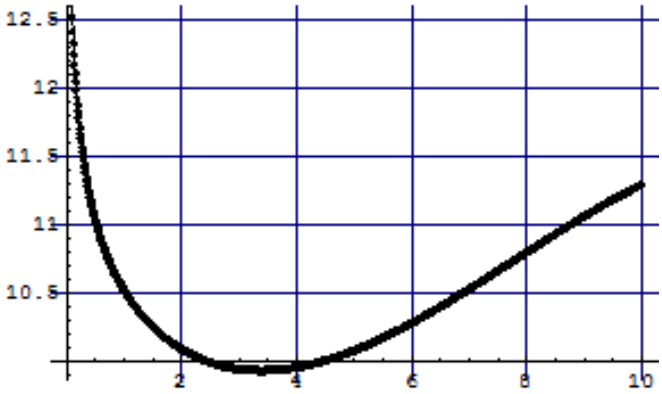

Figure 10: EM Solution ($\beta = 0.7$).

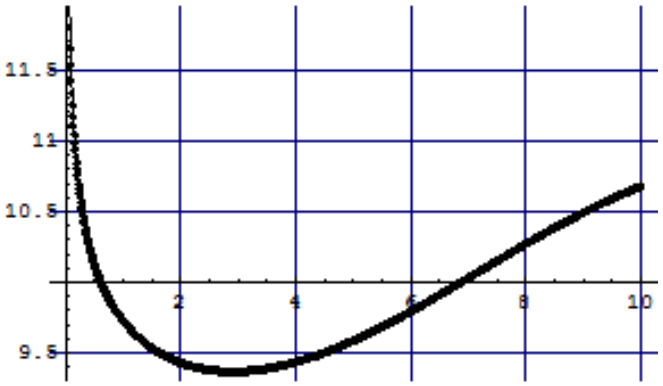

Figure 11: EM Solution ($\beta = 0.6$).

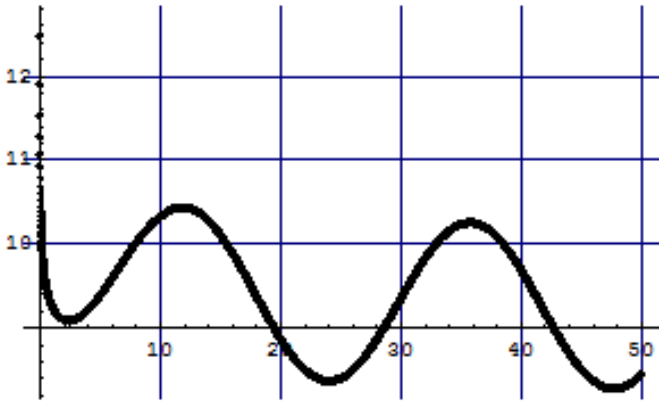

Figure 12: EM Solution ($\beta = 0.5$).

Figures 7-12: Similar to the previous set, these figures provide additional EM solutions at fractional orders of (1, 0.9,0.8, 0.7, 0.6, and 0.5). They emphasize the continuity and stability of the solutions, reinforcing the model's robustness across different fractional orders.

Example 7.3. Consider the FBM:

$$\begin{aligned} {}^{C}\check{D}^{\beta}\chi + \frac{0.004\chi}{0.8+\chi} &= 0.29\sin(0.265\tau + 1), \\ \chi(0) &= 19.09. \end{aligned} \tag{7.5}$$

The solution algorithm of FBM (7.5) using EM will be

$$X_m = \frac{1}{h^{-\beta}}\left(0.29\sin(0.265\tau_m + 1) - \frac{0.004\,(X_{m-1} - \chi_0)}{0.8 + (X_{m-1} - \chi_0)} - h^{-\beta}\sum_{j=1}^{m} w_j^{(\beta)} X_{m-j}\right), \qquad m = 2, 3, \cdots. \tag{7.6}$$

Figures 13-18 show EM solutions at different values of β ($\beta = 1, 0.9$, 0.8, 0.7, 0.6, and 0.5), respectively.

