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Some integral inequalities involving q-h fractional integral operator

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Abstract

In this work, we extend several well-known classical inequalities by applying a newly defined q-h operator over finite intervals. Specifically, we generalize the Cauchy-Schwarz integral inequality for double integrals, Grüss integral inequality, Korkine identity, and Grüss-Čebyšev integral inequality. These generalizations provide tighter bounds and enhanced applicability in the framework of quantum calculus. The q-h-integral, which combines features of the q-integral and h-integral, serves as a unifying tool to connect and extend existing results. Furthermore, we examine special cases to demonstrate the broader scope of these inequalities. Our findings highlight the versatility of the q-h-operator in refining and expanding the mathematical framework of integral inequalities in quantum calculus.

Keywords: q-h-Integral, q-h-integral inequalities, Cauchy-Schwarz integral inequality, Grüss-Čebyšev integral inequality, Grüss integral inequality.

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1. Introduction

The study of inequalities is fundamental to mathematical analysis with applications extending to fields such as physics, engineering, and finance. This paper explores a novel approach to extending classical integral inequalities by utilizing the recently developed q-h-operator. Building on the principles of q-calculus and h-calculus, the q-h-integral offers a unified framework that bridges established results with

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new extensions. Quantum calculus, also known as q-calculus, offers a contemporary framework for calculus that focuses on functions and their derivatives without relying on limits. Originally conceptualized by Euler in the 18th century, this branch of mathematics has been extensively studied in the works of Kac and Cheung [10]. In 1910, Jackson introduced the q-Jackson integral, a form of definite q-integral, which became a significant contribution to the field, and subsequent research has focused on its properties and those of q-derivatives. Notable contributions include the work by Tariboon and Ntouyas in 2013, which detailed the features of q-derivatives and q-integrals on finite intervals [20].

In recent years, the application of quantum calculus theory has gained considerable attention from researchers, leading to a growing body of significant contributions in this vibrant field. Notable contributions include advancements in fractional calculus, optimal control problems, and the exploration of q-difference and q-integral equations within q-analysis. To facilitate these studies, the concept of q-h-derivatives has been introduced, accompanied by the formulation of fundamental calculus rules. Additionally, the q-h-integral has been defined over a finite interval, further enriching the theoretical framework [17]. This new integral incorporates aspects of both q-calculus and h-calculus, and under specific conditions, it directly relates to standard analysis results. The increasing significance of mathematics in quantum calculus has drawn considerable attention to q-integral inequalities. Scholars have investigated Grüss-type inequalities within the realms of both classical analysis and time-scale calculus, expanding the scope of these mathematical concepts. The study of Grüss type integral inequalities has been expanded in various contexts. For instance, Peng and Miao [16] extended the inequality to functions with absolutely continuous first and second derivatives, assuming bounded third derivatives. Dragomir [4] investigated several variations of Grüss-type inequalities and their applications in inner product spaces. Further enhancements were made by Mercer [13], who provided additional refinements to the original Grüss inequality. In 2014, Tariboon and Ntouyas extended the Grüss inequality to quantum calculus [21]. For a detailed overview of related inequalities, key contributions can be found in [1, 3, 5–9, 14, 15, 19], with application of these inequalities discussed in [2, 11, 12, 18].

In this paper, we generalize several key inequalities, including the Cauchy-Schwarz integral inequality for double integrals, Grüss integral inequality, Korkine identity and Grüss-Čebyšev integral inequality, using the innovative q-h-operator. This generalization not only broadens the scope of Grüss- type inequalities but also offers a more flexible and comprehensive approach to tackling problems in quantum analysis. The inclusion of q-h-operator allows us to address scenarios with variable quantum steps and provides refined bounds for integrals over finite intervals. This advancement is expected to open new avenues for research in areas such as fractional quantum calculus, q-difference equations, and optimization in quantum systems. We also discuss special cases to highlight the connections between q-integrals, h-integrals, and standard integral inequalities.

The structure of this paper is as follows. Section 2 reviews the essential concepts and definitions related to the q-h-integral on finite intervals, laying the groundwork for our main findings. Section 3 outlines the detailed derivation of our findings. To the best of our knowledge, this work is the first to investigate Grüss-type integral inequalities employing the q-h-operator on finite intervals.

2. Generalization of q and h integral on finite intervals

A unified formulation of the q-integral and h-integral, referred to as the q-h-integral, was introduced by Shi et al. in their work on inequalities [17]. In this section, we outline the fundamental definitions and examples of the q-h-integral, which serve as the foundation for deriving our main results. For additional results on the q-h-integral, the reader may refer to [17].

Definition 2.1 ([17])**.** The q-h-derivative of a continuous function $f(\varphi)$, denoted as ${}_hD_q$, such that $f : \mathbb{I} = [\vartheta_1, \vartheta_2] \to \mathbb{R}$, with $0 < q < 1$, $h \in \mathbb{R}$, and $\varphi \in \mathbb{I}$, is defined as:

$$ {}_hD_q f(\varphi) = \frac{f\big((1-q)\vartheta_1 + q(\varphi + h)\big) - f(\varphi)}{(1-q)(\vartheta_1 - \varphi) + qh}; \quad \varphi \neq \frac{qh + (1-q)\vartheta_1}{1-q} := \varphi_o. $$

When $h = 0$, the formula for the q-derivative on interval $[\vartheta_1, \vartheta_2]$ is obtained, as discussed in [21]. Additionally, ${}_hD_q f(\varphi_o) = \lim_{\varphi \to \varphi_o} {}_hD_q f(\varphi)$.

Definition 2.2 ([17])**.** The q-h-integral of a continuous function $f(\varphi)$, denoted as $I_{q,h} f(\varphi)$, such that $f : \mathbb{I} = [\vartheta_1, \vartheta_2] \to \mathbb{R}$, with $0 < q < 1$, and $h \in \mathbb{R}$, is defined as:

$$I_{q,h} f(\varphi) := \int_{\vartheta_1}^{\vartheta_2} f(\varphi)\, {}_hD_q \varphi = \Big((1-q)(\vartheta_2 - \vartheta_1) + qh\Big) \sum_{p=0}^{\infty} q^p f\Big(q^p \vartheta_1 + (1-q^p)\vartheta_2 + pq^p h\Big), \quad \vartheta_2 > \vartheta_1.$$

Example 2.3. Take a constant function $f(\varphi) = \alpha$, the q-h-integral for a constant function is determined as:

$$I_{q,h} f(\varphi) := \int_{\vartheta_1}^{\vartheta_2} \alpha\, {}_hD_q \varphi = \Big((1-q)(\vartheta_2 - \vartheta_1) + qh\Big) \sum_{p=0}^{\infty} q^p \cdot \alpha = \alpha\Big((1-q)(\vartheta_2 - \vartheta_1) + qh\Big)\frac{1}{1-q},$$

$$\int_{\vartheta_1}^{\vartheta_2} \alpha\, {}_hD_q \varphi = \alpha\Big((\vartheta_2 - \vartheta_1) + \frac{qh}{1-q}\Big).$$

When $h = 0$, we have $\alpha(\vartheta_2 - \vartheta_1)$, which is the standard integration result of a constant function $f(\varphi) = \alpha$.

