J. Math. Computer Sci., 38 (2025), 313–329

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Analytical study of existence, uniqueness, and stability in impulsive neutral fractional Volterra-Fredholm equations

Prabakaran Raghavendran[a], Tharmalingam Gunasekar[a,b], Shyam Sundar Santra[c,*], Dumitru Baleanu[d,e], Debasish Majumder[c]

[a]Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology, Chennai - 600062, Tamil Nadu, India.

[b]School of Artifiial Intelligence and Data Science, Indian Institute of Technology (IIT), Jodhpur 342030, India.

[c]Department of Mathematics, JIS College of Engineering, Kalyani, West Bengal 741235, India.

[d]Department of Computer Science and Mathematics, Lebanese American University, Beirut-11022801, Lebanon.

[e]Instiute of Space Sciences-Subsidiary of INFLPR, Magurele-Bucharest, 077125 Magurele, Romania.

Abstract

This investigation focuses on an impulsive Volterra-Fredholm integro-differential equation enriched with fractional Caputo derivatives and subject to specific order conditions. The study establishes both the existence and uniqueness of analytical solutions using the Banach principle. Moreover, it reveals a distinctive outcome regarding the existence of at least one solution, supported by conditions derived from the Krasnoselskii fixed point theorem. Additionally, the paper extends its examination to impulsive neutral Volterra-Fredholm integro-differential equations, providing insights into their long-term behavior through Ulam stability. The inclusion of an illustrative example emphasizes the practical significance and reliability of the results.

Keywords: Caputo fractional derivative, Volterra-Fredholm integro-differential equation (IDE), Arzela-Ascoli theorem, Banach contraction principle, Krasnoselskii fixed point theorem, Ulam stability.

2020 MSC: 26A33, 34A12, 26D10, 26E50, 45G10, 45J05.

1. Introduction

In recent times, fractional calculus and fractional-order differential equations have attracted considerable interest owing to their profound effectiveness in representing and scrutinizing real-world phenomena [29]. This emerging field has witnessed substantial progress, particularly in the study of differential equations with fractional orders, which have proven to be indispensable tools for comprehending and representing various physical processes. Traditionally, integer-order differential equations have served as the cornerstone of mathematical modeling in science and engineering [31, 34, 42]. However, as researchers delved deeper into the intricacies of natural phenomena, it became increasingly evident that

*Corresponding author

Email addresses: rockypraba55@gmail.com (Prabakaran Raghavendran), tguna84@gmail.com or m23air514@iitj.ac.in (Tharmalingam Gunasekar), shyam01.math@gmail.com, shyamsundar.santra@jiscollege.ac.in (Shyam Sundar Santra), dumitru.baleanu@gmail.com (Dumitru Baleanu), debamath@rediffmail.com (Debasish Majumder)

doi: 10.22436/jmcs.038.03.03

Received: 2024-03-21 *Revised:* 2024-04-13 *Accepted:* 2024-07-25

many systems exhibit memory-dependent behaviors that cannot be adequately captured by integer-order derivatives alone [2, 5–7, 10, 24–26, 38]. This insight sparked an investigation into fractional calculus, a mathematical discipline that extends differentiation and integration to non-integer orders. The theory of differential equations of various orders, including fractional orders, has subsequently evolved into an essential framework for representing a diverse range of physical phenomena. The realm of fractional differential equations encompasses a broad spectrum of inquiry, encompassing both analytical and numerical methodologies. In this domain, researchers not only investigate practical computational methods but also delve into essential theoretical aspects. These theoretical facets encompass investigations into the existence, uniqueness, periodicity, and asymptotic behavior of solutions. For a comprehensive view of recent developments in fractional differential equations, valuable insights can be found in the works referenced [8, 12, 14, 29, 36, 40, 43]. In the past few years, fractional IDEs have risen to prominence as powerful instruments for describing intricate phenomena in a multitude of applied sciences and engineering fields. These equations combine differential and integral operators, providing a versatile framework for addressing a diverse range of phenomena. Fractional IDE finds significant utility across various domains, including but not limited to acoustic manipulation, signal analysis, electrochemical processes, viscoelastic materials, polymer behavior, electromagnetism, optics, medical science, economic modeling, chemical engineering, chaotic systems, and statistical physics. This extensive application scope underscores the relevance and versatility of fractional IDE in addressing real-world challenges and advancing our understanding of complex systems. Given the growing importance of these equations in both theoretical and practical contexts, it is imperative to delve deeper into their properties and solutions, which forms the core objective of this study [3, 19–21, 23, 37, 41]. Impulsive differential systems are employed to represent phenomena influenced by brief disturbances that are significantly shorter in duration compared to the overall time frame of the phenomenon. For a deeper understanding of this concept and its practical uses, you can explore the comprehensive works by Lakshmikantham et al. in their monographs [39], as well as see references [27, 28]. Ulam stability in the context of fractional calculus examines the resistance of solutions to fractional functional equations when subjected to small variations. It has gained prominence as a crucial framework for characterizing the long-term behavior of fractional-order systems, particularly in modeling complex phenomena. This area of research is pivotal in understanding the stability and dependability of fractional differential equations in diverse applications, one may find valuable insights in the works presented in references [4, 13, 39]. Columbu et al. [11] refined criteria for boundedness in chemotaxis systems by considering attraction-repulsion dynamics with nonlinear productions, providing insights into the stability of such systems. Li et al. [30] investigated chemotaxis models, this study emphasizes the combination of factors to maintain boundedness, particularly focusing on production and consumption dynamics, offering valuable contributions to understanding system stability.

Li et al. [31] examined properties of solutions to porous medium problems with different sources and boundary conditions. Agarwal et al. [1] provided remarks on oscillation of second order neutral differential equations. Bohner and Li [9] addressed oscillation of second-order p-Laplace dynamic equations with a nonpositive neutral coefficient. Li and Rogovchenko [32] examined the oscillation of second-order neutral differential equations. Zafer [42] presented oscillation criteria for even-order neutral differential equations. Li and Rogovchenko [33] discussed oscillation criteria for second-order superlinear Emden-Fowler neutral differential equations. Li et al. [34] investigated the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations.

With motivation drawn from the work of Hamoud and Ghadle [22] as well as Ndiaye and Mansal [35], in this paper, we investigate the existence and uniqueness of solutions for Impulsive Volterra-Fredholm IDEs. We employ both the Banach fixed-point theorem and the Krasnoselskii fixed-point theorem to address this challenge. Additionally, we explore the concept of Ulam stability for the derived solutions, providing valuable insights into their enduring characteristics. In order to extend the relevance of our findings to diverse systems and phenomena, we broaden our inquiry to include impulsive neutral Volterra-Fredholm IDEs. This expansion is supported by suitable examples in the relevant area, which serve to illustrate the underlying concepts. In conclusion, we provide a summary that is systematically organized and concisely summarizes the key findings of our investigation.

2. Preliminaries

In this section, our focus is on the commonly employed definitions in fractional calculus, specifically the Riemann-Liouville fractional derivative and the Caputo derivative, as discussed in prior academic literature [16, 17]. Let's examine the Banach space $M(\Lambda, \mathbb{R})$, equipped with the infinity norm defined as $|\aleph|_\infty = \sup\{|\aleph(\eta)| : \eta \in \Lambda = [\xi_0, „]\}$, where $\aleph$ is a member of $M(\Lambda, \mathbb{R})$.

Definition 2.1 ([29]). The fractional integral of a function δ under the Riemann-Liouville definition with order $\zeta > 0$ is defined as

$$J^\zeta \delta(\eta) = \frac{1}{\Gamma(\zeta)} \int_0^\eta (\eta - \xi)^{\zeta - 1} \delta(\xi)\, d\xi \quad \text{for } \eta > 0, \zeta \in \mathbb{R}^+,$$

where $\mathbb{R}^+$ represents the set of positive real numbers, and $J^0\delta(\eta) = \delta(\eta)$.

Definition 2.2 ([43]). The Caputo derivative of order ζ, where ζ is within the range 0 to 1, is applicable to a function $\delta : [0, 1) \to \mathbb{R}$ and it can be expressed as

$$D^\zeta \delta(\eta) = \frac{1}{\Gamma(1-\zeta)} \int_0^\eta \frac{\delta^{(0)}(\xi)}{(\eta - \xi)^\zeta}\, d\xi, \quad \eta > 0.$$

Theorem 2.3 (Arzela-Ascoli theorem, [17, 43]). *A sequence of functions that is both bounded and equicontinuous within the closed and bounded interval $[a, b]$ possesses a subsequence converging uniformly.*

Theorem 2.4 (Banach's fixed point theorem, [17, 19, 43]). *Consider a nonempty closed subset Υ in a Banach space φ. Then, for any contraction mapping T from Υ to itself, a unique fixed point exists.*

Theorem 2.5 (Krasnoselskii fixed point theorem, [17, 43]). *In a Banach space φ, let ξ be a nonempty closed and convex subset. Within ξ, there exist two functions $\mathcal{C}$ and $\mathcal{D}$ with the following properties.*

1. *$\mathcal{C}$ is a contraction mapping;*
2. *$\mathcal{D}$ is compact and continuous;*
3. *for all η and ν in ξ, $\mathcal{C}\eta + \mathcal{D}\nu$ remains within ξ.*

Under these conditions, there exists a ν in ξ for which $\mathcal{C}\nu + \mathcal{D}\nu = \nu$.

3. Impulsive Volterra-Fredholm integro-differential equation

In this section, we delve into the exploration of solutions existence and uniqueness, along with Ulam stability results, for impulsive Volterra-Fredholm IDE. Our investigation aims to provide valuable insights into the theoretical foundations of these equations. Through illustrative examples, we will demonstrate the significance of our findings in understanding the reliable behavior of fractional-order systems.

3.1. Existence and uniqueness results

In this subsection, we delve into the Caputo fractional Impulsive Volterra-Fredholm IDE, which is formulated as

$$\begin{aligned} {}^cD^\zeta \aleph(\xi) &= \delta(\xi)\aleph(\xi) + \upsilon(\xi, \aleph(\xi)) + \int_{\xi_0}^{\xi} Z_1(\xi, \vartheta, \aleph(\vartheta))d\vartheta + \int_{\xi_0}^{\theta} Z_2(\xi, \vartheta, \aleph(\vartheta))d\vartheta, \\ \Delta\aleph(\xi_k) &= I_k(\aleph(\xi_k)). \end{aligned} \tag{3.1}$$

This equation is supplemented by the initial condition

$$\aleph(\xi_0) = \aleph_0. \tag{3.2}$$

In the given expressions, ${}^cD^\zeta$ represents Caputo's fractional derivative with $0 < \zeta \leqslant 1$. The function $\aleph : \Lambda \to \mathbb{R}$, where $\Lambda = [\xi_0, \theta]$, denotes the continuous function under consideration. Additionally, $\upsilon : \Lambda \times \mathbb{R} \to \mathbb{R}$ and $Z_n : \Lambda \times \Lambda \times \mathbb{R} \to \mathbb{R}$, where $n = 1, 2$, are continuous functions. Here, $\Delta = \{(\xi, \vartheta) : 0 \leqslant \xi_0 \leqslant \vartheta \leqslant \xi \leqslant \theta\}$, $0 < \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_m < \xi_{m+1} = \theta$, and $\Delta\aleph(\xi_k) = \aleph(\xi_k^+) - \aleph(\xi_k^-)$, where $\aleph(\xi_k^+)$ and $\aleph(\xi_k^-)$ denote the left and right limits of $\aleph$ at ξ_k, respectively.

