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Multi-criteria decision-making method under the complex intuitionistic fuzzy environment

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Abstract

Complex intuitionistic fuzzy sets, which are distinguished by both membership and non-membership values, provide sophisticated decision-making tools by helping decision-makers better handle ambiguity, reluctance, and uncertainty in complex situations. In this paper, we introduce several novel operations on complex intuitionistic fuzzy sets, including a complex intuitionistic fuzzy distance measure, which is utilized to define σ-equalities of complex intuitionistic fuzzy sets. This method is applied to a decision-making scenario that takes place in the real world in order to show its practical value. This scenario illustrates the potential of complex intuitionistic fuzzy sets in tackling difficult and ambiguous choice issues with increased flexibility and accuracy. Finally, in order to demonstrate the efficacy of the suggested strategy, we compare it to other methods that have been taken in the past.

Keywords: Complex intuitionistic fuzzy set, operations on complex intuitionistic fuzzy set, decision-making matrices, decision-making method.

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1. Introduction

Uncertainty challenges develop when there is insufficient accurate knowledge or when the outcome of a situation is not predictable. In numerous fields, dealing with uncertainty is a common difficulty that requires specialized methods to handle inaccurate or incomplete information. Fuzzy sets (FSs) [43] are a mathematical concept introduced by Zadeh in 1965 as an extension of classical (crisp) set theory. Fuzzy sets are designed to more flexibly reflect vagueness and uncertainty than classical sets. This is especially helpful when working with imperfect information or when there are unclear boundaries between

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categories. Applications for fuzzy sets can be found in many domains, such as artificial intelligence, Decision-Making (D-M), expert systems, pattern recognition, and control systems. Al-Shami introduced aggregation operators based on $(2,1)$-Fuzzy sets [4]. He utilized them in multi-criteria decision making (MCDM) problems. Wang et al. [40] presented a MCDM problem in the environment of three-way FSs. Gholamizadeh et al. [19] discussed the contributions and applications of FSs theory in human reliability analysis. Zhou et al. [46] used a hybrid FMEA framework in a hesitant fuzzy environment to evaluate power transformers, providing another viewpoint on MCDM adaptation. Deivanayagampillai et al. [9] examined Industry 5.0 and found difficulties in developing economies before providing mitigation measures based on TODIM aggregation operators in a single-value neutrosophic environment.

Intuitionistic fuzzy sets (IFS) [7] are generalizations of FSs introduced by Atanassov in 1983. By considering both membership (M-S) and non-membership (N-M-S) degrees in grade value of IFSs, they provide a more comprehensive representation of uncertainty. This can be very helpful when discussing with situations where the researchers is uncertain about both the M-S and N-M-S status of an element in a set. Intuitionistic fuzzy sets provide a more comprehensive framework for capturing and modeling uncertainty, making them a useful extension of classical fuzzy sets in cases where a more precise representation of uncertainty is required. Intuitionistic fuzzy sets have been used in a variety of domains, including data mining, D-M problems, pattern recognition, artificial intelligence, and expert systems. A recognition concept based on the intuitionistic fuzzy correlation coefficient was presented by Ejegwa et al. [14], offering a fresh method for allocating courses in higher education. In a similar vein, Ejegwa et al. [12] showed encouraging implications for healthcare analytics by using Spearman's correlation coefficient to suggest a technique for identifying medical crises. Anum et al. [5] expanded on IFS applications by creating a weighted distance measure based on tendency coefficients that improves intelligent control system decision-making. Ejegwa et al. [11] introduced a novel distance metric for IFS in a different research that may find use in college admissions processes. Previous research by Ejegwa and Onyeke [13] further shown the usefulness of IFS in healthcare by analysing patient diagnoses using a modified correlation coefficient. Gul [20] enhanced the VIKOR technique with a bipolar fuzzy preference model, which allows for more flexibility in complex decision-making scenarios. Mishra et al. [30] employed interval-valued intuitionistic fuzzy distance metrics to evaluate sustainable wastewater treatment methods, highlighting the importance of MCDM in sustainability initiatives. Imran et al. [25] employed interval-valued intuitionistic fuzzy approaches for group decision-making in robot selection, demonstrating the utility of fuzzy aggregation techniques in technology-related applications. Kousar et al. [27] used MCDM to evaluate smog mitigation techniques, providing a thorough evaluation of health, economic, and ecological implications. Patel et al. [33] introduced similarity measure in the environment of IFSs. They presented their applications in software quality evaluation and face recognition. Hao et al. [22] proposed a new D-M algorithm for design scheme utilizing IFSs. Zeng et al. [45] defined some new intuitionistic fuzzy distance measures and utilized them in pattern recognition problems. Sahoo et al. [36] conducted a comprehensive bibliometric analysis on material selection, revealing significant trends and insights into the evolution of MCDM methods.

While IFSs are quite good at dealing with ambiguous viewpoints, they have limits when faced with circumstances in which the total values of M-S and N-M-S surpass 1. To address this limitation, Yager [42] introduced the Pythagorean Fuzzy Set (PFS). PFSs provide information in the form of M-S and N-M-S relative to values drawn from a set of attributes, with a rule constraining the sum of the squares of pairs to the unit interval. PFSs have been employed by numerous researchers across various scientific domains. The wide range of applications for MCDM in the industrial, environmental, and healthcare sectors is shown by Arora et al. [6] suggested trigonometric similarity metrics in Pythagorean fuzzy sets. Garg investigated IVPFSs and their applications [15]. Wei examined MCDM problems using PF information [41]. Tešić and Marinković [39] used fermatean fuzzy weight operators for combat system selection, demonstrating MCDM's versatility across industries. The intelligent decision support system developed by Hussain and Ullah [24] improves the accuracy of decision-making in uncertain situations by using spherical fuzzy Sugeno-Weber aggregation operators for practical applications.

Three different types of sets FSs, IFSs, and PFSs have been used in pattern recognition, image processing, and D-M. They all provide a more dynamic and nuanced way to represent ambiguity and imprecision than standard set theory. However, these three models are unable to solve problems in the two dimensions. To discuss two-dimensional phenomena, Ramot et al. [35] gave the idea of complex fuzzy sets (CFSs). The range of a CFS is the unit disk in a complex plane, and it is the generalization of an FS. The degree of M-S of a member in the CFS is described by the complex-valued function using both phase and magnitude information. Applications for CFSs can be identified in a number of domains, including D-M, image processing, pattern recognition, and control systems. Ma et al. [29] proposed a new algorithm in the environment of CFSs and identified a known signal among different signals. Zeeshan et al. [44] proposed a new D-M algorithm based on complex fuzzy distance measures. For large scale learning problems, Sobhi and Dick designed a new neuro-fuzzy architecture based on CFSs [37].

Alkouri and Salleh [2] gave the notions of CIFSs. A CIFS $\bar{I}$ is defined on a universal set U, and is characterized by a function $\bar{I} = \{(\rho, \kappa_{\bar{I}}(\rho), \tau_{\bar{I}}(\rho)) : \rho \in \Bbbk\}$, where $\kappa_{\bar{I}}(\rho) = \chi_{\bar{I}}(\rho)e^{i\mathfrak{I}_{\bar{I}}(\rho)}$ and $\tau_{\bar{I}}(\rho) = \hbar_{\bar{I}}(\rho)e^{i\mho_{\bar{I}}(\rho)}$, $i = \sqrt{(-1)}$, are M-S and N-M-S functions. The terms $\chi_{\bar{I}}(\rho)$ and $\hbar_{\bar{I}}(\rho)$ belong to closed interval $[0, 1]$, and both functions are real-valued with $0 \leqslant \chi_{\bar{I}}(\rho) + \hbar_{\bar{I}}(\rho) \leqslant 1$. The values of M-S and N-M-S of CIFSs lie in the unit disc in a complex plane. The amplitude components linked with M-S and N-M-S indicate the degree to which an element belongs or does not belong to CIFSs. The supplementary information includes phase terms for M-S and N-M-S, with the phase term typically utilized as a secondary dimension for representing periodicity. CIFSs has many applications in various fields. Garg and Rani elucidate a methodology for addressing MCDM problems within a CIFS environment [17]. Jan et al. [26] utilized CIFSs to explore the connections between different types of cybersecurity and the origins of cyberattacks. Hao et al. [23] introduced several operators for handling CIF information.

1.1. *Motivation*

In D-M problems, the motivation behind using CIFSs is to address classical and standard FSs. Some of key motivations are given below.

1. The classical FSs cannot fully addressed the D-M problems that involve uncertainties. CIFSs play a major role in D-M problems by incorporatin complex numbers in M-S function and N-M-S function.
2. In real-life world, many D-M problems are inherently complex and dynamic. CIFSs discuss such problems with more nuanced way and enabling decision-makers to overcome the intracacies of decision environments with evolving uncertainties.
3. CIFSs provide a powerful framework in MCDM problems where more than one criteria are considered. The complex representation of M-S function and N-M-S function in a CIFS discuss models of diverse criteria in a more comprehensive way.
4. CIFSs play a vital role in MCDM problems that involves the integration of information from different sources.
5. In dynamic environments, the D-M problems may change over time. The CIFSs are well suited for such D-M problems due to complex representation in M-S function and N-M-S function.

1.2. *Contributions/significance*

The role of CIFSs in D-M problems is significant and multifaceted. Some of key contributions of CIFSs in D-M problems are given below.

1. First of all we will define some novel operations on CIFSs including simple and bounded difference, disjoint and disjunctive sum and other essential operations.
2. We aim to extend the quasi-triangular mapping to CIFSs and prove that union, probabilistic sum, and bold sum are s-norms, while intersection, product, and bold intersection are t-norms in CIFSs.
3. We introduce a novel CIF weighted average similarity measure and demonstrate its practical applicability by solving a real-world decision-making problem: optimizing business investment portfolios across diverse enterprises.

4. To evaluate the effectiveness and efficiency of our proposed method, we will conduct a comparative analysis with existing approaches, assessing their strengths, weaknesses, and performance metrics to demonstrate the advantages and improvements of our methodology.

2. Complex intuitionistic fuzzy sets

In this section we shall review the concepts of CIFSs.

Definition 2.1 ([21])**.** A CIFS $\bar{I}$ defined on U is given as

$$\bar{I} = \{(\rho, \kappa_{\bar{I}}(\rho), \tau_{\bar{I}}(\rho)) : \rho \in \Bbbk\},$$

where $\kappa_{\bar{I}}(\rho), \tau_{\bar{I}}(\rho) : \Bbbk \longrightarrow \{a : a \in C, |a| \leqslant 1\}$ are complex-valued M-S and N-M-S, respectively, and representing as $\kappa_{\bar{I}}(\rho) = \chi_{\bar{I}}(\rho)e^{i\mathfrak{I}_{\bar{I}}(\rho)}$ and $\tau_{\bar{I}}(\rho) = \hbar_{\bar{I}}(\rho)e^{i\eth_{\bar{I}}(\rho)}$, where $0 \leqslant \chi_{\bar{I}}(\rho), \hbar_{\bar{I}}(\rho) \leqslant 1$, $0 \leqslant \chi_{\bar{I}}(\rho) + \hbar_{\bar{I}}(\rho) \leqslant 1$ and $0 \leqslant \mathfrak{I}_{\bar{I}}(\rho), \eth_{\bar{I}}(\rho) \leqslant 2\pi$, $0 \leqslant \mathfrak{I}_{\bar{I}}(\rho) + \eth_{\bar{I}(\rho)} \leqslant 2\pi$ for all $\rho \in \Bbbk$.

Definition 2.2 ([21])**.** Let $\bar{I}_1$ and $\bar{I}_2$ be two CIFSs on $\Bbbk$, and

$$\mu_{\bar{I}_1} = \left\{\left\langle \rho, \chi_{\bar{I}_1}(\rho)e^{i\mathfrak{I}_{\bar{I}_1}(\rho)}, \hbar_{\bar{I}_1}(\rho)e^{i\eth_{\bar{I}_1}(\rho)} \right\rangle : \rho \in \Bbbk\right\}$$

and

$$\mu_{\bar{I}_2} = \left\{\left\langle \rho, \chi_{\bar{I}_2}(\rho)e^{i\mathfrak{I}_{\bar{I}_2}(\rho)}, \hbar_{\bar{I}_2}(\rho)e^{i\eth_{\bar{I}_2}(\rho)} \right\rangle : \rho \in \Bbbk\right\},$$

the grade's of $\bar{I}_1$ and $\bar{I}_2$, respectively. Then, the union of these two CIFSs $\bar{I}_1$ and $\bar{I}_2$, denoted by $\bar{I}_1 \cup \bar{I}_2$, is defined as

$$\mu_{\bar{I}_1 \cup \bar{I}_2}(\rho) = \left\langle \begin{array}{l} \max(\chi_{\bar{I}_1}(\rho), \chi_{\bar{I}_2}(\rho))e^{i\max(\mathfrak{I}_{\bar{I}_1}(\rho), \mathfrak{I}_{\bar{I}_2}(\rho))}, \\ \min(\hbar_{\bar{I}_1}(\rho), \hbar_{\bar{I}_2}(\rho))e^{i\min(\eth_{\bar{I}_1}(\rho), \eth_{\bar{I}_2}(\rho))} \end{array} \right\rangle.$$

Example 2.3. If

$$\bar{I}_1 = \{\langle \rho_1, 0.5e^{i0.5\pi}, 0.4e^{i\pi} \rangle, \langle \rho_2, 0.6e^{i0.3\pi}, 0.2e^{i\pi} \rangle, \langle \rho_3, 0.2e^{i1.5\pi}, 0.7e^{i0.3\pi} \rangle\}$$

and

$$\bar{I}_2 = \{\langle \rho_1, 0.2e^{i\pi}, 0.6e^{i0.5\pi} \rangle, \langle \rho_2, 0.5e^{i0.8\pi}, 0.2e^{i0.5\pi} \rangle, \langle \rho_3, 0.2e^{i0.5\pi}, 0.2e^{i0.5\pi} \rangle\}$$

are CIFNs, then

$$\bar{I}_1 \cup \bar{I}_2 = \{\langle \rho_1, 0.5e^{i\pi}, 0.4e^{i0.5\pi} \rangle, \langle \rho_2, 0.6e^{i0.8\pi}, 0.2e^{i0.5\pi} \rangle, \langle \rho_3, 0.2e^{i1.5\pi}, 0.2e^{i0.3\pi} \rangle\}.$$

Definition 2.4 ([21])**.** Let $\bar{I}_1$ and $\bar{I}_2$ be two CIFSs on $\Bbbk$, and

$$\mu_{\bar{I}_1} = \left\{\left\langle \rho, \chi_{\bar{I}_1}(\rho)e^{i\mathfrak{I}_{\bar{I}_1}(\rho)}, \hbar_{\bar{I}_1}(\rho)e^{i\eth_{\bar{I}_1}(\rho)} \right\rangle : \rho \in \Bbbk\right\}$$

and

$$\mu_{\bar{I}_2} = \left\{\left\langle \rho, \chi_{\bar{I}_2}(\rho)e^{i\mathfrak{I}_{\bar{I}_2}(\rho)}, \hbar_{\bar{I}_2}(\rho)e^{i\eth_{\bar{I}_2}(\rho)} \right\rangle : \rho \in \Bbbk\right\},$$

the grade's of $\bar{I}_1$ and $\bar{I}_2$, respectively. Then, the intersection of these two CIFSs $\bar{I}_1$ and $\bar{I}_2$, denoted by $\bar{I}_1 \cap \bar{I}_2$, is defined as

$$\mu_{\bar{I}_1 \cap \bar{I}_2}(\rho) = \left\langle \begin{array}{l} \min(\chi_{\bar{I}_1}(\rho), \chi_{\bar{I}_2}(\rho))e^{i\min(\mathfrak{I}_{\bar{I}_1}(\rho), \mathfrak{I}_{\bar{I}_2}(\rho))}, \\ \max(\hbar_{\bar{I}_1}(\rho), \hbar_{\bar{I}_2}(\rho))e^{i\max(\eth_{\bar{I}_1}(\rho), \eth_{\bar{I}_2}(\rho))} \end{array} \right\rangle.$$

Example 2.5. Consider $\bar{I}_1$ and $\bar{I}_2$ in Example 2.3. Then, the intersection of these two CIFSs is

$$\bar{I}_1 \cap \bar{I}_2 = \{\langle \rho_1, 0.2e^{i0.5\pi}, 0.6e^{i\pi} \rangle, \langle \rho_2, 0.5e^{i0.3\pi}, 0.2e^{i\pi} \rangle, \langle \rho_3, 0.2e^{i0.5\pi}, 0.7e^{i0.5\pi} \rangle\}.$$

Definition 2.6 ([2]). Let $\bar{I}$ be a CIFS on $\Bbbk$, and $\mu_{\bar{I}} = \{\langle \rho, \chi_{\bar{I}}(\rho).e^{i\Im_{\bar{I}}(\rho)},\ \hbar_{\bar{I}}(\rho).e^{i\eth_{\bar{I}}(\rho)}\rangle : \rho \in \Bbbk\}$ the TG and FG values of $\bar{I}$, respectively. Then, the complement of $\bar{I}$ denoted $\bar{I}^c$, is defined as

$$\bar{I}^c = \left\{\left\langle \rho, \chi_{\bar{I}}(\rho).e^{i\Im_{\bar{I}}(\rho)},\ \hbar_{\bar{I}}(\rho).e^{i\eth_{\bar{I}}(\rho)}\right\rangle : \rho \in \Bbbk\right\}.$$

Example 2.7. If $\bar{I} = \{\langle \rho_1, 0.5e^{i0.5\pi}, 0.4e^{i\pi}\rangle, \langle \rho_2, 0.6e^{i0.3\pi}, 0.2e^{i\pi}\rangle, \langle \rho_3, 0.2e^{i1.5\pi}, 0.7e^{i0.3\pi}\rangle\}$, then complement of $\bar{I}$ is

$$\bar{I}^c = \{\langle \rho_1, 0.4e^{i\pi}, 0.5e^{i0.5\pi}\rangle, \langle \rho_2, 0.2e^{i\pi}, 0.6e^{i0.3\pi}\rangle, \langle \rho_3, 0.7e^{i0.3\pi}, 0.2e^{i1.5\pi}\rangle\}.$$

Definition 2.8. Let $\bar{I}_1$ and $\bar{I}_2$ be two CIFSs on $\Bbbk$, and

$$\mu_{\bar{I}_1} = \left\{\left\langle \rho, \chi_{\bar{I}_1}(\rho)e^{i\Im_{\bar{I}_1}(\rho)},\ \hbar_{\bar{I}_1}(\rho)e^{i\eth_{\bar{I}_1}(\rho)}\right\rangle :\ \rho \in \Bbbk\right\}$$

and

$$\mu_{\bar{I}_2} = \left\{\left\langle \rho, \chi_{\bar{I}_2}(\rho)e^{i\Im_{\bar{I}_2}(\rho)}, \hbar_{\bar{I}_2}(\rho)e^{i\eth_{\bar{I}_2}(\rho)}\right\rangle : \rho \in \Bbbk\right\}$$

the grade's of $\bar{I}_1$ and $\bar{I}_2$, respectively. Then, the simple difference of these two CIFSs $\bar{I}_1$ and $\bar{I}_2$, denoted by $\bar{I}_1 \backslash \bar{I}_2$, is defined as

$$\begin{aligned}\bar{I}_1 \backslash \bar{I}_2 = \mu_{(\bar{I}_1 \cap \bar{I}_2^c)}(\rho) &= \left\langle \chi_{\bar{I}_1}(\rho)e^{i\Im_{\bar{I}_1}(\rho)}, \hbar_{\bar{I}_1}(\rho)e^{i\eth_{\bar{I}_1}(\rho)}\right\rangle \cap \left\langle \hbar_{\bar{I}_2}(\rho)e^{i\eth_{\bar{I}_2}(\rho)}, \chi_{\bar{I}_2}(\rho)e^{i\Im_{\bar{I}_2}(\rho)}\right\rangle \\ &= \left\langle \begin{array}{c} \min(\chi_{\bar{I}_1}(\rho), \hbar_{\bar{I}_2}(\rho))e^{i\min(\Im_{\bar{I}_1}(\rho), \eth_{\bar{I}_2}(\rho)}, \\ \max(\hbar_{\bar{I}_1}(\rho), \chi_{\bar{I}_2}(\rho))e^{i\max(\eth_{\bar{I}_1}(\rho), \Im_{\bar{I}_2}(\rho))} \end{array} \right\rangle.\end{aligned}$$

Example 2.9. Consider $\bar{I}_1$ and $\bar{I}_2$ in Example 2.3. Then, the simple difference of these two CIFSs is

$$\begin{aligned}\bar{I}_1 \backslash \bar{I}_2 = \bar{I}_1 \cap \bar{I}_2^c &= \{\langle \rho_1, 0.5e^{i0.5\pi}, 0.4e^{i\pi}\rangle, \langle \rho_2, 0.6e^{i0.3\pi}, 0.3e^{i\pi}\rangle, \langle \rho_3, 0.2e^{i1.5\pi}, 0.7e^{i0.3\pi}\rangle\} \\ &\cap \{(\rho_1, 0.6e^{i0.5\pi}, 0.2e^{i\pi}), (\rho_2, 0.2e^{i0.5\pi}, 0.5e^{i0.8\pi}), (\rho_3, 0.2e^{i0.5\pi}, 0.2e^{i0.5\pi})\} \\ &= \{[c]c(\rho_1, 0.5e^{i0.5\pi}, 0.4e^{i\pi}), (\rho_2, 0.2e^{i0.3\pi}, 0.5e^{i\pi}), (\rho_3, 0.2e^{i0.5\pi}, 0.7e^{i0.5\pi})\}.\end{aligned}$$

Definition 2.10. Let $\bar{I}_1$ and $\bar{I}_2$ be two CIFSs on $\Bbbk$, and

$$\mu_{\bar{I}_1} = \left\{\left\langle \rho, \chi_{\bar{I}_1}(\rho)e^{i\Im_{\bar{I}_1}(\rho)}, \hbar_{\bar{I}_1}(\rho)e^{i\eth_{\bar{I}_1}(\rho)}\right\rangle : \rho \in \Bbbk\right\} \quad \text{and} \quad \mu_{\bar{I}_2} = \left\{\left\langle \rho, \chi_{\bar{I}_2}(\rho)e^{i\Im_{\bar{I}_2}(\rho)}, \hbar_{\bar{I}_2}(\rho)e^{i\eth_{\bar{I}_2}(\rho)}\right\rangle : \rho \in \Bbbk\right\}$$

the grade's of $\bar{I}_1$ and $\bar{I}_2$, respectively. Then, the bounded difference of these two CIFSs $\bar{I}_1$ and $\bar{I}_2$, denoted by $\bar{I}_1 \ominus \bar{I}_2$, is defined as

$$\mu_{(\bar{I}_1 \ominus \bar{I}_2)}(\rho) = \left\{ \begin{array}{c} \max(0, \chi_{\bar{I}_1}(\rho) \backslash \chi_{\bar{I}_2}(\rho)e^{i\max(0, \Im_{\bar{I}_1}(\rho) \backslash \Im_{\bar{I}_2}(\rho))}, \\ \max(0, \hbar_{\bar{I}_1}(\rho) \backslash \hbar_{\bar{I}_2}(\rho))\}e^{i\max(0, \eth_{\bar{I}_1}(\rho) \backslash \eth_{\bar{I}_2}(\rho))} \end{array} \right\}.$$

Here $[\chi_{\bar{I}_1}(\rho) \backslash \chi_{\bar{I}_2}(\rho)]$, $[\hbar_{\bar{I}_1}(\rho) \backslash \hbar_{\bar{I}_2}(\rho)]$ are similar to the bounded difference for standard FSs, we employ the following functions to determine the phase term.