Example 2.4. Take $f(\varphi) = \varphi$, the q-h-integral of $f(\varphi) = \varphi$ is determined as:

$$\begin{aligned} I_{q,h} f(\varphi) := \int_{\vartheta_1}^{\vartheta_2} \varphi\, {}_hD_q \varphi &= \Big((1-q)(\vartheta_2 - \vartheta_1) + qh\Big) \sum_{p=0}^{\infty} q^p \Big(q^p \vartheta_1 + (1-q^p)\vartheta_2 + pq^p h\Big) \\ &= \Big((1-q)(\vartheta_2 - \vartheta_1) + qh\Big)\Big(\frac{\vartheta_1}{1-q^2} + \frac{\vartheta_2}{1-q} - \frac{\vartheta_2}{1-q^2} + \sum_{p=0}^{\infty} pq^{2p} h\Big), \\ \int_{\vartheta_1}^{\vartheta_2} f(\varphi)\, {}_hD_q \varphi &= \Big((\vartheta_2 - \vartheta_1) + \frac{qh}{1-q}\Big)\Big(\frac{\vartheta_1 + \vartheta_2 q}{1+q} + \mathbf{S}'\Big), \end{aligned}$$

where $\mathbf{S}' = \sum_{p=0}^{\infty} pq^{2p} h$, when $h = 0$, we have

$$\int_{\vartheta_1}^{\vartheta_2} \varphi\, D_q \varphi = (\vartheta_2 - \vartheta_1)\Big(\frac{\vartheta_1 + \vartheta_2 q}{1+q}\Big),$$

the q-integral of function $f(\varphi) = \varphi$ on interval $[\vartheta_1, \vartheta_2]$. And we have $\frac{(\vartheta_2 - \vartheta_1)^2}{2}$, the classical integration of function $f(\varphi) = \varphi$, when $q \to 1$.

Example 2.5. Take $f(\varphi) = \varphi^2$, the q-h-integral of function $f(\varphi) = \varphi^2$ is evaluated as:

$$\int_{\vartheta_1}^{\vartheta_2} \varphi^2_h D_q \varphi = \Big((1-q)(\vartheta_2 - \vartheta_1) + qh\Big) \sum_{p=0}^{\infty} q^p \Big(q^p \vartheta_1 + (1-q^p)\vartheta_2 + pq^p h\Big)^2.$$

By direct computation, we have:

$$= \Big((\vartheta_2 - \vartheta_1) + \frac{qh}{1-q}\Big)\Big(\frac{{\vartheta_1}^2(1+q) + 2\vartheta_1\vartheta_2 q^2 + {\vartheta_2}^2 q(1+q^2)}{(1+q+q^2)(1+q)^2} + \mathbf{T}'\Big),$$

where, $\mathbf{T}' = \sum_{p=0}^{\infty} phq^{2p}\Big(2\vartheta_1\vartheta_2(1-q) + 2\vartheta_1 + pqh\Big)$, the sum $\mathbf{T}' \to 0$, when $h = 0$.

We obtain the q-integration of a function $f(\varphi) = \varphi^2$, when $h = 0$, given by:

$$\int_{\vartheta_1}^{\vartheta_2} \varphi^2 D_q \varphi = (\vartheta_2 - \vartheta_1)\left(\frac{{\vartheta_1}^2(1+q) + 2\vartheta_1\vartheta_2 q^2 + {\vartheta_2}^2 q(1+q^2)}{(1+q+q^2)(1+q)^2}\right). \tag{2.1}$$

Equation (2.1) transforms to the following classical integral, when $q \to 1$,

$$\int_{\vartheta_1}^{\vartheta_2} \varphi^2 d\varphi = \frac{{\vartheta_2}^3 - {\vartheta_1}^3}{3}.$$

Definition 2.6. The double q-h-integral of continuous function $f(\varphi, \rho)$, denoted as $I_{q,h} f(\varphi, \rho)$, such that $f : I = [\vartheta_1, \vartheta_2] \to \mathbb{R}$, with $0 < q < 1$ and $h \in \mathbb{R}$, is then defined as:

$$\begin{aligned} &I_{q,h} f(\varphi, \rho) \\ &:= \int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} f(\varphi, \rho)\, {}_h D_q \varphi\, {}_h D_q \rho \\ &= \int_{\vartheta_1}^{\vartheta_2} \Big((1-q)(\vartheta_2 - \vartheta_1) + qh\Big) \sum_{p=0}^{\infty} q^p f\Big(q^p \vartheta_1 + (1-q^p)\vartheta_2 + pq^p h, \rho\Big) \\ &= \Big((1-q)(\vartheta_2 - \vartheta_1) + qh\Big)^2 \sum_{u=0}^{\infty}\sum_{p=0}^{\infty} q^{p+u} f\Big(q^p \vartheta_1 + (1-q^p)\vartheta_2 + pq^p h, q^u \vartheta_1 + (1-q^u)\vartheta_2 + uq^u h\Big). \end{aligned}$$

3. Main results

In this section, we generalize several key inequalities of classical analysis using the innovative q-h-operator. We begin by proving the following lemmas.

Lemma 3.1. *For continuous functions* f *and* g *defined on the interval* $[\vartheta_1, \vartheta_2] \to \mathbb{R}$, *with* $0 < q < 1$ *and* $h \in \mathbb{R}$, *the* q-h-*integral adheres to the following inequality:*

$$\begin{aligned} &\frac{1}{2}\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} \big(f(\rho) - f(\varphi)\big)\big(g(\rho) - g(\varphi)\big)\, {}_h D_q \rho\, {}_h D_q \varphi \\ &\quad = \left(\vartheta_2 - \vartheta_1 + \frac{qh}{1-q}\right)\int_{\vartheta_1}^{\vartheta_2} f(\rho) g(\rho)\, {}_h D_q \rho - \int_{\vartheta_1}^{\vartheta_2} f(\rho)\, {}_h D_q \rho \int_{\vartheta_1}^{\vartheta_2} g(\rho)\, {}_h D_q \rho. \end{aligned} \tag{3.1}$$

Proof. We have

$$\big(f(\rho) - f(\varphi)\big)\big(g(\rho) - g(\varphi)\big) = f(\rho)f(\rho) - f(\rho)g(\varphi) - f(\varphi)g(\rho) + f(\varphi)g(\varphi).$$

