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Spectrum and pseudspectrum of D-stable matrices of economy models

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Abstract

The computation and analysis of structured singular values and D-stable structured matrices have an important and crucial role in system theory to study the stability and D-stability of dynamical systems. In this study, novel theoretical results are obtained to analyze interconnections between D-stability, D(α)-stability, H-stability, a rank-1 perturbation to D-semi-stable matrices, and the computation of the bounds of structured singular values for structured and unstructured matrices. The proposed methodology is based on various tools from linear algebra, matrix analysis and system theory. The pseudo-spectrum of structured matrices appearing in economic models provides insights to analyze and characterize stability and instability analysis and sensitivity of linear dynamical models. The numerical experimentation's on the computation and comparison of lower bounds of structured singular values show the effectiveness of the proposed methodology. The Matlab EigTool is used on the computation of pseudo-spectrum for structured matrices arising from economy models.

Keywords: Singular values, structured singular values, D-stable matrix, H-stable matrix.

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1. Introduction

The D-stability theory for a class of real valued square matrices introduced in [2, 9] was to analysis the models to study and analyze competitiveness in the markets. The D-stability theory is an important and fundamental concept from system theory which generalizes the classical notions of stability to define the stability regions in complex plane $\mathbb{C}$. The theory of D-stability further generalized left-half plane for the continuous time dynamical system or unit circle for discrete time dynamical systems. The D-stability

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theory allows to assess the stability of a dynamical system conditions subject to structured or unstructured constrains of uncertainties.

The D-stable matrices appears in input-output economy models, for instance, Leontief production model. The notion of D-stability helps in the analysis to the robustness of equilibrium states in an economy. The input-output matrix of an economy model is D-stable suggests that the economy can withstand across various proportional changes to factor productivity while maintaining stability. The D-stable matrices also appears in economy growth models, particularly in the multisector growth models. The D-stable matrices represent the Jacobian of the equilibrium state of macroeconomic systems to study stability, and to analyze the interactions between many sectors of the economy, for instance, consumption, production, etc.

The structured D-stable matrices, and structured additive D-stable matrices are of great interests of research in various areas of science and engineering, for instance, mathematical economics, dynamics of population, neural networks, electrical and control engineering [1, 15–18, 24, 36]. The closer connections between the computation of structured singular values and D-stability theory for a real valued n-dimensional square matrix were presented in [6]. The simpler condition to strongly D-stability were derived in [23].

The H-stability and H-semistability being as two strong and effective notions of structured matrix stability were studied in [5], and both necessary and sufficient conditions for a matrix to be structured H-stable or structured H-semi-stable were also presented. The notation of H-stability ensures the stability for continuous-time linear dynamical systems. The H-stability has important applications in engineering: control engineering, electrical engineering, mechanical engineering, and also in the modeling and analysis economic problems. The H-stability ensures that the matrix corresponding to linear dynamical system have negative real parts for all eigenvalues.

The notion of $D(\alpha)$ structured stability for $\alpha = (\alpha_1, \alpha_1, \cdots, \alpha_p)$ studied by Khalil and Kokotovic [19, 20] and is useful to study the problems such as time-invariant multi-parameter singular perturbations. In particular, the study systems represented by mathematical equation $E(\epsilon)\dot{z} = Dz$.

The computation of structured singular value, also knows as μ-value [31], is a well-known mathematical tool in control to study and analyze indicators like as robustness, performance, and stability of linear time invariant (LTI) dynamical systems. The necessary and sufficient conditions on structured singular values allow us to develop its connection with D-stability theory. In the study economic models as well as financial models, structured and unstructured uncertainties occurs due to unpredictable variations in market conditions, the policies developed by government officials, or some unpredictable external shocks. The computation and analysis of structured singular values (μ-values) is applicable to analyze both robustness as well as the performance of an economic policies. In an economy model system, structured singular value allows to test robustness of the system to uncertain external disturbances.

The study of matrix analysis, in particular, matrices play an important and diverse role in economy [29]. The economy models such as the models for market equilibrium, the models of macro economy to study equilibrium, IS-LM model, IO model developed by Leontief are widely consist of matrix based approaches. The IS-LM model of monetarist controversy was developed by [4]. The electricity market equilibrium model was developed by Baldick [3].