(i) Sum:
$$\Im_{\bar{I}_1}(\rho) \backslash \Im_{\bar{I}_2}(\rho) = \Im_{\bar{I}_1}(\rho) + \Im_{\bar{I}_2}(\rho), \qquad \eth_{\bar{I}_1}(\rho) \backslash \eth_{\bar{I}_2}(\rho) = \eth_{\bar{I}_1}(\rho) + \eth_{\bar{I}_2}(\rho).$$

(ii) Maximun:
$$\Im_{\bar{I}_1}(\rho) \backslash \Im_{\bar{I}_2}(\rho) = \max\{\Im_{\bar{I}_1}(\rho), \Im_{\bar{I}_2}(\rho)\}, \qquad \eth_{\bar{I}_1}(\rho) \backslash \eth_{\bar{I}_2}(\rho) = \max\{\eth_{\bar{I}_1}(\rho), \eth_{\bar{I}_2}(\rho)\}.$$

(iii) Minimun:
$$\Im_{\bar{I}_1}(\rho) \backslash \Im_{\bar{I}_2}(\rho) = \min\{\Im_{\bar{I}_1}(\rho), \Im_{\bar{I}_2}(\rho)\}, \qquad \eth_{\bar{I}_1}(\rho) \backslash \eth_{\bar{I}_2}(\rho) = \min\{\eth_{\bar{I}_1}(\rho), \eth_{\bar{I}_2}(\rho)\}.$$

(iv) "Winner take all":

$$\mathfrak{I}_{\bar{I}_1}(\rho)\backslash\mathfrak{I}_{\bar{I}_2}(\rho) = \left\{ \begin{array}{l} \mathfrak{I}_{\bar{I}_1}(\rho); \chi_{\bar{I}_1}(\rho) > \chi_{\bar{I}_2}(\rho) \\ \mathfrak{I}_{\bar{I}_2}(\rho); \chi_{\bar{I}_2}(\rho) > \chi_{\bar{I}_1}(\rho) \end{array} \right\}, \quad \eth_{\bar{I}_1}(\rho)\backslash\eth_{\bar{I}_2}(\rho) = \left\{ \begin{array}{l} \eth_{\bar{I}_1}(\rho); \hbar_{\bar{I}_1}(\rho) > \hbar_{\bar{I}_2}(\rho) \\ \eth_{\bar{I}_2}(\rho); \hbar_{\bar{I}_2}(\rho) > \hbar_{\bar{I}_1}(\rho) \end{array} \right\}.$$

The subsequent options are equally valid for determining the phase terms:

(i) Difference:

$$\mathfrak{I}_{\bar{I}_1}(\rho)\backslash\mathfrak{I}_{\bar{I}_2}(\rho) = \mathfrak{I}_{\bar{I}_1}(\rho) - \mathfrak{I}_{\bar{I}_2}(\rho), \quad \eth_{\bar{I}_1}(\rho)\backslash\eth_{\bar{I}_2}(\rho) = \eth_{\bar{I}_1}(\rho) - \eth_{\bar{I}_2}(\rho).$$

(ii) Average:

$$\mathfrak{I}_{\bar{I}_1}(\rho)\backslash\mathfrak{I}_{\bar{I}_2}(\rho) = \frac{\mathfrak{I}_{\bar{I}_1}(\rho) + \mathfrak{I}_{\bar{I}_2}(\rho)}{2}, \quad \eth_{\bar{I}_1}(\rho)\backslash\eth_{\bar{I}_2}(\rho) = \frac{\eth_{\bar{I}_1}(\rho) + \eth_{\bar{I}_2}(\rho)}{2}.$$

(iii) Weighted average:

$$\mathfrak{I}_{\bar{I}_1}(\rho)\backslash\mathfrak{I}_{\bar{I}_2}(\rho) = \frac{\chi_{\bar{I}_1}(\rho)\mathfrak{I}_{\bar{I}_1}(\rho) + \chi_{\bar{I}_2}(\rho)\mathfrak{I}_{\bar{I}_2}(\rho)}{\chi_{\bar{I}_1}(\rho) + \chi_{\bar{I}_2}(\rho)}, \quad \eth_{\bar{I}_1}(\rho)\backslash\eth_{\bar{I}_2}(\rho) = \frac{\hbar_{\bar{I}_1}(\rho)\eth_{\bar{I}_1}(\rho) + \hbar_{\bar{I}_2}(\rho)\eth_{\bar{I}_2}(\rho)}{\hbar_{\bar{I}_1}(\rho) + \hbar_{\bar{I}_2}(\rho)}.$$

Example 2.11. Consider $\bar{I}_1$ and $\bar{I}_2$ in Example 2.3. The bounded difference is

$$(\bar{I}_1 \ominus \bar{I}_2) = \left\{[c]c(\rho_1, 0.3e^{i0}, 0e^{i0.5\pi}), (\rho_2, 0.1e^{i0}, 0e^{i0.5\pi}), (\rho_3, 0e^{i\pi}, 0.5e^{i0})\right\}.$$

Definition 2.12. Let $\bar{I}_1$ and $\bar{I}_2$ be two CIFSs on $\Bbbk$, and

$$\mu_{\bar{I}_1} = \left\{\left\langle \rho, \chi_{\bar{I}_1}(\rho)e^{i\mathfrak{I}_{\bar{I}_1}(\rho)}, \hbar_{\bar{I}_1}(\rho)e^{i\eth_{\bar{I}_1}(\rho)} \right\rangle : \rho \in \Bbbk\right\}$$

and

$$\mu_{\bar{I}_2} = \left\{\left\langle \rho, \chi_{\bar{I}_2}(\rho)e^{i\mathfrak{I}_{\bar{I}_2}(\rho)}, \hbar_{\bar{I}_2}(\rho)e^{i\eth_{\bar{I}_2}(\rho)} \right\rangle : \rho \in \Bbbk\right\}$$

the grade's of $\bar{I}_1$ and $\bar{I}_2$, respectively. So the disjoint sum of these two CIFSs $\bar{I}_1$ and $\bar{I}_2$, denoted by $\bar{I}_1 \boxplus \bar{I}_2$, is defined as

$$\mu_{(\bar{I}_1 \boxplus \bar{I}_2)}(\rho) = \left|\chi_{\bar{I}_1}(\rho)\backslash\chi_{\bar{I}_2}(\rho)\right| e^{i\left|\mathfrak{I}_{\bar{I}_1}(\rho)\backslash\mathfrak{I}_{\bar{I}_2}(\rho)\right|}, \left|\hbar_{\bar{I}_1}(\rho)\backslash\hbar_{\bar{I}_2}(\rho)\right| e^{i\left|\eth_{\bar{I}_1}(\rho)\backslash\eth_{\bar{I}_2}(\rho)\right|}.$$

The operation " \" denotes the simple difference.

Example 2.13. Consider $\bar{I}_1$ and $\bar{I}_2$ in Example 2.3. The disjoint sum of these two CIFSs is.

$$(\bar{I}_1 \boxplus \bar{I}_2) = \{(\rho_1, 0.3e^{i0.5\pi}, 0.2e^{i0.5\pi}), (\rho_2, 0.1e^{i0.5\pi}, 0e^{i0.5\pi}), (\rho_3, 0e^{i\pi}, 0.5e^{i0.2\pi})\}.$$

Definition 2.14. Let $\bar{I}_1$ and $\bar{I}_2$ be two CIFSs on $\Bbbk$, and

$$\mu_{\bar{I}_1} = \left\{\left\langle \rho, \chi_{\bar{I}_1}(\rho)e^{i\mathfrak{I}_{\bar{I}_1}(\rho)}, \hbar_{\bar{I}_1}(\rho)e^{i\eth_{\bar{I}_1}(\rho)} \right\rangle : \rho \in \Bbbk\right\} \quad \text{and} \quad \mu_{\bar{I}_2} = \left\{\left\langle \rho, \chi_{\bar{I}_2}(\rho)e^{i\mathfrak{I}_{\bar{I}_2}(\rho)}, \hbar_{\bar{I}_2}(\rho)e^{i\eth_{\bar{I}_2}(\rho)} \right\rangle : \rho \in \Bbbk\right\}$$

the grade's of $\bar{I}_1$ and $\bar{I}_2$, respectively. Then, the disjunctive sum of these two CIFSs $\bar{I}_1$ and $\bar{I}_2$, denoted by $\bar{I}_1 \uplus \bar{I}_2$, is defined as

$$\bar{I}_1 \uplus \bar{I}_2 = (\bar{I}_1 \cap \bar{I}_2^c) \cup (\bar{I}_1^c \cap \bar{I}_2),$$

$$\mu_{(\bar{I}_1 \uplus \bar{I}_2)}(\rho) = \left\langle \begin{array}{l} \min(\chi_{\bar{I}_1}(\rho), \hbar_{\bar{I}_2}(\rho))e^{i\min(\mathfrak{I}_{\bar{I}_1}(\rho), \eth_{\bar{I}_2}(\rho))}, \\ \max(\hbar_{\bar{I}_1}(\rho), \chi_{\bar{I}_2}(\rho))e^{i\max(\eth_{\bar{I}_1}(\rho), \mathfrak{I}_{\bar{I}_2}(\rho))} \end{array} \right\rangle \cup \left\langle \begin{array}{l} \min(\hbar_{\bar{I}_1}(\rho), \chi_{\bar{I}_2}(\rho))e^{i\min(\eth_{\bar{I}_1}(\rho), \mathfrak{I}_{\bar{I}_2}(\rho))}, \\ \max(\chi_{\bar{I}_1}(\rho), \hbar_{\bar{I}_2}(\rho))e^{i\max(\mathfrak{I}_{\bar{I}_1}(\rho), \eth_{\bar{I}_2}(\rho))} \end{array} \right\rangle,$$

"c" is representation of standard complement of CIFSs.

Example 2.15. Consider $\bar{I}_1$ and $\bar{I}_2$ in Example 2.3. The disjunctive sum is,

$$\begin{aligned}\bar{I}_1 \uplus \bar{I}_2 &= (\bar{I}_1 \cap \bar{I}_2^c) \cup (\bar{I}_1^c \cap \bar{I}_2)\\ &= \left\{\left\langle \rho_1, 0.6e^{i0.5\pi}, 0.4e^{i\pi}\right\rangle, \left\langle \rho_2, 0.2e^{i0.3\pi}, 0.5e^{i\pi}\right\rangle, \left\langle \rho_3, 0.2e^{i0.5\pi}, 0.7e^{i0.5\pi}\right\rangle\right\}\\ &\cup \left\{\left\langle \rho_1, 0.2e^{i\pi}, 0.6e^{i0.5\pi}\right\rangle, \left\langle \rho_2, 0.2e^{i0.8\pi}, 0.6e^{i0.5\pi}\right\rangle, \left\langle \rho_3, 0.2e^{i0.3\pi}, 0.2e^{i1.5\pi}\right\rangle\right\},\\ \bar{I}_1 \uplus \bar{I}_2 &= \left\{\left\langle \rho_1, 0.6e^{i\pi}, 0.4e^{i0.5\pi}\right\rangle, \left\langle \rho_2, 0.2e^{i0.8\pi}, 0.5e^{i0.5\pi}\right\rangle, \left\langle \rho_3, 0.2e^{i0.5\pi}, 0.2e^{i0.5\pi}\right\rangle\right\}.\end{aligned}$$

Definition 2.16.

(1). A quasi-triangular norm $\digamma$ is a mapping $(0,1] \times (0,1] \to [0,1]$ that satisfies the following conditions:

(a) $\digamma(1,1) = 1$;
(b) $\digamma(\rho, \rho') = \digamma(\rho', \rho)$;
(c) $\digamma(\rho, \rho') \leqslant \digamma(\rho'', \rho''')$, whenever $\rho \leqslant \rho''$,$\rho' \leqslant \rho'''$;
(d) $\digamma(\digamma(\rho, \rho'), \rho'') = \digamma(\rho, \digamma(\rho', \rho''))$.

(2). A triangular norm $\digamma$ is a mapping $[0,1] \times [0,1] \longrightarrow [0,1]$ and satisfies condition (a)→(d) and the following conditions.

(a) $\digamma(0,0) = 0$. We say that $\digamma$ is an s-norm, if $\digamma$ is satisfied.
(b) $\digamma(\rho, 0) = \rho$. We say that T is a t-norm $\digamma$ is satisfied.
(c) $\digamma(\rho, 1) = \rho$. We say that a binary function $\check{T} : \hat{H}(\Bbbk) \times \hat{H}(\Bbbk) \longrightarrow \hat{H}(\Bbbk)$, where $\hat{H}(\Bbbk)$ denotes the collection of CIFSs, and defined as

$$\check{T}(\bar{I}_1, \bar{I}_2) \longrightarrow \left\{ \left\langle \begin{array}{c} \sup\limits_{\rho\in\Bbbk} \digamma_1(\chi_{\bar{I}_1}(\rho), \chi_{\bar{I}_2}(\rho)) e^{i2\pi \sup\limits_{\rho\in\Bbbk} \digamma_2\left(\frac{\Im_{\bar{I}_1}(\rho)}{2\pi}, \frac{\Im_{\bar{I}_2}(\rho)}{2\pi}\right)}, \\ \inf\limits_{\rho\in\Bbbk} \digamma_1(\hbar_{\bar{I}_1}(\rho), \hbar_{\bar{I}_2}(\rho)) e^{i2\pi \inf\limits_{\rho\in\Bbbk} \digamma_2\left(\frac{\eth_{\bar{I}_1}(\rho)}{2\pi}, \frac{\eth_{\bar{I}_2}(\rho)}{2\pi}\right)} \end{array} \right\rangle \right\},$$

is a triangular norm if $\digamma_1$ is triangular norm and $\digamma_2$ is a quasi-triangular norm. We say that $\check{T}$ is s-norm if $\digamma_1$ is s-norm. We say that $\check{T}$ is a t-norm if $\digamma_1$ is t-norm.

Proposition 2.17. *The complex intuitionistic fuzzy union on $\hat{H}(\Bbbk)$ is an s-norm.*

Proof. For the proof of this proposition see appendix AppendixA. □

Proposition 2.18. *The CIF intersection on $\hat{H}(\Bbbk)$ is a t-norm.*

Proof. For the proof of this proposition see appendix AppendixB. □

Definition 2.19. Let $\bar{I}_1$ and $\bar{I}_2$ be two CIFSs on $\Bbbk$, and

$$\mu_{\bar{I}_1} = \left\{\left\langle \rho, \chi_{\bar{I}_1}(\rho) e^{i\Im_{\bar{I}_1}(\rho)}, \hbar_{\bar{I}_1}(\rho) e^{i\eth_{\bar{I}_1}(\rho)} \right\rangle : \rho \in \Bbbk \right\} \text{ and } \mu_{\bar{I}_2} = \left\{\left\langle \rho, \chi_{\bar{I}_2}(\rho) e^{i\Im_{\bar{I}_2}(\rho)}, \hbar_{\bar{I}_2}(\rho) e^{i\eth_{\bar{I}_2}(\rho)} \right\rangle : \rho \in \Bbbk \right\}$$

the grade's of $\bar{I}_1$ and $\bar{I}_2$, respectively. Then, *the CIF product of* $\bar{I}_1$ and $\bar{I}_2$, is denoted by $\bar{I}_1 \boxtimes \bar{I}_2$ and defined as

$$\begin{aligned}\mu_{\bar{I}_1 \boxtimes \bar{I}_2}(\rho) &= \left\langle \chi_{\bar{I}_1}(\rho) e^{i\Im_{\bar{I}_1}(\rho)}, \hbar_{\bar{I}_1}(\rho) e^{i\eth_{\bar{I}_1}(\rho)} \right\rangle \boxtimes \left\langle \chi_{\bar{I}_2}(\rho) e^{i\Im_{\bar{I}_2}(\rho)}, \hbar_{\bar{I}_2}(\rho) e^{i\eth_{\bar{I}_2}(\rho)} \right\rangle\\ &= \left\{ (\chi_{\bar{I}_1}(\rho).\chi_{\bar{I}_2}(\rho)) e^{i2\pi\left(\frac{\Im_{\bar{I}_1}(\rho)}{2\pi}.\frac{\Im_{\bar{I}_2}(\rho)}{2\pi}\right)}, (\hbar_{\bar{I}_1}(\rho).\hbar_{\bar{I}_2}(\rho)) e^{i2\pi\left(\frac{\eth_{\bar{I}_1}(\rho)}{2\pi}.\frac{\eth_{\bar{I}_2}(\rho)}{2\pi}\right)} \right\}.\end{aligned}$$

Example 2.20. Consider $\bar{I}_1$ and $\bar{I}_2$ in Example 2.3. Then, *the CIF product is*

$$\mu_{\bar{I}_1 \boxtimes \bar{I}_2}(\rho) = \left\{(\rho_1, 0.1e^{i0.25\pi}, 0.24e^{i0.25\pi}), (\rho_2, 0.3^{i0.12\pi}, 0.04e^{i0.25\pi}), (\rho_3, 0.06e^{i0.375\pi}, 0.14e^{i0.075\pi})\right\}.$$

Proposition 2.21. *The CIF product on $\hat{\mathrm{H}}(\Bbbk)$ is a t-norm.*

Proof. For the proof of this proposition see appendix AppendixC. □

Definition 2.22. Let $\bar{\mathrm{I}}_1$ and $\bar{\mathrm{I}}_2$ be two CIFSs on $\Bbbk$, and

$$\mu_{\bar{\mathrm{I}}_1} = \left\{\left\langle \rho, \chi_{\bar{\mathrm{I}}_1}(\rho)e^{i\Im_{\bar{\mathrm{I}}_1}(\rho)}, \hbar_{\bar{\mathrm{I}}_1}(\rho)e^{i\eth_{\bar{\mathrm{I}}_1}(\rho)} \right\rangle : \rho \in \Bbbk\right\} \text{ and } \mu_{\bar{\mathrm{I}}_2} = \left\{\left\langle \rho, \chi_{\bar{\mathrm{I}}_2}(\rho)e^{i\Im_{\bar{\mathrm{I}}_2}(\rho)}, \hbar_{\bar{\mathrm{I}}_2}(\rho)e^{i\eth_{\bar{\mathrm{I}}_2}(\rho)} \right\rangle : \rho \in \Bbbk\right\}$$

the grade's of $\bar{\mathrm{I}}_1$ and $\bar{\mathrm{I}}_2$, respectively. Then, *the CIF probabilistic sum of* $\bar{\mathrm{I}}_1$ and $\bar{\mathrm{I}}_2$, is denoted by $\bar{\mathrm{I}}_1 \dotplus \bar{\mathrm{I}}_2$ and defined as

$$\mu_{\bar{\mathrm{I}}_1 \dotplus \bar{\mathrm{I}}_2}(\rho) = \left\{ \begin{array}{l} \left(\frac{\chi_{\bar{\mathrm{I}}_1}(\rho)+\chi_{\bar{\mathrm{I}}_2}(\rho)-\chi_{\bar{\mathrm{I}}_1}(\rho).\chi_{\bar{\mathrm{I}}_2}(\rho)}{2}\right) e^{i2\pi\left(\frac{\Im_{\bar{\mathrm{I}}_1}(\rho)+\Im_{\bar{\mathrm{I}}_2}(\rho)-\Im_{\bar{\mathrm{I}}_1}(\rho).\Im_{\bar{\mathrm{I}}_2}(\rho)}{2\pi}\right)}, \\ \left(\frac{\hbar_{\bar{\mathrm{I}}_1}(\rho)+\hbar_{\bar{\mathrm{I}}_3}(\rho)-\hbar_{\bar{\mathrm{I}}_1}(\rho).\hbar_{\bar{\mathrm{I}}_3}(\rho)}{2}\right) e^{i2\pi\left(\frac{\eth_{\bar{\mathrm{I}}_1}(\rho)+\eth_{\bar{\mathrm{I}}_2}(\rho)-\eth_{\bar{\mathrm{I}}_1}(\rho).\eth_{\bar{\mathrm{I}}_2}(\rho)}{2\pi}\right)} \end{array} \right\}.$$

Proposition 2.23. *The CIF probolistic sum on $\hat{\mathrm{H}}(\Bbbk)$ is an s-norm.*

Proof. For the proof of this proposition see appendix AppendixD. □

Definition 2.24. Let $\bar{\mathrm{I}}_1$ and $\bar{\mathrm{I}}_2$ be two CIFSs on $\Bbbk$, and

$$\mu_{\bar{\mathrm{I}}_1} = \left\{\left\langle \rho, \chi_{\bar{\mathrm{I}}_1}(\rho)e^{i\Im_{\bar{\mathrm{I}}_1}(\rho)}, \hbar_{\bar{\mathrm{I}}_1}(\rho)e^{i\eth_{\bar{\mathrm{I}}_1}(\rho)} \right\rangle : \rho \in \Bbbk\right\} \text{ and } \mu_{\bar{\mathrm{I}}_2} = \left\{\left\langle \rho, \chi_{\bar{\mathrm{I}}_2}(\rho)e^{i\Im_{\bar{\mathrm{I}}_2}(\rho)}, \hbar_{\bar{\mathrm{I}}_2}(\rho)e^{i\eth_{\bar{\mathrm{I}}_2}(\rho)} \right\rangle : \rho \in \Bbbk\right\}$$

the grade's of $\bar{\mathrm{I}}_1$ and $\bar{\mathrm{I}}_2$, respectively. Then, *the CIF bold sum of* $\bar{\mathrm{I}}_1$ and $\bar{\mathrm{I}}_2$, denoted by $\bar{\mathrm{I}}_1 \oplus \bar{\mathrm{I}}_2$ is defined as

$$\mu_{\bar{\mathrm{I}}_1 \oplus \bar{\mathrm{I}}_2}(\rho) = \left\langle \begin{array}{c} \min(1, \chi_{\bar{\mathrm{I}}_1}(\rho) + \chi_{\bar{\mathrm{I}}_2}(\rho))e^{i\min(2\pi, \Im_{\bar{\mathrm{I}}_1}(\rho)+\Im_{\bar{\mathrm{I}}_2}(\rho))}, \\ \max(0, \hbar_{\bar{\mathrm{I}}_1}(\rho) - \hbar_{\bar{\mathrm{I}}_2}(\rho))e^{i\max(0, \eth_{\bar{\mathrm{I}}_1}(\rho)-\eth_{\bar{\mathrm{I}}_1}(\rho))} \end{array} \right\rangle.$$