Applying the double q-h-integration to the above inequality over the region $[\vartheta_1, \vartheta_2]^2$ results in:

$$\begin{aligned} &\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} \big(f(\rho) - f(\varphi)\big)\big(g(\rho) - g(\varphi)\big)\big)\, {}_h D_q \rho\, {}_h D_q \varphi \\ &\quad = \int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} f(\rho) g(\rho)\, {}_h D_q \rho\, {}_h D_q \varphi - \int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} f(\rho) g(\varphi)\, {}_h D_q \rho\, {}_h D_q \varphi \\ &\qquad - \int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} f(\varphi) g(\rho)\, {}_h D_q \rho\, {}_h D_q \varphi + \int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} f(\varphi) g(\varphi)\, {}_h D_q \rho\, {}_h D_q \varphi. \end{aligned}$$

By applying the definition of the q-h-integral on the interval $[\vartheta_1, \vartheta_2]$ and referring to Example 2.3, we obtain the following result:

$$\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} \big(f(\rho) - f(\varphi)\big)\big(g(\rho) - g(\varphi)\big)\big)\, {}_h D_q \rho\, {}_h D_q \varphi int_{\vartheta_1}^{\vartheta_2} \int_{\vartheta_1}^{\vartheta_2} \big(f(\rho) - f(\varphi)\big)\big(g(\rho) - g(\varphi)\big)\big)\, {}_h D_q \rho\, {}_h D_q \varphi$$

$$
\begin{aligned}
&= \Big((1-q)(\vartheta_2-\vartheta_1)+qh\Big)\Big(\sum_{r=0}^{\infty} q^r f(q^r\vartheta_1+(1-q^r)\vartheta_2+rq^rh) g(q^r\vartheta_1+(1-q^r)\vartheta_2+rq^rh)\Big)\Big((\vartheta_2-\vartheta_1)+\frac{qh}{1-q}\Big)\\
&\quad -\Big((1-q)(\vartheta_2-\vartheta_1)+qh\Big)^2\Big(\sum_{r=0}^{\infty} q^r f(q^r\vartheta_1+(1-q^r)\vartheta_2+rq^rh)\Big)\Big(\sum_{r=0}^{\infty} q^r g(q^r\vartheta_1+(1-q^r)\vartheta_2+rq^rh)\Big)\\
&\quad -\Big((1-q)(\vartheta_2-\vartheta_1)+qh\Big)^2\Big(\sum_{r=0}^{\infty} q^r f(q^r\vartheta_1+(1-q^r)\vartheta_2+rq^rh)\Big)\Big(\sum_{r=0}^{\infty} q^r g(q^r\vartheta_1+(1-q^r)\vartheta_2+rq^rh)\Big)\\
&\quad +\Big((1-q)(\vartheta_2-\vartheta_1)+qh\Big)\Big(\sum_{r=0}^{\infty} q^r f(q^r\vartheta_1+(1-q^r)\vartheta_2+rq^rh) g(q^r\vartheta_1+(1-q^r)\vartheta_2+rq^rh)\Big)\Big((\vartheta_2-\vartheta_1)+\frac{qh}{1-q}\Big)\\
&= 2\Big((1-q)(\vartheta_2-\vartheta_1)+qh\Big)\Big(\sum_{r=0}^{\infty} q^r f(q^r\vartheta_1+(1-q^r)\vartheta_2+rq^rh) g(q^r\vartheta_1+(1-q^r)\vartheta_2+rq^rh)\Big)\Big((\vartheta_2-\vartheta_1)+\frac{qh}{1-q}\Big)\\
&\quad -2\Big((1-q)(\vartheta_2-\vartheta_1)+qh\Big)^2\Big(\sum_{r=0}^{\infty} q^r f(q^r\vartheta_1+(1-q^r)\vartheta_2+rq^rh)\Big)\Big(\sum_{r=0}^{\infty} q^r g(q^r\vartheta_1+(1-q^r)\vartheta_2+rq^rh)\Big).
\end{aligned}
$$

From this, we can write:

$$
\begin{aligned}
&\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2}\big(f(\rho)-f(\varphi)\big)\big(g(\rho)-g(\varphi)\big)\,{}_hD_q\rho\,{}_hD_q\varphi\\
&\qquad = 2\Big(\vartheta_2-\vartheta_1+\frac{qh}{1-q}\Big)\int_{\vartheta_1}^{\vartheta_2} f(\rho)g(\rho)\,{}_hD_q\rho - 2\int_{\vartheta_1}^{\vartheta_2} f(\rho)\,{}_hD_q\rho\int_{\vartheta_1}^{\vartheta_2} g(\rho)\,{}_hD_q\rho.
\end{aligned}
$$

From this we have our required equality:

$$
\begin{aligned}
&\frac{1}{2}\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2}\big(f(\rho)-f(\varphi)\big)\big(g(\rho)-g(\varphi)\big)\,{}_hD_q\rho\,{}_hD_q\varphi\\
&\qquad = \Big(\vartheta_2-\vartheta_1+\frac{qh}{1-q}\Big)\int_{\vartheta_1}^{\vartheta_2} f(\rho)g(\rho)\,{}_hD_q\rho - \int_{\vartheta_1}^{\vartheta_2} f(\rho)\,{}_hD_q\rho\int_{\vartheta_1}^{\vartheta_2} g(\rho)\,{}_hD_q\rho.
\end{aligned}
$$

This concludes the proof. □

Remark 3.2. We have q-Korkine identity [21], when $h = 0$:

$$
\frac{1}{2}\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2}\big(f(\rho)-f(\varphi)\big)\big(g(\rho)-g(\varphi)\big)d_q\rho d_q\varphi = (\vartheta_2-\vartheta_1)\int_{\vartheta_1}^{\vartheta_2} f(\rho)g(\rho)d_q\rho - \int_{\vartheta_1}^{\vartheta_2} f(\rho)d_q\rho\int_{\vartheta_1}^{\vartheta_2} g(\rho)d_q\rho.
$$

And, we have classical Korkine identity, when $q \to 1$.