A new study on the demand side factors which effect foreign investment in African countries was presented in [12]. The revenue and cost management of re-manufactured products was introduced in [30]. The most recent literature on the interconnection between matrix analysis and models from economy are developed in [7, 8, 13, 28, 41, 42].

A comprehensive analysis to health, economic, and ecological impacts were discussed in a greater detail in [21]. The analysis particularly concentrate to study and analyze the smog's health issues, economic problems, social problems and the solutions were employed to mitigate these effects. In [26], a new concept was developed to study the intuitionistic fuzzy bipolar metric space. In addition, the authors have given proof to fixed point theorems. A new study to picture fuzzy Lie algebra was presented in order to provide a mathematical scheme which opens new doors for scientific discoveries and their outcomes in a wider sphere of disciplines, were refer interested read to see [22] and the references therein. Some new concepts on orthogonal modified F-contraction type-I and type-II are presented in [33]. Furthermore, the results on fixed-point theorems for the case of self-mapping in orthogonal metrics were proved. In [27], new results were introduced on orthogonal concepts on the concerning F-contraction mappings. The authors have demonstrated some new fixed-point theorems for self-mapping while working on a complete orthogonal metric space.

The proposed methodology presented in this article covers the gaps in the sense that how to quantify the interconnection between D-stability to structured singular values characterization under various constraints. As per our best knowledge there is not a single article available in literature about strong interconnections amongst structured H-stability, $D(\alpha)$ structured stability and μ-values. Our proposed methodology discuss the stability, D-stability, H-stability, and $D(\alpha)$-stability and structured singular values for structured matrices appearing across the economy models.

The proposed methodology presented in this article is applicable to n-dimensional complex valued, and real valued matrices. We have used various tools from linear algebra, matrix analysis and system theory to give an accessible approach in order to build a new mathematical technique rather than developing a complex geometric approach to discuss and analyze the interconnections between structured D-stability, structured H-stability, $D(\alpha)$ structured stability and the computation of structured singular values. Our proposed methodology ensures that equilibrium states of an economy models will remains stable under structured or unstructured uncertainties. The proposed methodology allows to analyze the robust analysis of performance of economy models under structured or unstructured uncertainties. The economic models particularly those based on linear assumptions, can benefit from the proposed methodology to study D-stability and its connection with structured singular values to ensure that a number of economic indicators will remain within acceptable range.

In this article, we have developed some new and interesting results on the interconnections between the computation of μ-values and structured matrices like H-stable, $D(\alpha)$-stable matrices. Furthermore, we have developed novel results for rank-1 perturbations in D-semi-stable matrices, and computation of μ-values.

This paper is organized as follows. Section 2 is on preliminaries, where we have recalled notations, basic definitions, and results concerning our proposed study. The novel results on the interconnections amongst structured H-stable matrices and μ-values for structured matrices from economy models in Section 3. In the Section 4, a number of new results on the interconnection between $D(\alpha)$-stable matrices and structured singular values. These results are obtained with the help of the usage of a number of mathematical tools from linear algebra, matrix analysis, and system theory. The results and analysis on interconnection between structured singular values and a rank-1 perturbations to D-stable matrices are presented in Section 5. The numerical experimentation's and comparison on the computation of lower bounds of structured singular values (μ-values) for structured and unstructured matrices is carried out in Section 6. Finally, the conclusion is presented in the Section 8 of this articel.

2. Preliminaries

Throughout this paper, $\mathbb{R}$, and $\mathbb{C}$ denotes the field of real and complex numbers. The notations $\mathbb{R}^{n\times n}$, and $\mathbb{C}^{n\times n}$ represents n-dimensional real and complex valued matrices. The symbols $\lambda(\cdot)$ and $\mu(\cdot)$ denotes eigenvalues and structured singular values corresponding to a given matrix. The real part of the eigenvalues is denoted by $\mathrm{Re}(\lambda(\cdot))$. The family of positive definite structured matrices is represented as H^+ and $H > 0$ means that H is a positive definite matrix. The following Definitions 2.1-2.3, Theorem 2.10, and Theorem 2.11 are taken from [14].