Prior to exploring our primary results and their proofs, we present a lemma along with following crucial hypotheses.

(A1) Consider continuous functions Z_1 and $Z_2 : \Lambda \times \Lambda \times \mathbb{R} \to \mathbb{R}$, defined on the set $D = (\xi, \vartheta) : 0 \leqslant \xi_0 \leqslant \vartheta \leqslant \xi \leqslant \theta$. They satisfy the conditions

$$|Z_1(\eta, \vartheta, \aleph_1(\vartheta)) - Z_1(\eta, \vartheta, \aleph_2(\vartheta))| \leqslant \chi^*_{z_1} \|\aleph_1(\vartheta) - \aleph_2(\vartheta)\|,$$
$$\text{and}$$
$$|Z_2(\eta, \vartheta, \aleph_1(\vartheta)) - Z_2(\eta, \vartheta, \aleph_2(\vartheta))| \leqslant \chi^*_{z_2} \|\aleph_1(\vartheta) - \aleph_2(\vartheta)\|.$$

(A2) The continuous function $\upsilon : \Lambda \times \mathbb{R} \to \mathbb{R}$ adheres to the condition $|\upsilon(\xi, \aleph_1) - \upsilon(\xi, \aleph_2)| \leqslant \chi^*_\upsilon \|\aleph_1 - \aleph_2\|$. The continuous function $\delta : \Lambda \to \mathbb{R}$ is smooth, and the positive constants $\chi^*_{z_1}$, $\chi^*_{z_2}$, and χ^*_υ are defined.

(A3) There exist constants $\chi^*_1 > 0$ and $\chi^*_2 > 0$ such that $\|I_k(\aleph_1) - I_k(\aleph_2)\| \leqslant \chi^*_1 \|\aleph_1 - \aleph_2\|$ and $\|I_k(\aleph)\| \leqslant \chi^*_2$ for each $\aleph$, $\aleph_1$, $\aleph_2 \in \varphi$ and $k = 1, 2, \ldots, m$.

(A4) There exist constants μ, σ, and ρ in $M(\Lambda, \mathbb{R}^+)$ such that the following conditions hold: $|\upsilon(\xi, \aleph)| \leqslant \mu(\xi)$ for all $(\xi, \aleph) \in \Lambda \times \varphi$, $|Z_1(\xi, \vartheta, \aleph)| \leqslant \sigma(\xi)$ for all $(\xi, \vartheta, \aleph) \in \Lambda \times \Lambda \times \varphi$, $|Z_2(\xi, \vartheta, \aleph)| \leqslant \rho(\xi)$ for all $(\xi, \vartheta, \aleph) \in \Lambda \times \Lambda \times \varphi$.

Lemma 3.1. *Assuming $\aleph_0(\xi) \in M(\Lambda, \mathbb{R})$, the function $\aleph(\xi) \in M(\Lambda, \mathbb{R}^+)$ is a solution to problem (3.1)-(3.2) if and only if it satisfies the condition*

$$\begin{aligned}\aleph(\xi) = \aleph_0 &+ \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi - \vartheta)^{\zeta - 1} \delta(\vartheta) \aleph(\vartheta) d\vartheta + \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi - \vartheta)^{\zeta - 1} \upsilon(\vartheta, \aleph(\vartheta)) d\vartheta \\ &+ \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi - \vartheta)^{\zeta - 1} \int_{\vartheta}^{\xi} Z_1(\eta, \vartheta, \aleph(\vartheta)) d\eta d\vartheta \\ &+ \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi - \vartheta)^{\zeta - 1} \int_{\vartheta}^{\theta} Z_2(\eta, \vartheta, \aleph(\vartheta)) d\eta d\vartheta + \sum_{i=0}^{\infty} I_i(\aleph(\xi_i))\end{aligned} \tag{3.3}$$

for $\xi \in (\xi_k, \xi_{k+1}]$, $k = 1, 2, \ldots, m$.

Proof. By applying the integral operator (2.1) on both sides of equation (3.1), we obtain the integral equation (3.3). □

Theorem 3.2. *Suppose conditions* (A1)-(A3) *are met, and for two positive real numbers α and λ with $0 < \alpha < 1$, assume they satisfy the equations*

$$\left[\frac{\|\delta\|_\infty + \chi^*_\upsilon}{\Gamma(\zeta + 1)} + \frac{(\chi^*_{z_1} + \chi^*_{z_2})\theta}{(\zeta + 1)\Gamma(\zeta)} \right] \theta^\zeta = \alpha,$$

and

$$|\aleph_0| + m\chi^*_2 + \left[\frac{\upsilon_0}{\Gamma(\zeta + 1)} + \frac{(z^*_1 + z^*_2)\theta}{(\zeta + 1)\Gamma(\zeta)} \right] \theta^\zeta = (1 - \alpha)\lambda.$$

Then, the initial value problem (3.1)-(3.2) has a unique continuous solution over the interval $[\xi_0, \theta]$, where $\upsilon_0 = \max\{|\upsilon(\vartheta, 0)| : \vartheta \in \Lambda\}$, $z^*_1 = \max\{|Z_1(\eta, \vartheta, 0)| : (\eta, \vartheta) \in D\}$, and $z^*_2 = \max\{|Z_2(\eta, \vartheta, 0)| : (\eta, \vartheta) \in D\}$.

Proof. Consider the operator $T : M(\Lambda, \mathbb{R}) \to M(\Lambda, \mathbb{R})$ defined by

$$(T\aleph)(\xi) = \aleph_0 + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph(\vartheta)d\vartheta + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\upsilon(\vartheta,\aleph(\vartheta))d\vartheta$$
$$+\frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\left(\int_{\vartheta}^{\xi} Z_1(\eta,\vartheta,\aleph(\vartheta))d\eta + \int_{\vartheta}^{\theta} Z_2(\eta,\vartheta,\aleph(\vartheta))d\eta\right)d\vartheta + \sum_{i=0}^{\infty} I_i(\aleph(\xi_i)).$$

Additionally, let Ψ_λ be the set of functions $\aleph \in M(\Lambda, \mathbb{R})$ such that $\|\aleph\|_\infty \leqslant \lambda$ for some $\lambda > 0$. Our goal is to establish the existence of a fixed point for the operator T within the subset $\Psi_\lambda \subset M(\Lambda, \mathbb{R})$. This fixed point corresponds to the unique solution of the initial value problem (3.1)-(3.2). To accomplish this, we will present the proof in two distinct steps.

Step 1. Our objective is to show that the operator T maintains functions within the set Ψ_λ. Given the previously mentioned assumptions, if $\aleph$ is a function within the set Ψ_λ and ξ belongs to the interval Λ, we have

$$\begin{aligned}
|(T\aleph)(\xi)| \leqslant{}& |\aleph_0| + \frac{1}{\Gamma(\zeta)}\int_{t_0}^{\xi}(\xi-\vartheta)^{\zeta-1}|\delta(\vartheta)||\aleph(\vartheta)|d\vartheta + \frac{1}{\Gamma(\zeta)}\int_{t_0}^{\xi}(\xi-\vartheta)^{\zeta-1}|\upsilon(\vartheta,\aleph(\vartheta))|d\vartheta \\
&+\frac{1}{\Gamma(\zeta)}\int_{t_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\left(\int_{\vartheta}^{\xi}|Z_1(\eta,\vartheta,\aleph(\vartheta))|d\eta + \int_{t_0}^{\theta}|Z_2(\eta,\vartheta,\aleph(\vartheta))|d\eta\right)d\vartheta + \sum_{i=0}^{\infty}|I_i(\aleph(\xi_i))| \\
\leqslant{}& |\aleph_0| + \frac{1}{\Gamma(\zeta)}\int_{t_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\|\delta\|_\infty\|\aleph\|_\infty d\vartheta + \frac{1}{\Gamma(\zeta)}\int_{t_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\big(|\upsilon(\vartheta,\aleph(\vartheta))-\upsilon(\vartheta,0)| \\
&+|\upsilon(\vartheta,0)|\big)d\vartheta + \frac{1}{\Gamma(\zeta)}\int_{t_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\left(\int_{\vartheta}^{\xi}\big(|Z_1(\eta,\vartheta,\aleph(\vartheta))-Z_1(\eta,\vartheta,0)| + |Z_1(\eta,\vartheta,0)|\big)d\eta\right. \\
&\left.+\int_{t_0}^{\theta}\big(|Z_2(\eta,\vartheta,\aleph(\vartheta))-Z_2(\eta,\vartheta,0)| + |Z_2(\eta,\vartheta,0)|\big)d\eta\right)d\vartheta + m\chi_2^* \\
\leqslant{}& |\aleph_0| + \frac{\|\delta\|_\infty\theta^\zeta\lambda}{\Gamma(\zeta+1)} + \frac{\theta^\zeta}{\Gamma(\zeta+1)}(\chi_\upsilon^*\lambda+\upsilon_0) + \frac{\theta^{(\zeta+1)}}{(\zeta+1)\Gamma(\zeta)}(\chi_{z_1}^*\lambda + z_1^*) + \frac{\theta^{(\zeta+1)}}{(\zeta+1)\Gamma(\zeta)}(\chi_{z_2}^*\lambda+z_2^*) + m\chi_2^* \\
\leqslant{}& |\aleph_0| + \frac{\|\delta\|_\infty\theta^\zeta\lambda}{\Gamma(\zeta+1)} + \frac{\theta^\zeta}{\Gamma(\zeta+1)}\chi_\upsilon^*\lambda + \frac{\theta^\zeta}{\Gamma(\zeta+1)}\upsilon_0 + \frac{\theta^{(\zeta+1)}}{(\zeta+1)\Gamma(\zeta)}\chi_{z_1}^*\lambda + \frac{\theta^{(\zeta+1)}}{(\zeta+1)\Gamma(\zeta)}z_1^* \\
&+\frac{\theta^{(\zeta+1)}}{(\zeta+1)\Gamma(\zeta)}\chi_{z_2}^*\lambda + \frac{\theta^{(\zeta+1)}}{(\zeta+1)\Gamma(\zeta)}z_2^* + m\chi_2^* \\
\leqslant{}& |\aleph_0| + m\chi_2^* + \theta^\zeta\left(\frac{\upsilon_0}{\Gamma(\zeta+1)} + \frac{(z_1^*+z_2^*)\theta}{(\zeta+1)\Gamma(\zeta)}\right) + \theta^\zeta\lambda\left(\frac{\|\delta\|_\infty+\chi_\upsilon^*}{\Gamma(\zeta+1)} + \frac{(\chi_{z_1}^*+\chi_{z_2}^*)\theta}{(\zeta+1)\Gamma(\zeta)}\right) \\
={}& (1-\alpha)\lambda + \alpha\lambda = \lambda.
\end{aligned}$$

Therefore, we can deduce that $|T\aleph| \leqslant \lambda$, indicating that $T\aleph$ belongs to Ψ_λ. This establishes that $T\Psi_\lambda$ is a subset of Ψ_λ.