Lemma 3.3. *For continuous functions* f *and* g *defined on the interval* $[\vartheta_1, \vartheta_2] \to \mathbb{R}$*, satisfying the conditions:*

$$
\theta_1 \leqslant f(\varphi) \leqslant \theta_2, \quad \lambda_1 \leqslant g(\varphi) \leqslant \lambda_2, \quad \forall \varphi \in [\vartheta_1, \vartheta_2], \theta_1, \theta_2, \lambda_1, \lambda_2 \in \mathbb{R},
$$

with $0 < q < 1$ *and* $h \in \mathbb{R}$*, the* q-h*-integral adheres to the following inequality:*

$$
\begin{aligned}
&\Big((\vartheta_2-\vartheta_1)+\frac{qh}{1-q}\Big)\int_{\vartheta_1}^{\vartheta_2} f^2(\varphi)\,{}_hD_q\varphi - \Big(\int_{\vartheta_1}^{\vartheta_2} f(\varphi)\,{}_hD_q\varphi\Big)^2\\
&\quad \leqslant \Big(\int_{\vartheta_1}^{\vartheta_2} f(\varphi)\,{}_hD_q\varphi - \theta_1\Big((\vartheta_2-\vartheta_1)+\frac{qh}{1-q}\Big)\Big)\Big(\theta_2\Big((\vartheta_2-\vartheta_1)+\frac{qh}{1-q}\Big) - \int_{\vartheta_1}^{\vartheta_2} f(\varphi)\,{}_hD_q\varphi\Big),
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
&\Big((\vartheta_2-\vartheta_1)+\frac{qh}{1-q}\Big)\int_{\vartheta_1}^{\vartheta_2} g^2(\varphi)\,{}_hD_q\varphi - \Big(\int_{\vartheta_1}^{\vartheta_2} g(\varphi)\,{}_hD_q\varphi\Big)^2\\
&\quad \leqslant \Big(\int_{\vartheta_1}^{\vartheta_2} g(\varphi)\,{}_hD_q\varphi - \lambda_1\Big((\vartheta_2-\vartheta_1)+\frac{qh}{1-q}\Big)\Big)\Big(\lambda_2\Big((\vartheta_2-\vartheta_1)+\frac{qh}{1-q}\Big) - \int_{\vartheta_1}^{\vartheta_2} g(\varphi)\,{}_hD_q\varphi\Big).
\end{aligned}
\tag{3.3}
$$

Proof. Given the condition $\theta_1 \leqslant f(\varphi) \leqslant \theta_2$, from the given conditions, we have $(\theta_2 - f(\varphi)) \geqslant 0$, $(f(\varphi) - \theta_1) \geqslant 0$. By combining these two inequalities, we obtain

$$(\theta_2 - f(\varphi))(f(\varphi) - \theta_1) \geqslant 0.$$

Taking the q-h-integral with respect to φ over $[\vartheta_1, \vartheta_2]$ and then multiplying by $\left(\vartheta_2 - \vartheta_1 + \frac{qh}{1-q}\right)$ yields:

$$\left(\vartheta_2 - \vartheta_1 + \frac{qh}{1-q}\right) \int_{\vartheta_1}^{\vartheta_2} (\theta_2 - f(\varphi))(f(\varphi) - \theta_1)\,{}_hD_q\varphi \geqslant 0. \tag{3.4}$$

We have

$$\begin{aligned}&(\theta_2 - f(\varphi))(f(\rho) - \theta_1) + (\theta_2 - f(\rho)(f(\varphi) - \theta_1))\\ &\quad - (\theta_2 - f(\rho))(f(\rho) - \theta_1) - (\theta_2 - f(\varphi))(f(\varphi) - \theta_1) = f^2(\rho) + f^2(\varphi) - 2f(\varphi)f(\rho).\end{aligned}$$

By performing the double q-h-integral with respect to φ and ρ over the interval from ϑ_1 to ϑ_2, we obtain

$$\begin{aligned}&\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} (\theta_2 - f(\varphi))(f(\rho) - \theta_1)\,{}_hD_q\varphi\,{}_hD_q\rho + \int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} (\theta_2 - f(\rho)(f(\varphi) - \theta_1))\,{}_hD_q\varphi\,{}_hD_q\rho\\ &\quad - \int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} (\theta_2 - f(\rho))(f(\rho) - \theta_1)\,{}_hD_q\varphi\,{}_hD_q\rho - \int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} (\theta_2 - f(\varphi))(f(\varphi) - \theta_1)\,{}_hD_q\varphi\,{}_hD_q\rho - (???)\\ &\quad = \int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} f^2(\rho)\,{}_hD_q\varphi\,{}_hD_q\rho + \int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} f^2(\varphi)\,{}_hD_q\varphi\,{}_hD_q\rho - 2\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} f(\varphi)f(\rho)\,{}_hD_q\varphi\,{}_hD_q\rho.\end{aligned}$$

By applying the definition of the q-h-integral and simplifying, we obtain:

$$\begin{aligned}&2\left(\theta_2(\vartheta_2 - \vartheta_1 + \frac{qh}{1-q}) - \int_{\vartheta_1}^{\vartheta_2} f(\varphi)\,{}_hD_q\varphi\right)\left(\int_{\vartheta_1}^{\vartheta_2} f(\varphi)\,{}_hD_q\varphi - \theta_1(\vartheta_2 - \vartheta_1 + \frac{qh}{1-q})\right) - 2\left(\vartheta_2 - \vartheta_1 + \frac{qh}{1-q}\right)\\ &\quad \times \int_{\vartheta_1}^{\vartheta_2} (\theta_2 - f(\varphi))(f(\varphi) - \theta_1)\,{}_hD_q\varphi = 2((\vartheta_2 - \vartheta_1) + \frac{qh}{1-q}) \int_{\vartheta_1}^{\vartheta_2} f^2(\varphi)\,{}_hD_q\varphi - 2\left(\int_{\vartheta_1}^{\vartheta_2} f(\varphi)\,{}_hD_q\varphi\right)^2,\\ &\Longrightarrow \left(\theta_2(\vartheta_2 - \vartheta_1 + \frac{qh}{1-q}) - \int_{\vartheta_1}^{\vartheta_2} f(\varphi)\,{}_hD_q\varphi\right)\left(\int_{\vartheta_1}^{\vartheta_2} f(\varphi)\,{}_hD_q\varphi - \theta_1(\vartheta_2 - \vartheta_1 + \frac{qh}{1-q})\right) - \left(\vartheta_2 - \vartheta_1 + \frac{qh}{1-q}\right)\\ &\quad \times \int_{\vartheta_1}^{\vartheta_2} (\theta_2 - f(\varphi))(f(\varphi) - \theta_1)\,{}_hD_q\varphi = ((\vartheta_2 - \vartheta_1) + \frac{qh}{1-q}) \int_{\vartheta_1}^{\vartheta_2} f^2(\varphi)\,{}_hD_q\varphi - \left(\int_{\vartheta_1}^{\vartheta_2} f(\varphi)\,{}_hD_q\varphi\right)^2.\end{aligned}$$

By using inequality (3.4) we have:

$$\begin{aligned}&((\vartheta_2 - \vartheta_1) + \frac{qh}{1-q}) \int_{\vartheta_1}^{\vartheta_2} f^2(\varphi)\,{}_hD_q\varphi - \left(\int_{\vartheta_1}^{\vartheta_2} f(\varphi)\,{}_hD_q\varphi\right)^2\\ &\quad \leqslant \left(\int_{\vartheta_1}^{\vartheta_2} f(\varphi)\,{}_hD_q\varphi - \theta_1\left((\vartheta_2 - \vartheta_1) + \frac{qh}{1-q}\right)\right)\left(\theta_2\left((\vartheta_2 - \vartheta_1) + \frac{qh}{1-q}\right) - \int_{\vartheta_1}^{\vartheta_2} f(\varphi)\,{}_hD_q\varphi\right).\end{aligned}$$