Example 2.25. Consider $\bar{\mathrm{I}}_1$ and $\bar{\mathrm{I}}_2$ in Example 2.3. Then, the CIF bold sum is

$$\bar{\mathrm{I}}_1 \oplus \bar{\mathrm{I}}_2 = \left\{(\rho_1, 0.7e^{i1.5\pi}, 0e^{i0.5\pi}), (\rho_2, 1e^{i1.1\pi}, 0e^{i0.5\pi}), (\rho_3, 0.4e^{i2\pi}, 0.5e^{i(0)})\right\}.$$

Proposition 2.26. *The CIF bold sum on $\hat{\mathrm{H}}(\Bbbk)$ is an s-norm.*

Proof. For the proof of this proposition see appendix AppendixE. □

Definition 2.27. Let $\bar{\mathrm{I}}_1$ and $\bar{\mathrm{I}}_2$ be two CIFSs on $\Bbbk$, and

$$\mu_{\bar{\mathrm{I}}_1} = \left\{\left\langle \rho, \chi_{\bar{\mathrm{I}}_1}(\rho)e^{i\Im_{\bar{\mathrm{I}}_1}(\rho)}, \hbar_{\bar{\mathrm{I}}_1}(\rho)e^{i\eth_{\bar{\mathrm{I}}_1}(\rho)} \right\rangle : \rho \in \Bbbk\right\} \text{ and } \mu_{\bar{\mathrm{I}}_2} = \left\{\left\langle \rho, \chi_{\bar{\mathrm{I}}_2}(\rho)e^{i\Im_{\bar{\mathrm{I}}_2}(\rho)}, \hbar_{\bar{\mathrm{I}}_2}(\rho)e^{i\eth_{\bar{\mathrm{I}}_2}(\rho)} \right\rangle : \rho \in \Bbbk\right\}$$

the grade's of $\bar{\mathrm{I}}_1$ and $\bar{\mathrm{I}}_2$, respectively. Then, the CIF bold intersection of $\bar{\mathrm{I}}_1$ and $\bar{\mathrm{I}}_2$, denoted by $\bar{\mathrm{I}}_1 \Cap \bar{\mathrm{I}}_2$ is defined as

$$\mu_{\bar{\mathrm{I}}_1 \Cap \bar{\mathrm{I}}_2}(\rho) = \left\langle \begin{array}{c} \max(0, \chi_{\bar{\mathrm{I}}_1}(\rho) + \chi_{\bar{\mathrm{I}}_2}(\rho) - 1)e^{i\max(0, \Im_{\bar{\mathrm{I}}_1}(\rho)+\Im_{\bar{\mathrm{I}}_2}(\rho)-2\pi)}, \\ \max(0, \hbar_{\bar{\mathrm{I}}_1}(\rho) + \hbar_{\bar{\mathrm{I}}_2}(\rho) - 1)e^{i\max(0, \eth_{\bar{\mathrm{I}}_1}(\rho)+\eth_{\bar{\mathrm{I}}_2}(\rho)-2\pi)} \end{array} \right\rangle.$$

Proposition 2.28. *The CIF bold intersection on $\hat{\mathrm{H}}(\Bbbk)$ is a t-norm.*

Proof. For the proof of this proposition see appendix AppendixF. □

Definition 2.29. The distance of CIFSs is a function $\Xi : \hat{\mathrm{H}}(\Bbbk) \times \hat{\mathrm{H}}(\Bbbk) \longrightarrow [0, 1]$ with the property: for any $\bar{\mathrm{I}}_1, \bar{\mathrm{I}}_2, \bar{\mathrm{I}}_3 \in \hat{\mathrm{H}}(\Bbbk)$,

(1) $\Xi(\bar{\mathrm{I}}_1, \bar{\mathrm{I}}_2) \geqslant 0$, and $\Xi(\bar{\mathrm{I}}_1, \bar{\mathrm{I}}_2) = 0$ if and only if $\bar{\mathrm{I}}_1 = \bar{\mathrm{I}}_2$;
(2) $\Xi(\bar{\mathrm{I}}_1, \bar{\mathrm{I}}_2) = \Xi(\bar{\mathrm{I}}_2, \bar{\mathrm{I}}_1)$;

(3) $\Xi(\bar{I}_1,\bar{I}_3) \leqslant \Xi(\bar{I}_1,\bar{I}_2)+\Xi(\bar{I}_2,\bar{I}_3)$.

The distance function Ξ between any two CIFSs $\bar{I}_1$ and $\bar{I}_2$ is defined as follows:

$$\Xi(\bar{I}_1,\bar{I}_2)=\max\left\{\begin{array}{l}\sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_1}(\rho)-\chi_{\bar{I}_2}(\rho)\right|,\frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_1}(\rho)-\Im_{\bar{I}_2}(\rho)\right|,\\ \sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_1}(\rho)-\hbar_{\bar{I}_2}(\rho)\right|,\frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\eth_{\bar{I}_1}(\rho)-\eth_{\bar{I}_2}(\rho)\right|\end{array}\right\}.$$

Example 2.30. Consider $\bar{I}_1$ and $\bar{I}_2$ in Example 2.3. Then, the distance between $\bar{I}_1$ and $\bar{I}_2$ is

$$\Xi(\bar{I}_1,\bar{I}_2)=\max\langle 0.4,0.4,0.5,0.25\rangle,\quad \Xi(\bar{I}_1,\bar{I}_2)=0.5.$$

Theorem 2.31. *The function Ξ is a distance function of CIFSs on $\Bbbk$.*

Proof. For the proof of this theorem see appendix AppendixG. □

Definition 2.32. Let E and L be two real FSs on U, and their M-S functions $\mu_E(\rho)$ and $\mu_L(\rho)$, respectively. Then, E and L are said to be ϖ-equal, denoted by $E=(\hbar)L$, if and only if $\sup\limits_{\rho\in\Bbbk}|\mu_E(\rho)-\mu_L(\rho)|\leqslant 1-\varpi$, $0\leqslant\varpi\leqslant 1$. In this way, we say E and L construct a ϖ-equality.

Definition 2.33. Let $\bar{I}_1$ and $\bar{I}_2$ be two CIFSs on $\Bbbk$, and $\mu_{\bar{I}_1}=\left\{\left\langle\rho,\chi_{\bar{I}_1}(\rho)e^{i\Im_{\bar{I}_1}(\rho)},\hbar_{\bar{I}_1}(\rho)e^{i\eth_{\bar{I}_1}(\rho)}\right\rangle:\rho\in\Bbbk\right\}$ and $\mu_{\bar{I}_2}=\left\{\left\langle\rho,\chi_{\bar{I}_2}(\rho)e^{i\Im_{\bar{I}_2}(\rho)},\hbar_{\bar{I}_2}(\rho)e^{i\eth_{\bar{I}_2}(\rho)}\right\rangle:\rho\in\Bbbk\right\}$ the grade's of $\bar{I}_1$ and $\bar{I}_2$, respectively. Then $\bar{I}_1$ and $\bar{I}_2$ are said to be ϖ-equal, denoted by $\bar{I}_1=(\varpi)\bar{I}_2$, if and only if $\Xi(\bar{I}_1,\bar{I}_1)\leqslant 1-\varpi$, $0\leqslant\varpi\leqslant 1$.

Proposition 2.34. *Let $\bar{I}_1$ and $\bar{I}_2$ be two CIFSs on U. Then,*

(1) $\bar{I}_1=(0)\bar{I}_2$;

(2) $\bar{I}_1=(1)\bar{I}_2 \Longleftrightarrow \bar{I}_1=\bar{I}_2$;

(3) $\bar{I}_1=(\varpi)\bar{I}_2 \Longleftrightarrow \bar{I}_1=\bar{I}_2$;

(4) $\bar{I}_1=(\varpi_1)\bar{I}_2$ *and* $\varpi_2\leqslant\varpi_1 \Longleftrightarrow \bar{I}_1=(\varpi_2)\bar{I}_1$;

(5) *if* $\bar{I}_1=(\varpi_\alpha)\bar{I}_1$, *then* $\bar{I}_1=(\sup\limits_{\bar{I}\in P}\varpi_\alpha)\bar{I}_2$, *where* P *denotes the index set;*

(6) $\forall\bar{I}_1,\bar{I}_2$, *there exist a unique* ϖ *such that* $\bar{I}_1=(\varpi)\bar{I}_2$ *and if* $\bar{I}_1=(\varpi')\bar{I}_2$, *then* $\varpi'\leqslant\varpi$.

Proof. For the proof of this proposition see appendix AppendixH. □

Proposition 2.35. *If* $\bar{I}_1=(\varpi_1)\bar{I}_2$ *and* $\bar{I}_2=(\varpi_2)\bar{I}_3$, *then* $\bar{I}_1=(\varpi)\bar{I}_3$, *where* $\varpi=\varpi_1*\varpi_2=\varpi_1+\varpi_2-1$.

Proof. For the proof of this proposition see appendix AppendixI. □

Theorem 2.36. *If* $\bar{I}_1=(\varpi)\bar{I}_2$, *then* $\bar{I}_1^c=(\varpi)\bar{I}_2^c$.

Proof. For the proof of this theorem see appendix AppendixJ. □

Theorem 2.37. *If* $\bar{I}_1=(\varpi_1)\bar{I}_1^*$ *and* $\bar{I}_2=(\varpi_2)\bar{I}_2^*$, *then* $\bar{I}_1\boxtimes\bar{I}_2=(\varpi_1*\varpi_2)\,\bar{I}_1^*\boxtimes\bar{I}_2^*$.

Proof. For the proof of this theorem see appendix AppendixK. □

Theorem 2.38. *If* $\bar{I}_1=(\varpi_1)\bar{I}_1^*$ *and* $\bar{I}_2=(\varpi_2)\bar{I}_2^*$, *then* $\bar{I}_1\dotplus\bar{I}_2=(\varpi_1*\varpi_2)\,\bar{I}_1^*\dotplus\bar{I}_2^*$.

Proof. For the proof of this theorem see appendix AppendixL. □

Corollary 2.39. *If* $\bar{I}_\alpha=(\varpi_\alpha)\,\bar{I}^*$, *where* $\alpha\in P$, *and P is the index set, then*

$$\bar{I}_1\dotplus\bar{I}_1\dotplus\cdots\dotplus\bar{I}_\alpha=(\varpi_1*\varpi_2*\cdots*\varpi_\alpha)\,\bar{I}_1^*\dotplus\bar{I}_2^*\dotplus\cdots\dotplus\bar{I}_\alpha^*.$$

Proof. Trivial from Theorem 2.38. □

3. Decision-making methodology under complex intuitionistic fuzzy sets

Complex intuitionistic fuzzy sets provide a richer and more adaptable framework for addressing decision-making problems, especially in situations where uncertainty, imprecision, and hesitation are influential factors. CIFS enable the integration of various sources of information in decision-making through suitable aggregation methods. The utilization of complex numbers enables a more intricate integration of M-S and N-M-S information. For D-M algorithm, we considered the following steps with flowchart of CIFSs.

Step 1. Assume there are n different alternatives, $\mathcal{L} = \{\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n\}$ in a multiattributes $\digamma$ problem and m is different criteria form the index set $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m\}$. The alternatives' attribute value $\mathcal{L}_r$ for index $\mathcal{F}_p$ is $\bar{I}_{rp} = \langle \chi_{\bar{I}_{rp}}(\rho) e^{i\Im_{\bar{I}_{rp}}(\rho)}, \hbar_{\bar{I}_{rp}}(\rho) e^{i\eth_{\bar{I}_{rp}}(\rho)} \rangle$ $(r = 1, 2, \ldots, n;\ p = 1, 2, \ldots, m)$. So the CIFSs $\digamma$ matrix is

$$\digamma = \begin{pmatrix} \bar{I}_{11} & \bar{I}_{12} & \cdots & \bar{I}_{1m} \\ \bar{I}_{21} & \bar{I}_{22} & \cdots & \bar{I}_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \bar{I}_{n1} & \bar{I}_{n2} & \cdots & \bar{I}_{nm} \end{pmatrix},$$

where

$$\bar{I}_{rp} = \left\langle \chi_{\bar{I}_{rp}}(\rho) e^{i\Im_{\bar{I}_{rp}}(\rho)}, \hbar_{\bar{I}_{rp}}(\rho) e^{i\eth_{\bar{I}_{rp}}(\rho)} \right\rangle.$$

Step 2. Assume that

$$\chi_{\bar{I}_{r1}}(\rho) = \max\left\{\chi_{\bar{I}_{rp}}(\rho),\ \text{for fix } r \text{ and } p = 1, 2, \ldots, m\right\},$$
$$\hbar_{\bar{I}_{r1}}(\rho) = \min\left\{\hbar_{\bar{I}_{rp}}(\rho),\ \text{for fix } r \text{ and } p = 1, 2, \ldots, m\right\},$$
$$\Im_{\bar{I}_{r1}}(\rho) = \max\left\{\Im_{\bar{I}_{rp}}(\rho),\ \text{for fix } r \text{ and } p = 1, 2, \ldots, m\right\},$$
$$\eth_{\bar{I}_{r1}}(\rho) = \min\left\{\eth_{\bar{I}_{rp}}(\rho),\ \text{for fix } r \text{ and } p = 1, 2, \ldots, m\right\},$$

and thus

$$\digamma_{l\max} = \begin{pmatrix} \chi_{\bar{I}_{11}}(\rho) e^{r\Im_{\bar{I}_{11}}(\rho)}, \hbar_{\bar{I}_{11}}(\rho) e^{r\eth_{r_{11}}(\rho)} \\ \chi_{\bar{I}_{21}}(\rho) e^{r\Im_{\bar{I}_{21}}(\rho)}, \hbar_{\bar{I}_{21}}(\rho) e^{r\eth_{\bar{I}_{21}}(\rho)} \\ . \\ . \\ . \\ \chi_{\bar{I}_{n1}}(\rho) e^{r\Im_{\bar{I}_{n1}}(\rho)}, \hbar_{\bar{I}_{n1}}(\rho) e^{r\eth_{\bar{I}_{n1}}(\rho)} \end{pmatrix},\ l = 1, 2, 3, \ldots, \tag{3.1}$$

is called CIFS max $\digamma$ matrix.

Step 3. Assume that

$$\chi'_{\bar{I}_{r1}}(\rho) = \min\left\{\chi'_{\bar{I}_{rp}}(\rho),\ \text{for fix } r \text{ and } p = 1, 2, \ldots, m\right\},$$
$$\hbar'_{\bar{I}_{r1}}(\rho) = \max\left\{\hbar'_{\bar{I}_{rp}}(\rho),\ \text{for fix } r \text{ and } p = 1, 2, \ldots, m\right\},$$
$$\Im'_{\bar{I}_{r1}}(\rho) = \min\left\{\Im'_{\bar{I}_{rp}}(\rho),\ \text{for fix } r \text{ and } p = 1, 2, \ldots, m\right\},$$
$$\eth'_{\bar{I}_{r1}}(\rho) = \max\left\{\eth'_{\bar{I}_{rp}}(\rho),\ \text{for fix } r \text{ and } p = 1, 2, \ldots, m\right\},$$

and thus

$$\digamma_{l\min} = \begin{pmatrix} \chi'_{\bar{I}_{11}}(\rho) e^{r\Im'_{\bar{I}_{11}}(\rho)}, \hbar'_{\bar{I}_{11}}(\rho) e^{r\eth'_{\bar{I}_{11}}(\rho)} \\ \chi'_{\bar{I}_{21}}(\rho) e^{r\Im'_{\bar{I}_{21}}(\rho)}, \hbar'_{\bar{I}_{21}}(\rho) e^{r\eth'_{\bar{I}_{21}}(\rho)} \\ . \\ . \\ . \\ \chi'_{\bar{I}_{n1}}(\rho) e^{r2\pi\Im'_{\bar{I}_{n1}}(\rho)}, \hbar'_{\bar{I}_{n1}}(\rho) e^{r2\pi\eth'_{\bar{I}_{n1}}(\rho)} \end{pmatrix},\ p = 1, 2, 3 \ldots, \tag{3.2}$$

is called CIFS min $\digamma$ matrix.

Step 4. The average similarity measure operator $\check{S}_{l}$ between CIFS max and min $\digamma$ matrices are defined as

$$\check{S}_{l}(\digamma_{l\max}, \digamma_{l\min}) = \begin{bmatrix} \max\left(\frac{\chi_{\bar{I}_{r1}}(\rho)+\chi'_{\bar{I}_{r1}}(\rho)}{2}\right) e^{r\max\left(\frac{\Im_{\bar{I}_{r1}}(\rho)+\Im'_{\bar{I}_{r1}}(\rho)}{2}\right)}, \\ \min\left(\frac{\hbar_{\bar{I}_{r1}}(\rho)+\hbar'_{\bar{I}_{r1}}(\rho)}{2}\right) e^{r\min\left(\frac{\eth_{\bar{I}_{r1}}(\rho)+\eth'_{\bar{I}_{r1}}(\rho)}{2}\right)} \end{bmatrix}, \; r = 1,2,3\ldots,n, \tag{3.3}$$

$$\check{S}_{l}(\digamma_{l\max}, \digamma_{l\min}) = \left[\check{D}_{\check{S}_{l1}}(\rho)e^{r\kappa_{\check{S}_{l1}}(\rho)}, \check{D}'_{\check{S}_{l1}}(\rho)e^{r\kappa'_{\check{S}_{l1}}(\rho)}\right].$$

Step 5. The average weighted similarity measure operator D is defined as:

$$D = \left[\frac{w_r \max(\check{D}_{\check{S}_{l1}}(\rho))}{\sqrt{\sum\left(\check{D}_{\check{S}_{l1}}(\rho)\right)^2}} e^{r\max\left(\kappa_{\check{S}_{l1}}(\rho)\right)}, \frac{w_r \min(\check{D}'_{\check{S}_{l1}}(\rho))}{\sqrt{\sum\left(\check{D}'_{\check{S}_{l1}}(\rho)\right)^2}} e^{r\min\left(\kappa'_{\check{S}_{l1}}(\rho)\right)}\right],$$

$$l = 1,2,3,\ldots, \quad w_r : r = 1,2,3,\ldots,m,$$

where $w_r = (w_1, w_2, \ldots, w_m)$ is the index weigh vector and $\sum_{r=1}^{n} w_r = 1$. The values of M-S and N-M-S of D lie in a unit disc.

Step 6. Rank the alternatives by evaluating the values of the average weighted similarity measure operator, ultimately, choosing the most preferable option(s).

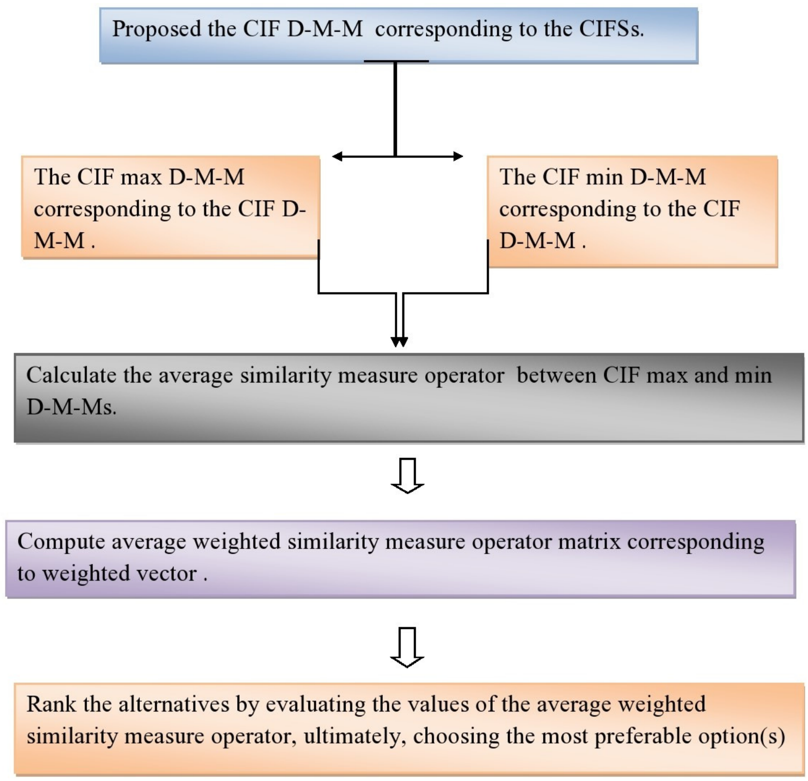
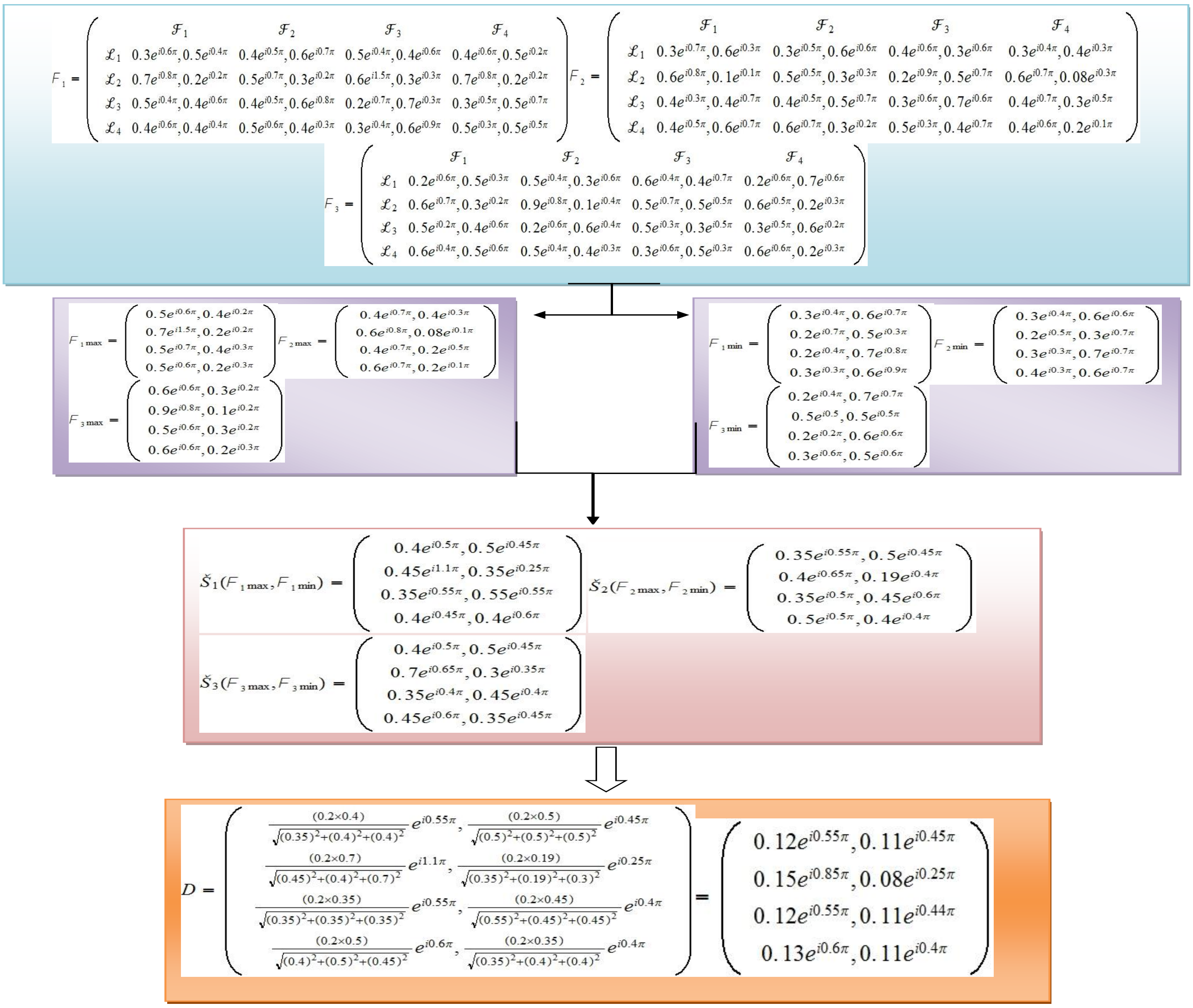


Figure 1: Flowchart of the proposed methodology.