Step 2. Our aim is to prove that T is a contraction mapping. Let's consider two functions, $\aleph_1$ and $\aleph_2$, both belonging to the set Ψ_λ. Therefore,

$$\begin{aligned}
|(T\aleph_1)(\xi) - (T\aleph_2)(\xi)| \leqslant{}& \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}|\delta(\vartheta)||\aleph_1(\vartheta)-\aleph_2(\vartheta)|d\vartheta \\
&+\frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}|\upsilon(\vartheta,\aleph_1(\vartheta)) - \upsilon(\vartheta,\aleph_2(\vartheta))|d\vartheta \\
&+\frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\left(\int_{\vartheta}^{\xi}|Z_1(\eta,\vartheta,\aleph_1(\vartheta)) - Z_1(\eta,\vartheta,\aleph_2(\vartheta))|d\eta\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\vartheta}^{\theta} |Z_2(\eta,\vartheta,\aleph_1(\vartheta)) - Z_2(\eta,\vartheta,\aleph_2(\vartheta))| d\eta \Bigg) d\vartheta + \sum_{i=0}^{\infty} |I_i(\aleph_1(\xi_i)) - I_i(\aleph_2(\xi_i))| \\
&\leqslant \frac{\|\delta\|_\infty \theta^\zeta}{\Gamma(\zeta+1)} \|\aleph_1 - \aleph_2\| + \frac{\chi_\upsilon^* \theta^\zeta}{\Gamma(\zeta+1)} \|\aleph_1 - \aleph_2\| \\
&\quad + \frac{\chi_{z_1}^* \theta^{\zeta+1} + \chi_{z_2}^* \theta^{\zeta+1}}{(\zeta+1)\Gamma(\zeta)} \|\aleph_1 - \aleph_2\| + m\chi_1^* \|\aleph_1 - \aleph_2\| \\
&= \left[\left(\frac{\|\delta\|_\infty + \chi_\upsilon^*}{\Gamma(\zeta+1)} + \frac{(\chi_{z_1}^* + \chi_{z_2}^*)\theta}{(\zeta+1)\Gamma(\zeta)} \right) \theta^\zeta + m\chi_1^* \right] \|\aleph_1 - \aleph_2\| = \beta \|\aleph_1 - \aleph_2\|,
\end{aligned}
$$

where $\beta = \left[\frac{\|\delta\|_\infty + \chi_\upsilon^*}{\Gamma(\zeta+1)} + \frac{(\chi_{z_1}^* + \chi_{z_2}^*)\theta}{(\zeta+1)\Gamma(\zeta)}\right]\theta^\zeta + m\chi_1^* < 1$, we get $\|T\aleph_1 - T\aleph_2\| \leqslant \beta\|\aleph_1 - \aleph_2\|$. This confirms that T functions as a contraction mapping. Consequently, in alignment with Theorem 2.4, there exists a fixed point denoted as $\aleph \in M(\Lambda, \mathbb{R})$ such that $T\aleph = \aleph$. This fixed point serves as the unique solution to the initial value problem defined by equations (3.1)-(3.2). This concludes the proof of the theorem. □

Theorem 3.3. *Under the assumption (A4), it follows that there exists at least one solution to the problem outlined in* (3.1)–(3.2) *over* $[\xi_0, \theta]$.

Proof. Consider a fixed $\left[|\aleph_0| + \frac{\theta^\zeta |\delta|\infty |\aleph|_\infty}{\Gamma(\zeta+1)} + \frac{\theta^\zeta |\mu| M}{\Gamma(\zeta+1)} + \frac{\theta^{\zeta+1}|\sigma|M}{(\zeta+1)\Gamma(\zeta)} + \frac{\theta^{\zeta+1}|\rho|M}{(\zeta+1)\Gamma(\zeta)} + \sum_{i=0}^{\infty} |I_i(\aleph(\xi_i))|\right] \leqslant r$ and define the set $\Psi_\lambda = \aleph \in M; |\aleph|_\infty \leqslant \lambda$. In this context, we introduce the operators ϵ and Υ on Ψ_λ as

$$
\begin{aligned}
(\epsilon\aleph)(\xi) =& \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi - \vartheta)^{\zeta-1} \upsilon(\vartheta, \aleph(\vartheta)) d\vartheta + \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi - \vartheta)^{\zeta-1} \int_{\vartheta}^{\xi} Z_1(\eta, \vartheta, \aleph(\vartheta)) d\eta d\vartheta \\
&+ \int_{\xi_0}^{\xi} (\xi - \vartheta)^{\zeta-1} \int_{\vartheta}^{\theta} Z_2(\eta, \vartheta, \aleph(\vartheta)) d\eta d\vartheta, \\
(\Upsilon\aleph)(\xi) =& \aleph_0 + \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi - \vartheta)^{\zeta-1} \delta(\vartheta) \aleph(\vartheta) d\vartheta + \sum_{i=0}^{\infty} I_i(\aleph(\xi_i)).
\end{aligned}
$$

After examining $\aleph_1$ and $\aleph_2$ from the set Ψ_λ, we can write

$$
\begin{aligned}
\|\epsilon\aleph_1 + \Upsilon\aleph_2\| \leqslant& \Bigg\| \aleph_0 + \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi - \vartheta)^{\zeta-1} \delta(\vartheta) \aleph(\vartheta) d\vartheta + \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi - \vartheta)^{\zeta-1} \upsilon(\vartheta, \aleph(\vartheta)) d\vartheta \\
&+ \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi - \vartheta)^{\zeta-1} \left(\int_{\vartheta}^{\xi} Z_1(\eta, \vartheta, \aleph(\vartheta)) d\eta + \int_{\vartheta}^{\theta} Z_2(\eta, \vartheta, \aleph(\vartheta)) d\eta \right) d\vartheta + \sum_{i=0}^{\infty} I_i(\aleph(\xi_i)) \Bigg\| \\
\leqslant& \|\aleph_0\| + \frac{\theta^\zeta \|\delta\|_\infty \|\aleph\|_\infty}{\Gamma(\zeta+1)} + \frac{\theta^\zeta \|\mu\|_M}{\Gamma(\zeta+1)} + \frac{\theta^{\zeta+1} \|\sigma\|_M}{(\zeta+1)\Gamma(\zeta)} + \frac{\theta^{\zeta+1} \|\rho\|_M}{(\zeta+1)\Gamma(\zeta)} + \sum_{i=0}^{\infty} \|I_i(\aleph(\xi_i))\| \leqslant r.
\end{aligned}
$$

Therefore, $\epsilon\aleph_1 + \Upsilon\aleph_2 \in \Psi_\lambda$. Importantly, the assumption (A4) guarantees that Υ functions as a contraction mapping. The continuity of functions υ, Z_1, and Z_2 outlined in (3.1)-(3.2) implies the continuity of the operator Ψ. Additionally, it is crucial to emphasize that ϵ remains uniformly bounded on Ψ_λ as

$$
\|\epsilon\aleph\| \leqslant \frac{\theta^\zeta \|\mu\|_M}{\Gamma(\zeta+1)} + \frac{\theta^{\zeta+1} \|\sigma\|_M}{(\zeta+1)\Gamma(\zeta)} + \frac{\theta^{\zeta+1} \|\rho\|_M}{(\zeta+1)\Gamma(\zeta)}.
$$

Now, we establish the compactness of the operator ϵ. Given that υ, Z_1, and Z_2 are bounded on the compact sets $\Phi_1 = \Lambda \times \varphi$, $\Phi_2 = \Lambda \times \Lambda \times \varphi$, let us define $\sup_{(\xi,\aleph)\in\Phi_1} |\upsilon(\xi,\aleph)| = C_1$ and $\sup_{(\xi,\vartheta,\aleph)\in\Phi_2} |Z_i(\xi,\vartheta,\aleph)| = C_2$, where $i = 1, 2$. For $\xi_1, \xi_2 \in [\xi_0, \theta]$, and $\aleph \in \Psi_\lambda$, it is evident that