Definition 2.1. Let $\alpha = \{p_1, p_2, \ldots, p_r\}$ be the partition of $\{1, 2, \ldots, n\}$. A diagonal matrix having diagonal blocks indexed by $p_1, p_2, \ldots, p_r$ is called α-diagonal.

Definition 2.2. Let $\alpha = \{p_1, p_2, \ldots, p_r\}$ be the partition of $\{1, 2, \ldots, n\}$. An n-dimensional complex valued matrix is known as $H(\alpha)$-stable matrix if the product AH denotes a stable matrix to each α-diagonal structured positive definite matrix, that is, the matrix H.

Definition 2.3. Let $\alpha = \{p_1, p_2, \ldots, p_r\}$ be the partition of $\{1, 2, \ldots, n\}$. An n-dimensional matrix A is known as to be real Lyapunov α-scalar stable matrix if $\exists$ a structured positive definite matrix D such that $(AD + DA^*) > 0$, where $*$ denotes the complex conjugate transpose of matrix A.

Definition 2.4 ([40]). Let M be an n-dimensional matrix, $\epsilon > 0$, a small perturbation. The pseusospectrum is the set of eigenvalues $\lambda \in \mathbb{C}$ such that

$$\|(\lambda I_n - M)^{-1}\| > \frac{1}{\epsilon}.$$

Definition 2.5. Let M be an n-dimensional matrix, then M is a stable matrix if $\text{Re}(\lambda_i(A)) > 0$, $\forall i = 1 : n$.

Definition 2.6 ([2]). Let M be an n-dimensional matrix, then M is structured D-stable matrix if $\text{Re}(\lambda_i(DA)) > 0$, $\forall i = 1 : n$ and for each positive diagonal matrix D.

Definition 2.7. The set of repeated number of real or complex scalar blocks is denoted by $\mathbb{B}_1$, and is defined as

$$\mathbb{B}_1 := \{\text{Diag}(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \cdots, \delta_S I_{r_S}) : \delta_i \in \mathbb{R} \text{ (or } \mathbb{C}), \ \forall\, i = 1 : n\}.$$

Definition 2.8. The set of full number of blocks is denoted by $\mathbb{B}_2$, and is defined as

$$\mathbb{B}_2 := \{\text{Diag}(\Delta_1, \Delta_2, \cdots, \Delta_S) : \Delta_1 \in \mathbb{R}^{m,n} \text{ (or } \ \mathbb{C}^{m,n}), \ \forall\, i = 1 : n\}.$$

Definition 2.9 ([31]). The structured singular value (or μ-values) for an n-dimensional structured matrix M with respect to a set of block-diagonal uncertainties $\mathbb{B}$ is defined as

$$\mu_{\mathbb{B}}(M) := (\min \|\Delta\|_2 : \det(I_n - M\Delta) = 0, \ \forall\, \Delta \in \mathbb{B})^{-1},$$

otherwise $\mu_{\mathbb{B}}(M) = 0$ for $\det(I_n - M\Delta) \neq 0, \ \forall\, \Delta \in \mathbb{B}$.

Theorem 2.10. *Let α be the partition of $\{1, 2, \ldots, n\}$. Then real Lyapunov α-scalar stable matrix is $H(\alpha)$ stable matrix.*

Theorem 2.11. *$A \in \mathbb{C}^{n\times n}$ is structured H-stable matrix if and only if the conditions given below holds true.*

(i) *$(A + A^*)$ is structured positive semi-definite matrix.*
(ii) *$x^*(A + A^*)x = 0$. In turn, $x^*(A - A^*)x = 0$ corresponding to each vector x.*
(iii) *$\lambda_i(A) \neq 0, \ \forall i = 1 : n$, where $\lambda(\cdot)$ is an eigenvalue from the spectrum of structured matrix A.*

3. New results on interconnection between H-stable matrices and μ-values

In this section of the article, we aim to develop novel results on interconnections between structured H-stability and structured singular values (or μ-values) for a family of squared real or complex valued structured matrices. The following Definition 3.1 is taken from [2].

Definition 3.1. A structured matrix $A \in \mathbb{R}^{n\times n}$ is known as structured H-stable matrix for stable HA to each structured symmetric positive-definite matrix H.