4. Implementation of the proposed methodology on a decision making problem

We can outline the algorithm for addressing a conventional D-M problem using the CIF D-M-M, max CIF D-M-M, min CIF D-M-M, and average similarity operators with the following steps with flowchart.

Assume that a businessman choosing to invest in three different enterprises (alternatives). Let $\mathcal{L} = \{\mathcal{L}_1 = \text{dell laptop company}, \mathcal{L}_2 = \text{Samsung company}, \mathcal{L}_3 = \text{Toyota company}, \mathcal{L}_4 = \text{oil refinery company}\}$ and let $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4\}$ be a class of criteria. The expert team conducts an evaluation of these four enterprises from 2018 to 2020. The expert team expresses the evaluation results for each index in the form of CIFSs as in the following Figure.



Step 1. We have the following CIF D-M-Ms F_1, F_2, and F_3 as follows.

Matrix 1: evaluation of enterprises in 2018:

$$F_1 = \begin{array}{c} \\ \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{L}_3 \\ \mathcal{L}_4 \end{array} \begin{pmatrix} \mathcal{F}_1 & \mathcal{F}_2 & \mathcal{F}_3 & \mathcal{F}_4 \\ 0.3e^{i0.6\pi},0.5e^{i0.4\pi} & 0.4e^{i0.5\pi},0.6e^{i0.7\pi} & 0.5e^{i0.4\pi},0.4e^{i0.6\pi} & 0.4e^{i0.6\pi},0.5e^{i0.2\pi} \\ 0.7e^{i0.8\pi},0.2e^{i0.2\pi} & 0.5e^{i0.7\pi},0.3e^{i0.2\pi} & 0.6e^{i1.5\pi},0.3e^{i0.3\pi} & 0.7e^{i0.8\pi},0.2e^{i0.2\pi} \\ 0.5e^{i0.4\pi},0.4e^{i0.6\pi} & 0.4e^{i0.5\pi},0.6e^{i0.8\pi} & 0.2e^{i0.7\pi},0.7e^{i0.3\pi} & 0.3e^{i0.5\pi},0.5e^{i0.7\pi} \\ 0.4e^{i0.6\pi},0.4e^{i0.4\pi} & 0.5e^{i0.6\pi},0.4e^{i0.3\pi} & 0.3e^{i0.4\pi},0.6e^{i0.9\pi} & 0.5e^{i0.3\pi},0.5e^{i0.5\pi} \end{pmatrix}.$$

Matrix 2: evaluation of enterprises in 2019:

$$F_2 = \begin{array}{c} \\ \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{L}_3 \\ \mathcal{L}_4 \end{array} \begin{pmatrix} \mathcal{F}_1 & \mathcal{F}_2 & \mathcal{F}_3 & \mathcal{F}_4 \\ 0.3e^{i0.7\pi}, 0.6e^{i0.3\pi} & 0.3e^{i0.5\pi}, 0.6e^{i0.6\pi} & 0.4e^{i0.6\pi}, 0.3e^{i0.6\pi} & 0.3e^{i0.4\pi}, 0.4e^{i0.3\pi} \\ 0.6e^{i0.8\pi}, 0.1e^{i0.1\pi} & 0.5e^{i0.5\pi}, 0.3e^{i0.3\pi} & 0.2e^{i0.9\pi}, 0.5e^{i0.7\pi} & 0.6e^{i0.7\pi}, 0.08e^{i0.3\pi} \\ 0.4e^{i0.3\pi}, 0.4e^{i0.7\pi} & 0.4e^{i0.5\pi}, 0.5e^{i0.7\pi} & 0.3e^{i0.6\pi}, 0.7e^{i0.6\pi} & 0.4e^{i0.7\pi}, 0.3e^{i0.5\pi} \\ 0.4e^{i0.5\pi}, 0.6e^{i0.7\pi} & 0.6e^{i0.7\pi}, 0.3e^{i0.2\pi} & 0.5e^{i0.3\pi}, 0.4e^{i0.7\pi} & 0.4e^{i0.6\pi}, 0.2e^{i0.1\pi} \end{pmatrix}.$$

Matrix 3: evaluation of enterprises in 2020:

$$F_3 = \begin{array}{c} \\ \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{L}_3 \\ \mathcal{L}_4 \end{array} \begin{pmatrix} \mathcal{F}_1 & \mathcal{F}_2 & \mathcal{F}_3 & \mathcal{F}_4 \\ 0.2e^{i0.6\pi}, 0.5e^{i0.3\pi} & 0.5e^{i0.4\pi}, 0.3e^{i0.6\pi} & 0.6e^{i0.4\pi}, 0.4e^{i0.7\pi} & 0.2e^{i0.6\pi}, 0.7e^{i0.6\pi} \\ 0.6e^{i0.7\pi}, 0.3e^{i0.2\pi} & 0.9e^{i0.8\pi}, 0.1e^{i0.4\pi} & 0.5e^{i0.7\pi}, 0.5e^{i0.5\pi} & 0.6e^{i0.5\pi}, 0.2e^{i0.3\pi} \\ 0.5e^{i0.2\pi}, 0.4e^{i0.6\pi} & 0.2e^{i0.6\pi}, 0.6e^{i0.4\pi} & 0.5e^{i0.3\pi}, 0.3e^{i0.5\pi} & 0.3e^{i0.5\pi}, 0.6e^{i0.2\pi} \\ 0.6e^{i0.4\pi}, 0.5e^{i0.6\pi} & 0.5e^{i0.4\pi}, 0.4e^{i0.3\pi} & 0.3e^{i0.6\pi}, 0.5e^{i0.3\pi} & 0.6e^{i0.6\pi}, 0.2e^{i0.3\pi} \end{pmatrix}.$$

Step 2. We construct the CIF max use equation (3.1) resulting in the matrices $F_{1\,\max}$, $F_{2\,\max}$, $F_{3\,\max}$, corresponding to the CIF F matrices F_1, F_2, F_3, as

$$F_{1\,\max} = \begin{pmatrix} 0.5e^{i0.6\pi}, 0.4e^{i0.2\pi} \\ 0.7e^{i1.5\pi}, 0.2e^{i0.2\pi} \\ 0.5e^{i0.7\pi}, 0.4e^{i0.3\pi} \\ 0.5e^{i0.6\pi}, 0.2e^{i0.3\pi} \end{pmatrix}, \quad F_{2\,\max} = \begin{pmatrix} 0.4e^{i0.7\pi}, 0.4e^{i0.3\pi} \\ 0.6e^{i0.8\pi}, 0.08e^{i0.1\pi} \\ 0.4e^{i0.7\pi}, 0.2e^{i0.5\pi} \\ 0.6e^{i0.7\pi}, 0.2e^{i0.1\pi} \end{pmatrix}, \quad F_{3\,\max} = \begin{pmatrix} 0.6e^{i0.6\pi}, 0.3e^{i0.2\pi} \\ 0.9e^{i0.8\pi}, 0.1e^{i0.2\pi} \\ 0.5e^{i0.6\pi}, 0.3e^{i0.2\pi} \\ 0.6e^{i0.6\pi}, 0.2e^{i0.3\pi} \end{pmatrix}.$$

Step 3. We construct the CIF min use equation (3.2) resulting in the matrices $F_{1\,\min}$, $F_{2\,\min}$, and $F_{3\,\min}$, corresponding to the CIF F matrices F_1, F_2, F_3 as

$$F_{1\,\min} = \begin{pmatrix} 0.3e^{i0.4\pi}, 0.6e^{i0.7\pi} \\ 0.2e^{i0.7\pi}, 0.5e^{i0.3\pi} \\ 0.2e^{i0.4\pi}, 0.7e^{i0.8\pi} \\ 0.3e^{i0.3\pi}, 0.6e^{i0.9\pi} \end{pmatrix}, \quad F_{2\,\min} = \begin{pmatrix} 0.3e^{i0.4\pi}, 0.6e^{i0.6\pi} \\ 0.2e^{i0.5\pi}, 0.3e^{i0.7\pi} \\ 0.3e^{i0.3\pi}, 0.7e^{i0.7\pi} \\ 0.4e^{i0.3\pi}, 0.6e^{i0.7\pi} \end{pmatrix}, \quad F_{3\,\min} = \begin{pmatrix} 0.2e^{i0.4\pi}, 0.7e^{i0.7\pi} \\ 0.5e^{i0.5}, 0.5e^{i0.5\pi} \\ 0.2e^{i0.2\pi}, 0.6e^{i0.6\pi} \\ 0.3e^{i0.6\pi}, 0.5e^{i0.6\pi} \end{pmatrix}.$$

Step 4. The average similarity measure operator $\check{S}_t$; $t = 1, 2, 3$, between the CIF max and min using equation (3.3), then F matrices is S

$$\check{S}_1(F_{1\,\max}, F_{1\,\min}) = \begin{pmatrix} 0.4e^{i0.5\pi}, 0.5e^{i0.45\pi} \\ 0.45e^{i1.1\pi}, 0.35e^{i0.25\pi} \\ 0.35e^{i0.55\pi}, 0.55e^{i0.55\pi} \\ 0.4e^{i0.45\pi}, 0.4e^{i0.6\pi} \end{pmatrix},$$

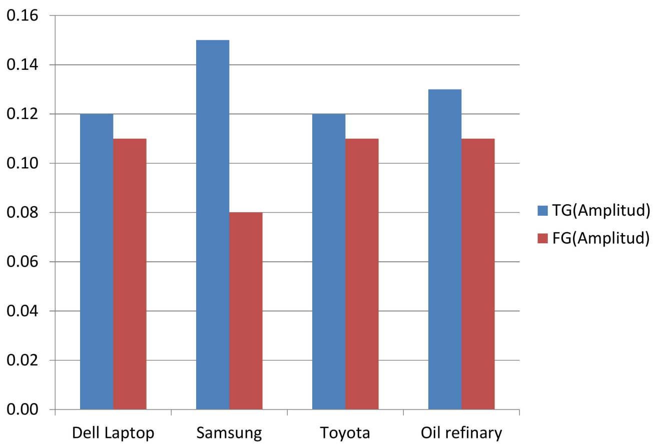
$$\check{S}_2(F_{2\,\max}, F_{2\,\min}) = \begin{pmatrix} 0.35e^{i0.55\pi}, 0.5e^{i0.45\pi} \\ 0.4e^{i0.65\pi}, 0.19e^{i0.4\pi} \\ 0.35e^{i0.5\pi}, 0.45e^{i0.6\pi} \\ 0.5e^{i0.5\pi}, 0.4e^{i0.4\pi} \end{pmatrix},$$

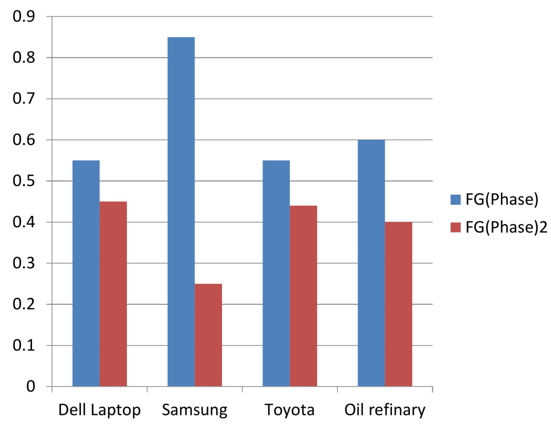
$$\check{S}_3(F_{3\,\max}, F_{3\,\min}) = \begin{pmatrix} 0.4e^{i0.5\pi}, 0.5e^{i0.45\pi} \\ 0.7e^{i0.65\pi}, 0.3e^{i0.35\pi} \\ 0.35e^{i0.4\pi}, 0.45e^{i0.4\pi} \\ 0.45e^{i0.6\pi}, 0.35e^{i0.45\pi} \end{pmatrix}.$$

Step 5. Now we compute the average weighted similarity measure operator D corresponding to the weighted vector $w = (0.2, 0.2, 0.2, 0.2)$, i.e.,

$$D = \begin{pmatrix} \frac{(0.2\times0.4)}{\sqrt{(0.35)^2+(0.4)^2+(0.4)^2}}e^{i0.55\pi}, \frac{(0.2\times0.5)}{\sqrt{(0.5)^2+(0.5)^2+(0.5)^2}}e^{i0.45\pi} \\ \frac{(0.2\times0.7)}{\sqrt{(0.45)^2+(0.4)^2+(0.7)^2}}e^{i1.1\pi}, \frac{(0.2\times0.19)}{\sqrt{(0.35)^2+(0.19)^2+(0.3)^2}}e^{i0.25\pi} \\ \frac{(0.2\times0.35)}{\sqrt{(0.35)^2+(0.35)^2+(0.35)^2}}e^{i0.55\pi}, \frac{(0.2\times0.45)}{\sqrt{(0.55)^2+(0.45)^2+(0.45)^2}}e^{i0.4\pi} \\ \frac{(0.2\times0.5)}{\sqrt{(0.4)^2+(0.5)^2+(0.45)^2}}e^{i0.6\pi}, \frac{(0.2\times0.35)}{\sqrt{(0.35)^2+(0.4)^2+(0.4)^2}}e^{i0.4\pi} \end{pmatrix} = \begin{pmatrix} 0.12e^{i0.55\pi}, 0.11e^{i0.45\pi} \\ 0.15e^{i0.85\pi}, 0.08e^{i0.25\pi} \\ 0.12e^{i0.55\pi}, 0.11e^{i0.44\pi} \\ 0.13e^{i0.6\pi}, 0.11e^{i0.4\pi} \end{pmatrix}.$$

Note: The weighted vector varies depending on the problem and criteria. However, in this case, we are assigning the same weight to each company.





Step 6. From the average weighted similarity measure operator D, we conclude that the N-M-S of Samsung company is greater than the M-S of all other companies and N-M-S of Samsung company is smaller than the M-S of all other companies. So, $\mathcal{L}_2$ = samsung company is the best choice for investment.

5. Comparison analysis

Akram et al. [1] proposed the method which is generalized operators for complex intuitionistic fuzzy sets named complex intuitionistic fuzzy Hamacher (complex intuitionistic fuzzy Hamacher weighted average (CIFHWA), complex intuitionistic fuzzy Hamacher geometric aggregation (CIFHWG), complex intuitionistic fuzzy Hamacher ordered weighted average (CIFHOWA), and complex intuitionistic fuzzy Hamacher ordered weighted geometric (CIFHOWG) operator. We have analyzed the appearance of the proposed model to the Akram et al. [1] model. They work on the existing operators for complex intuitionistic fuzzy sets, they fail some times, where a CIF set is not in its range and follows the operator.

Here we presented the application of CIFSs in D-M problems. In this application, we identified a best choice for investment under the CIF environment. The model that presented in this paper for selecting a best choice, is different than the methods previously developed. Here we used new kinds of operators for CIFSs to develop an algorithm for further use in D-M problems. The proposed method analysis the two-dimensional problem by two sides, which is M-S and N-M-S having amplitude and phase terms. The proposed method is more superior to developed in a CIF environment, as others FS, IFS, CFS, which is using for one dimension or two dimensional but only for membership values. Here $\mathcal{L}_2$ is a best choice as in Table 1.

Table 1:

Proposed method	$\mathcal{L}_1$	$\mathcal{L}_2$	$\mathcal{L}_3$	$\mathcal{L}_4$	Ranking alternatives
National market profit rate $\left(\varkappa_{\bar{I}_{rp}}\right)$	0.12	0.15	0.12	0.13	$\mathcal{L}_2 > \mathcal{L}_4 > \mathcal{L}_1 \geqslant \mathcal{L}_3$
International market profit rate $\left(\mathfrak{I}_{\bar{I}_{rp}}(\ell)\right)$	0.55	0.85	0.55	0.6	$\mathcal{L}_2 > \mathcal{L}_4 > \mathcal{L}_1 \geqslant \mathcal{L}_3$
National market loss rate $\left(\hbar_{\bar{I}_{rp}}(\ell)\right)$	0.11	0.08	0.11	0.11	$\mathcal{L}_2 < \mathcal{L}_4 \leqslant \mathcal{L}_1 \leqslant \mathcal{L}_3$
International market loss rate $\left(\eth_{\bar{I}_{rp}}(\ell)\right)$	0.45	0.25	0.44	0.4	$\mathcal{L}_2 < \mathcal{L}_4 < \mathcal{L}_3 \leqslant \mathcal{L}_1$

Akram et al. [1] method solutions for Matrix 1 (2018), Matrix 2 (2019), Matrix 3 (2020) are shown in Tables 2, 3, and 4, respectively.

Table 2:

	$\mathcal{L}_1$	$\mathcal{L}_2$	$\mathcal{L}_3$	$\mathcal{L}_4$	Ranking alternatives
CIFHWG	0.444 23	0.908 54	0.080 4	0.329 28	$\mathcal{L}_2 > \mathcal{L}_1 > \mathcal{L}_4 > \mathcal{L}_3$

Table 3:

	$\mathcal{L}_1$	$\mathcal{L}_2$	$\mathcal{L}_3$	$\mathcal{L}_4$	Ranking alternatives
CIFHWG	0.068 97	0.375 21	−0.069 33	0.205 78	$\mathcal{L}_2 > \mathcal{L}_4 > \mathcal{L}_1 > \mathcal{L}_3$

Table 4:

	$\mathcal{L}_1$	$\mathcal{L}_2$	$\mathcal{L}_3$	$\mathcal{L}_4$	Ranking alternatives
CIFHWG	−0.028 59	0.388 31	0.304 65	0.241 61	$\mathcal{L}_2 > \mathcal{L}_3 > \mathcal{L}_4 > \mathcal{L}_1$

$\mathcal{L}_2$ is best choice ever but Akram et al. [1] method calculate one by one matrix separately, haven't any formula to combine or calculate the data which is in parts/couple of matrix to give a single solution.

6. Conclusion

In multi-criteria D-M, where multiple conflicting criteria are considered, CIFS offer a versatile approach. They can effectively model diverse criteria and their associated uncertainties, providing a comprehensive solution to complex decision problems. In this paper, we defined some novel operations on CIFSs. We proposed a novel CIF distance measure and used them to define σ-qualities of CIFSs. Moreover, we proposed a new D-M algorithm in the environment of CIFSs. We introduced a new CIF average weighted similarity measure and utilized them in decision-making problems. Further, we discussed a real-life application based on CIF average weighted similarity measure.

The proposed method offers a robust solution for multi criteria decision-making challenges, yet it has certain limitations that we address here. This approach utilizes CIFSs, which are defined by membership and non-membership functions with the condition that their sum must not exceed 1. However, there are cases where this sum surpasses the limit, leading to the failure of CIFSs. Additionally, while CIFSs include membership and non-membership functions, they lack a neutral function, which can be a drawback in specific situations. To overcome these limitations, future work will focus on generalizing this approach using complex Pythagorean fuzzy sets and complex q-rung orthopair fuzzy sets. These sets ensure that the sum of the squares of membership and non-membership functions, or the q^{th} power of these functions, remains within the required limit of 1. We also plan to extend this to CPFSs, which incorporate membership, non-membership, and neutral functions.

AppendixA.

Proof of preposition 2.17. Clearly the axioms (a), (b), (e), and (f) are obvious from Definition 2.16. We have to prove (c) and (d).