$$\|(\epsilon\aleph)(\xi_1) - (\epsilon\aleph)(\xi_2)\|$$

$$
\begin{aligned}
&= \Big\| \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi_1} (\xi_1-\vartheta)^{\zeta-1} \upsilon(\vartheta,\aleph(\vartheta))d\vartheta + \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi_1} (\xi_1-\vartheta)^{\zeta-1} \int_{\vartheta}^{\xi_1} Z_1(\eta,\vartheta,\aleph(\vartheta))d\eta d\vartheta \\
&\quad + \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi_1} (\xi_1-\vartheta)^{\zeta-1} \int_{\vartheta}^{\theta} Z_2(\eta,\vartheta,\aleph(\vartheta))d\eta d\vartheta - \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi_2} (\xi_2-\vartheta)^{\zeta-1} \upsilon(\vartheta,\aleph(\vartheta))d\vartheta \\
&\quad - \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi_2} (\xi_2-\vartheta)^{\zeta-1} \int_{\vartheta}^{\xi_2} Z_1(\eta,\vartheta,\aleph(\vartheta))d\eta d\vartheta - \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi_2} (\xi_2-\vartheta)^{\zeta-1} \int_{\vartheta}^{\theta} Z_2(\eta,\vartheta,\aleph(\vartheta))d\eta d\vartheta \Big\| \\
&\leqslant \frac{1}{\Gamma(\zeta)} \Big\| \int_{\xi_2}^{\xi_1} (\xi_1-\vartheta)^{\zeta-1} \upsilon(\vartheta,\aleph(\vartheta))d\vartheta + \int_{\xi_2}^{\xi_1} (\xi_1-\vartheta)^{\zeta-1} \int_{\vartheta}^{\xi_1} Z_1(\eta,\vartheta,\aleph(\vartheta))d\eta d\vartheta \\
&\quad + \int_{\xi_2}^{\xi_1} (\xi_1-\vartheta)^{\zeta-1} \int_{\vartheta}^{\theta} Z_2(\eta,\vartheta,\aleph(\vartheta))d\eta d\vartheta - \int_{\xi_0}^{\xi_2} (\xi_2-\vartheta)^{\zeta-1} \upsilon(\vartheta,\aleph(\vartheta))d\vartheta \\
&\quad - \int_{\xi_0}^{\xi_2} (\xi_2-\vartheta)^{\zeta-1} \int_{\vartheta}^{\xi_2} Z_1(\eta,\vartheta,\aleph(\vartheta))d\eta d\vartheta - \int_{\xi_0}^{\xi_2} (\xi_2-\vartheta)^{\zeta-1} \int_{\vartheta}^{\theta} Z_2(\eta,\vartheta,\aleph(\vartheta))d\eta d\vartheta \\
&\quad + \int_{\xi_0}^{\xi_2} (\xi_1-\vartheta)^{\zeta-1} \upsilon(\vartheta,\aleph(\vartheta))d\vartheta + \int_{\xi_0}^{\xi_2} (\xi_1-\vartheta)^{\zeta-1} \int_{\vartheta}^{\xi_1} Z_1(\eta,\vartheta,\aleph(\vartheta))d\eta d\vartheta \\
&\quad + \int_{\xi_0}^{\xi_2} (\xi_1-\vartheta)^{\zeta-1} \int_{\vartheta}^{\theta} Z_2(\eta,\vartheta,\aleph(\vartheta))d\eta d\vartheta \Big\| \\
&\leqslant \frac{1}{\Gamma(\zeta)} \Big\| \int_{\xi_2}^{\xi_1} (\xi_1-\vartheta)^{\zeta-1} \big[\upsilon(\vartheta,\aleph(\vartheta)) + \int_{\vartheta}^{\xi_1} Z_1(\eta,\vartheta,\aleph(\vartheta))d\eta + \int_{\vartheta}^{\theta} Z_2(\eta,\vartheta,\aleph(\vartheta))d\eta\big] d\vartheta \\
&\quad - \int_{\xi_0}^{\xi_2} \big[(\xi_2-\vartheta)^{\zeta-1} - (\xi_1-\vartheta)^{\zeta-1}\big] \upsilon(\vartheta,\aleph(\vartheta))d\vartheta \\
&\quad - \int_{\xi_0}^{\xi_2} \big[(\xi_2-\vartheta)^{\zeta-1} \int_{\vartheta}^{\xi_2} Z_1(\eta,\vartheta,\aleph(\vartheta))d\eta - (\xi_1-\vartheta)^{\zeta-1} \int_{\vartheta}^{\xi_1} Z_1(\eta,\vartheta,\aleph(\vartheta))d\eta\big] d\vartheta \\
&\quad - \int_{\xi_0}^{\xi_2} \big[(\xi_2-\vartheta)^{\zeta-1} \int_{\vartheta}^{\theta} Z_2(\eta,\vartheta,\aleph(\vartheta))d\eta - (\xi_1-\vartheta)^{\zeta-1} \int_{\vartheta}^{\theta} Z_2(\eta,\vartheta,\aleph(\vartheta))d\eta\big] d\vartheta \Big\| \\
&\leqslant \frac{C_1}{\Gamma(\zeta+1)} |2(\xi_1-\xi_2)^{\zeta}| + \frac{C_2\zeta}{\Gamma(\zeta+2)} |2(\xi_1-\xi_2)^{\zeta+1}| + \frac{C_3\zeta}{\Gamma(\zeta+2)} |2(\xi_1-\xi_2)^{\zeta+1}| \\
&\quad + \frac{C_1}{\Gamma(\zeta+1)} |\xi_2^{\zeta}| + \frac{C_2\zeta}{\Gamma(\zeta+2)} |\xi_2^{\zeta+1}| + \frac{C_3\zeta}{\Gamma(\zeta+2)} |\xi_2^{\zeta+1}| - \frac{C_1}{\Gamma(\zeta+1)} |\xi_1^{\zeta}| - \frac{C_2\zeta}{\Gamma(\zeta+2)} |\xi_1^{\zeta+1}| - \frac{C_3\zeta}{\Gamma(\zeta+2)} |\xi_1^{\zeta+1}| \\
&\leqslant \frac{C_1}{\Gamma(\zeta+1)} |2(\xi_1-\xi_2)^{\zeta} + \xi_2^{\zeta} - \xi_1^{\zeta}| + \frac{C_2\zeta}{\Gamma(\zeta+2)} |2(\xi_1-\xi_2)^{\zeta} + \xi_2^{\zeta} - \xi_1^{\zeta}| + \frac{C_3\zeta}{\Gamma(\zeta+2)} |2(\xi_1-\xi_2)^{\zeta} + \xi_2^{\zeta} - \xi_1^{\zeta}|.
\end{aligned}
$$

This quantity is independent of the choice of $\aleph$. Thus, ϵ demonstrates relative compactness on Ψ_λ. Consequently, following the Arzela-Ascoli theorem, ϵ is a compact operator on Ψ_λ. The conditions specified in Theorem 2.5 are all fulfilled. Consequently, the conclusion derived from Theorem 2.5 is applicable, signifying the existence of at least one solution to the problem (3.1)-(3.2). This marks the completion of the theorem's proof. □

3.2. *Ulam stability results*

In this subsection, we delve into the Ulam stability of the problem (3.1)-(3.2). Let's examine the inequality

$$
\left| {}^{c}D^{\zeta}\aleph(\xi) - \delta(\xi)\aleph(\xi) - \upsilon(\xi,\aleph(\xi)) - \int_{\xi_0}^{\xi} Z_1(\xi,\vartheta,\aleph(\vartheta))d\vartheta - \int_{\xi_0}^{\theta} Z_2(\xi,\vartheta,\aleph(\vartheta))d\vartheta \right| \leqslant \varepsilon, \ \ \Delta\aleph(\xi_k) = I_k(\aleph(\xi_k)). \tag{3.4}
$$

Definition 3.4 ([17])**.** Consider a positive constant $C_\delta > 0$ that validates the Ulam-Hyers stability of (3.1)-(3.2). For any given $\varepsilon > 0$ and a solution $\nu \in M(\Lambda, \mathbb{R})$ satisfying inequality (3.4), there exists a corresponding solution $\eta \in M(\Lambda, \mathbb{R})$ to problem (3.1)-(3.2). This solution guarantees $|\nu(\xi) - \eta(\xi)| \leqslant \varepsilon C_\delta$ for all $\xi \in \Lambda$.

Theorem 3.5. *If* (A1)-(A3) *are satisfied, then problem* (3.1)-(3.2) *demonstrates Ulam-Hyers stability when* $\beta < 1$.

Proof. Take $\varepsilon > 0$, and suppose $\nu \in M(\Lambda, \mathbb{R})$ satisfies inequality (3.4). Let $\eta \in M(\Lambda, \mathbb{R})$ be the unique solution to the following problem. In this context, we recall that

$$\begin{aligned}\aleph(\xi) = \aleph_0 &+ \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph(\vartheta)d\vartheta + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\upsilon(\vartheta,\aleph(\vartheta))d\vartheta \\ &+ \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\left(\int_{\vartheta}^{\xi} Z_1(\eta,\vartheta,\aleph(\vartheta))d\eta + \int_{\vartheta}^{\theta} Z_2(\eta,\vartheta,\aleph(\vartheta))d\eta\right)d\vartheta + \sum_{i=0}^{\infty} I_i(\aleph(\xi_i)).\end{aligned}$$

Integrating inequality (3.4) and including the initial condition from problem (3.2), we get

$$\begin{aligned}\Bigg|\aleph(\xi) - \aleph_0 &- \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph(\vartheta)d\vartheta - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\upsilon(\vartheta,\aleph(\vartheta))d\vartheta \\ &- \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\int_{\vartheta}^{\xi} Z_1(\eta,\vartheta,\aleph(\vartheta))d\eta d\vartheta \\ &- \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\int_{\vartheta}^{\theta} Z_2(\eta,\vartheta,\aleph(\vartheta))d\eta d\vartheta - \sum_{i=0}^{\infty} I_i(\aleph(\xi_i))\Bigg| \leqslant \varepsilon\frac{\theta^{\zeta}}{\Gamma(\zeta+1)}.\end{aligned}$$

Moreover, let's consider

$$\begin{aligned}|\aleph_1(\xi) - \aleph_2(\xi)| &\leqslant \Bigg|\aleph_1(\xi) - \aleph_0 - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph_2(\vartheta)d\vartheta - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\upsilon(\vartheta,\aleph_2(\vartheta))d\vartheta \\ &\quad - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\int_{\vartheta}^{\xi} Z_1(\eta,\vartheta,\aleph_2(\vartheta))d\eta d\vartheta \\ &\quad - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\int_{\vartheta}^{\theta} Z_2(\eta,\vartheta,\aleph_2(\vartheta))d\eta d\vartheta - \sum_{i=0}^{\infty} I_i(\aleph_2(\xi_i))\Bigg| \\ &\leqslant \Bigg|\aleph_1(\xi) - \aleph_0 - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph_1(\vartheta)d\vartheta + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph_1(\vartheta)d\vartheta \\ &\quad - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph_2(\vartheta)d\vartheta - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\upsilon(\vartheta,\aleph_1(\vartheta))d\vartheta \\ &\quad + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\upsilon(\vartheta,\aleph_1(\vartheta))d\vartheta - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\upsilon(\vartheta,\aleph_2(\vartheta))d\vartheta \\ &\quad - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\int_{\vartheta}^{\xi} Z_1(\eta,\vartheta,\aleph_1(\vartheta))d\eta d\vartheta + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\int_{\vartheta}^{\xi} Z_1(\eta,\vartheta,\aleph_1(\vartheta))d\eta d\vartheta \\ &\quad - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\int_{\vartheta}^{\xi} Z_1(\eta,\vartheta,\aleph_2(\vartheta))d\eta d\vartheta - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\int_{\vartheta}^{\theta} Z_2(\eta,\vartheta,\aleph_1(\vartheta))d\eta d\vartheta \\ &\quad + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\int_{\vartheta}^{\theta} Z_2(\eta,\vartheta,\aleph_1(\vartheta))d\eta d\vartheta - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\int_{\vartheta}^{\theta} Z_2(\eta,\vartheta,\aleph_2(\vartheta))d\eta d\vartheta \\ &\quad - \sum_{i=0}^{\infty} I_i(\aleph_1(\xi_i)) + \sum_{i=0}^{\infty} I_i(\aleph_1(\xi_i)) - \sum_{i=0}^{\infty} I_i(\aleph_2(\xi_i))\Bigg| \\ &\leqslant \Bigg|\aleph_1(\xi) - \aleph_0 - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph_1(\vartheta)d\vartheta - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\upsilon(\vartheta,\aleph_1(\vartheta))d\vartheta\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \int_{\vartheta}^{\xi} Z_1(\eta,\vartheta,\aleph_1(\vartheta))d\eta d\vartheta - \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \int_{\vartheta}^{\theta} Z_2(\eta,\vartheta,\aleph_1(\vartheta))d\eta d\vartheta \\
&- \sum_{i=0}^{\infty} I_i(\aleph_1(\xi_i)) \Big| + \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} |\delta(\vartheta)|\, |\aleph_1(\vartheta)-\aleph_2(\vartheta)| d\vartheta \\
&+ \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} |\upsilon(\vartheta,\aleph_1(\vartheta)) - \upsilon(\vartheta,\aleph_2(\vartheta)| d\vartheta \\
&+ \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \int_{\vartheta}^{\xi} |Z_1(\eta,\vartheta,\aleph_1(\vartheta)) - Z_1(\eta,\vartheta,\aleph_2(\vartheta))| d\eta d\vartheta \\
&+ \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \int_{\vartheta}^{\theta} |Z_2(\eta,\vartheta,\aleph_1(\vartheta)) - Z_2(\eta,\vartheta,\aleph_2(\vartheta))| d\eta d\vartheta \\
&+ \sum_{i=0}^{\infty} |I_i(\aleph_1(\xi_i)) - I_i(\aleph_2(\xi_i))|,
\end{aligned}
$$