This result is the desired inequality (3.2). Similarly, by applying the condition $\lambda_1 \leqslant g(\varphi) \leqslant \lambda_2$, we obtain the second required inequality (3.3). □

Lemma 3.4. *For continuous functions* f *and* g *defined on the interval* $[\vartheta_1, \vartheta_2] \to \mathbb{R}$, *with* $0 < q < 1$ *and* $h \in \mathbb{R}$, *the* q-h*-integral adheres to the following inequality:*

$$\begin{aligned}&\left|\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} f(\varphi, \rho)g(\varphi, \rho)\,{}_hD_q\varphi\,{}_hD_q\rho\right|\\ &\quad \leqslant \left(\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} f^2(\varphi, \rho)\,{}_hD_q\varphi\,{}_hD_q\rho\right)^{\frac{1}{2}}\left(\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} g^2(\varphi, \rho)\,{}_hD_q\varphi\,{}_hD_q\rho\right)^{\frac{1}{2}}.\end{aligned} \tag{3.5}$$

Proof. Using the definition of the double q-h integral, it follows that:

$$\begin{aligned}&\left(\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} f(\varphi,\rho)g(\varphi,\rho)\,{}_hD_q\varphi\,{}_hD_q\rho\right)^2\\&=\Bigg(\big((1-q)(\vartheta_2-\vartheta_1)+qh\big)^2\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}q^{r+k}f\big(q^r\vartheta_1+(1-q^r)\vartheta_2+rq^rh,q^k\vartheta_1+(1-q^k)\vartheta_2+kq^rh\big)\\&\quad\times g\big(q^r\vartheta_1+(1-q^r)\vartheta_2+rq^rh,q^k\vartheta_1+(1-q^k)\vartheta_2+kq^kh\big)\Bigg)^2.\end{aligned}$$

Applying the discrete Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned}&\left(\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} f(\varphi,\rho)g(\varphi,\rho)\,{}_hD_q\varphi\,{}_hD_q\rho\right)^2\\&\leqslant\Big((1-q)(\vartheta_2-\vartheta_1)+qh\Big)^2\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}q^{r+k}f^2\big(q^r\vartheta_1+(1-q^r)\vartheta_2+rq^rh,q^k\vartheta_1+(1-q^k)\vartheta_2+kq^kh\big)\\&\quad\times\Big((1-q)(\vartheta_2-\vartheta_1)+qh\Big)^2\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}q^{r+k}g^2\big(q^r\vartheta_1+(1-q^r)\vartheta_2+rq^rh,q^k\vartheta_1+(1-q^k)\vartheta_2+kq^kh\big)\\&=\left(\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} f^2(\varphi,\rho)\,{}_hD_q\varphi\,{}_hD_q\rho\right)\left(\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} g^2(\varphi,\rho)\,{}_hD_q\varphi\,{}_hD_q\rho\right).\end{aligned}$$

Taking the square root of both sides yields the required inequality:

$$\left|\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} f(\varphi,\rho)g(\varphi,\rho)\,{}_hD_q\varphi\,{}_hD_q\rho\right|\leqslant\left(\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} f^2(\varphi,\rho)\,{}_hD_q\varphi\,{}_hD_q\rho\right)^{\frac{1}{2}}\left(\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} g^2(\varphi,\rho)\,{}_hD_q\varphi\,{}_hD_q\rho\right)^{\frac{1}{2}}.$$

□

Remark 3.5. Inequality (3.5) reduces to q-Cauchy-Bunyakovsky-Schwarz integral inequality for double integral [21], when $h = 0$,

$$\begin{aligned}&\left|\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} f(\varphi,\rho)g(\varphi,\rho)\,{}_hD_q\varphi\, d_q\rho\right|\\&\qquad\leqslant\left(\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} f^2(\varphi,\rho)\,d_q\varphi\,{}_hD_q\rho\right)^{\frac{1}{2}}\left(\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} g^2(\varphi,\rho)\,{}_hD_q\varphi\, d_q\rho\right)^{\frac{1}{2}}.\end{aligned}\tag{3.6}$$

Inequality (3.6) transformes to classical Cauchy-Bunyakovsky-Schwarz integral inequality, when $q \to 1$.

The following theorem gives the Generalization of Grüss integral inequality on finite interval $[\vartheta_1, \vartheta_2]$ using novel q-h-operator.

Theorem 3.6. *For continuous functions* f *and* g *defined on the interval* $[\vartheta_1, \vartheta_2] \to \mathbb{R}$*, satisfying the conditions:*

$$\theta_1 \leqslant f(\varphi) \leqslant \theta_2, \quad \lambda_1 \leqslant g(\varphi) \leqslant \lambda_2, \quad \forall\varphi \in [\vartheta_1, \vartheta_2], \theta_1, \theta_2, \lambda_1, \lambda_2 \in \mathbb{R},$$

with $0 < q < 1$ *and* $h \in \mathbb{R}$*, the* q-h*-integral adheres to the following inequality:*

$$\begin{aligned}&\Bigg|\frac{1-q}{((\vartheta_2-\vartheta_1)(1-q)+qh)}\int_{\vartheta_1}^{\vartheta_2} f(\rho)g(\rho)\,{}_hD_q\rho-\left(\frac{1-q}{((\vartheta_2-\vartheta_1)(1-q)+qh)}\int_{\vartheta_1}^{\vartheta_2} f(\rho)\,{}_hD_q\rho\right)\\&\times\left(\frac{1-q}{((\vartheta_2-\vartheta_1)(1-q)+qh)}\int_{\vartheta_1}^{\vartheta_2} g(\rho)\,{}_hD_q\rho\right)\Bigg|\leqslant\frac{1}{4}(\theta_2-\theta_1)(\lambda_2-\lambda_1).\end{aligned}\tag{3.7}$$

Proof. From q-h-Korkine identity we have:

$$\left(\vartheta_2-\vartheta_1+\frac{qh}{1-q}\right)\int_{\vartheta_1}^{\vartheta_2} f(\rho)g(\rho)\,{}_hD_q\rho-\int_{\vartheta_1}^{\vartheta_2} f(\rho)\,{}_hD_q\rho\int_{\vartheta_1}^{\vartheta_2} g(\rho)\,{}_hD_q\rho$$
$$=\frac{1}{2}\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2}\left(f(\rho)-f(\varphi)\right)\left(g(\rho)-g(\varphi)\right)\,{}_hD_q\rho\,{}_hD_q\varphi.$$