(c). Let $\bar{I}_2$, $\bar{I}_2$, and $\bar{I}_3$ be three CIFSs on $\Bbbk$ and

$$\mu_{\bar{I}_1} = \left\{\left\langle \rho, \chi_{\bar{I}_1}(\rho)e^{i\Im_{\bar{I}_1}(\rho)}, \hbar_{\bar{I}_1}(\rho)e^{i\eth_{\bar{I}_1}(\rho)}\right\rangle : \rho \in \Bbbk\right\},$$
$$\mu_{\bar{I}_2} = \left\{\left\langle \rho, \chi_{\bar{I}_2}(\rho)e^{i\Im_{\bar{I}_2}(\rho)}, \hbar_{\bar{I}_2}(\rho)e^{i\eth_{\bar{I}_2}(\rho)}\right\rangle : \rho \in \Bbbk\right\},$$

and

$$\mu_{\bar{I}_3} = \left\{\left\langle \rho, \chi_{\bar{I}_3}(\rho)e^{i\Im_{\bar{I}_3}(\rho)}, \hbar_{\bar{I}_3}(\rho)e^{i\eth_{\bar{I}_3}(\rho)}\right\rangle : \rho \in \Bbbk\right\},$$

denote their grade's, respectively. We suppose $\chi_{\bar{I}_1}(\rho) \leqslant \chi_{\bar{I}_2}(\rho), \hbar_{\bar{I}_1}(\rho) \geqslant \hbar_{\bar{I}_2}(\rho)$, and $\Im_{\bar{I}_1}(\rho) \leqslant \Im_{\bar{I}_2}(\rho)$, $\eth_{\bar{I}_1}(\rho) \geqslant \eth_{\bar{I}_2}(\rho)$, $\forall \rho \in \Bbbk$. Then,

$$\left|\mu_{\bar{I}_1\cup\bar{I}_3}(\rho)\right| = \left(\left|\chi_{\bar{I}_1}(\rho).\chi_{\bar{I}_3}(\ell)\right|, \left|\hbar_{\bar{I}_1}(\rho), \hbar_{\bar{I}_3}(\rho)\right|\right) \leqslant \left(\left|\chi_{\bar{I}_2}(\rho).\chi_{\bar{I}_3}(\rho)\right|, \left|\hbar_{\bar{I}_2}(\rho), \hbar_{\bar{I}_3}(\rho)\right|\right) = \left|\mu_{\bar{I}_2\cup\bar{I}_3}(\rho)\right|, \forall\, \rho \in \Bbbk,$$

and

$$\Im_{(\bar{I}_1\cup\bar{I}_3)}(\rho) = \max(\Im_{\bar{I}_1}(\rho), \Im_{\bar{I}_3}(\rho)) \leqslant \max(\Im_{\bar{I}_2}(\rho), \Im_{\bar{I}_3}(\rho)) = \Im_{(\bar{I}_2\cup\bar{I}_3)}(\rho),$$
$$\eth_{(\bar{I}_1\cup\bar{I}_3)}(\ell) = \min(\eth_{\bar{I}_1}(\rho), \eth_{\bar{I}_3}(\rho)) \geqslant \min(\eth_{\bar{I}_2}(\rho), \eth_{\bar{I}_3}(\rho)) = \eth_{(\bar{I}_2\cup\bar{I}_3)}(\rho).$$

(d).

$$\mu_{\bar{I}_1\cup(\bar{I}_2\cup\bar{I}_3)}(\rho) = \left\{\begin{array}{c} \max\{\chi_{\bar{I}_1}(\rho), \max(\chi_{\bar{I}_2}(\ell), \chi_{\bar{I}_3}(\rho))\} \\ e^{i\max\{\Im_{\bar{I}_1}(\rho), \max(\Im_{\bar{I}_2}(\rho), \Im_{\bar{I}_3}(\rho))\}}, \\ \min\{\hbar_{\bar{I}_1}(\rho), \min(\hbar_{\bar{I}_2}(\rho), \hbar_{\bar{I}_3}(\rho))\} \\ e^{i\min\{\eth_{\bar{I}_1}(\rho), \min(\eth_{\bar{I}_2}(\rho), \eth_{\bar{I}_3}(\rho))\}} \end{array}\right\}$$
$$= \left\{\begin{array}{c} \max\{\chi_{\bar{I}_1}(\rho), \chi_{\bar{I}_2}(\rho), \chi_{\bar{I}_3}(\rho)\} \\ e^{i\max\{\Im_{\bar{I}_1}(\rho), \Im_{\bar{I}_2}(\rho), \Im_{\bar{I}_3}(\rho)\}}, \\ \min\{\hbar_{\bar{I}_1}(\rho), \hbar_{\bar{I}_2}(\rho), \hbar_{\bar{I}_3}(\rho)\}] \\ e^{i\min\{\eth_{\bar{I}_1}(\rho), \eth_{\bar{I}_2}(\rho), \eth_{\bar{I}_3}(\rho)\}} \end{array}\right\} = \left\{\begin{array}{c} \max\{\max(\chi_{\bar{I}_1}(\rho), \chi_{\bar{I}_2}(\rho)), \chi_{\bar{I}_3}(\rho)\} \\ e^{i\max\{\max(\Im_{\bar{I}_1}(\rho), \Im_{\bar{I}_2}(\rho)), \Im_{\bar{I}_3}(\rho)\}}, \\ \min\{\min(\hbar_{\bar{I}_1}(\rho), \hbar_{\bar{I}_2}(\rho)), \hbar_{\bar{I}_3}(\rho)\} \\ e^{i\min\{\min(\eth_{\bar{I}_1}(\rho), \eth_{\bar{I}_2}(\rho)), \eth_{\bar{I}_3}(\rho)\}} \end{array}\right\} = \mu_{(\bar{I}_1\cup\bar{I}_2)\cup\bar{I}_3}(\rho).$$

□

AppendixB.

Proof of Proposition 2.18. Clearly the axioms (a), (b), and (g) are obvious from Definition 2.16. We have to prove (c) and (d).

(c). Let $\bar{I}_2$, $\bar{I}_2$, and $\bar{I}_3$ be three CIFSs on $\Bbbk$ and

$$\mu_{\bar{I}_1} = \left\{\left\langle \rho, \chi_{\bar{I}_1}(\rho)e^{i\Im_{\bar{I}_1}(\rho)}, \hbar_{\bar{I}_1}(\rho)e^{i\eth_{\bar{I}_1}(\rho)}\right\rangle : \rho \in \Bbbk\right\},$$
$$\mu_{\bar{I}_2} = \left\{\left\langle \rho, \chi_{\bar{I}_2}(\rho)e^{i\Im_{\bar{I}_2}(\rho)}, \hbar_{\bar{I}_2}(\rho)e^{i\eth_{\bar{I}_2}(\rho)}\right\rangle : \rho \in \Bbbk\right\},$$

and

$$\mu_{\bar{I}_3} = \left\{\left\langle \rho, \chi_{\bar{I}_3}(\rho)e^{i\Im_{\bar{I}_3}(\rho)}, \hbar_{\bar{I}_3}(\rho)e^{i\eth_{\bar{I}_3}(\rho)}\right\rangle : \rho \in \Bbbk\right\},$$

denote their grade's, respectively. We suppose $\chi_{\bar{I}_1}(\rho) \leqslant \chi_{\bar{I}_2}(\rho)$, $\hbar_{\bar{I}_1}(\rho) \geqslant \hbar_{\bar{I}_2}(\rho)$, and $\Im_{\bar{I}_1}(\rho) \leqslant \Im_{\bar{I}_2}(\rho)$, $\eth_{\bar{I}_1}(\rho) \geqslant \eth_{\bar{I}_2}(\rho)$, $\forall \rho \in \Bbbk$. Then,

$$\left|\mu_{\bar{I}_1\cap\bar{I}_3}(\rho)\right| = \left(\left|\chi_{\bar{I}_1}(\rho).\chi_{\bar{I}_3}(\ell)\right|, \left|\hbar_{\bar{I}_1}(\rho), \hbar_{\bar{I}_3}(\rho)\right|\right) \leqslant \left(\left|\chi_{\bar{I}_2}(\rho).\chi_{\bar{I}_3}(\rho)\right|, \left|\hbar_{\bar{I}_2}(\rho), \hbar_{\bar{I}_3}(\rho)\right|\right) = \left|\mu_{\bar{I}_2\cap\bar{I}_3}(\rho)\right|, \forall\, \rho \in \Bbbk,$$

and

$$\mathfrak{I}_{(\bar{I}_1\cap\bar{I}_3)}(\rho) = \min(\mathfrak{I}_{\bar{I}_1}(\rho), \mathfrak{I}_{\bar{I}_3}(\rho)) \leqslant \min(\mathfrak{I}_{\bar{I}_2}(\rho), \mathfrak{I}_{\bar{I}_3}(\rho)) = \mathfrak{I}_{(\bar{I}_2\cap\bar{I}_3)}(\rho),$$
$$\eth_{(\bar{I}_1\cap\bar{I}_3)}(\rho) = \max(\eth_{\bar{I}_1}(\rho), \eth_{\bar{I}_3}(\rho)) \geqslant \max(\eth_{\bar{I}_2}(\rho), \eth_{\bar{I}_3}(\rho)) = \eth_{(\bar{I}_2\cap\bar{I}_3)}(\rho).$$

(d).

$$\mu_{\bar{I}_1\cap(\bar{I}_2\cap\bar{I}_3)}(\rho) = \left\{ \begin{array}{c} [\min\{\chi_{\bar{I}_1}(\rho), \min(\chi_{\bar{I}_2}(\rho), \chi_{\bar{I}_3}(\rho))\}, \\ \max\{\hbar_{\bar{I}_1}(\rho), \max(\hbar_{\bar{I}_2}(\rho), \hbar_{\bar{I}_3}(\rho))\}] \\ e^{i[\min\{\mathfrak{I}_{\bar{I}_1}(\rho), \min(\mathfrak{I}_{\bar{I}_2}(\rho), \mathfrak{I}_{\bar{I}_3}(\rho))\}, \max\{\eth_{\bar{I}_1}(\rho), \max(\eth_{\bar{I}_2}(\rho), \eth_{\bar{I}_3}(\rho))\}]} \end{array} \right\}$$
$$= \left\{ \begin{array}{c} [\min\{\chi_{\bar{I}_1}(\rho), \chi_{\bar{I}_2}(\rho), \chi_{\bar{I}_3}(\rho)\}, \\ \max\{\hbar_{\bar{I}_1}(\rho), \hbar_{\bar{I}_2}(\rho), \hbar_{\bar{I}_3}(\rho)\}] \\ e^{i[\min\{\mathfrak{I}_{\bar{I}_1}(\rho), \mathfrak{I}_{\bar{I}_2}(\rho), \mathfrak{I}_{\bar{I}_3}(\rho)\}, \max\{\eth_{\bar{I}_1}(\rho, \eth_{\bar{I}_2}(\rho), \eth_{\bar{I}_3}(\rho)\}]} \end{array} \right\}$$
$$= \left\{ \begin{array}{c} [\min\{\min(\chi_{\bar{I}_1}(\rho), \chi_{\bar{I}_2}(\rho)), \chi_{\bar{I}_3}(\rho)\}, \\ \max\{\max(\hbar_{\bar{I}_1}(\rho), \hbar_{\bar{I}_2}(\rho)), \hbar_{\bar{I}_3}(\rho)\}] \\ e^{i[\min\{\min(\mathfrak{I}_{\bar{I}_1}(\rho), \mathfrak{I}_{\bar{I}_2}(\rho)), \mathfrak{I}_{\bar{I}_3}(\rho)\}, \max\{\max(\eth_{\bar{I}_1}(\rho), \eth_{\bar{I}_2}(\rho)), \eth_{\bar{I}_3}(\rho)\}]} \end{array} \right\} = \mu_{(\bar{I}_1\cap\bar{I}_2)\cap\bar{I}_3}(\rho).$$

□

AppendixC.

Proof of Proposition 2.21. Clearly the axioms (a), (b), (e), and (g) are obvious from definition Definition 2.16. We have to prove (c) and (d).

(c). Let $\bar{I}_2, \bar{I}_2$, and $\bar{I}_3$ be three CIFSs on $\Bbbk$ and

$$\mu_{\bar{I}_1} = \left\{\left\langle \rho, \chi_{\bar{I}_1}(\rho)e^{i\mathfrak{I}_{\bar{I}_1}(\ell)}, \hbar_{\bar{I}_1}(\rho)e^{i\eth_{\bar{I}_1}(\rho)}\right\rangle : \rho \in \Bbbk\right\},$$
$$\mu_{\bar{I}_2} = \left\{\left\langle \rho, \chi_{\bar{I}_2}(\rho)e^{i\mathfrak{I}_{\bar{I}_2}(\rho)}, \hbar_{\bar{I}_2}(\rho)e^{i\eth_{\bar{I}_2}(\rho)}\right\rangle : \ell \in \Bbbk\right\},$$

and

$$\mu_{\bar{I}_3} = \left\{\left\langle \rho, \chi_{\bar{I}_3}(\rho)e^{i\mathfrak{I}_{\bar{I}_3}(\rho)}, \hbar_{\bar{I}_3}(\rho)e^{i\eth_{\bar{I}_3}(\rho)}\right\rangle : \rho \in \Bbbk\right\},$$

denote their grade's, respectively. We suppose $|\chi_{\bar{I}_1}(\rho)| \leqslant |\chi_{\bar{I}_2}(\rho)|$, $|\hbar_{\bar{I}_1}(\rho)| \geqslant |\hbar_{\bar{I}_2}(\rho)|$, and $\mathfrak{I}_{\bar{I}_1}(\rho) \leqslant \mathfrak{I}_{\bar{I}_2}(\rho)$, $\eth_{\bar{I}_1}(\rho) \geqslant \eth_{\bar{I}_2}(\rho)$, $\forall\rho \in \Bbbk$. Then,

$$|\mu_{\bar{I}_1\boxtimes\bar{I}_3}(\rho)| = (|\chi_{\bar{I}_1}(\rho).\chi_{\bar{I}_3}(\rho)|, |\hbar_{\bar{I}_1}(\rho), \hbar_{\bar{I}_3}(\rho)|) \leqslant (|\chi_{\bar{I}_2}(\rho).\chi_{\bar{I}_3}(\rho)|, |\hbar_{\bar{I}_2}(\rho), \hbar_{\bar{I}_3}(\rho)|) = |\mu_{\bar{I}_2\boxtimes\bar{I}_3}(\rho)|, \forall\ \rho \in \Bbbk,$$
$$\mathfrak{I}_{\bar{I}_1\boxtimes\bar{I}_3}(\rho) = 2\pi\left(\frac{\mathfrak{I}_{\bar{I}_1}(\rho)}{2\pi}.\frac{\mathfrak{I}_{\bar{I}_3}(\rho)}{2\pi}\right) \leqslant 2\pi\left(\frac{\mathfrak{I}_{\bar{I}_2}(\ell)}{2\pi}.\frac{\mathfrak{I}_{\bar{I}_3}(\rho)}{2\pi}\right) = \mathfrak{I}_{\bar{I}_2\boxtimes\bar{I}_3}(\rho), \forall\ \rho \in \Bbbk,$$
$$|\eth_{\bar{I}_1\boxtimes\bar{I}_3}(\rho)| = 2\pi\left(\frac{\eth_{\bar{I}_1}(\rho)}{2\pi}.\frac{\eth_{\bar{I}_3}(\rho)}{2\pi}\right) \geqslant 2\pi\left(\frac{\eth_{\bar{I}_2}(\rho)}{2\pi}.\frac{\eth_{\bar{I}_3}(\rho)}{2\pi}\right) = |\eth_{\bar{I}_2\boxtimes\bar{I}_3}(\rho)|, \forall\ \rho \in \Bbbk,$$

(d).

$$\mu_{\bar{I}_1\boxtimes(\bar{I}_2\boxtimes\bar{I}_3)}(\rho) = \left\{ \begin{array}{c} (\chi_{\bar{I}_1}(\rho)(\chi_{\bar{I}_2}(\rho).\varkappa_{\bar{I}_3}(\rho)))e^{i2\pi\left(\frac{\mathfrak{I}_{\bar{I}_1}(\rho)}{2\pi}.\left(\frac{\frac{\mathfrak{I}_{\bar{I}_2}(\rho)}{2\pi}.\frac{\mathfrak{I}_{\bar{I}_3}(\rho)}{2\pi}}{2\pi}\right)\right)}, \\ (\hbar_{\bar{I}_1}(\rho)(\hbar_{\bar{I}_2}(\rho).\hbar_{\bar{I}_3}(\rho)))e^{i2\pi\left(\frac{\eth_{\bar{I}_1}(\rho)}{2\pi}.\left(\frac{\frac{\eth_{\bar{I}_2}(\rho)}{2\pi}.\frac{\eth_{\bar{I}_3}(\rho)}{2\pi}}{2\pi}\right)\right)} \end{array} \right\}$$

$$= \left\{ \begin{array}{l} (\chi_{\bar{I}_1}(\rho).\chi_{\bar{I}_2}(\rho).\chi_{\bar{I}_3}(\rho))e^{i2\pi\left(\left(\frac{\frac{\mathfrak{I}_{\bar{I}_1}(\rho)}{2\pi}.\frac{\mathfrak{I}_{\bar{I}_2}(\rho)}{2\pi}}{2\pi}\right)\frac{\mathfrak{I}_{\bar{I}_3}(\rho)}{2\pi}\right)}, \\ (\hbar_{\bar{I}_1}(\rho).\hbar_{\bar{I}_2}(\rho).\hbar_{\bar{I}_3}(\rho))e^{i2\pi\left(\left(\frac{\frac{\eth_{\bar{I}_1}(\rho)}{2\pi}.\frac{\eth_{\bar{I}_2}(\rho)}{2\pi}}{2\pi}\right).\frac{\eth_{\bar{I}_3}(\ell)}{2\pi}\right)} \end{array} \right\}$$

$$= \left\{ \begin{array}{l} ((\chi_{\bar{I}_1}(\rho).\chi_{\bar{I}_2}(\rho))\chi_{\bar{I}_3}(\rho))e^{i2\pi\left(\left(\frac{\frac{\mathfrak{I}_{\bar{I}_1}(\rho)}{2\pi}.\frac{\mathfrak{I}_{\bar{I}_2}(\rho)}{2\pi}}{2\pi}\right)\frac{\mathfrak{I}_{\bar{I}_3}(\rho)}{2\pi}\right)}, \\ ((\hbar_{\bar{I}_1}(\rho).\hbar_{\bar{I}_2}(\rho))\hbar_{\bar{I}_3}(\rho))e^{i2\pi\left(\left(\frac{\frac{\eth_{\bar{I}_1}(\rho)}{2\pi}.\frac{\eth_{\bar{I}_2}(\rho)}{2\pi}}{2\pi}\right).\frac{\eth_{\bar{I}_3}(\rho)}{2\pi}\right)} \end{array} \right\} = \mu_{((\bar{I}_1 \boxtimes \bar{I}_2)\boxtimes \bar{I}_3)}(\ell),$$

where $C = \mathfrak{I}_{\bar{I}_i}(\rho) + \mathfrak{I}_{\bar{I}_j}(\rho) + \mathfrak{I}_{\bar{I}_k}(\rho), D = \eth_{\bar{I}_i}(\rho) + \eth_{\bar{I}_j}(\rho) + \eth_{\bar{I}_k}(\rho)$, $i, j, k = 1, 2, 3$. □

AppendixD.

Proof of proposition 2.23. Clearly the axioms (a), (b), (e), and (f) are obvious from Definition 2.16. We have to prove (c) and (d).

(c). Let $\bar{I}_2, \bar{I}_2$, and $\bar{I}_3$ be three CIFSs on $\Bbbk$ and

$$\mu_{\bar{I}_1} = \left\{ \left\langle \rho, \chi_{\bar{I}_1}(\rho)e^{i\mathfrak{I}_{\bar{I}_1}(\rho)}, \hbar_{\bar{I}_1}(\rho)e^{i\eth_{\bar{I}_1}(\rho)} \right\rangle : \rho \in \Bbbk \right\},$$
$$\mu_{\bar{I}_2} = \left\{ \left\langle \rho, \chi_{\bar{I}_2}(\rho)e^{i\mathfrak{I}_{\bar{I}_2}(\rho)}, \hbar_{\bar{I}_2}(\ell)e^{i\eth_{\bar{I}_2}(\rho)} \right\rangle : \rho \in \Bbbk \right\},$$

and

$$\mu_{\bar{I}_3} = \left\{ \left\langle \rho, \chi_{\bar{I}_3}(\rho)e^{i\mathfrak{I}_{\bar{I}_3}(\ell)}, \hbar_{\bar{I}_3}(\rho)e^{i\eth_{\bar{I}_3}(\rho)} \right\rangle : \rho \in \Bbbk \right\},$$

denote their grade values, respectively. We suppose $|\chi_{\bar{I}_1}(\rho)| \leqslant |\chi_{\bar{I}_2}(\rho)|$, $|\hbar_{\bar{I}_1}(\rho)| \geqslant |\hbar_{\bar{I}_2}(\rho)|$, and $\mathfrak{I}_{\bar{I}_1}(\rho) \leqslant \mathfrak{I}_{\bar{I}_2}(\rho)$, $\eth_{\bar{I}_1}(\rho) \geqslant \eth_{\bar{I}_2}(\rho)$, $\forall \rho \in \Bbbk$. Then,

$$\begin{aligned}
|\mu_{\bar{I}_1 \dot{+} \bar{I}_3}(\rho)| &= \left\langle \frac{1}{2} \left\{ \begin{array}{l} |\chi_{\bar{I}_1}(\rho) + \chi_{\bar{I}_3}(\rho) - \chi_{\bar{I}_1}(\rho).\chi_{\bar{I}_3}(\rho)|, \\ |\hbar_{\bar{I}_1}(\rho) + \hbar_{\bar{I}_3}(\rho) - \hbar_{\bar{I}_1}(\rho).\hbar_{\bar{I}_3}(\rho)| \end{array} \right\} \right\rangle \\
&\leqslant \left\langle \frac{1}{2} \left\{ \begin{array}{l} |\chi_{\bar{I}_2}(\rho) + \chi_{\bar{I}_3}(\rho) - \chi_{\bar{I}_2}(\rho).\chi_{\bar{I}_3}(\rho)|, \\ |\hbar_{\bar{I}_2}(\rho) + \hbar_{\bar{I}_3}(\rho) - \hbar_{\bar{I}_2}(\rho).\hbar_{\bar{I}_3}(\rho)| \end{array} \right\} \right\rangle \\
&= \left\langle \frac{1}{2} \left\{ \begin{array}{l} |\chi_{\bar{I}_2}(\rho) + \chi_{\bar{I}_3}(\rho)(1 - \chi_{\bar{I}_2}(\rho))|, \\ |\hbar_{\bar{I}_2}(\rho) + \hbar_{\bar{I}_3}(\ell)(1 - \hbar_{\bar{I}_2}(\rho))| \end{array} \right\} \right\rangle = |\mu_{\bar{I}_2 \dot{+} \bar{I}_3}(\rho)|, \ \forall \rho \in \Bbbk, \\
\mathfrak{I}_{\bar{I}_1 \dot{+} \bar{I}_3}(\rho) &= 2\pi \left(\frac{\mathfrak{I}_{\bar{I}_1}(\rho) + \mathfrak{I}_{\bar{I}_3}(\rho) - \mathfrak{I}_{\bar{I}_1}(\rho).\mathfrak{I}_{\bar{I}_3}(\rho)}{2\pi} \right) \\
&\leqslant 2\pi \left(\frac{\mathfrak{I}_{\bar{I}_2}(\rho) + \mathfrak{I}_{\bar{I}_3}(\rho) - \mathfrak{I}_{\bar{I}_2}(\rho).\mathfrak{I}_{\bar{I}_3}(\rho)}{2\pi} \right) = 2\pi \left(\frac{\mathfrak{I}_{\bar{I}_2}(\rho) + \mathfrak{I}_{\bar{I}_3}(\rho)(1 - \mathfrak{I}_{\bar{I}_2}(\rho))}{2\pi} \right) = \mathfrak{I}_{\bar{I}_2 \dot{+} \bar{I}_3}(\rho), \\
\eth_{\bar{I}_1 \dot{+} \bar{I}_3}(\ell) &= 2\pi \left(\frac{\eth_{\bar{I}_1}(\rho) + \eth_{\bar{I}_3}(\rho) - \eth_{\bar{I}_1}(\rho).\eth_{\bar{I}_3}(\rho)}{2\pi} \right) \\
&\leqslant 2\pi \left(\frac{\eth_{\bar{I}_2}(\rho) + \eth_{\bar{I}_3}(\rho) - \eth_{\bar{I}_2}(\rho).\eth_{\bar{I}_3}(\rho)}{2\pi} \right) = 2\pi \left(\frac{\eth_{\bar{I}_2}(\rho) + \eth_{\bar{I}_3}(\rho)(1 - \eth_{\bar{I}_2}(\rho))}{2\pi} \right) = \eth_{\bar{I}_2 \dot{+} \bar{I}_3}(\rho).
\end{aligned}$$