Theorem 3.2 shows matrix $A \in \mathbb{C}^{n\times n}$ being as a H-stable matrix whenever H is a structured positive definite matrix.

Theorem 3.2. *Let $A, H \in \mathbb{C}^{n\times n}, H > 0$, a structured positive definite matrix. If A is structured H-stable for every $H > 0$, then $H > 0$, a structured positive definite matrix whenever A is structured H-stable.*

Proof. Suppose that given $A \in \mathbb{C}^{n\times n}$ is structured H-stable matrix to each H, a structured positive definite matrix. This means that $\mathrm{Re}(\lambda_i(AH)) > 0, \forall i$. In fact, $H > 0$ causes $\mathrm{Re}(\lambda_i(AH)) > 0, \forall i$ for given $A \in \mathbb{C}^{n\times n}$. We conclude that for given $A \in \mathbb{C}^{n\times n}$, $\mathrm{Re}(\lambda_i(AH)) > 0, \forall i$ and this is very much possible only if $H > 0$, a positive definite matrix. □

The following gives an interaction between H-stability and μ-values.

Theorem 3.3. *Let $A \in \mathbb{R}^{n\times n}$ be H-stable. Then for every $H \in H^+, 0 \leqslant \mu_{\mathbb{B}}\left(\frac{1}{A^2}\right) < 1$, with*

$$H^+ = \{H : \lambda_i(H) > 0, \forall i = 1 : n\},$$

and the set $\mathbb{B}$ denotes the set of uncertainties having a block-diagonal structure.

Proof. Our aim is to show $0 \leqslant \mu_{\mathbb{B}}\left(\frac{1}{A^2}\right) < 1$ holds true for each $H \in H^+$. As D-stability of a matrix implies its stability, in turn this helps us to prove our desired result. $A \in \mathbb{R}^{n\times n}$ is D-stable iff A is stable and

$$\lambda_i \begin{pmatrix} A & -H \\ H & A \end{pmatrix} \neq 0, \quad \forall i.$$

Since, $\lambda_i \begin{pmatrix} A & -H \\ H & A \end{pmatrix} \neq 0,\ \forall i$. In turn this implies that $\lambda_i\left(A^2 - HA^{-1}HA\right) \neq 0,\ \forall i$. Furthermore,

$$\lambda_i\left(A^2 - HA^{-1}HA\right) \neq 0 \Rightarrow \lambda_i\left(I_n - \frac{1}{A^2}H\right) \neq 0,\ \forall H \in H^+.$$

Finally, we conclude that

$$\lambda_i\left(I_n - \frac{1}{A^2}H\right) \neq 0, \Rightarrow 0 \leqslant \mu_{\mathbb{B}}\left(\frac{1}{A^2}\right) < 1.$$

□

Theorem 3.4. *Let $A, H \in \mathbb{C}^{n\times n}, H \geqslant 0$, a Hermitian positive semi-definite matrix. If $\mathrm{Re}(\lambda_i(AH)) = \mathrm{Re}(\lambda_i(H))$ for every $H \geqslant 0$, then $\mathrm{Re}(\lambda_i(A)) > 0, \forall i$ and*

$$0 \leqslant \mu_{\mathbb{B}}\left((\boldsymbol{i}I_n + A)^{-1}(\boldsymbol{i}I_n - A)\right) < 1,\ \boldsymbol{i} = \sqrt{-1}.$$

Proof. Firstly, we aim to show that $\mathrm{Re}(\lambda_i(A)) > 0, \forall i$ if $\mathrm{Re}(\lambda_i(AH)) = \mathrm{Re}(\lambda_i(H)), \forall i, \forall H \geqslant 0$. From lemma [5], we have that, if $\mathrm{Re}(\lambda_i(A)) \geqslant 0, \forall i$ and $A \in \mathbb{C}^{n\times n}$ is $n \times n$-singular matrix, then $\exists\, U$, a unitary matrix such that

$$U^*AU = \begin{pmatrix} M_{11} + \mathbf{i}N_{11} & \mathbf{i}N_{12} \\ \mathbf{i}N_{21} & \cdot \end{pmatrix},$$

with $M_{11} > 0$; for $U^*AU = \begin{pmatrix} \cdot & \cdot \\ \cdot & I_n \end{pmatrix} \geqslant 0$. In turn, this yields $\mathrm{Re}(\lambda_i(AH)) = \mathrm{Re}(\lambda_i(H)), \forall i$. Secondly, the aim is to show that

$$0 \leqslant \mu_{\mathbb{B}}\left((\mathbf{i}I_n + A)^{-1}(\mathbf{i}I_n - A)\right) < 1.$$