Applying q-h-Cauchy-Schwarz integral inequality for double integral it follows:

$$\begin{aligned}&\left(\frac{1}{2}\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2}\left(f(\rho)-f(\varphi)\right)\left(g(\rho)-g(\varphi)\right)\,{}_hD_q\rho\,{}_hD_q\varphi\right)^2\\ &\leqslant\left(\frac{1}{2}\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2}\left(f(\rho)-f(\varphi)\right)^2_h D_q\rho\,{}_hD_q\varphi\right)\left(\frac{1}{2}\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2}\left(g(\rho)-g(\varphi)\right)^2_h D_q\rho\,{}_hD_q\varphi\right)\\ &=\left(\vartheta_2-\vartheta_1+\frac{qh}{1-q}\right)\int_{\vartheta_1}^{\vartheta_2} f^2(\rho)\,{}_hD_q\rho-\left(\int_{\vartheta_1}^{\vartheta_2} f(\rho)\,{}_hD_q\rho\right)^2\\ &\quad\times\left(\vartheta_2-\vartheta_1+\frac{qh}{1-q}\right)\int_{\vartheta_1}^{\vartheta_2} g^2(\rho)\,{}_hD_q\rho-\left(\int_{\vartheta_1}^{\vartheta_2} g(\rho)\,{}_hD_q\rho\right)^2.\end{aligned}$$

Utilizing inequalities (3.2) and (3.3), we obtain:

$$\begin{aligned}&\left(\vartheta_2-\vartheta_1+\frac{qh}{1-q}\right)\int_{\vartheta_1}^{\vartheta_2} f^2(\rho)\,{}_hD_q\rho-\left(\int_{\vartheta_1}^{\vartheta_2} f(\rho)\,{}_hD_q\rho\right)^2\\ &\quad\times\left(\vartheta_2-\vartheta_1+\frac{qh}{1-q}\right)\int_{\vartheta_1}^{\vartheta_2} g^2(\rho)\,{}_hD_q\rho-\left(\int_{\vartheta_1}^{\vartheta_2} g(\rho)\,{}_hD_q\rho\right)^2\\ &\leqslant\left(\int_{\vartheta_1}^{\vartheta_2} f(\varphi)\,{}_hD_q\varphi-\theta_1\left((\vartheta_2-\vartheta_1)+\frac{qh}{1-q}\right)\right)\left(\theta_2\left((\vartheta_2-\vartheta_1)+\frac{qh}{1-q}\right)-\int_{\vartheta_1}^{\vartheta_2} f(\varphi)\,{}_hD_q\varphi\right)\\ &\quad\times\left(\int_{\vartheta_1}^{\vartheta_2} g(\varphi)\,{}_hD_q\varphi-\lambda_1\left((\vartheta_2-\vartheta_1)+\frac{qh}{1-q}\right)\right)\left(\lambda_2\left((\vartheta_2-\vartheta_1)+\frac{qh}{1-q}\right)-\int_{\vartheta_1}^{\vartheta_2} g(\varphi)\,{}_hD_q\varphi\right).\end{aligned}$$

Now, applying the AM-GM inequality $\alpha\beta\leqslant\frac{1}{4}(\alpha+\beta)^2$, we obtain:

$$\begin{aligned}&\left(\int_{\vartheta_1}^{\vartheta_2} f(\varphi)\,{}_hD_q\varphi-\theta_1\left((\vartheta_2-\vartheta_1)+\frac{qh}{1-q}\right)\right)\left(\theta_2\left((\vartheta_2-\vartheta_1)+\frac{qh}{1-q}\right)-\int_{\vartheta_1}^{\vartheta_2} f(\varphi)\,{}_hD_q\varphi\right)\\ &\quad\times\left(\int_{\vartheta_1}^{\vartheta_2} g(\varphi)\,{}_hD_q\varphi-\lambda_1\left((\vartheta_2-\vartheta_1)+\frac{qh}{1-q}\right)\right)\left(\lambda_2\left((\vartheta_2-\vartheta_1)+\frac{qh}{1-q}\right)-\int_{\vartheta_1}^{\vartheta_2} g(\varphi)\,{}_hD_q\varphi\right)\\ &\leqslant\frac{(\theta_2-\theta_1)^2}{4}\frac{(\lambda_2-\lambda_1)^2}{4}\left((\vartheta_2-\vartheta_1)+\frac{qh}{1-q}\right)^4.\end{aligned}$$

Now by taking square root on both sides, we get

$$\begin{aligned}&\left|\left(\vartheta_2-\vartheta_1+\frac{qh}{1-q}\right)\int_{\vartheta_1}^{\vartheta_2} f(\rho)g(\rho)\,{}_hD_q\rho-\int_{\vartheta_1}^{\vartheta_2} f(\rho)\,{}_hD_q\rho\int_{\vartheta_1}^{\vartheta_2} g(\rho)\,{}_hD_q\rho\right|\\ &\qquad\leqslant\frac{1}{4}(\theta_2-\theta_1)(\lambda_2-\lambda_1)\left((\vartheta_2-\vartheta_1)+\frac{qh}{1-q}\right)^2.\end{aligned}$$

From this, we derive the required inequality

$$\left|\frac{1-q}{((\vartheta_2-\vartheta_1)(1-q)+qh)}\int_{\vartheta_1}^{\vartheta_2} f(\rho)g(\rho)\,{}_hD_q\rho-\left(\frac{1-q}{((\vartheta_2-\vartheta_1)(1-q)+qh)}\int_{\vartheta_1}^{\vartheta_2} f(\rho_h D_q\rho\right)\right.$$

$$\left(\frac{1-q}{((\vartheta_2-\vartheta_1)(1-q)+qh)}\int_{\vartheta_1}^{\vartheta_2} g(\rho)\,{}_hD_q\rho\right)\Bigg| \leqslant \frac{1}{4}(\theta_2-\theta_1)(\lambda_2-\lambda_1).$$

This concludes the proof. □

Corollary 3.7. *By setting* $h=0$*, the inequality* (3.7) *transforms to:*

$$\left|\frac{1}{(\vartheta_2-\vartheta_1)}\int_{\vartheta_1}^{\vartheta_2} f(\rho)g(\rho)d_q\rho - \left(\frac{1}{(\vartheta_2-\vartheta_1)}\int_{\vartheta_1}^{\vartheta_2} f(\rho)d_q\rho\right)\left(\frac{1}{(\vartheta_2-\vartheta_1)}\int_{\vartheta_1}^{\vartheta_2} g(\rho)d_q\rho\right)\right| \leqslant \frac{1}{4}(\theta_2-\theta_1)(\lambda_2-\lambda_1).$$

This results is the q*-Grüss integral inequality in quantum calculus over a finite interval* [21]*. Furthermore, as* $q \to 1$*, we recover the classical Grüss integral inequality*

$$\left|\frac{1}{(\vartheta_2-\vartheta_1)}\int_{\vartheta_1}^{\vartheta_2} f(\rho)g(\rho)d\rho - \left(\frac{1}{(\vartheta_2-\vartheta_1)}\int_{\vartheta_1}^{\vartheta_2} f(\rho)d\rho\right)\left(\frac{1}{(\vartheta_2-\vartheta_1)}\int_{\vartheta_1}^{\vartheta_2} g(\rho)d\rho\right)\right| \leqslant \frac{1}{4}(\theta_2-\theta_1)(\lambda_2-\lambda_1).$$

The following theorem gives the generalization of Grüss-Čebyšev integral inequality on finite interval $[\vartheta_1, \vartheta_2]$ using novel q-h-operator for lipschitzian mappings.