(d).

$$
\begin{aligned}
\mu_{\bar{I}_1 \dotplus (\bar{I}_2 \dotplus \bar{I}_3)}(\rho) &= \left\{ \left\langle \begin{array}{c} \left(\frac{\chi_{\bar{I}_1}(\rho)+\chi_{\bar{I}_2\dotplus\bar{I}_3}(\rho)-\chi_{\bar{I}_1}(\rho).\chi_{\bar{I}_2\dotplus\bar{I}_3}(\rho)}{2}\right) \\ e^{i2\pi\left(\frac{\Im_{\bar{I}_1}(\rho)+\Im_{\bar{I}_2\dotplus\bar{I}_3}(\rho)-\Im(\ell)_{\bar{I}_1}.\Im_{\bar{I}_2\dotplus\bar{I}_3}(\ell)}{2\pi}\right)}, \\ \left(\frac{\left(\hbar_{\bar{I}_1}(\rho)+\hbar_{\bar{I}_2\dotplus\bar{I}_3}(\rho)-\hbar_{\bar{I}_1}(\rho).\hbar_{\bar{I}_2\dotplus\bar{I}_3}(\rho)\right)}{2}\right) \\ e^{i2\pi\left(\frac{\eth_{\bar{I}_1}(\rho)+\eth_{\bar{I}_2\dotplus\bar{I}_3}(\rho)-\eth_{\bar{I}_1}(\rho).\eth_{\bar{I}_2\dotplus\bar{I}_3}(\rho)}{2\pi}\right)} \end{array} \right\rangle \right\} \\
&= \left\{ \begin{array}{c} \frac{1}{2}\left(\begin{array}{c} \chi_{\bar{I}_1}(\rho)+(\chi_{\bar{I}_2}(\rho)+\chi_{\bar{I}_3}(\rho)-\chi_{\bar{I}_2}(\rho).\chi_{\bar{I}_3}(\rho) \\ \chi_{\bar{I}_2}(\rho)+\chi_{\bar{I}_3}(\rho)-\chi_{\bar{I}_2}(\rho).\chi_{\bar{I}_3}(\rho)) \end{array} \right) \\ e^{i2\pi\left(\frac{\Im_{\bar{I}_1}(\rho)+\Im_{\bar{I}_2\dotplus\bar{I}_3}(\rho)-\Im(\rho)_{\bar{I}_1}.\Im_{\bar{I}_2\dotplus\bar{I}_3}(\rho)}{2\pi}\right)}, \\ \frac{1}{2}\left(\begin{array}{c} \hbar_{\bar{I}_1}(\rho)+(\hbar_{\bar{I}_2}(\rho)+\hbar_{\bar{I}_3}(\rho)-\hbar_{\bar{I}_2}(\rho).\hbar_{\bar{I}_3}(\rho) \\ -\hbar_{\bar{I}_1}(\rho).(\hbar_{\bar{I}_2}(\rho)+\hbar_{\bar{I}_3}(\rho)-\hbar_{\bar{I}_2}(\rho).\hbar_{\bar{I}_3}(\rho)) \end{array} \right) \\ e^{i2\pi\left(\frac{\eth_{\bar{I}_1}(\rho)+\eth_{\bar{I}_2\dotplus\bar{I}_3}(\rho)-\eth_{\bar{I}_1}(\rho).\eth_{\bar{I}_2\dotplus\bar{I}_3}(\rho)}{2\pi}\right)} \end{array} \right\} \\
&= \left\{ \begin{array}{c} \frac{1}{2}\left(\begin{array}{c} (\chi_{\bar{I}_1}(\rho)+\chi_{\bar{I}_2}(\rho)+\chi_{\bar{I}_3}(\rho)-\chi_{\bar{I}_2}(\rho).\chi_{\bar{I}_3}(\rho) \\ -\chi_{\bar{I}_1}(\rho)\chi_{\bar{I}_2}(\rho)-\chi_{\bar{I}_1}(\rho)\chi_{\bar{I}_3}(\rho)+\chi_{\bar{I}_1}(\rho)\chi_{\bar{I}_2}(\rho).\chi_{\bar{I}_3}(\rho) \end{array} \right) \\ e^{i2\pi\left(\frac{\Im_{\bar{I}_1\dotplus\bar{I}_2}(\ell)+\Im_{\bar{I}_3}(\rho)-\Im(\rho)_{\bar{I}_1\dotplus\bar{I}_2}.\Im_{\bar{I}_3}(\rho)}{2\pi}\right)}, \\ \frac{1}{2}\left(\begin{array}{c} (\hbar_{\bar{I}_1}(\rho)+\hbar_{\bar{I}_2}(\rho)+\hbar_{\bar{I}_3}(\rho)-\hbar_{\bar{I}_2}(\rho).\hbar_{\bar{I}_3}(\rho) \\ -\hbar_{\bar{I}_1}(\rho)\hbar_{\bar{I}_2}(\rho)-\hbar_{\bar{I}_1}(\rho)\hbar_{\bar{I}_3}(\rho)+\hbar_{\bar{I}_1}(\rho)\hbar_{\bar{I}_2}(\rho).\hbar_{\bar{I}_3}(\rho)) \end{array} \right) \\ e^{i2\pi\left(\frac{\eth_{\bar{I}_1\dotplus\bar{I}_2}(\rho)+\eth_{\bar{I}_3}(\rho)-\eth_{\bar{I}_1\dotplus\bar{I}_2}(\ell).\eth_{\bar{I}_3}(\rho)}{2\pi}\right)} \end{array} \right\} \\
&= \left\{ \begin{array}{c} \frac{1}{2}\left(\begin{array}{c} (\chi_{\bar{I}_1}(\rho)+\chi_{\bar{I}_2}(\rho)-\chi_{\bar{I}_1}(\rho)\chi_{\bar{I}_2}(\rho))+\chi_{\bar{I}_3}(\rho) \\ -\chi_{\bar{I}_1}(\rho)\chi_{\bar{I}_3}(\rho)-\chi_{\bar{I}_2}(\rho)\chi_{\bar{I}_3}(\rho)+\chi_{\bar{I}_1}(\rho)\chi_{\bar{I}_2}(\rho).\chi_{\bar{I}_3}(\rho) \end{array} \right) \\ e^{i2\pi\left(\frac{\Im_{\bar{I}_1\dotplus\bar{I}_2}(\rho)+\Im_{\bar{I}_3}(\rho)-\Im(\ell)_{\bar{I}_1\dotplus\bar{I}_2}.\Im_{\bar{I}_3}(\rho)}{2\pi}\right)}, \\ \frac{1}{2}\left(\begin{array}{c} (\hbar_{\bar{I}_1}(\rho)+\hbar_{\bar{I}_2}(\rho)-\hbar_{\bar{I}_1}(\rho)\hbar_{\bar{I}_2}(\rho))+\hbar_{\bar{I}_3}(\rho)-\hbar_{\bar{I}_1}(\rho)\hbar_{\bar{I}_3}(\rho) \\ -\hbar_{\bar{I}_2}(\rho).\hbar_{\bar{I}_3}(\rho)+\hbar_{\bar{I}_1}(\rho)\hbar_{\bar{I}_2}(\rho).\hbar_{\bar{I}_3}(\ell) \end{array} \right) \\ e^{i2\pi\left(\frac{\eth_{\bar{I}_1\dotplus\bar{I}_2}(\rho)+\eth_{\bar{I}_3}(\rho)-\eth_{\bar{I}_1\dotplus\bar{I}_2}(\rho).\eth_{\bar{I}_3}(\rho)}{B}\right)} \end{array} \right\} \\
&= \left\{ \begin{array}{c} \frac{1}{2}\left((\chi_{\bar{I}_1\dotplus\bar{I}_2}(\rho)+\chi_{\bar{I}_3}(\rho)-(\chi_{\bar{I}_1}(\rho)+\chi_{\bar{I}_2}(\rho)-\chi_{\bar{I}_1}(\rho)\chi_{\bar{I}_2}(\rho)\chi_{\bar{I}_3}(\rho))\right) \\ e^{i2\pi\left(\frac{\Im_{\bar{I}_1\dotplus\bar{I}_2}(\rho)+\Im_{\bar{I}_3}(\rho)-\Im(\rho)_{\bar{I}_1\dotplus\bar{I}_2}.\Im_{\bar{I}_3}(\rho)}{2\pi}\right)}, \\ \frac{1}{2}\left(\hbar_{\bar{I}_1\dotplus\bar{I}_2}(\rho)+\hbar_{\bar{I}_3}(\rho)-(\hbar_{\bar{I}_1}(\rho)+\hbar_{\bar{I}_2}(\rho)-\hbar_{\bar{I}_1}(\rho)\hbar_{\bar{I}_2}(\rho)\hbar_{\bar{I}_3}(\rho))\right) \\ e^{i2\pi\left(\frac{\eth_{\bar{I}_1\dotplus\bar{I}_2}(\rho)+\eth_{\bar{I}_3}(\rho)-\eth_{\bar{I}_1\dotplus\bar{I}_2}(\rho).\eth_{\bar{I}_3}(\rho)}{2\pi}\right)} \end{array} \right\} \\
&= \left\{ \begin{array}{c} \left(\frac{\chi_{\bar{I}_1\dotplus\bar{I}_2}(\rho)+\chi_{\bar{I}_3}(\rho)-\chi_{\bar{I}_1\dotplus\bar{I}_2}(\rho)\chi_{\bar{I}_3}(\rho)}{2}\right) \\ e^{i2\pi\left(\frac{\Im_{\bar{I}_1\dotplus\bar{I}_2}(\rho)+\Im_{\bar{I}_3}(\rho)-\Im(\rho)_{\bar{I}_1\dotplus\bar{I}_2}.\Im_{\bar{I}_3}(\rho)}{2\pi}\right)}, \\ \left(\frac{\hbar_{\bar{I}_1\dotplus\bar{I}_2}(\rho)+\hbar_{\bar{I}_3}(\rho)-\hbar_{\bar{I}_1\dotplus\bar{I}_2}(\rho)\hbar_{\bar{I}_3}(\rho)}{2}\right) \\ e^{i2\pi\left(\frac{\eth_{\bar{I}_1\dotplus\bar{I}_2}(\rho)+\eth_{\bar{I}_3}(\rho)-\eth_{\bar{I}_1\dotplus\bar{I}_2}(\rho).\eth_{\bar{I}_3}(\rho)}{2\pi}\right)} \end{array} \right\} = \mu_{(\bar{I}_1\dotplus\bar{I}_2)\dotplus\bar{I}_3}(\rho).
\end{aligned}
$$

$\square$

AppendixE.

Proof of proposition 2.26. Clearly the axioms (a), (b), (e), and (f) are obvious from Definition 2.16. We have to prove (c) and (d).

(c). Let $\bar{I}_2, \bar{I}_2$, and $\bar{I}_3$ be three CIFSs on $\Bbbk$ and

$$\mu_{\bar{I}_1} = \left\{\left\langle \rho, \chi_{\bar{I}_1}(\rho)e^{i\mathfrak{I}_{\bar{I}_1}(\rho)}, \hbar_{\bar{I}_1}(\rho)e^{i\eth_{\bar{I}_1}(\rho)}\right\rangle : \rho \in \Bbbk\right\},$$
$$\mu_{\bar{I}_2} = \left\{\left\langle \rho, \chi_{\bar{I}_2}(\rho)e^{i\mathfrak{I}_{\bar{I}_2}(\rho)}, \hbar_{\bar{I}_2}(\rho)e^{i\eth_{\bar{I}_2}(\rho)}\right\rangle : \rho \in \Bbbk\right\},$$

and

$$\mu_{\bar{I}_3} = \left\{\left\langle \rho, \chi\bar{I}3(\rho)e^{i\mathfrak{I}_{\bar{I}_3}(\rho)}, \hbar_{\bar{I}_3}(\rho)e^{i\eth_{\bar{I}_3}(\rho)}\right\rangle : \rho \in \Bbbk\right\},$$

denote their grade values, respectively. We suppose $|\chi_{\bar{I}_1}(\rho)| \leqslant |\chi_{\bar{I}_2}(\rho)|$, $|\hbar_{\bar{I}_1}(\rho)| \geqslant |\hbar_{\bar{I}_2}(\ell)|$, and $\mathfrak{I}_{\bar{I}_1}(\rho) \leqslant \mathfrak{I}_{\bar{I}_2}(\rho)$, $\eth_{\bar{I}_1}(\rho) \geqslant \eth_{\bar{I}_2}(\rho)$, $\forall \rho \in \Bbbk$. Then,

$$\begin{aligned}
&\left|\mu_{(\bar{I}_1\oplus\bar{I}_2)}(\rho)\right| = \left\langle \begin{matrix} \min(1, \chi_{\bar{I}_1}(\rho) + \chi_{\bar{I}_3}(\rho)), \\ \max(0, \hbar_{\bar{I}_1}(\rho) - \hbar_{\bar{I}_3}(\rho)) \end{matrix} \right\rangle \leqslant \left\langle \begin{matrix} \min(1, \chi_{\bar{I}_2}(\rho) + \chi_{\bar{I}_3}(\rho)), \\ \max(0, \hbar_{\bar{I}_2}(\rho) - \hbar_{\bar{I}_3}(\rho)) \end{matrix} \right\rangle = \left|\mu_{(\bar{I}_2\oplus\bar{I}_3)}(\rho)\right|, \ \forall\ \rho \in \Bbbk, \\
&\mathfrak{I}_{(\bar{I}_1\oplus\bar{I}_3)}(\rho) = \min(2\pi, \mathfrak{I}_{\bar{I}_1}(\rho) + \mathfrak{I}_{\bar{I}_3}(\rho)) \leqslant \min(2\pi, \mathfrak{I}_{\bar{I}_2}(\rho) + \mathfrak{I}_{\bar{I}_3}(\rho)) = \mathfrak{I}_{(\bar{I}_2\oplus\bar{I}_3)}(\rho), \\
&\eth_{(\bar{I}_1\oplus\bar{I}_3)}(\rho) = \max(0, \eth_{\bar{I}_1}(\ell) - \eth_{\bar{I}_3}(\rho)) \geqslant \max(0, \eth_{\bar{I}_2}(\rho) - \eth_{\bar{I}_3}(\rho)) = \eth_{(\bar{I}_2\oplus\bar{I}_3)}(\rho).
\end{aligned}$$

(d).

$$\begin{aligned}
\mu_{\bar{I}_1\oplus(\bar{I}_2\oplus\bar{I}_3)}(\rho) &= \left\{ \begin{matrix} \min\{1, \chi_{\bar{I}_1}(\rho) + (\min(1, \chi_{\bar{I}_2}(\rho) + \chi_{\bar{I}_3}(\rho)))\} \\ e^{i\min\{2\pi, \mathfrak{I}_{\bar{I}_1}(\rho) + (\min(2\pi, \mathfrak{I}_{\bar{I}_2}(\rho) + \mathfrak{I}_{\bar{I}_3}(\rho)))\}}, \\ \max\{0, (\hbar_{\bar{I}_1}(\rho) - (\max(0, \hbar_{\bar{I}_2}(\rho) - \hbar_{\bar{I}_3}(\rho))\} \\ e^{i\max\{0, \eth_{\bar{I}_1}(\rho) - (\max(0, \eth_{\bar{I}_2}(\rho) - \eth_{\bar{I}_3}(\rho))\}} \end{matrix} \right\} \\
&= \left\{ \begin{matrix} \min\{1, \chi_{\bar{I}_1}(\rho) + \chi_{\bar{I}_2}(\rho) + \chi_{\bar{I}_3}(\rho) \\ e^{i\min\{2\pi, \mathfrak{I}_{\bar{I}_1}(\rho) + \mathfrak{I}_{\bar{I}_2}(\rho) + \mathfrak{I}_{\bar{I}_3}(\rho)\}}, \\ \max\{0, \hbar_{\bar{I}_1}(\rho) - \hbar_{\bar{I}_2}(\rho) - \hbar_{\bar{I}_3}(\rho)\} \\ e^{i\max\{0, \eth_{\bar{I}_1}(\rho) - \eth_{\bar{I}_2}(\rho) - \eth_{\bar{I}_3}(\rho)\}} \end{matrix} \right\} \\
&= \left\{ \begin{matrix} \min\{\min(1, \chi_{\bar{I}_1}(\rho) + \chi_{\bar{I}_2}(\rho)) + \chi_{\bar{I}_3}(\rho), 1\} \\ e^{i\min\{\min(2\pi, \mathfrak{I}_{\bar{I}_1}(\rho) + \mathfrak{I}_{\bar{I}_2}(\rho)) + \mathfrak{I}_{\bar{I}_3}(\rho), 2\pi\}}, \\ \max\{\max(0, \hbar_{\bar{I}_1}(\rho) - \hbar_{\bar{I}_2}(\rho)) - \hbar_{\bar{I}_3}(\rho), 0\} \\ e^{i\max\{\max(0, \eth_{\bar{I}_1}(\rho) - \eth_{\bar{I}_2}(\rho)) - \eth_{\bar{I}_3}(\rho), 0\}} \end{matrix} \right\} = \mu_{(\bar{I}_1\oplus\bar{I}_2)\oplus\bar{I}_3}(\rho).
\end{aligned}$$

□

AppendixF.

Proof of proposition 2.28. Clearly the axioms (a), (b), (e), and (f) are obvious from Definition 2.16. We have to prove (c) and (d).

(c). Let $\bar{I}_2, \bar{I}_2$, and $\bar{I}_3$ be three CIFSs on $\Bbbk$ and

$$\mu_{\bar{I}_1} = \left\{\left\langle \rho, \chi_{\bar{I}_1}(\rho)e^{i\mathfrak{I}_{\bar{I}_1}(\rho)}, \hbar_{\bar{I}_1}(\rho)e^{i\eth_{\bar{I}_1}(\rho)}\right\rangle : \rho \in \Bbbk\right\},$$
$$\mu_{\bar{I}_2} = \left\{\left\langle \rho, \chi_{\bar{I}_2}(\rho)e^{i\mathfrak{I}_{\bar{I}_2}(\rho)}, \hbar_{\bar{I}_2}(\rho)e^{i\eth_{\bar{I}_2}(\rho)}\right\rangle : \rho \in \Bbbk\right\},$$

and

$$\mu_{\bar{I}_3} = \left\{\left\langle \rho, \chi_{\bar{I}_3}(\rho)e^{i\mathfrak{I}_{\bar{I}_3}(\rho)}, \hbar_{\bar{I}_3}(\rho)e^{i\eth_{\bar{I}_3}(\rho)}\right\rangle : \rho \in \Bbbk\right\},$$

denote their grade values, respectively. We suppose $\left|\chi_{\bar{I}_1}(\rho)\right| \leqslant \left|\chi_{\bar{I}_2}(\rho)\right|$, $\left|\hbar_{\bar{I}_1}(\rho)\right| \geqslant \left|\hbar_{\bar{I}_2}(\rho)\right|$, and $\mathfrak{I}_{\bar{I}_1}(\rho) \leqslant \mathfrak{I}_{\bar{I}_2}(\rho)$, $\eth_{\bar{I}_1}(\rho) \geqslant \eth_{\bar{I}_2}(\rho)$, $\forall \rho \in \Bbbk$. Then,

$$\left|\mu_{(\bar{I}_1 \Cap \bar{I}_2)}(\rho)\right| = \left\langle \begin{array}{c} \max(0, \chi_{\bar{I}_1}(\rho) + \chi_{\bar{I}_3}(\rho) - 1), \\ \max(0, \hbar_{\bar{I}_1}(\rho) + \hbar_{\bar{I}_3}(\rho) - 1) \end{array} \right\rangle \leqslant \left\langle \begin{array}{c} \max(0, \chi_{\bar{I}_2}(\rho) + \chi_{\bar{I}_3}(\rho) - 1), \\ \max(0, \hbar_{\bar{I}_2}(\rho) + \hbar_{\bar{I}_3}(\rho) - 1) \end{array} \right\rangle = \left|\mu_{(\bar{I}_2 \Cap \bar{I}_3)}(\rho)\right|, \ \forall \rho \in \Bbbk,$$

and

$$\begin{aligned} \mathfrak{I}_{(\bar{I}_1 \Cap \bar{I}_3)}(\rho) &= \max(0, \mathfrak{I}_{\bar{I}_1}(\rho) + \mathfrak{I}_{\bar{I}_3}(\rho) - 2\pi) \leqslant \max(0, \mathfrak{I}_{\bar{I}_2}(\rho) + \mathfrak{I}_{\bar{I}_3}(\rho) - 2\pi) = \mathfrak{I}_{(\bar{I}_2 \Cap \bar{I}_3)}(\rho), \\ \eth_{(\bar{I}_1 \Cap \bar{I}_3)}(\rho) &= \max(0, \eth_{\bar{I}_1}(\rho) + \eth_{\bar{I}_3}(\rho) - 2\pi) \geqslant \max(0, \eth_{\bar{I}_2}(\rho) + \eth_{\bar{I}_3}(\rho) - 2\pi) = \eth_{(\bar{I}_2 \Cap \bar{I}_3)}(\rho). \end{aligned}$$

(d).

$$\begin{aligned} \mu_{\bar{I}_1 \Cap (\bar{I}_2 \Cap \bar{I}_3)}(\rho) &= \left\{ \begin{array}{c} \max\{0, \chi_{\bar{I}_1}(\rho) + \max(0, \chi_{\bar{I}_2}(\rho) + \chi_{\bar{I}_3}(\rho) - 1) - 1\} \\ e^{i \max\{0, \mathfrak{I}_{\bar{I}_1}(\rho) + \max(0, \mathfrak{I}_{\bar{I}_2}(\rho) + \mathfrak{I}_{\bar{I}_3}(\rho) - 2\pi) - 2\pi\}}, \\ \max\{0, \hbar_{\bar{I}_1}(\rho) + \max(0, \hbar_{\bar{I}_2}(\rho) + \hbar_{\bar{I}_3}(\rho) - 1) - 1\} \\ e^{i \max\{0, \eth_{\bar{I}_1}(\rho) + \min(0, \eth_{\bar{I}_2}(\rho) + \eth_{\bar{I}_3}(\rho) - 2\pi) - 2\pi\}} \end{array} \right\} \\ &= \left\{ \begin{array}{c} \max\{0, \chi_{\bar{I}_1}(\rho) + \chi_{\bar{I}_2}(\rho) + \chi_{\bar{I}_3}(\rho) - 2\} \\ e^{i \max\{0, \mathfrak{I}_{\bar{I}_1}(\rho) + \mathfrak{I}_{\bar{I}_2}(\rho) + \mathfrak{I}_{\bar{I}_3}(\rho) - 4\pi\}}, \\ \max\{0, \hbar_{\bar{I}_1}(\rho) + \hbar_{\bar{I}_2}(\rho) + \hbar_{\bar{I}_3}(\rho) - 2\} \\ e^{i \max\{0, \eth_{\bar{I}_1}(\rho) + \eth_{\bar{I}_2}(\rho) + \eth_{\bar{I}_3}(\rho) - 4\pi)\}} \end{array} \right\} \\ &= \left\{ \begin{array}{c} \max\{0, \max(0, \chi_{\bar{I}_1}(\rho) + \chi_{\bar{I}_2}(\rho) - 1) + \chi_{\bar{I}_3}(\rho) - 1\} \\ e^{i \max\{0, \max(0, \mathfrak{I}_{\bar{I}_1}(\rho) + \mathfrak{I}_{\bar{I}_2}(\rho) - 2\pi) + \mathfrak{I}_{\bar{I}_3}(\rho) - 2\pi\}}, \\ \max\{1, \max(0, \hbar_{\bar{I}_1}(\rho) + \hbar_{\bar{I}_2}(\rho) - 1) + \hbar_{\bar{I}_3}(\rho) - 1\} \\ e^{i \max\{0, \min(0, \eth_{\bar{I}_1}(\rho) + \eth_{\bar{I}_2}(\rho) - 2\pi) + \eth_{\bar{I}_3}(\rho) - 2\pi)\}} \end{array} \right\} = \mu_{(\bar{I}_1 \Cap \bar{I}_2) \Cap \bar{I}_3}(\rho). \end{aligned}$$

□

AppendixG.