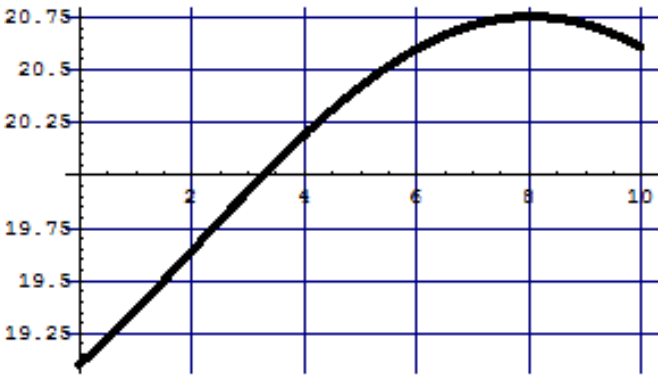

Figure 13: EM Solution ($\beta = 1$).

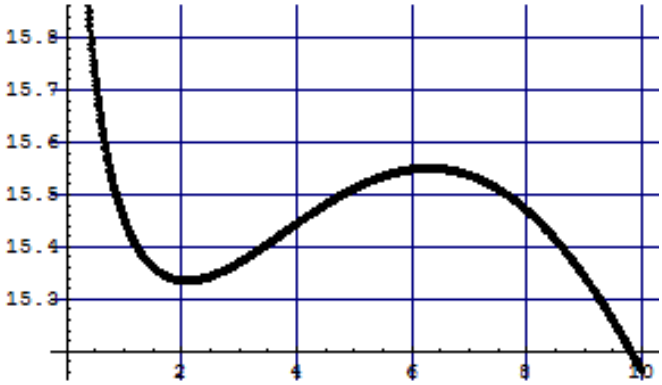

Figure 14: EM Solution ($\beta = 0.9$).

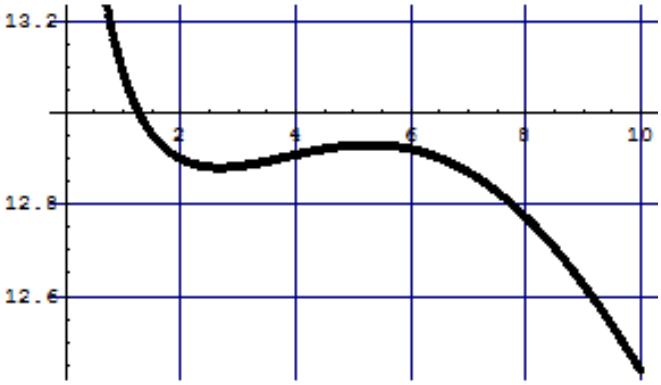

Figure 15: EM Solution ($\beta = 0.8$).

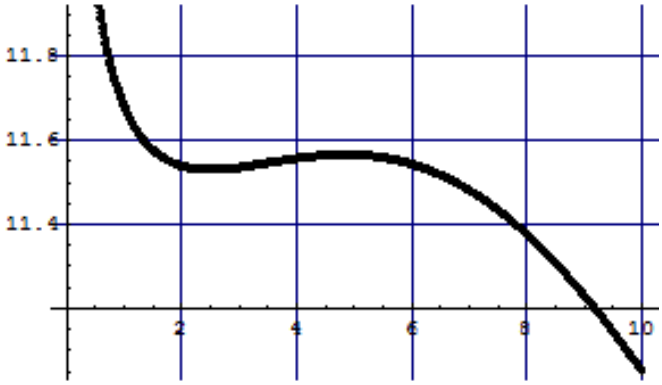

Figure 16: EM Solution ($\beta = 0.7$).

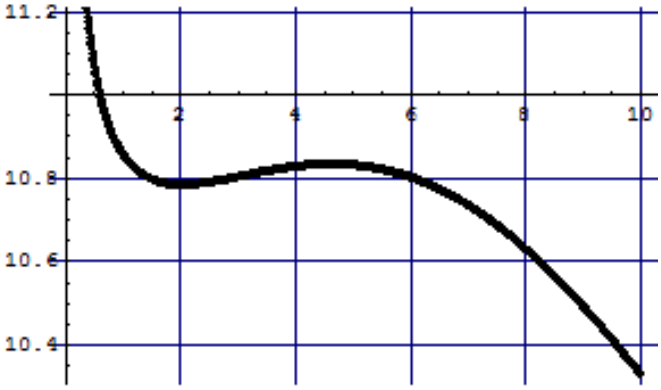

Figure 17: EM Solution ($\beta = 0.6$).