Theorem 3.8. *For functions* f *and* g *defined on the interval* $[\vartheta_1, \vartheta_2] \to \mathbb{R}$*, being Lipschitzian continuous functions on* $[\vartheta_1, \vartheta_2]$ *with* $\Lambda_1 > 0$*,* $\Lambda_2 > 0$*, with* $0 < q < 1$*, and* $h \in \mathbb{R}$*, such that*

$$\left|f(\rho)-f(\varphi)\right| \leqslant \Lambda_1|\rho-\varphi| \quad \text{and} \quad \left|g(\rho)-g(\varphi)\right| \leqslant \Lambda_2|\rho-\varphi|, \tag{3.8}$$

for all $\varphi, \rho \in [\vartheta_1, \vartheta_2]$*, then* q*-*h*-integral adheres to the following inequality:*

$$\begin{aligned}\Bigg|\frac{1-q}{((\vartheta_2-\vartheta_1)(1-q)+qh)}\int_{\vartheta_1}^{\vartheta_2} f(\rho)g(\rho)\,{}_hD_q\rho - \left(\frac{1-q}{((\vartheta_2-\vartheta_1)(1-q)+qh)}\int_{\vartheta_1}^{\vartheta_2} f(\rho)\,{}_hD_q\rho\right)\\ \times\left(\frac{1-q}{((\vartheta_2-\vartheta_1)(1-q)+qh)}\int_{\vartheta_1}^{\vartheta_2} g(\rho)\,{}_hD_q\rho\right)\Bigg| \leqslant \Lambda_1\Lambda_2\left(\frac{q(\vartheta_2-\vartheta_1)^2}{(1+q+q^2)(1+q)^2}+S\right).\end{aligned} \tag{3.9}$$

Here, S *denotes the sum of series.*

Proof. Using the q-h-Korkine identity, we have:

$$\begin{aligned}\left(\vartheta_2-\vartheta_1+\frac{qh}{1-q}\right)\int_{\vartheta_1}^{\vartheta_2} f(\rho)g(\rho)\,{}_hD_q\rho - \int_{\vartheta_1}^{\vartheta_2} f(\rho)\,{}_hD_q\rho\int_{\vartheta_1}^{\vartheta_2} g(\rho)\,{}_hD_q\rho\\ = \frac{1}{2}\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2}\big(f(\rho)-f(\varphi)\big)\big(g(\rho)-g(\varphi)\big)\,{}_hD_q\rho\,{}_hD_q\varphi.\end{aligned} \tag{3.10}$$

From (3.8), we have

$$\left|\big(f(\rho)-f(\varphi)\big)\big(g(\rho)-g(\varphi)\big)\right| \leqslant \Lambda_1\Lambda_2\big(\vartheta\varepsilon-\varphi\big)^2, \quad \forall\varphi,\rho\in[\vartheta_1,\vartheta_2].$$

Taking double q-h-integrals on $[\vartheta_1, \vartheta_2]$, it follows

$$\begin{aligned}&\frac{1}{2}\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2}\left|\big(f(\rho)-f(\varphi)\big)\big(g(\rho)-g(\varphi)\big)\,{}_hD_q\rho\,{}_hD_q\varphi\right|\\ &\leqslant \Lambda_1\Lambda_2\frac{1}{2}\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2}(\vartheta\varepsilon-\varphi)^2_h D_q\rho\,{}_hD_q\varphi = \Lambda_1\Lambda_2\left[\left(\vartheta_2-\vartheta_1+\frac{qh}{1-q}\right)\int_{\vartheta_1}^{\vartheta_2}(\rho)^2_h D_q\rho - \left(\int_{\vartheta_1}^{\vartheta_2}\rho_h D_q\rho\right)^2\right].\end{aligned}$$

Based on Examples 2.4 and 2.5, the expression simplifies to:

$$= \Lambda_1\Lambda_2\left(\frac{(\vartheta_2-\vartheta_1)(1-q)+qh}{1-q}\right)^2\left(\frac{q(\vartheta_2-\vartheta_1)^2}{(1+q+q^2)(1+q)^2}+S\right),$$

where $\mathbf{S} = \mathbf{T}' - \mathbf{S}'^2 - 2\left(\frac{\vartheta_1+\vartheta_2 q}{1+q}\right)\mathbf{S}'$, notice $\mathbf{S} \to 0$, when $h = 0$. Thus, we arrive at the following inequality:

$$\begin{aligned} &\frac{1}{2}\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} \left|\big(\mathbf{f}(\boldsymbol{\rho}) - \mathbf{f}(\boldsymbol{\varphi})\big)\big(\mathbf{g}(\boldsymbol{\rho}) - \mathbf{g}(\boldsymbol{\varphi})\big)_h D_q \rho\, {}_h D_q \varphi\right| \\ &\quad \leqslant \Lambda_1\Lambda_2\left(\frac{(\vartheta_2-\vartheta_1)(1-q)+qh}{1-q}\right)^2\left(\frac{q(\vartheta_2-\vartheta_1)^2}{(1+q+q^2)(1+q)^2} + \mathbf{S}\right). \end{aligned} \tag{3.11}$$

By substituting (3.11) into (3.10), we obtain

$$\begin{aligned} &\left(\vartheta_2 - \vartheta_1 + \frac{qh}{1-q}\right)\int_{\vartheta_1}^{\vartheta_2} f(\rho)g(\rho)\,{}_h D_q\rho - \int_{\vartheta_1}^{\vartheta_2} f(\rho)\,{}_h D_q\rho \int_{\vartheta_1}^{\vartheta_2} g(\rho)\,{}_h D_q\rho \\ &\quad \leqslant \Lambda_1\Lambda_2\left(\frac{(\vartheta_2-\vartheta_1)(1-q)+qh}{1-q}\right)^2\left(\frac{q(\vartheta_2-\vartheta_1)^2}{(1+q+q^2)(1+q)^2} + S\right). \end{aligned}$$