Proof of Theorem 2.31. Properties (i) and (ii) can be easily verify. Here we only prove (iii).

(iii). Let $\bar{I}_1$, $\bar{I}_2$, and $\bar{I}_3$ be three CIFSs on U,

$$\begin{aligned} \mu_{\bar{I}_1} &= \left\{ \left\langle \rho, \chi_{\bar{I}_1}(\rho) e^{i \mathfrak{I}_{\bar{I}_1}(\rho)}, \hbar_{\bar{I}_1}(\rho) e^{i \eth_{\bar{I}_1}(\rho)} \right\rangle : \rho \in \Bbbk \right\}, \\ \mu_{\bar{I}_2} &= \left\{ \left\langle \rho, \chi_{\bar{I}_2}(\rho) e^{i \mathfrak{I}_{\bar{I}_2}(\rho)}, \hbar_{\bar{I}_2}(\rho) e^{i \eth_{\bar{I}_2}(\rho)} \right\rangle : \rho \in \Bbbk \right\} \end{aligned}$$

and

$$\mu_{\bar{I}_3}(\rho) = \left\langle \chi_{\bar{I}_3}(\rho) e^{i \mathfrak{I}_{\bar{I}_3}(\rho)}, \hbar_{\bar{I}_3}(\rho) e^{i \eth_{\bar{I}_3}(\rho)} \right\rangle,$$

be their grade values. Then,

$$\begin{aligned} \Xi(\bar{I}_1, \bar{I}_3) &= \max \left\{ \begin{array}{c} \sup\limits_{\rho \in \Bbbk} \left|\chi_{\bar{I}_1}(\rho) - \chi_{\bar{I}_3}(\rho)\right|, \frac{1}{2\pi} \sup\limits_{\rho \in \Bbbk} \left|\mathfrak{I}_{\bar{I}_1}(\rho) - \mathfrak{I}_{\bar{I}_3}(\rho)\right|, \\ \sup\limits_{\rho \in \Bbbk} \left|\hbar_{\bar{I}_1}(\rho) - \hbar_{\bar{I}_3}(\rho)\right|, \frac{1}{2\pi} \sup\limits_{\rho \in \Bbbk} \left|\eth_{\bar{I}_1}(\rho) - \eth_{\bar{I}_3}(\rho)\right| \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} \sup\limits_{\rho \in \Bbbk} |\chi_{\bar{I}_1}(\rho) - \chi_{\bar{I}_2}(\rho) + \chi_{\bar{I}_2}(\rho) - \chi_{\bar{I}_3}(\rho)|, \\ \frac{1}{2\pi} \sup\limits_{\rho \in \Bbbk} |\mathfrak{I}_{\bar{I}_1}(\rho) - \mathfrak{I}_{\bar{I}_2}(\rho) + \mathfrak{I}_{\bar{I}_2}(\rho) - \mathfrak{I}_{\bar{I}_3}(\rho)|, \\ \sup\limits_{\rho \in \Bbbk} |\hbar_{\bar{I}_1}(\rho) - \hbar_{\bar{I}_2}(\rho) + \hbar_{\bar{I}_2}(\rho) - \hbar_{\bar{I}_3}(\rho)|, \\ \frac{1}{2\pi} \sup\limits_{\rho \in \Bbbk} |\eth_{\bar{I}_1}(\rho) - \eth_{\bar{I}_2}(\rho) + \eth_{\bar{I}_2}(\rho) - \eth_{\bar{I}_3}(\rho)| \end{array} \right\} \end{aligned}$$

$$\leqslant \max\left\{\begin{array}{l}\sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_1}(\rho)-\chi_{\bar{I}_2}(\rho)\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_1}(\rho)-\Im_{\bar{I}_2}(\rho)\right|,\\ \sup\limits_{\ell\in\Bbbk}\left|\hbar_{\bar{I}_1}(\rho)-\hbar_{\bar{I}_2}(\rho)|\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\eth_{\bar{I}_1}(\rho)-\eth_{\bar{I}_2}(\rho)\right|\end{array}\right\}$$
$$+\max\left\{\begin{array}{l}\sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_2}(\rho)-\chi_{\bar{I}_3}(\rho)\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_2}(\rho)-\Im_{\bar{I}_3}(\rho)\right|,\\ \sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_2}(\rho)-\hbar_{\bar{I}_3}(\rho)\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\eth_{\bar{I}_2}(\rho)-\eth_{\bar{I}_3}(\rho)\right|\end{array}\right\} = \Xi(\bar{I}_1,\bar{I}_2)+\Xi(\bar{I}_2,\bar{I}_3).$$

□

AppendixH.

Proof of proposition 2.34. Properties (1)-(4) can be proved easily from the definition. Here we have to prove only properties (5) and (6).

(5). Since $\forall\alpha\in P$, $\bar{I}_1 = (\varpi_\alpha)\,\bar{I}_2$, we have

$$\Xi(\bar{I}_1,\bar{I}_2) = \max\left\{\begin{array}{l}\sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_1}(\rho)-\chi_{\bar{I}_2}(\rho)\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_1}(\rho)-\Im_{\bar{I}_2}(\rho)\right|,\\ \sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_1}(\rho)-\hbar_{\bar{I}_2}(\rho)\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\eth_{\bar{I}_1}(\rho)-\eth_{\bar{I}_2}(\rho)\right|\end{array}\right\} \leqslant 1-\varpi_\alpha,\ \forall\alpha\in P.$$

Therefore,

$$\sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_1}(\rho)-\chi_{\bar{I}_2}(\rho)\right| \leqslant 1-\sup\limits_{\rho\in\Bbbk}\varpi_\alpha, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_1}(\rho)-\Im_{\bar{I}_2}(\rho)\right| \leqslant 1-\sup\limits_{\rho\in\Bbbk}\varpi_\alpha,$$

and

$$\sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_1}(\rho)-\hbar_{\bar{I}_2}(\rho)\right| \leqslant 1-\sup\limits_{\rho\in\Bbbk}\varpi_\alpha, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\eth_{\bar{I}_1}(\rho)-\eth_{\bar{I}_2}(\rho)\right| \leqslant 1-\sup\limits_{\rho\in\Bbbk}\varpi_\alpha.$$

So,

$$\Xi(\bar{I}_1,\bar{I}_2) = \max\left\{\begin{array}{l}\sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_1}(\rho)-\chi_{\bar{I}_2}(\rho)\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_1}(\rho)-\Im_{\bar{I}_2}(\rho)\right|,\\ \sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_1}(\rho)-\hbar_{\bar{I}_2}(\rho)\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\eth_{\bar{I}_1}(\rho)-\eth_{\bar{I}_2}(\rho)\right|\end{array}\right\} \leqslant 1-\sup\limits_{\rho\in\Bbbk}\varpi_\alpha, \forall\alpha\in P.$$

Thus $A = (\sup\limits_{\rho\in\Bbbk}\varpi_\alpha)\bar{I}_1$.

(6). Let $\varpi = 1-\Xi(\bar{I}_1,\bar{I}_2)$. Then $\bar{I}_1 = (\hbar)\bar{I}_1$, we have $1-\varpi = \Xi(\bar{I}_1,\bar{I}_2) \leqslant 1-\varpi'$, consequently $\varpi' \leqslant \varpi$. Now we are supposing there two constants ϖ_1 and ϖ_2, which satisfy the required properties, then $\varpi_1 \leqslant \varpi_2$ and $\varpi_2 \leqslant \varpi_1$. This implies $\varpi_1 = \varpi_2$. So, the desired ϖ is unique. □

AppendixI.

Proof of proposition 2.35. Since $\bar{I}_1 = (\varpi_1)\bar{I}_2$ and $\bar{I}_2 = (\varpi_2)\bar{I}_3$, we have

$$\Xi(\bar{I}_1,\bar{I}_2) = \max\left\{\begin{array}{l}\sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_1}(\rho)-\chi_{\bar{I}_2}(\rho)\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_1}(\rho)-\Im_{\bar{I}_2}(\rho)\right|,\\ \sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_1}(\rho)-\hbar_{\bar{I}_2}(\rho)\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\eth_{\bar{I}_1}(\rho)-\eth_{\bar{I}_2}(\rho)\right|\end{array}\right\} \leqslant 1-\varpi_1$$

and

$$\Xi(\bar{I}_2,\bar{I}_3) = \max\left\{\begin{array}{l}\sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_2}(\rho)-\chi_{\bar{I}_3}(\rho)|\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_2}(\rho)-\Im_{\bar{I}_3}(\rho)\right|,\\ \sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_2}(\rho)-\hbar_{\bar{I}_3}(\rho)\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\eth_{\bar{I}_2}(\rho)-\eth_{\bar{I}_3}(\rho)\right|\end{array}\right\} \leqslant 1-\varpi_2,$$

therefore

$$\sup_{\rho\in\Bbbk}|\chi_{\bar{I}_1}(\rho)-\chi_{\bar{I}_3}(\rho)|\leqslant 1-\varpi_1, \qquad \sup_{\rho\in\Bbbk}|\hbar_{\bar{I}_1}(\rho)-\hbar_{\bar{I}_3}(\rho)|\leqslant 1-\varpi_1,$$

$$\frac{1}{2\pi}\sup_{\rho\in\Bbbk}|\Im_{\bar{I}_1}(\rho)-\Im_{\bar{I}_3}(\rho)|\leqslant 1-\varpi_1, \qquad \frac{1}{2\pi}\sup_{\rho\in\Bbbk}|\eth_{\bar{I}_1}(\rho)-\eth_{\bar{I}_3}(\rho)|\leqslant 1-\varpi_1,$$

and

$$\sup_{\rho\in\Bbbk}\left|\chi_{\bar{I}_2}(\rho)-\chi_{\bar{I}_3}(\rho)\right|\leqslant 1-\varpi_2, \qquad \sup_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_2}(\rho)-\hbar_{\bar{I}_3}(\rho)\right|\leqslant 1-\varpi_2,$$

$$\frac{1}{2\pi}\sup_{\rho\in\Bbbk}\left|\Im_{\bar{I}_2}(\rho)-\Im_{\bar{I}_3}(\rho)\right|\leqslant 1-\varpi_2, \qquad \frac{1}{2\pi}\sup_{\rho\in\Bbbk}\left|\eth_{\bar{I}_2}(\rho)-\eth_{\bar{I}_3}(\rho)\right|\leqslant 1-\varpi_2.$$

Consequently we have

$$\begin{aligned}
\Xi(\bar{I}_1,\bar{I}_3) &= \max\left\{\begin{array}{c}\sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_1}(\rho)-\chi_{\bar{I}_3}(\rho)\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_1}(\rho)-\Im_{\bar{I}_3}(\rho)\right|,\\ \sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_1}(\rho)-\hbar_{\bar{I}_3}(\rho)\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\eth_{\bar{I}_1}(\rho)-\eth_{\bar{I}_3}(\rho)\right|\end{array}\right\}\\
&= \max\left\{\begin{array}{c}\sup\limits_{\rho\in\Bbbk}|\chi_{\bar{I}_1}(\rho)-\chi_{\bar{I}_2}(\rho)+\chi_{\bar{I}_2}(\rho)-\chi_{\bar{I}_3}(\rho)|,\\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}|\Im_{\bar{I}_1}(\rho)-\Im_{\bar{I}_2}(\rho)+\Im_{\bar{I}_2}(\rho)-\Im_{\bar{I}_3}(\rho)|,\\ \sup\limits_{\rho\in\Bbbk}|\hbar_{\bar{I}_1}(\rho)-\hbar_{\bar{I}_2}(\rho)+\hbar_{\bar{I}_2}(\rho)-\hbar_{\bar{I}_3}(\rho)|,\\ \frac{1}{2\pi}\sup\limits_{\ell\in\Bbbk}|\eth_{\bar{I}_1}(\rho)-\eth_{\bar{I}_2}(\rho)+\eth_{\bar{I}_2}(\rho)-\eth_{\bar{I}_3}(\rho)|\end{array}\right\},\\
\Xi(\bar{I}_1,\bar{I}_3) &\leqslant \max\left\{\begin{array}{c}\sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_1}(\rho)-\chi_{\bar{I}_2}(\rho)\right|+\sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_2}(\ell)-\chi_{\bar{I}_3}(\rho)\right|,\\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}|\Im_{\bar{I}_1}(\rho)-\Im_{\bar{I}_2}(\rho)|+\frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_2}(\rho)-\Im_{\bar{I}_3}(\rho)\right|,\\ \sup\limits_{\rho\in\Bbbk}|\hbar_{\bar{I}_1}(\rho)-\hbar_{\bar{I}_2}(\rho)|+\sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_2}(\rho)-\hbar_{\bar{I}_3}(\rho)\right|,\\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}|\eth_{\bar{I}_1}(\ell)-\eth_{\bar{I}_2}(\rho)|+\frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\eth_{\bar{I}_2}(\rho)-\eth_{\bar{I}_3}(\rho)\right|\end{array}\right\}\\
&= \max\left\{\begin{array}{c}(1-\varpi_1)(1-\varpi_2),(1-\varpi_1)(1-\varpi_2),\\ (1-\varpi_1)(1-\varpi_2),(1-\varpi_1)(1-\varpi_2)\end{array}\right\} = (1-\varpi_1)(1-\varpi_2) = 1-(\varpi_1+\varpi_2-1),
\end{aligned}$$

and distance $\Xi(\bar{I}_1,\bar{I}_3)\leqslant 1$ from the definition. Therefore

$$\Xi(\bar{I}_1,\bar{I}_3)\leqslant 1-\varpi_1*\varpi_2=1+\varpi.$$

Thus $\bar{I}_1=(\varpi)\bar{I}_3$. □

AppendixJ.

Proof of Theorem 2.36. This is because

$$\Xi(\bar{I}_1^c,\bar{I}_2^c)=\max\left\{\begin{array}{c}\sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_2^c}(\rho)-\chi_{\bar{I}_1^c}(\rho)\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_2^c}(\rho)-\Im_{\bar{I}_1^c}(\rho)\right|,\\ \sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_2^c}(\rho)-\hbar_{\bar{I}_1^c}(\rho)\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\eth_{\bar{I}_2^c}(\rho)-\eth_{\bar{I}_1^c}(\rho)\right|\end{array}\right\},$$

$$\Xi(\bar{I}_1^c,\bar{I}_2^c)=\max\left\{\begin{array}{c}\sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_1^c}(\rho)-\chi_{\bar{I}_2^c}(\rho)\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_1^c}(\rho)-\Im_{\bar{I}_2^c}(\rho)\right|,\\ \sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_1^c}(\rho)-\hbar_{\bar{I}_2^c}(\rho)\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\eth_{\bar{I}_1^c}(\rho)-\eth_{\bar{I}_2^c}(\ell)\right|\end{array}\right\}\leqslant 1-\varpi.$$

□

AppendixK.

Proof of Theorem 2.37. Since $\bar{I}_1 = (\varpi_1)\bar{I}_1^*$ and $\bar{I}_2 = (\varpi_2)\bar{I}_2^*$, we have

$$\Xi(\bar{I}_1, \bar{I}_1^*) = \max\left\{ \begin{array}{l} \sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_1}(\rho) - \chi_{\bar{I}_1^*}(\rho)\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_1}(\rho) - \Im_{\bar{I}_1^*}(\rho)\right|, \\ \sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_1}(\rho) - \hbar_{\bar{I}_1^*}(\rho)\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\eth_{\bar{I}_1}(\rho) - \eth_{\bar{I}_1^*}(\rho)\right| \end{array} \right\} \leqslant 1 - \varpi_1,$$

and

$$\Xi(\bar{I}_2, \bar{I}_2^*) = \max\left\{ \begin{array}{l} \sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_2}(\rho) - \varkappa_{\bar{I}_2^*}(\rho)\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_2}(\rho) - \Im_{\bar{I}_2^*}(\rho)\right|, \\ \sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_2}(\rho) - \hbar_{\bar{I}_2^*}(\rho)\right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\eth_{\bar{I}_2}(\rho) - \eth_{\bar{I}_2^*}(\rho)\right| \end{array} \right\} \leqslant 1 - \varpi_2,$$

therefore

$$\begin{aligned} &\sup_{\rho\in\Bbbk}\left|\chi_{\bar{I}_1}(\rho) - \chi_{\bar{I}_1^*}(\rho)\right| \leqslant 1 - \varpi_1, && \frac{1}{2\pi}\sup_{\rho\in\Bbbk}\left|\Im_{\bar{I}_1}(\rho) - \Im_{\bar{I}_1^*}(\rho)\right| \leqslant 1 - \varpi_1, \\ &\sup_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_1}(\rho) - \hbar_{\bar{I}_1^*}(\rho)\right| \leqslant 1 - \varpi_1, && \frac{1}{2\pi}\sup_{\rho\in\Bbbk}\left|\eth_{\bar{I}_1}(\rho) - \eth_{\bar{I}_1^*}(\rho)\right| \leqslant 1 - \varpi_1, \end{aligned}$$

and

$$\begin{aligned} &\sup_{\rho\in\Bbbk}\left|\chi_{\bar{I}_2}(\rho) - \chi_{\bar{I}_2^*}(\rho)\right| \leqslant 1 - \varpi_2, && \frac{1}{2\pi}\sup_{\rho\in\Bbbk}\left|\Im_{\bar{I}_2}(\rho) - \Im_{\bar{I}_2^*}(\rho)\right| \leqslant 1 - \varpi_2, \\ &\sup_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_2}(\rho) - \hbar_{\bar{I}_2^*}(\rho)\right| \leqslant 1 - \varpi_2, && \frac{1}{2\pi}\sup_{\rho\in\Bbbk}\left|\eth_{\bar{I}_2}(\rho) - \eth_{\bar{I}_2^*}(\rho)\right| \leqslant 1 - \varpi_2, \end{aligned}$$

consequently

$$\begin{aligned} \Xi\left(\bar{I}_1 \boxtimes \bar{I}_2, \bar{I}_1^* \boxtimes \bar{I}_2^*\right) &= \max\left\{ \begin{array}{c} \sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_1\boxtimes\bar{I}_2}(\rho) - \chi_{\bar{I}_1^*\boxtimes\bar{I}_2^*}(\rho)\right|, \\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_1\boxtimes\bar{I}_2}(\rho) - \Im_{\bar{I}_1^*\boxtimes\bar{I}_2^*}(\rho)\right|, \\ \sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_1\boxtimes\bar{I}_2}(\rho) - \hbar_{\bar{I}_1^*\boxtimes\bar{I}_2^*}(\rho)\right|, \\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\eth_{\bar{I}_1\boxtimes\bar{I}_2}(\rho) - \eth_{\bar{I}_1^*\boxtimes\bar{I}_2^*}(\rho)\right| \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} \sup\limits_{\rho\in\Bbbk}\left| \begin{array}{c} \chi_{\bar{I}_1}(\rho).\chi_{\bar{I}_2}(\rho) - \\ \chi_{\bar{I}_1^*}(\rho).\chi_{\bar{I}_2^*}(\rho) \end{array} \right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left| \begin{array}{c} 2\pi\left(\frac{\Im_{\bar{I}_1}(\rho)}{2\pi}.\frac{\Im_{\bar{I}_2}(\rho)}{2\pi}\right) - \\ 2\pi\left(\frac{\Im_{\bar{I}_1^*}(\rho)}{2\pi}.\frac{\Im_{\bar{I}_2^*}(\rho)}{2\pi}\right) \end{array} \right|, \\ \sup\limits_{\rho\in\Bbbk}\left| \begin{array}{c} \hbar_{\bar{I}_1}(\rho).\hbar_{\bar{I}_2}(\rho) - \\ \hbar_{\bar{I}_1^*}(\rho).\hbar_{\bar{I}_2^*}(\rho) \end{array} \right|, \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left| \begin{array}{c} 2\pi\left(\frac{\eth_{\bar{I}_1}(\rho)}{2\pi}.\frac{\eth_{\bar{I}_2}(\rho)}{2\pi}\right) - \\ 2\pi\left(\frac{\eth_{\bar{I}_1^*}(\rho)}{2\pi}.\frac{\eth_{\bar{I}_2^*}(\rho)}{2\pi}\right) \end{array} \right| \end{array} \right\} \end{aligned}$$