$$
\begin{aligned}
\|\aleph_1-\aleph_2\| &\leqslant \frac{\varepsilon}{\Gamma(\zeta+1)} + \left(\left[\frac{\|\delta\|_\infty + \chi_\upsilon^*}{\Gamma(\zeta+1)} + \frac{(\chi_{z_1}^* + \chi_{z_2}^*)\theta}{(\zeta+1)\Gamma(\zeta)} \right] \theta^\zeta + m\chi_1^* \right) \|\aleph_1-\aleph_2\|, \\
\|\aleph_1-\aleph_2\| &\leqslant \frac{\varepsilon}{\Gamma(\zeta+1)} + \beta\|\aleph_1-\aleph_2\|, \\
\|\aleph_1-\aleph_2\| &\leqslant \frac{\varepsilon}{\Gamma(\zeta+1)(1-\beta)} = \varepsilon M, \text{ where } M = \frac{1}{\Gamma(\zeta+1)(1-\beta)}.
\end{aligned}
$$

Therefore, we can deduce that the problem (3.1)-(3.2) demonstrates Ulam-Hyers stability. This concludes the proof of the theorem. □

Example 3.6. We examine the Caputo fractional Volterra-Fredholm IDE (3.1)-(3.2) under the following parameters: $\zeta = 0.5$, $\theta = 0.5$, $\chi_\upsilon^* = 0.2$, $\chi_{Z_1}^* = 0.3$, $\chi_{Z_2}^* = 0.3$, $m\chi_1^* = \frac{1}{12}$, and $\|\delta\|_\infty = 0.1$. Now, it follows that

$$
\begin{aligned}
\beta &= \left[\frac{\|\delta\|_\infty + \chi_\upsilon^*}{\Gamma(\zeta+1)} + \frac{(\chi_{z_1}^* + \chi_{z_2}^*)\theta}{(\zeta+1)\Gamma(\zeta)} \right] \theta^\zeta + m\chi_1^* \\
&= \left[\frac{0.1+0.2}{\Gamma(\frac{1}{2}+1)} + \frac{(0.3+0.3)(0.5)}{(\frac{1}{2}+1)\Gamma(\frac{1}{2})} \right] (0.5)^{\frac{1}{2}} + \frac{1}{12} \\
&= \left[\frac{0.3}{\Gamma(\frac{3}{2})} + \frac{(0.6)(0.5)}{(\frac{3}{2})\Gamma(\frac{1}{2})} \right] (0.5)^{\frac{1}{2}} + 0.084 \\
&= \left[\frac{0.3}{0.886} + \frac{0.30}{(1.5)(1.772)} \right] (0.707) = (0.45)(0.707) + 0.084 = 0.4014 < 1.
\end{aligned}
$$

If we set $\varepsilon = 0.5$, the value of M can be computed as follows:

$$
M = \frac{1}{\Gamma(\zeta+1)(1-\beta)} = \frac{1}{\Gamma(\frac{3}{2})(1-0.4014)} = \frac{1}{0.5303} = 1.8857.
$$

Now, when we multiply ε by M, we get $\varepsilon M = 0.5 \times 1.8857 = 0.9428$. As all the conditions of Theorem 3.2 are met, there exists a unique and stable solution to the given equation.

4. Impulsive neutral Volterra-Fredholm integro-differential equation

In this section, we delve into examining the presence and singularity of solutions, along with investigating the Ulam stability outcomes for impulsive neutral Volterra-Fredholm IDE. This exploration provides valuable insights into the theoretical underpinnings, and we will highlight the importance of our results through illustrative examples.

4.1. Existence and uniqueness results

In this subsection, we delve into the Caputo fractional impulsive neutral Volterra-Fredholm IDE, given by

$$\begin{aligned} {}^{c}D^{\zeta}\Big[\aleph(\xi)-\upsilon_1(\xi,\aleph(\xi))\Big] &= \delta(\xi)\aleph(\xi)+\upsilon_2(\xi,\aleph(\xi))+\int_{\xi_0}^{\xi} Z_1(\xi,\vartheta,\aleph(\vartheta))d\vartheta+\int_{\xi_0}^{\theta} Z_2(\xi,\vartheta,\aleph(\vartheta))d\vartheta, \\ \Delta\aleph(\xi_k) &= I_k(\aleph(\xi_k)). \end{aligned} \tag{4.1}$$

This equation comes with the initial condition

$$\aleph(\xi_0)=\aleph_0. \tag{4.2}$$

In the given expressions, ${}^{c}D^{\zeta}$ denotes Caputo's fractional derivative with $0<\zeta\leqslant 1$, and $\aleph:\Lambda\to\mathbb{R}$, where $\Lambda=[\xi_0,\theta]$, represents the considered continuous function. Moreover, $\upsilon_n:\Lambda\times\mathbb{R}\to\mathbb{R}$ and $Z_n:\Lambda\times\Lambda\times\mathbb{R}\to\mathbb{R}$, where $n=1,2$, are continuous functions. Here, $\Delta=(\xi,\vartheta):0\leqslant\xi_0\leqslant\vartheta\leqslant\xi\leqslant\theta$, $0<\xi_0<\xi_1<\xi_2<\cdots<\xi_m<\xi_{m+1}=\theta$, $\Delta\aleph(\xi_k)=\aleph(\xi_k^+)-\aleph(\xi_k^-)$, $\aleph(\xi_k^+)$, $\aleph(\xi_k^-)$ denote the left and right limits of $\aleph$ at ξ_k, respectively. Before delving into our main results and their proofs, we present the following lemma along with some essential hypotheses.

(B1) Let Z_1 and $Z_2:\Lambda\times\Lambda\times\mathbb{R}\to\mathbb{R}$ be continuous functions defined on $D=\{(\xi,\vartheta):0\leqslant\xi_0\leqslant\vartheta\leqslant\xi\leqslant\theta\}$. They satisfy the conditions

$$|Z_1(\tau,\vartheta,\aleph_1(\vartheta))-Z_1(\tau,\vartheta,\aleph_2(\vartheta))|\leqslant\chi^*_{z_1}\|\aleph_1(\vartheta)-\aleph_2(\vartheta)\|,$$

and

$$|Z_2(\tau,\vartheta,\aleph_1(\vartheta))-Z_2(\tau,\vartheta,\aleph_2(\vartheta))|\leqslant\chi^*_{z_2}\|\aleph_1(\vartheta)-\aleph_2(\vartheta)\|.$$

(B2) The functions υ_1 and $\upsilon_2:\Lambda\times\mathbb{R}\to\mathbb{R}$ are continuous, and they satisfy the conditions

$$|\upsilon_1(\xi,\aleph_1)-\upsilon_1(\xi,\aleph_2)|\leqslant\chi^*_{\upsilon_1}\|\aleph_1-\aleph_2\|,\quad \text{and}\quad |\upsilon_2(\xi,\aleph_1)-\upsilon_2(\xi,\aleph_2)|\leqslant\chi^*_{\upsilon_2}\|\aleph_1-\aleph_2\|.$$

(B3) The function $\delta:\Lambda\to\mathbb{R}$ is continuous, and the constants $\chi^*_{z_1}$, $\chi^*_{z_2}$, $\chi^*_{\upsilon_1}$, and $\chi^*_{\upsilon_2}$ are all positive.

(B4) There exist constants $\chi_1^*>0$ and $\chi_2^*>0$ such that $\|I_k(\aleph_1)-I_k(\aleph_2)\|\leqslant\chi_1^*\|\aleph_1-\aleph_2\|$ and $\|I_k(\aleph)\|\leqslant\chi_2^*$ for each $\aleph$, $\aleph_1$, $\aleph_2\in\varphi$ and $k=1,2,\ldots,m$.

(B5) There exist constants ω, μ, σ, and ρ in $M(\Lambda,\mathbb{R}^+)$ such that: $|\upsilon_1(\xi,\aleph)|\leqslant\omega(\xi),\forall(\xi,\aleph)\in\Lambda\times\varphi$, $|\upsilon_2(\xi,\aleph)|\leqslant\mu(\xi)$, $\forall(\xi,\aleph)\in\Lambda\times\varphi$, $|Z_1(\xi,\vartheta,\aleph)|\leqslant\sigma(\xi)$, $\forall(\xi,\vartheta,\aleph)\in\Lambda\times\Lambda\times\varphi$, and $|Z_2(\xi,\vartheta,\aleph)|\leqslant\rho(\xi),\forall(\xi,\vartheta,\aleph)\in\Lambda\times\Lambda\times\varphi$.

Lemma 4.1. *If $\aleph_0(\xi)\in M(\Lambda,\mathbb{R})$, then $\aleph(\xi)\in M(\Lambda,\mathbb{R}^+)$ constitutes a solution to problem (4.1)-(4.2) if and only if it satisfies the condition*

$$\begin{aligned} \aleph(\xi) &= \aleph_0-\upsilon_1(\xi_0,\aleph_0)+\upsilon_1(\xi,\aleph(\xi))+\frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph(\vartheta)d\vartheta \\ &\quad+\frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\upsilon_2(\vartheta,\aleph(\vartheta))d\vartheta+\frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\int_{\vartheta}^{\xi}Z_1(\tau,\vartheta,\aleph(\vartheta))d\tau d\vartheta \\ &\quad+\frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\int_{\vartheta}^{\theta}Z_2(\tau,\vartheta,\aleph(\vartheta))d\tau d\vartheta+\sum_{i=0}^{\infty}I_i(\aleph(\xi_i)) \end{aligned} \tag{4.3}$$

for $\xi\in(\xi_k,\xi_{k+1}]$, $k=1,2,\ldots,m$.