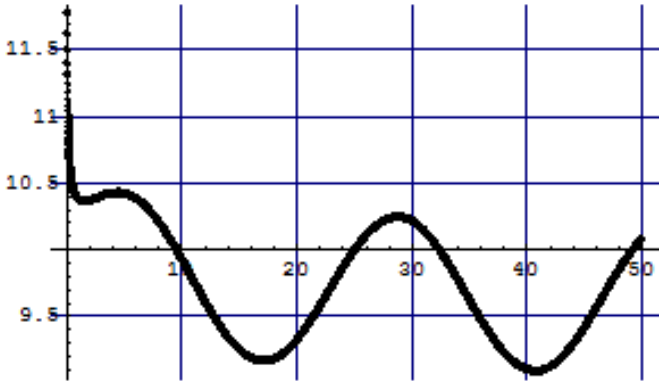

Figure 18: EM Solution ($\beta = 0.5$).

Figures 13-18: These figures further extend the analysis by presenting EM solutions at fractional orders ranging from 1 to 0.5. They collectively illustrate the convergence of results with integer-order models, validating the fractional approach against established medical data.

Example 7.4. Consider the FBM:

$$\begin{aligned} {}^{C}\check{D}^{\beta}\chi + \frac{0.004\chi}{7+\chi} &= 0.3\sin(0.265\tau + 2.8), \\ \chi(0) &= 15.25. \end{aligned} \tag{7.7}$$

The solution algorithm of FBM (7.7) using EM will be

$$X_m = \frac{1}{h^{-\beta}} \left(0.3 \sin(0.265\tau_m + 2.8) - \frac{0.004\,(X_{m-1} - \chi_0)}{7 + (X_{m-1} - \chi_0)} - h^{-\beta} \sum_{j=1}^{m} w_j^{(\beta)} X_{m-j} \right), \quad m = 2, 3, \cdots . \tag{7.8}$$

Figures 19-24 show EM solutions at different values of β ($\beta = 1, 0.9, 0.8, 0.7, 0.6$, and 0.5), respectively.

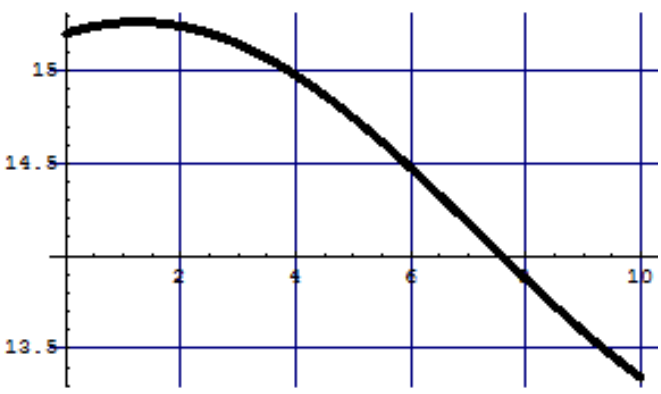

Figure 19: EM Solution ($\beta = 1$).

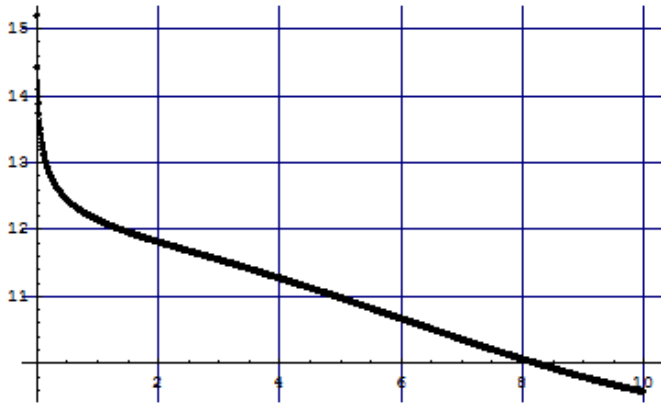

Figure 20: EM Solution ($\beta = 0.9$).

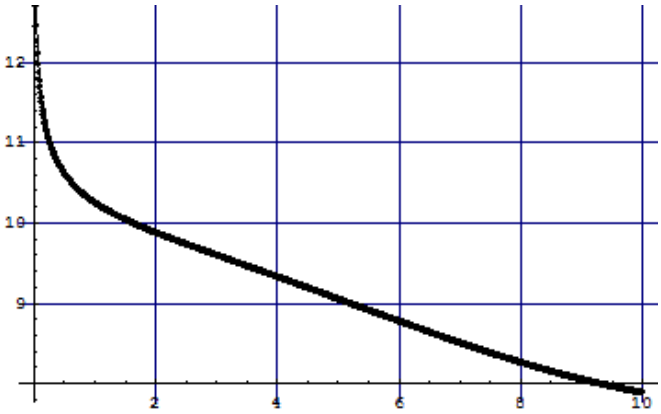

Figure 21: EM Solution ($\beta = 0.8$).