$$= \max\left\{ \begin{array}{c} \sup\limits_{\rho\in\Bbbk}\left| \begin{array}{c} \chi_{\bar{I}_1}(\rho).\chi_{\bar{I}_2}(\rho) - \chi_{\bar{I}_1}(\rho.\chi_{\bar{I}_2^*}(\rho) \\ +\chi_{\bar{I}_1}(\rho).\chi_{\bar{I}_2^*}(\rho) - \chi_{\bar{I}_1^*}(\rho).\chi_{\bar{I}_2^*}(\rho) \end{array} \right|, \\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left| \begin{array}{c} \left(\frac{\Im_{\bar{I}_1}(\rho).\Im_{\bar{I}_2}(\rho)}{2\pi}\right) - \left(\frac{\Im_{\bar{I}_1}(\rho).\Im_{\bar{I}_2^*}(\rho)}{2\pi}\right) \\ + \left(\frac{\Im_{\bar{I}_1}(\rho).\Im_{\bar{I}_2^*}(\rho)}{2\pi}\right) - \left(\frac{\Im_{\bar{I}_1^*}(\rho).\Im_{\bar{I}_2^*}(\rho)}{2\pi}\right) \end{array} \right|, \\ \sup\limits_{\rho\in\Bbbk}\left| \begin{array}{c} \hbar_{\bar{I}_1}(\rho).\hbar_{\bar{I}_2}(\rho) - \hbar_{\bar{I}_1}(\rho).\hbar_{\bar{I}_2^*}(\rho) \\ +\hbar_{\bar{I}_1}(\rho).\hbar_{\bar{I}_2^*}(\rho) - \hbar_{\bar{I}_1^*}(\rho).\hbar_{\bar{I}_2^*}(\rho) \end{array} \right|, \\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left| \begin{array}{c} \left(\frac{\eth_{\bar{I}_1}(\rho).\eth_{\bar{I}_2}(\rho)}{2\pi}\right) - \left(\frac{\eth_{\bar{I}_1}(\rho).\eth_{\bar{I}_2^*}(\rho)}{2\pi}\right) \\ + \left(\frac{\eth_{\bar{I}_1}(\rho).\eth_{\bar{I}_2^*}(\rho)}{2\pi}\right) - \left(\frac{\eth_{\bar{I}_1^*}(\rho).\eth_{\bar{I}_2^*}(\rho)}{2\pi}\right) \end{array} \right| \end{array} \right\}$$

$$= \max\left\{ \begin{array}{c} \sup\limits_{\rho\in\Bbbk}\left| \begin{array}{c} \chi_{\bar{I}_1}(\rho)\left(\chi_{\bar{I}_2}(\rho) - \chi_{\bar{I}_2^*}(\rho)\right) \\ + \left(\chi_{\bar{I}_1}(\rho) - \varkappa_{\bar{I}_1^*}(\rho)\right)\chi_{\bar{I}_2^*}(\rho) \end{array} \right|, \\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left| \begin{array}{c} \frac{\Im_{\bar{I}_1}(\rho)}{2\pi}\left(\Im_{\bar{I}_2}(\rho) - \Im_{\bar{I}_2^*}(\rho)\right) \\ +\frac{\Im_{\bar{I}_2^*}(\rho)}{2\pi}\left(\Im_{\bar{I}_1}(\rho) - \Im_{\bar{I}_1^*}(\rho)\right) \end{array} \right|, \\ \sup\limits_{\rho\in\Bbbk}\left| \begin{array}{c} \hbar_{\bar{I}_1}(\rho)\left(\hbar_{\bar{I}_2}(\rho) - \hbar_{\bar{I}_2^*}(\rho)\right) \\ + \left(\hbar_{\bar{I}_1}(\rho) - \hbar_{\bar{I}_1^*}(\rho)\right)\hbar_{\bar{I}_2^*}(\rho) \end{array} \right|, \\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left| \begin{array}{c} \frac{\eth_{\bar{I}_1}(\rho)}{2\pi}\left(\eth_{\bar{I}_2}(\rho) - \eth_{\bar{I}_2^*}(\rho)\right) \\ +\frac{\eth_{\bar{I}_2^*}(\rho)}{2\pi}\left(\eth_{\bar{I}_1}\rho - \eth_{\bar{I}_1^*}(\rho)\right) \end{array} \right| \end{array} \right\}$$

$$\leqslant \max\left\{ \begin{array}{c} \sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_2}(\rho) - \chi_{\bar{I}_2^*}(\rho)\right| + \sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_1}(\rho) - \chi_{\bar{I}_1^*}(\rho)\right|, \\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_2}(\rho) - \Im_{\bar{I}_2^*}(\ell)\right| + \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_1}(\rho) - \Im_{\bar{I}_1^*}(\rho)\right|, \\ \sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_2}(\rho) - \hbar_{\bar{I}_2^*}(\rho)\right| + \sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_1}(\rho) - \hbar_{\bar{I}_1^*}(\rho)\right|, \\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\rho_{\bar{I}_2}(\rho) - \eth_{\bar{I}_2^*}(\rho)\right| + \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\eth_{\bar{I}_1}(\rho) - \eth_{\bar{I}_1^*}(\rho)\right| \end{array} \right\}$$

$$\leqslant \max\left(\begin{array}{c} (1-\varpi_2)(1-\varpi_1), (1-\varpi_2)(1-\varpi_1), \\ (1-\varpi_2)(1-\varpi_1), (1-\varpi_2)(1-\varpi_1) \end{array} \right) = 1 - (\varpi_1 + \varpi_2 - 1).$$

We note that $\Xi\left(\bar{I}_1 \boxtimes \bar{I}_2, \bar{I}_1^* \boxtimes \bar{I}_2^*\right) \leqslant 1$, so $\Xi\left(\bar{I}_1 \boxtimes \bar{I}_2, \bar{I}_1^* \boxtimes \bar{I}_2^*\right) \leqslant 1 - \varpi_1 * \varpi_2$. □

AppendixL.

Proof of Theorem 2.38. Since $\bar{I}_1 = (\varpi_1)\bar{I}_1^*$ and $\bar{I}_2 = (\varpi_2)\bar{I}_2^*$, we have

$$\Xi(\bar{I}_1, \bar{I}_1^*) = \max\left\{ \begin{array}{c} \sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_1}(\rho) - \chi_{\bar{I}_1^*}(\rho)\right|, \\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_1}(\rho) - \Im_{\bar{I}_1^*}(\rho)\right|, \\ \sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_1}(\rho) - \hbar_{\bar{I}_1^*}(\rho)\right|, \\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\eth_{\bar{I}_1}(\rho) - \eth_{\bar{I}_1^*}(\rho)\right| \end{array} \right\} \leqslant 1 - \varpi_1$$

and

$$\Xi(\bar{I}_2,\bar{I}_2^*)=\max\left\{\begin{array}{c}\sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_2}(\rho)-\chi_{\bar{I}_2^*}(\rho)\right|,\frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_2}(\rho)-\Im_{\bar{I}_2^*}(\rho)\right|,\\ \sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_2}(\rho)-\hbar_{\bar{I}_2^*}(\rho)\right|,\frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\eth_{\bar{I}_2}(\rho)-\eth_{\bar{I}_2^*}(\rho)\right|\end{array}\right\}\leqslant 1-\varpi_2,$$

therefore

$$\begin{aligned}&\sup_{\rho\in\Bbbk}\left|\chi_{\bar{I}_1}(\rho)-\chi_{\bar{I}_1^*}(\rho)\right|\leqslant 1-\varpi_1, &&\frac{1}{2\pi}\sup_{\rho\in\Bbbk}\left|\Im_{\bar{I}_1}(\rho)-\Im_{\bar{I}_1^*}(\rho)\right|\leqslant 1-\varpi_1,\\ &\sup_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_1}(\rho)-\hbar_{\bar{I}_1^*}(\rho)\right|\leqslant 1-\varpi_1, &&\frac{1}{2\pi}\sup_{\rho\in\Bbbk}\left|\eth_{\bar{I}_1}(\rho)-\eth_{\bar{I}_1^*}(\rho)\right|\leqslant 1-\varpi_1,\end{aligned}$$

and

$$\begin{aligned}&\sup_{\rho\in\Bbbk}\left|\chi_{\bar{I}_2}(\rho)-\chi_{\bar{I}_2^*}(\rho)\right|\leqslant 1-\varpi_2, &&\frac{1}{2\pi}\sup_{\rho\in\Bbbk}\left|\Im_{\bar{I}_2}(\rho)-\Im_{\bar{I}_2^*}(\rho)\right|\leqslant 1-\varpi_2,\\ &\sup_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_2}(\rho)-\hbar_{\bar{I}_2^*}(\rho)\right|\leqslant 1-\varpi_2, &&\frac{1}{2\pi}\sup_{\rho\in\Bbbk}\left|\eth_{\bar{I}_2}(\rho)-\eth_{\bar{I}_2^*}(\rho)\right|\leqslant 1-\varpi_2,\end{aligned}$$

consequently

$$\begin{aligned}\Xi\left(\bar{I}_1\dot{+}\bar{I}_2,\bar{I}_1^*\dot{+}\bar{I}_2^*\right)&=\max\left\{\begin{array}{c}\sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_1\dot{+}\bar{I}_2}(\rho)-\chi_{\bar{I}_1^*\dot{+}\bar{I}_2^*}(\rho)\right|,\\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_1\dot{+}\bar{I}_2}(\rho)-\Im_{\bar{I}_1^*\dot{+}\bar{I}_2^*}(\rho)\right|,\\ \sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_1\dot{+}\bar{I}_2}(\rho)-\hbar_{\bar{I}_1^*\dot{+}\bar{I}_2^*}(\rho)\right|,\\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\eth_{\bar{I}_1\dot{+}\bar{I}_2}(\rho)-\eth_{\bar{I}_1^*\dot{+}\bar{I}_2^*}(\rho)\right|\end{array}\right\}\\ &=\max\left\{\begin{array}{c}\sup\limits_{\rho\in\Bbbk}\left|\begin{array}{c}\left(\frac{\chi_{\bar{I}_1}(\rho)+\chi_{\bar{I}_2}(\rho)-\chi_{\bar{I}_1}(\rho).\chi_{\bar{I}_2}(\rho)}{2}\right)\\ -\left(\frac{\chi_{\bar{I}_1^*}(\rho)+\chi_{\bar{I}_2^*}(\rho)-\chi_{\bar{I}_1^*}(\rho).\chi_{\bar{I}_2^*}(\rho)}{2}\right)\end{array}\right|,\\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\begin{array}{c}2\pi\left(\frac{\Im_{\bar{I}_1}(\rho)+\Im_{\bar{I}_2}(\rho)-\Im_{\bar{I}_1}(\rho).\Im_{\bar{I}_2}(\rho)}{2\pi}\right)\\ -2\pi\left(\frac{\Im_{\bar{I}_1^*}(\rho)+\Im_{\bar{I}_2^*}(\rho)-\Im_{\bar{I}_1^*}(\rho).\Im_{\bar{I}_2^*}(\rho)}{2\pi}\right)\end{array}\right|,\\ \sup\limits_{\rho\in\Bbbk}\left|\begin{array}{c}\left(\frac{\hbar_{\bar{I}_1}(\rho)+\hbar_{\bar{I}_2}(\rho)-\hbar_{\bar{I}_1}(\rho).\hbar_{\bar{I}_2}(\rho)}{2}\right)\\ -\left(\frac{\hbar_{\bar{I}_1^*}(\rho)+\hbar_{\bar{I}_2^*}(\rho)-\hbar_{\bar{I}_1^*}(\rho).\hbar_{\bar{I}_2^*}(\rho)}{2}\right)\end{array}\right|,\\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\begin{array}{c}\pi\left(\frac{\eth_{\bar{I}_1}(\rho)+\eth_{\bar{I}_2}(\rho)-\eth_{\bar{I}_1}(\rho).\eth_{\bar{I}_2}(\rho)}{2\pi}\right)\\ -2\pi\left(\frac{\eth_{\bar{I}_1^*}(\rho)+\eth_{\bar{I}_2^*}(\rho)-\eth_{\bar{I}_1^*}(\rho).\eth_{\bar{I}_2^*}(\rho)}{2\pi}\right)\end{array}\right|\end{array}\right\}\end{aligned}$$

$$= \max \left\{ \begin{array}{c} \sup\limits_{\rho \in \Bbbk} \left| \begin{array}{c} \chi_{\bar{I}_1}(\rho) + \chi_{\bar{I}_2}(\rho) - \chi_{\bar{I}_1}(\rho).\chi_{\bar{I}_2}(\rho) - \\ \chi_{\bar{I}_1}(\rho).\chi_{\bar{I}_2^*}(\rho) \\ +\chi_{\bar{I}_1}(\rho).\chi_{\bar{I}_2^*}(\rho) - \chi_{\bar{I}_1^*}(\rho) - \chi_{\bar{I}_2^*}(\rho) + \\ \chi_{\bar{I}_1^*}(\rho).\chi_{\bar{I}_2^*}(\rho) \end{array} \right|, \\ \frac{1}{2\pi} \sup\limits_{\rho \in \Bbbk} \left| \begin{array}{c} \Im_{\bar{I}_1}(\rho) + \Im_{\bar{I}_2}(\rho) - \Im_{\bar{I}_1}(\ell).\Im_{\bar{I}_2}(\rho) + \\ \Im_{\bar{I}_1}(\rho).\Im_{\bar{I}_2^*}(\rho) \\ -\Im_{\bar{I}_1}(\rho).\Im_{\bar{I}_2^*}(\rho) - \Im_{\bar{I}_1^*}(\rho) - \Im_{\bar{I}_2^*}(\rho) + \\ \Im_{\bar{I}_1^*}(\rho).\Im_{\bar{I}_2^*}(\rho) \end{array} \right|, \\ \sup\limits_{\rho \in \Bbbk} \left| \frac{1}{2} \left\{ \begin{array}{c} \left(\hbar_{\bar{I}_1}(\rho) + \hbar_{\bar{I}_2}(\rho) - \hbar_{\bar{I}_1}(\rho).\hbar_{\bar{I}_2}(\rho) \right) \\ -\hbar_{\bar{I}_1}(\rho).\hbar_{\bar{I}_2^*}(\rho) \\ +\hbar_{\bar{I}_1}(\rho).\hbar_{\bar{I}_2^*}(\rho) - \hbar_{\bar{I}_1^*}(\rho) - \hbar_{\bar{I}_2^*}(\rho) + \\ \hbar_{\bar{I}_1^*}(\rho).\hbar_{\bar{I}_2^*}(\rho) \end{array} \right\} \right|, \\ \frac{1}{2\pi} \sup\limits_{\rho \in \Bbbk} \left| \begin{array}{c} \left(\eth_{\bar{I}_1}(\rho) + \eth_{\bar{I}_2}(\rho) - \eth_{\bar{I}_1}(\rho).\eth_{\bar{I}_2}(\rho) \right) + \\ \eth_{\bar{I}_1}(\rho).\eth_{\bar{I}_2^*}(\rho) \\ -\eth_{\bar{I}_1}(\rho).\eth_{\bar{I}_2^*}(\rho) - \eth_{\bar{I}_1^*}(\rho) - \eth_{\bar{I}_2^*}(\rho) + \\ \eth_{\bar{I}_1^*}(\rho).\eth_{\bar{I}_2^*}(\rho) \end{array} \right| \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} \frac{1}{2} \sup\limits_{\rho \in \Bbbk} \left| \begin{array}{c} \chi_{\bar{I}_1}(\rho) \left(1 - \chi_{\bar{I}_2^*}(\rho)\right) + \chi_{\bar{I}_2}(\rho) \left(1 - \chi_{\bar{I}_1}(\rho)\right) \\ -\chi_{\bar{I}_2^*}(\rho) \left(1 - \chi_{\bar{I}_1}(\rho)\right) - \chi_{\bar{I}_1^*}(\rho) \left(1 - \chi_{\bar{I}_2^*}(\rho)\right) \end{array} \right|, \\ \frac{1}{2\pi} \sup\limits_{\rho \in \Bbbk} \left| \begin{array}{c} \Im_{\bar{I}_1}(\rho) \left(1 - \Im_{\bar{I}_2^*}(\rho)\right) + \Im_{\bar{I}_2}(\rho) \left(1 - \Im_{\bar{I}_1}(\ell)\right) \\ -\Im_{\bar{I}_2^*}(\rho) \left(1 - \Im_{\bar{I}_1}(\rho)\right) - \Im_{\bar{I}_1^*}(\rho) \left(1 - \Im_{\bar{I}_2^*}(\rho)\right) \end{array} \right|, \\ \frac{1}{2} \sup\limits_{\rho \in \Bbbk} \left| \begin{array}{c} \hbar_{\bar{I}_1}(\rho) \left(1 - \hbar_{\bar{I}_2^*}(\rho)\right) + \hbar_{\bar{I}_2}(\rho) \left(1 - \hbar_{\bar{I}_1}(\rho)\right) \\ +\hbar_{\bar{I}_2^*}(\rho) \left(1 - \hbar_{\bar{I}_1}(\rho)\right) - \hbar_{\bar{I}_1^*}(\rho) \left(1 - \hbar_{\bar{I}_2^*}(\rho)\right) \end{array} \right|, \\ \frac{1}{2\pi} \sup\limits_{\rho \in \Bbbk} \left| \begin{array}{c} \eth_{\bar{I}_1}(\rho) \left(1 - \eth_{\bar{I}_2^*}(\rho)\right) + \eth_{\bar{I}_2}(\rho) \left(1 - \eth_{\bar{I}_1}(\rho)\right) \\ -\eth_{\bar{I}_2^*}(\rho) \left(1 - \eth_{\bar{I}_1}(\rho)\right) - \eth_{\bar{I}_1^*}(\rho) \left(1 - \eth_{\bar{I}_2^*}(\rho)\right) \end{array} \right| \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} \frac{1}{2} \sup\limits_{\rho \in \Bbbk} \left| \begin{array}{c} \left(1 - \chi_{\bar{I}_2^*}(\rho)\right) \left(\chi_{\bar{I}_1}(\rho) - \chi_{\bar{I}_1^*}(\rho)\right) \\ + \left(1 - \chi_{\bar{I}_1}(\rho)\right) \left(\chi_{\bar{I}_2}(\rho) - \chi_{\bar{I}_2^*}(\rho)\right) \end{array} \right|, \\ \frac{1}{2\pi} \sup\limits_{\rho \in \Bbbk} \left| \begin{array}{c} \left(1 - \Im_{\bar{I}_2^*}(\rho)\right) \left(\Im_{\bar{I}_1}(\rho) - \Im_{\bar{I}_1^*}(\rho)\right) \\ + \left(1 - \Im_{\bar{I}_1}(\rho)\right) \left(\Im_{\bar{I}_2}(\rho) - \Im_{\bar{I}_2^*}(\rho)\right) \end{array} \right|, \\ \frac{1}{2} \sup\limits_{\rho \in \Bbbk} \left| \begin{array}{c} \left(1 - \hbar_{\bar{I}_2^*}(\rho)\right) \left(\hbar_{\bar{I}_1}(\rho) - \hbar_{\bar{I}_1^*}(\rho)\right) \\ + \left(1 - \hbar_{\bar{I}_1}(\rho)\right) \left(\hbar_{\bar{I}_2}(\rho) - \hbar_{\bar{I}_2^*}(\rho)\right) \end{array} \right|, \\ \frac{1}{2\pi} \sup\limits_{\rho \in \Bbbk} \left| \begin{array}{c} \left(1 - \eth_{\bar{I}_2^*}(\rho)\right) \left(\eth_{\bar{I}_1}(\rho) - \eth_{\bar{I}_1^*}(\rho)\right) \\ + \left(1 - \eth_{\bar{I}_1}(\rho)\right) \left(\eth_{\bar{I}_2}(\rho) - \eth_{\bar{I}_2^*}(\rho)\right) \end{array} \right| \end{array} \right\}$$

$$\leqslant \max \left\{ \begin{array}{c} \frac{1}{2}\sup\limits_{\rho\in\Bbbk}\left|\left(1-\chi_{\bar{I}_2^*}(\rho)\right)\left(\chi_{\bar{I}_1}(\rho)-\chi_{\bar{I}_1^*}(\rho)\right)\right| \\ +\frac{1}{2}\sup\limits_{\rho\in\Bbbk}\left|\left(1-\chi_{\bar{I}_1}(\rho)\right)\left(\chi_{\bar{I}_2}(\ell)-\chi_{\bar{I}_2^*}(\rho)\right)\right|, \\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\left(1-\Im_{\bar{I}_2^*}(\rho)\right)\left(\Im_{\bar{I}_1}(\rho)-\Im_{\bar{I}_1^*}(\rho)\right)\right| \\ +\frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\left(1-\Im_{\bar{I}_1}(\rho)\right)\left(\Im_{\bar{I}_2}(\rho)-\Im_{\bar{I}_2^*}(\rho)\right)\right|, \\ \frac{1}{2}\sup\limits_{\rho\in\Bbbk}\left|\left(1-\hbar_{\bar{I}_2^*}(\rho)\right)\left(\hbar_{\bar{I}_1}(\rho)-\hbar_{\bar{I}_1^*}(\rho)\right)\right| \\ +\frac{1}{2}\sup\limits_{\rho\in\Bbbk}\left|\left(1-\hbar_{\bar{I}_1}(\rho)\right)\left(\hbar_{\bar{I}_2}(\rho)-\hbar_{\bar{I}_2^*}(\rho)\right)\right|, \\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\left(1-\eth_{\bar{I}_2^*}(\rho)\right)\left(\eth_{\bar{I}_1}(\rho)-\eth_{\bar{I}_1^*}(\rho)\right)\right| \\ +\frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\left(1-\eth_{\bar{I}_1}(\rho)\right)\left(\eth_{\bar{I}_2}(\rho)-\eth_{\bar{I}_2^*}(\rho)\right)\right| \end{array} \right\}$$

$$\leqslant \max \left\{ \begin{array}{c} \sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_1}(\rho)-\chi_{\bar{I}_1^*}(\rho)\right| + \sup\limits_{\rho\in\Bbbk}\left|\chi_{\bar{I}_2}(\rho)-\chi_{\bar{I}_2^*}(\rho)\right|, \\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_1}(\rho)-\Im_{\bar{I}_1^*}(\rho)\right| + \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\Im_{\bar{I}_2}(\rho)-\Im_{\bar{I}_2^*}(\rho)\right|, \\ \sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_1}(\rho)-\hbar_{\bar{I}_1^*}(\rho)\right| + \sup\limits_{\rho\in\Bbbk}\left|\hbar_{\bar{I}_2}(\rho)-\hbar_{\bar{I}_2^*}(\rho)\right|, \\ \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\left(\eth_{\bar{I}_1}(\rho)-\eth_{\bar{I}_1^*}(\rho)\right)\right| + \frac{1}{2\pi}\sup\limits_{\rho\in\Bbbk}\left|\left(\eth_{\bar{I}_2}(\rho)-\eth_{\bar{I}_2^*}(\rho)\right)\right| \end{array} \right\},$$

$$\Xi\left(\bar{I}_1 \dotplus \bar{I}_2, \bar{I}_1^* \dotplus \bar{I}_2^*\right) \leqslant \max \left(\begin{array}{c} (1-\varpi_1)+(1-\varpi_2),(1-\varpi_1)+(1-\varpi_2), \\ (1-\varpi_1)+(1-\varpi_2),(1-\varpi_1)+(1-\varpi_2) \end{array} \right) = 1-(\varpi_1+\varpi_2)-1.$$

We note that $\Xi\left(\bar{I}_1 \dotplus \bar{I}_2, \bar{I}_1^* \dotplus \bar{I}_2^*\right) \leqslant 1$. So $\Xi\left(\bar{I}_1 \dotplus \bar{I}_2, \bar{I}_1^* \dotplus \bar{I}_2^*\right) \leqslant 1-\varpi_1 * \varpi_2$. □

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