Proof. This can be easily illustrated by applying the integral operator (2.1) to both sides of equation (4.1), leading to the integral equation (4.3). □

Theorem 4.2. *Assuming conditions* (B1)-(B4) *are satisfied, and considering two positive real numbers* α *and* λ *with* $0 < \alpha < 1$, *if they fulfill the following equations*

$$\chi^*_{\upsilon_1} + \left[\frac{\|\delta\|_\infty + \chi^*_{\upsilon_2}}{\Gamma(\zeta+1)} + \frac{(\chi^*_{z_1} + \chi^*_{z_2})\theta}{(\zeta+1)\Gamma(\zeta)}\right]\theta^\zeta = \alpha,$$

and

$$|\aleph_0| + |\upsilon_1(\xi_0, \aleph_0)| + \upsilon_1^* + \left[\frac{\upsilon_2^*}{\Gamma(\zeta+1)} + \frac{(z_1^* + z_2^*)\theta}{(\zeta+1)\Gamma(\zeta)}\right]\theta^\zeta + m\chi_2^* = (1-\alpha)\lambda,$$

then, the initial value problem (4.1)-(4.2) *has a unique continuous solution over the interval* $[\xi_0, \theta]$, *where* $\upsilon_1^* = \max\{|\upsilon_1(\vartheta, 0)| : \vartheta \in \Lambda\}$, $\upsilon_2^* = \max\{|\upsilon_2(\vartheta, 0)| : \vartheta \in \Lambda\}$, $z_1^* = \max\{|Z_1(\tau, \vartheta, 0)| : (\tau, \vartheta) \in D\}$, *and* $z_2^* = \max\{|Z_2(\tau, \vartheta, 0)| : (\tau, \vartheta) \in D\}$.

Proof. Considerthe operator $T : M(\Lambda, \mathbb{R}) \to M(\Lambda, \mathbb{R})$ defined by

$$(Th)(\xi) = \aleph_0 - \upsilon_1(\xi_0, \aleph_0) + \upsilon_1(\xi, \aleph(\xi)) + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph(\vartheta)d\vartheta + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\upsilon_2(\vartheta, \aleph(\vartheta))d\vartheta$$
$$+ \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\left(\int_{\vartheta}^{\xi} Z_1(\tau, \vartheta, \aleph(\vartheta))d\tau + \int_{\vartheta}^{\theta} Z_2(\tau, \vartheta, \aleph(\vartheta))d\tau\right)d\vartheta + \sum_{i=0}^{\infty} I_i(\aleph(\xi_i)).$$

Moreover, let Ψ_λ be the set of functions $\aleph \in M(\Lambda, \mathbb{R})$ such that $\|\aleph\|_\infty \leqslant \lambda$ for some $\lambda > 0$. Our goal is to establish the existence of a fixed point for the operator T within the subset $\Psi_\lambda \subset M(\Lambda, \mathbb{R})$. This fixed point corresponds to the unique solution of the initial value problem (4.1)-(4.2). To accomplish this, we will present the proof in two distinct steps.

Step 1. Our goal is to show that the operator T preserves functions within the set Ψ_λ. Given the previously stated hypotheses, for any function $\aleph$ belonging to the set Ψ_λ and for all ξ in the interval Λ, we get

$$
\begin{aligned}
|(Th)(\xi)| \leqslant\ & |\aleph_0| + |\upsilon_1(\xi_0, \aleph_0)| + |\upsilon_1(\xi, \aleph(\xi))| + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}|\delta(\vartheta)||\aleph(\vartheta)|d\vartheta \\
& + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}|\upsilon_2(\vartheta, \aleph(\vartheta))|d\vartheta \\
& + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\left(\int_{\vartheta}^{\xi}|Z_1(\tau, \vartheta, \aleph(\vartheta))|d\tau + \int_{\xi_0}^{\theta}|Z_2(\tau, \vartheta, \aleph(\vartheta))|d\tau\right)d\vartheta + \sum_{i=0}^{\infty}|I_i(\aleph(\xi_i))| \\
\leqslant\ & |\aleph_0| + |\upsilon_1(\xi_0, \aleph_0)| + |\upsilon_1(\xi, \aleph(\xi)) - \upsilon_1(\xi, 0)| + |\upsilon_1(\xi, 0)| + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\|\delta\|_\infty\|\aleph\|_\infty d\vartheta \\
& + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\big(|\upsilon_2(\vartheta, \aleph(\vartheta)) - \upsilon_2(\vartheta, 0)| + |\upsilon_2(\vartheta, 0)|\big)d\vartheta \\
& + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\left(\int_{\vartheta}^{\xi}\big(|Z_1(\tau, \vartheta, \aleph(\vartheta)) - Z_1(\tau, \vartheta, 0)| + |Z_1(\tau, \vartheta, O)|\big)d\tau\right. \\
& \left. + \int_{\xi_0}^{\theta}\big(|Z_2(\tau, \vartheta, \aleph(\vartheta)) - Z_2(\tau, \vartheta, 0)| + |Z_2(\tau, \vartheta, 0)|\big)d\tau\right)d\vartheta + m\chi_2^* \\
\leqslant\ & |\aleph_0| + |\upsilon_1(\xi_0, \aleph_0)| + \chi^*_{\upsilon_1} + \upsilon_1^* + \frac{\|\delta\|_\infty\theta^\zeta\lambda}{\Gamma(\zeta+1)} + \frac{\theta^\zeta}{\Gamma(\zeta+1)}(\chi^*_{\upsilon_2}\lambda + \upsilon_2) + \frac{\theta^{(\zeta+1)}}{(\zeta+1)\Gamma(\zeta)}(\chi^*_{z_1}\lambda + z_1^*) \\
& + \frac{\theta^{(\zeta+1)}}{(\zeta+1)\Gamma(\zeta)}(\chi^*_{z_2}\lambda + z_2^*) + m\chi_2^* \\
\leqslant\ & |\aleph_0| + |\upsilon_1(\xi_0, \aleph_0)| + \chi^*_{\upsilon_1} + \upsilon_1^* + \frac{\|\delta\|_\infty\theta^\zeta\lambda}{\Gamma(\zeta+1)} + \frac{\theta^\zeta}{\Gamma(\zeta+1)}\chi^*_{\upsilon_2}\lambda + \frac{\theta^\zeta}{\Gamma(\zeta+1)}\upsilon_2 + \frac{\theta^{(\zeta+1)}}{(\zeta+1)\Gamma(\zeta)}\chi^*_{z_1}\lambda
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\theta^{(\zeta+1)}}{(\zeta+1)\Gamma(\zeta)}z_1^* + \frac{\theta^{(\zeta+1)}}{(\zeta+1)\Gamma(\zeta)}\chi_{z_2}^*\lambda + \frac{\theta^{(\zeta+1)}}{(\zeta+1)\Gamma(\zeta)}z_2^* + m\chi_2^* \\
&\leqslant |\aleph_0| + |\upsilon_1(\xi_0,\aleph_0)| + \upsilon_1^* + m\chi_2^* + \theta^\zeta\left[\frac{\upsilon_2^*}{\Gamma(\zeta+1)} + \frac{(z_1^*+z_2^*)\theta}{(\zeta+1)\Gamma(\zeta)}\right] \\
&+\lambda\left(\chi_{\upsilon_1}^* + \left[\frac{\|\delta\|_\infty + \chi_\upsilon^*}{\Gamma(\zeta+1)} + \frac{(\chi_{z_1}^*+\chi_{z_2}^*)\theta}{(\zeta+1)\Gamma(\zeta)}\right]\theta^\zeta\right) = (1-\alpha)\lambda + \alpha\lambda = \lambda.
\end{aligned}
$$

Thus, we can conclude that $|T\aleph| \leqslant \lambda$, implying that $T\aleph \in \Psi_\lambda$ and establishing $T\Psi_\lambda$ as a subset of Ψ_λ.

Step 2. Our goal is to show that T is a contraction mapping. Consider two functions, $\aleph_1$ and $\aleph_2$, both belonging to Ψ_λ,

$$
\begin{aligned}
|(Th_1)(\xi) - (Th_2)(\xi)| &\leqslant |\upsilon_1(\xi,\aleph_1(\xi)) - \upsilon_1(\xi,\aleph_2(\xi))| + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}|\delta(\vartheta)||\aleph_1(\vartheta)-\aleph_2(\vartheta)|d\vartheta \\
&+\frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}|\upsilon_2(\vartheta,\aleph_1(\vartheta)) - \upsilon_2(\vartheta,\aleph_2(\vartheta))|d\vartheta \\
&+\frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\left(\int_{\vartheta}^{\xi}|Z_1(\tau,\vartheta,\aleph_1(\vartheta)) - Z_1(\tau,\vartheta,\aleph_2(\vartheta))|d\tau\right. \\
&\left.+\int_{\vartheta}^{\theta}|Z_2(\tau,\vartheta,\aleph_1(\vartheta)) - Z_2(\tau,\vartheta,\aleph_2(\vartheta))|d\tau\right)d\vartheta + \sum_{i=0}^{\infty}|I_i(\aleph_1(\xi_i)) - I_i(\aleph_2(\xi_i))| \\
&\leqslant \chi_{\upsilon_1}^*\|\aleph_1-\aleph_2\| + \frac{\|\delta\|_\infty\theta^\zeta}{\Gamma(\zeta+1)}\|\aleph_1-\aleph_2\| + \frac{\chi_{\upsilon_2}^*\theta^\zeta}{\Gamma(\zeta+1)}\|\aleph_1-\aleph_2\| \\
&+\frac{\chi_{z_1}^*\theta^{\zeta+1}+\chi_{z_2}^*\theta^{\zeta+1}}{(\zeta+1)\Gamma(\zeta)}\|\aleph_1-\aleph_2\| + m\chi_1^*\|\aleph_1-\aleph_2\| \\
&= \left[\chi_{\upsilon_1}^* + \left(\frac{\|\delta\|_\infty+\chi_{\upsilon_2}^*}{\Gamma(\zeta+1)} + \frac{(\chi_{z_1}^*+\chi_{z_2}^*)\theta}{(\zeta+1)\Gamma(\zeta)}\right)\theta^\zeta + m\chi_1^*\right]\|\aleph_1-\aleph_2\| = \beta\|\aleph_1-\aleph_2\|.
\end{aligned}
$$

As $\beta = \left[\chi_{\upsilon_1}^* + \left(\frac{\|\delta\|_\infty+\chi_{\upsilon_2}^*}{\Gamma(\zeta+1)} + \frac{(\chi_{z_1}^*+\chi_{z_2}^*)\theta}{(\zeta+1)\Gamma(\zeta)}\right)\theta^\zeta + m\chi_1^*\right] < 1$, we get $\|Th_1 - Th_2\| \leqslant \beta\|\aleph_1-\aleph_2\|$. This confirms that T functions as a contraction mapping. As a result, according to Theorem 2.4, there exists a fixed point denoted as $\aleph \in M(\Lambda,\mathbb{R})$ such that $T\aleph = \aleph$. This fixed point represents the unique solution to the initial value problem outlined by equations (4.1)-(4.2). This concludes the proof of the theorem. □

Theorem 4.3. *If* (B5) *holds, then the inequality*

$$
P := \chi_{\upsilon_1}^* + \frac{\chi_{\upsilon_2}^*\theta^\zeta}{\Gamma(\zeta+1)} + \frac{\chi_{z_1}^*\theta^{\zeta+1}+\chi_{z_2}^*\theta^{\zeta+1}}{(\zeta+1)\Gamma(\zeta)} < 1
$$

ensures the existence of at least one solution to the problem outlined in (4.1)–(4.2) *over* $[\xi_0,\theta]$.