Hence, we conclude with

$$\begin{aligned} &\Bigg|\frac{1-q}{((\vartheta_2-\vartheta_1)(1-q)+qh)}\int_{\vartheta_1}^{\vartheta_2} f(\rho)g(\rho)\,{}_h D_q\rho - \left(\frac{1-q}{((\vartheta_2-\vartheta_1)(1-q)+qh)}\int_{\vartheta_1}^{\vartheta_2} f(\rho_h D_q\rho\right) \\ &\times\left(\frac{1-q}{((\vartheta_2-\vartheta_1)(1-q)+qh)}\int_{\vartheta_1}^{\vartheta_2} g(\rho)\,{}_h D_q\rho\right)\Bigg| \leqslant \Lambda_1\Lambda_2\left(\frac{q(\vartheta_2-\vartheta_1)^2}{(1+q+q^2)(1+q)^2} + S\right). \end{aligned}$$

This gives us the required inequality (3.9). □

Corollary 3.9. *By setting $h = 0$, inequality (3.9) transforms to:*

$$\left|\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} f(\rho)g(\rho)d_q\rho - \left(\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} f(\rho)d_q\varphi\right)\left(\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} g(\rho)d_q\rho\right)\right| \leqslant \frac{q\Lambda_1\Lambda_2(\vartheta_2-\vartheta_1)^2}{(1+q)^2(1+q+q^2)}. \tag{3.12}$$

This represents the q-Grüss-Čebyšev integral inequality in quantum calculus over a finite interval [21]. Additionally, inequality (3.12) transforms to the classical Grüss-Čebyšev integral inequality, as $q \to 1$,

$$\left|\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} f(\rho)g(\rho)d\rho - \left(\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} f(\rho)d\varphi\right)\left(\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} g(\rho)d\rho\right)\right| \leqslant \frac{\Lambda_1\Lambda_2(\vartheta_2-\vartheta_1)^2}{12}.$$

4. Discussion

In this work, we have extended classical as well as q-integral inequalities, including the Cauchy-Schwarz integral inequality for double integrals, Grüss integral inequality, Korkine identity and Grüss-Čebyšev integral inequality, by utilizing the q-h-operator. Additionally, several special cases were derived, which demonstrate the connections between standard, h, and q-integral inequalities. This study contributes to the growing body of research on q-h-integral inequalities and highlights their significance in the broader framework of integral inequalities. For future work, our aim is to expand this research by exploring additional inequalities from classical analysis and q calculus, such as Jensen-type, Hermite-Hadamard type, and Ostrowski-type inequalities. We will focus on utilizing the q-h-operator to derive novel results, such as tighter bounds, generalized forms, and broader applications in quantum calculus.

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References

[1] T. A. Aljaaidi, D. B. Pachpatte, *Some Grüss-type inequalities using generalized Katugampola fractional integral*, AIMS Math., **5** (2019), 1011–1024. 1

[2] A. Alkhazzan, J. Wang, Y. Nie, H. Khan, J. Alzabut, *A stochastic SIRS modeling of transport-related infection with three types of noises*, Alex. Eng. J., **76** (2023), 557–572. 1

[3] G. A. Anastassiou, *Intelligent Mathematics: Computational Analysis*, Springer-Verlag, Berlin, (2011). 1

[4] S. S. Dragomir, *Some integral inequalities of Grs̈s type*, RGMIA Res. Rep. Collect., **1** (1998), 95–111. 1

[5] S. S. Dragomir, N. T. Diamond, *Integral inequalities of Grüss type via Pólya-Szegö and Shisha-Mond results*, RGMIA Res. Rep. Collect., **5** (2002), 1–9. 1

[6] H. Gauchman, *Integral inequalities in* q*-calculus*, Comput. Math. Appl., **47** (2004), 281–300.

[7] K. Jangid, S. D. Purohit, R. Agarwal, *On Grs̈s type inequality involving a fractional integral operator with a multi-index Mittag-Leffler function as a kernel*, Appl. Math. Inf. Sci., **16** (2022), 269–276.

[8] M. B. Jeelani, A. M. Saeed, M. S. Abdo, K. Shah, *Positive solutions for fractional boundary value problems under a generalized fractional operator*, Math. Methods Appl. Sci., **44** (2021), 9524–9540.

[9] M. Jleli, B. Samet, *On some integral inequalities in Quantum calculus*, J. Funct. Spaces, **2020** (2020), 10 pages. 1

[10] V. G. Kac, P. Cheung, *Quantum Calculus*, Springer-Verlag, New York, (2002). 1

[11] H. Khan, J. Alzabut, H. Gulzar, O. Tunç, S. Pinelas, *On system of variable order nonlinear p-Laplacian fractional differential equations with biological application*, Mathematics, **11** (2023), 17 pages. 1

[12] H. Khan, J. Alzabut, A. Shah, Z.-Y. He, S. Etemad, S. Rezapour, A. Zada, *On fractal-fractional waterborne disease model: A study on theoretical and numerical aspects of solutions via simulations*, Fractals, **31** (2023), 16 pages. 1

[13] A. McD. Mercer, *An improvement of the Grüss inequality*, JIPAM. J. Inequal. Pure Appl. Math., **6** (2005), 4 pages. 1

[14] N. Minculete, L. Ciurdariu, *A generalized form of Grüss type inequality and other integral inequalities*, J. Inequal. Appl., **2014** (2014), 18 pages. 1

[15] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, *Classical and new inequalities in analysis*, Kluwer Academic Publishers Group, Dordrecht, (1993). 1

[16] G.-H. Peng, Y. Miao, *A note on Grüss type inequality*, Appl. Math. Sci. (Ruse), **3** (2009), 399–402. 1

[17] D. Shi, G. Farid, B. A. I. Younis, H. Abu-Zinadah, M. Anwar, *A unified representation of q-and h-integrals and consequences in inequalities*, Axioms, **13** (2024), 12 pages. 1, 2, 2.1, 2.2

[18] W. Shatanawi, A. Boutiara, M. S. Abdo, M. B. Jeelani, K. Abodayeh, *Nonlocal and multiple-point fractional boundary value problem in the frame of a generalized Hilfer derivative*, Adv. Differ. Equ., **2021** (2021), 19 pages. 1

[19] J. Sousa, D. S. Oliveira, E. C. de Oliveira, *Grs̈s-type inequality by means of a fractional integral*, arXiv preprint arXiv:1705.00965, (2017), 16 pages. 1

[20] J. Tariboon, S. K. Ntouyas, *Quantum calculus on finite intervals and applications to impulsive difference equations*, Adv. Differ. Equ., **2013** (2013), 19 pages. 1

[21] J. Tariboon, S. K. Ntouyas, *Quantum integral inequalities on finite intervals*, J. Inequal. Appl., **2014** (2014), 1–13. 1, 2.1, 3.2, 3.5, 3.7, 3.9