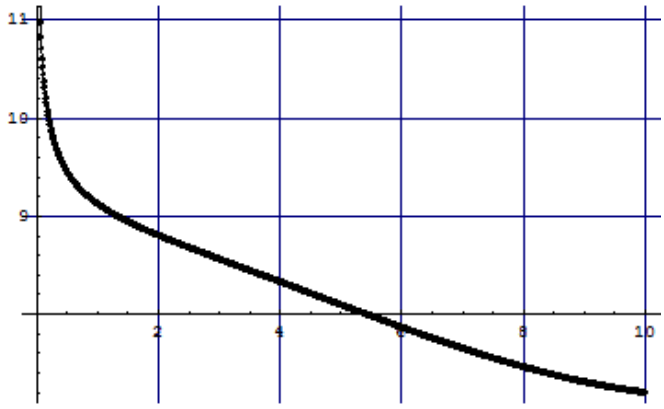

Figure 22: EM Solution ($\beta = 0.7$).

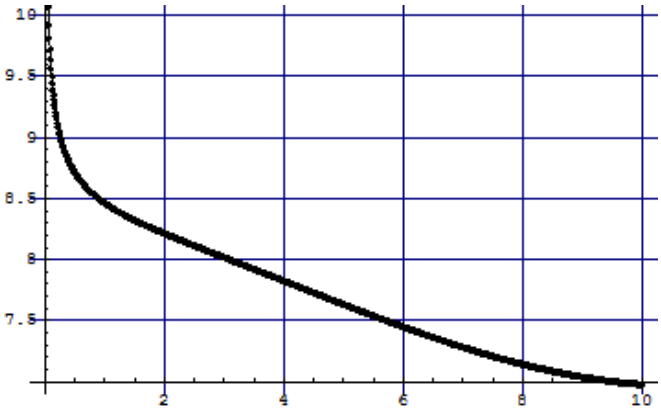

Figure 23: EM Solution ($\beta = 0.6$).

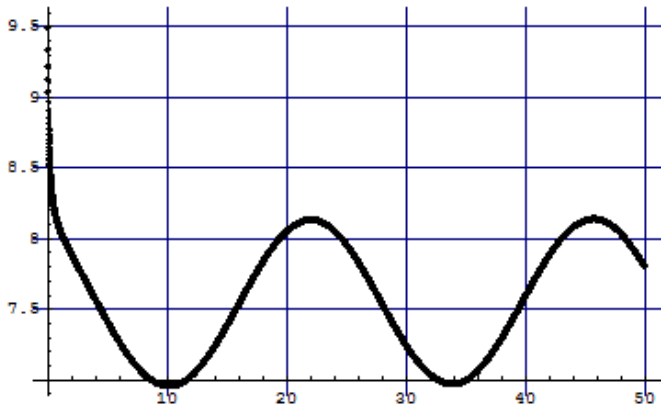

Figure 24: EM Solution ($\beta = 0.5$).

Figures 19-24: These figures further extend the analysis by presenting EM solutions at fractional orders ranging from 1 to 0.5. They collectively illustrate the convergence of results with integer-order models, validating the fractional approach against established medical data.

Overall, these figures serve to visually represent the mathematical findings and their implications for understanding the complex behavior of brain metabolite variations in relation to circadian rhythms.

8. Conclusion

In this research, we introduce an important mathematical model: the model of brain metabolite variations in the circadian rhythm. For the general form of this model, we prove the existence, uniqueness, and stability of its solution. In addition, we give the numerical solution in four different cases of this model. We see from their results that, they coincide with the results given in [13] for integer-order ($\beta = 1$) which matches well with the medical data (see [13], P. 223) and our solution in (Figures 1, 7, 13, and 19).

This study presents a significant advancement in understanding brain metabolite variations through the lens of fractional differential equations (FDEs), highlighting both theoretical and practical implications in neuroscience. The established existence, uniqueness, and stability of solutions not only deepen our comprehension of circadian rhythms but also pave the way for enhanced modeling techniques that could inform clinical practices for metabolic and sleep-related disorders. Future research should consider extending this work by exploring more intricate fractional models, incorporating real-time biological data, and applying these methodologies to other physiological systems to further enrich the field of fractional calculus and its applications in neuroscience.

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