Proof. Choose a fixed $r \geqslant \left[\|\aleph_0\| + \|\upsilon_1(0,\aleph(0))\| + \|\omega\|_C + \frac{\theta^\zeta\|\delta\|_\infty\|\aleph\|_\infty}{\Gamma(\zeta+1)} + \frac{\theta^\zeta\|\mu\|_C}{\Gamma(\zeta+1)} + \frac{\theta^{\zeta+1}\|\sigma\|_C}{(\zeta+1)\Gamma(\zeta)} + \frac{\theta^{\zeta+1}\|\rho\|_C}{(\zeta+1)\Gamma(\zeta)} + \sum_{i=0}^{\infty}\|I_i(\aleph(\xi_i))\|\right]$ and define the set $\Psi_\lambda = \{\aleph \in M; \|\aleph\|_\infty \leqslant \lambda\}$. In this context, we introduce the operators ϵ and Υ on Ψ_λ as

$$
\begin{aligned}
(\epsilon\aleph)(\xi) &= -\upsilon_1(\xi_0,\aleph_0) + \upsilon_1(\xi,\aleph(\xi)) + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\upsilon_2(\vartheta,\aleph(\vartheta))d\vartheta \\
&+\frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\int_{\vartheta}^{\xi}Z_1(\tau,\vartheta,\aleph(\vartheta))d\tau d\vartheta + \int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\int_{\vartheta}^{\theta}Z_2(\tau,\vartheta,\aleph(\vartheta))d\tau d\vartheta,
\end{aligned}
$$

$$(\Upsilon\aleph)(\xi) = \aleph_0 + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph(\vartheta)d\vartheta + \sum_{i=0}^{\infty} I_i(\aleph(\xi_i)).$$

When examining $\aleph_1$ and $\aleph_2$ within the set Ψ_λ, it is evident that

$$\begin{aligned}
\|\epsilon\aleph_1 + \Upsilon\aleph_2\| \leqslant & \Big\|\aleph_0 - \upsilon_1(\xi_0,\aleph_0) + \upsilon_1(\xi,\aleph(\xi)) + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph(\vartheta)d\vartheta \\
& + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\upsilon_2(\vartheta,\aleph(\vartheta))d\vartheta \\
& + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\left(\int_{\vartheta}^{\xi} Z_1(\tau,\vartheta,\aleph(\vartheta))d\tau + \int_{\vartheta}^{\theta} Z_2(\tau,\vartheta,\aleph(\vartheta))d\tau\right)d\vartheta + \sum_{i=0}^{\infty} I_i(\aleph(\xi_i))\Big\| \\
\leqslant & \|\aleph_0\| + \|\upsilon_1(\xi_0,\aleph_0)\| + \|\omega\|_C + \frac{\theta^{\zeta}\|\delta\|_\infty\|\aleph\|_\infty}{\Gamma(\zeta+1)} + \frac{\theta^{\zeta}\|\mu\|_C}{\Gamma(\zeta+1)} + \frac{\theta^{\zeta+1}\|\sigma\|_C}{(\zeta+1)\Gamma(\zeta)} \\
& + \frac{\theta^{\zeta+1}\|\rho\|_C}{(\zeta+1)\Gamma(\zeta)} + \sum_{i=0}^{\infty}\|I_i(\aleph(\xi_i))\| \leqslant r.
\end{aligned}$$

Next, we show that $(\epsilon\aleph)$ possesses contraction properties

$$\begin{aligned}
\|\epsilon\aleph_1 - \epsilon\aleph_2\| \leqslant & \|\upsilon_1(\xi,\aleph_1(\xi)) - \upsilon_1(\xi,\aleph_2(\xi))\| + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\|\upsilon_2(\vartheta,\aleph_1(\vartheta)) - \upsilon_2(\vartheta,\aleph_2(\vartheta))\|d\vartheta \\
& + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\Bigg(\int_{\vartheta}^{\xi}\|Z_1(\tau,\vartheta,\aleph_1(\vartheta)) - Z_1(\tau,\vartheta,\aleph_2(\vartheta))\|d\tau \\
& + \int_{\vartheta}^{\theta}\|Z_2(\tau,\vartheta,\aleph_1(\vartheta)) - Z_2(\tau,\vartheta,\aleph_2(\vartheta))\|d\tau\Bigg)d\vartheta \\
\leqslant & \chi_{\upsilon_1}^{*}\|\aleph_1-\aleph_2\| + \frac{\chi_{\upsilon_2}^{*}\theta^{\zeta}}{\Gamma(\zeta+1)}\|\aleph_1-\aleph_2\| + \frac{\chi_{z_1}^{*}\theta^{\zeta+1} + \chi_{z_2}^{*}\theta^{\zeta+1}}{(\zeta+1)\Gamma(\zeta)}\|\aleph_1-\aleph_2\| \\
\leqslant & \left(\chi_{\upsilon_1}^{*} + \frac{\chi_{\upsilon_2}^{*}\theta^{\zeta}}{\Gamma(\zeta+1)} + \frac{\chi_{z_1}^{*}\theta^{\zeta+1} + \chi_{z_2}^{*}\theta^{\zeta+1}}{(\zeta+1)\Gamma(\zeta)}\right)\|\aleph_1-\aleph_2\| \leqslant P\|\aleph_1-\aleph_2\|.
\end{aligned}$$

Hence, ϵ acts as a contraction. The continuity of δ implies the continuity of the operator Υ. Furthermore, Υ is uniformly bounded on Ψ_λ,

$$\|(\Upsilon\aleph)(\xi)\| \leqslant \|\aleph_0\| + \|\frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph(\vartheta)d\vartheta\| + \sum_{i=0}^{\infty}\|I_i(\aleph(\xi_i))\| \leqslant \|\aleph_0\| + \frac{\theta^{\zeta}\|\delta\|_\infty\lambda}{\Gamma(\zeta+1)} + m\chi_2^{*}.$$

To verify the compactness of the operator Υ, it is essential to illustrate its equicontinuity property. For this purpose, let's introduce $\bar{\delta}$ as the supremum of $|\delta(\vartheta)\aleph(\vartheta)|$. Now, considering any pair of points ξ_1 and ξ_2 within the interval $[\xi_0,\theta]$, where $\xi_1 > \xi_2$, and for any function $\aleph$ in the set Ψ_λ, we have

$$\begin{aligned}
&\|(\Upsilon\aleph)(\xi_1) - (\Upsilon\aleph)(\xi_2)\| \\
&\leqslant \Big\|\frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi_1}(\xi_1-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph(\vartheta)d\vartheta - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi_2}(\xi_2-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph(\vartheta)d\vartheta\Big\| \\
&\leqslant \Big\|\frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi_2}(\xi_2-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph(\vartheta)d\vartheta + \frac{1}{\Gamma(\zeta)}\int_{\xi_2}^{\xi_1}(\xi_1-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph(\vartheta)d\vartheta - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi_2}(\xi_2-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph(\vartheta)d\vartheta\Big\| \\
&\leqslant \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi_2}\Big\|\left[(\xi_2-\vartheta)^{\zeta-1} - (\xi_1-\vartheta)^{\zeta-1}\right]\delta(\vartheta)\aleph(\vartheta)d\vartheta\Big\| + \frac{1}{\Gamma(\zeta)}\int_{\xi_2}^{\xi_1}(\xi_1-\vartheta)^{\zeta-1}\|\delta(\vartheta)\aleph(\vartheta)d\vartheta\|d\vartheta
\end{aligned}$$

$$\leqslant \left[\frac{{\xi_2}^{\zeta}}{\Gamma(\zeta+1)} - \frac{{\xi_1}^{\zeta}}{\Gamma(\zeta+1)}\right]\|\delta\|_\infty\|\aleph\|_\infty + \frac{2(\xi_1-\xi_2)^{\zeta}}{\Gamma(\zeta+1)}\|\delta\|_\infty\|\aleph\|_\infty$$
$$\leqslant \frac{\bar{\delta}}{\Gamma(\zeta+1)}|2(\xi_1-\xi_2)^{\zeta} + {\xi_2}^{\zeta} - {\xi_1}^{\zeta}| \leqslant \frac{\bar{\delta}}{\Gamma(\zeta+1)}|\xi_1-\xi_2|^{\zeta} \rightarrow 0 \text{ as } \xi_1 \rightarrow \xi_2.$$

Therefore, Υ is equicontinuous. According to the Arzela-Ascoli Theorem, Υ is compact. All the conditions specified in Theorem 2.5 are met. Consequently, the conclusion of Theorem 2.5 holds, signifying that the problem (4.1)-(4.2) has at least one solution. This concludes the proof of the theorem. □

4.2. Ulam stability results

In this subsection, we will explore the Ulam stability of the problem (4.1)-(4.2). Let's introduce the following inequality.

$$\begin{aligned}
&\left|{}^{c}D^{\zeta}\left[\aleph(\xi) - \upsilon_1(\xi,\aleph(\xi))\right] - \delta(\xi)\aleph(\xi) - \upsilon_2(\xi,\aleph(\xi)) - \int_{\xi_0}^{\xi} Z_1(\xi,\vartheta,\aleph(\vartheta))d\vartheta - \int_{\xi_0}^{\theta} Z_2(\xi,\vartheta,\aleph(\vartheta))d\vartheta\right| \leqslant \varepsilon,\\
&\Delta\aleph(\xi_k) = I_k(\aleph(\xi_k)).
\end{aligned} \tag{4.4}$$

Theorem 4.4. *If* (B1)-(B4) *are satisfied, equation* (4.1)-(4.2) *demonstrates Ulam-Hyers stability when* $\beta < 1$.

Proof. Consider $\varepsilon > 0$, and let $\nu \in M(\Lambda,\mathbb{R})$ satisfy inequality (4.4). Additionally, let $\eta \in M(\Lambda,\mathbb{R})$ be the unique solution to the following problem. In this context, we note that

$$\begin{aligned}
\aleph(\xi) = \aleph_0 - \upsilon_1(\xi_0,\aleph_0) + \upsilon_1(\xi,\aleph(\xi)) + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph(\vartheta)d\vartheta + \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\upsilon_2(\vartheta,\aleph(\vartheta))d\vartheta\\
+ \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\left(\int_{\vartheta}^{\xi} Z_1(\tau,\vartheta,\aleph(\vartheta))d\tau + \int_{\vartheta}^{\theta} Z_2(\tau,\vartheta,\aleph(\vartheta))d\tau\right)d\vartheta + \sum_{i=0}^{\infty} I_i(\aleph(\xi_i)).
\end{aligned}$$

By integrating inequality (4.4) and including the initial condition of problem (4.2), we have

$$\begin{aligned}
&\Big|\aleph(\xi) - \aleph_0 + \upsilon_1(\xi_0,\aleph_0) - \upsilon_1(\xi,\aleph(\xi)) - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph(\vartheta)d\vartheta\\
&\quad - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\upsilon_2(\vartheta,\aleph(\vartheta))d\vartheta - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\int_{\vartheta}^{\xi} Z_1(\tau,\vartheta,\aleph(\vartheta))d\tau d\vartheta\\
&\quad - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\int_{\vartheta}^{\theta} Z_2(\tau,\vartheta,\aleph(\vartheta))d\tau d\vartheta - \sum_{i=0}^{\infty} I_i(\aleph(\xi_i))\Big| \leqslant \varepsilon\frac{\theta^{\zeta}}{\Gamma(\zeta+1)}.
\end{aligned}$$

Moreover, let's consider

$$\begin{aligned}
&|\aleph_1(\xi) - \aleph_2(\xi)|\\
&\leqslant \Big|\aleph_1(\xi) - \aleph_0 + \upsilon_1(\xi_0,\aleph_0) - \upsilon_1(\xi,\aleph_2(\xi)) - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\delta(\vartheta)\aleph_2(\vartheta)d\vartheta\\
&\quad - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\upsilon_2(\vartheta,\aleph_2(\vartheta))d\vartheta - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\int_{\vartheta}^{\xi} Z_1(\tau,\vartheta,\aleph_2(\vartheta))d\tau d\vartheta\\
&\quad - \frac{1}{\Gamma(\zeta)}\int_{\xi_0}^{\xi}(\xi-\vartheta)^{\zeta-1}\int_{\vartheta}^{\theta} Z_2(\tau,\vartheta,\aleph_2(\vartheta))d\tau d\vartheta - \sum_{i=0}^{\infty} I_i(\aleph_2(\xi_i))\Big|\\
&\leqslant \Big|\aleph_1(\xi) - \aleph_0 + \upsilon_1(\xi_0,\aleph_0) - \upsilon_1(\xi,\aleph_1(\xi)) + \upsilon_1(\xi,\aleph_1(\xi)) - \upsilon_1(\xi,\aleph_2(\xi))
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \delta(\vartheta) \aleph_1(\vartheta) d\vartheta + \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \delta(\vartheta) \aleph_1(\vartheta) d\vartheta \\
&- \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \delta(\vartheta) \aleph_2(\vartheta) d\vartheta - \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \upsilon_2(\vartheta, \aleph_1(\vartheta)) d\vartheta \\
&+ \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \upsilon_2(\vartheta, \aleph_1(\vartheta)) d\vartheta - \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \upsilon_2(\vartheta, \aleph_2(\vartheta)) d\vartheta \\
&- \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \int_{\vartheta}^{\xi} Z_1(\tau, \vartheta, \aleph_1(\vartheta)) d\tau d\vartheta + \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \int_{\vartheta}^{\xi} Z_1(\tau, \vartheta, \aleph_1(\vartheta)) d\tau d\vartheta \\
&- \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \int_{\vartheta}^{\xi} Z_1(\tau, \vartheta, \aleph_2(\vartheta)) d\tau d\vartheta - \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \int_{\vartheta}^{\theta} Z_2(\tau, \vartheta, \aleph_1(\vartheta)) d\tau d\vartheta \\
&+ \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \int_{\vartheta}^{\theta} Z_2(\tau, \vartheta, \aleph_1(\vartheta)) d\tau d\vartheta - \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \int_{\vartheta}^{\theta} Z_2(\tau, \vartheta, \aleph_2(\vartheta)) d\tau d\vartheta \\
&- \sum_{i=0}^{\infty} I_i(\aleph_1(\xi_i)) + \sum_{i=0}^{\infty} I_i(\aleph_1(\xi_i)) - \sum_{i=0}^{\infty} I_i(\aleph_2(\xi_i)) \Bigg| \\
\leqslant & \Bigg| \aleph_1(\xi) - \aleph_0 + \upsilon_1(\xi_0, \aleph_0) - \upsilon_1(\xi, \aleph_1(\xi)) - \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \delta(\vartheta) \aleph_1(\vartheta) d\vartheta \\
&- \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \upsilon_2(\vartheta, \aleph_1(\vartheta)) d\vartheta - \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \int_{\vartheta}^{\xi} Z_1(\tau, \vartheta, \aleph_1(\vartheta)) d\tau d\vartheta \\
&- \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \int_{\vartheta}^{\theta} Z_2(\tau, \vartheta, \aleph_1(\vartheta)) d\tau d\vartheta - \sum_{i=0}^{\infty} I_i(\aleph_1(\xi_i)) \Bigg| + |\upsilon_1(\xi, \aleph_1(\xi)) - \upsilon_2(\xi, \aleph_2(\xi))| \\
&+ \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} |\delta(\vartheta)| \, |\aleph_1(\vartheta) - \aleph_2(\vartheta)| d\vartheta + \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} |\upsilon_2(\vartheta, \aleph_1(\vartheta)) \\
&- \upsilon_2(\vartheta, \aleph_2(\vartheta))| d\vartheta + \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \int_{\vartheta}^{\xi} |Z_1(\tau, \vartheta, \aleph_1(\vartheta)) - Z_1(\tau, \vartheta, \aleph_2(\vartheta))| d\tau d\vartheta \\
&+ \frac{1}{\Gamma(\zeta)} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \int_{\xi_0}^{\xi} (\xi-\vartheta)^{\zeta-1} \int_{\vartheta}^{\theta} |Z_2(\tau, \vartheta, \aleph_1(\vartheta)) - Z_2(\tau, \vartheta, \aleph_2(\vartheta))| d\tau d\vartheta \\
&+ \sum_{i=0}^{\infty} |I_i(\aleph_1(\xi_i)) - I_i(\aleph_2(\xi_i))|,
\end{aligned}
$$

$$
\|\aleph_1 - \aleph_2\| \leqslant \frac{\varepsilon}{\Gamma(\zeta+1)} + \left[\chi^*_{\upsilon_1} + \left(\frac{\|\delta\|_\infty + \chi^*_{\upsilon_2}}{\Gamma(\zeta+1)} + \frac{(\chi^*_{z_1} + \chi^*_{z_2})\theta}{(\zeta+1)\Gamma(\zeta)} + m\chi^*_1 \right) \theta^{\zeta} \right] \|\aleph_1 - \aleph_2\|,
$$

$$
\|\aleph_1 - \aleph_2\| \leqslant \frac{\varepsilon}{\Gamma(\zeta+1)} + \beta \|\aleph_1 - \aleph_2\|,
$$

$$
\|\aleph_1 - \aleph_2\| \leqslant \frac{\varepsilon}{\Gamma(\zeta+1)(1-\beta)} = \varepsilon M_1, \text{ where } M_1 = \frac{1}{\Gamma(\zeta+1)(1-\beta)}.
$$

Therefore, we can deduce that the problem (4.1)-(4.2) demonstrates Ulam-Hyers stability. This concludes the proof of the theorem. □

Example 4.5. We examine the Caputo fractional Volterra-Fredholm IDE (4.1)-(4.2) under the following parameters: $\zeta = 0.5$, $\theta = 0.5$, $\chi^*_{\upsilon_1} = 0.2$, $\chi^*_{\upsilon_2} = 0.4$, $\chi^*_{Z_1} = 0.3$, $\chi^*_{Z_2} = 0.3$, $m\chi^*_1 = \frac{1}{14}$, and $\|\delta\|_\infty = 0.3$. Now, it follows that

$$
\beta = \left[\chi^*_{\upsilon_1} + \left(\frac{\|\delta\|_\infty + \chi^*_{\upsilon_2}}{\Gamma(\zeta+1)} + \frac{(\chi^*_{z_1} + \chi^*_{z_2})\theta}{(\zeta+1)\Gamma(\zeta)} \right) \theta^{\zeta} + m\chi^*_1 \right]
$$

$$
\begin{aligned}
&= \left[0.2 + \frac{0.3+0.4}{\Gamma(\frac{1}{2}+1)} + \frac{(0.3+0.3)(0.5)}{(\frac{1}{2}+1)\Gamma(\frac{1}{2})}\right](0.5)^{\frac{1}{2}} + \frac{1}{14}\\
&= \left[0.2 + \frac{0.7}{\Gamma(\frac{3}{2})} + \frac{(0.6)(0.5)}{(\frac{3}{2})\Gamma(\frac{1}{2})}\right](0.5)^{\frac{1}{2}} + 0.071\\
&= \left[0.2 + \frac{0.7}{0.886} + \frac{0.30}{(1.5)(1.772)}\right](0.707) + 0.071 = (1.102)(0.707) + 0.071 = 0.8501 < 1.
\end{aligned}
$$

Since all the conditions of Theorem 4.2 are met, there exists a unique and stable solution to the given equation.

5. Conclusion

The investigation conducted in this study has explored impulsive Volterra-Fredholm IDEs, incorporating fractional Caputo derivatives and adhering to specific order conditions. The research rigorously established the existence and uniqueness of analytical solutions through the application of the Banach principle. Notably, it revealed the existence of at least one solution, supported by precise conditions derived from the Krasnoselskii fixed point theorem. Furthermore, the research extended its scope to encompass impulsive neutral Volterra-Fredholm IDEs, thereby expanding the applicability of these findings within the mathematical domain. Additionally, this paper explored the notion of Ulam stability concerning the derived solutions, providing insight into their long-term behavior and enriching comprehension of their practical significance. To emphasize the real-world applicability and credibility of the results presented herein, an illustrative example has been thoughtfully included. This example effectively demonstrates how the theoretical findings can be applied in practical scenarios, highlighting the potential impact and value of this research in the field of IDEs and related areas.

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