Consider $A = e^H$, a stable matrix where $H \geqslant 0$, a positive semi-definite matrix. Let $P > 0$, a positive definite such that $\lambda_i(\mathbf{i}I_n + e^H P) \neq 0,\ \forall i$, where $P = (\mathbf{i}I_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta)$ for all $\Delta \in \mathbb{B}$. This allows us to have that

$$\lambda_i\left(\mathbf{i}I_n + e^H(iI_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta)\right) \neq 0, \quad \forall i, \forall \Delta \in \mathbb{B}.$$

Furthermore,

$$\lambda_i\left((\mathbf{i}I_n + e^H)^{-1}(\mathbf{i}I_n - e^H)\Delta\right) \neq 0, \quad \forall i, \forall \Delta \in \mathbb{B}.$$

Finally, this implies that

$$\lambda_i\left((\mathbf{i}I_n + A)^{-1}(\mathbf{i}I_n - A)\Delta\right) \neq 0, \quad \forall i, \forall \Delta \in \mathbb{B}.$$

Thus,

$$0 \leqslant \mu_{\mathbb{B}}\left((\mathbf{i}I_n + A)^{-1}(\mathbf{i}I_n - A)\right) < 1.$$

This completes the proof. □

The following Definitions are taken from [19, 20], respectively.

Definition 3.5. If the product DA is a stable matrix, then given structured matrix $A \in \mathbb{R}^{n\times n}$ is known as $D(\alpha)$ structured stable matrix to each positive α-scalar matrix D.

Definition 3.6. If $D[\alpha_k]$ is a scalar structured matrix to each $k = 1 : p$, that is,

$$D = \mathrm{Diag}(d_{11}I[\alpha_1], \ldots, d_{11}I[\alpha_p]),$$

then diagonal matrix D is called an α-scalar matrix.

Note: In Definition 3.6, $D[\alpha_k]$ represents a submatrix generated from number of rows and number of columns with indices α_k, where $\alpha = (\alpha_1, \ldots, \alpha_p)$ represents partition of indices set $\{1, \ldots, n\}$ and $1 \leqslant p \leqslant n$ being as a positive integer.

4. The novel results on interconnection between $D(\alpha)$-stable and structured singular matrices

The following theorem links the bridge for class of matrices like $D(\alpha)$-stable and structured singular values.

Theorem 4.1. *Let the structured matrix $A \in \mathbb{R}^{n\times n}$, then matrix A is $D(\alpha)$ structured stable matrix if and only if*

$$\mathrm{Re}\left(\lambda_i(\mathrm{Diag}(d_{kk}I[\alpha_k]))A + A^T(\mathrm{Diag}(d_{kk}I[\alpha_k]))\right) > 0,\ \forall i = 1 : n, \forall k = 1 : p, 1 \leqslant p \leqslant n,$$

and $0 \leqslant \mu_{\mathbb{B}}(M) < 1$, *where* M *is obtained from* A *as*

$$M = \left(\boldsymbol{i}I + \mathrm{Diag}(d_{kk}I[\alpha_k])A + A^T(\mathrm{Diag}(d_{kk}I[\alpha_k]))\right)^{-1}\left(\boldsymbol{i}I - \mathrm{Diag}(d_{kk}I[\alpha_k])A - A^T(\mathrm{Diag}(d_{kk}I[\alpha_k]))\right).$$

Proof. The aim is to prove that the structured matrix $A \in \mathbb{R}^{n\times n}$ is $D(\alpha)$-stable iff $0 \leqslant \mu_{\mathbb{B}}(M) < 1$. Let $\Delta \in \mathbb{B}$ a block-diagonal structure, that is,

$$\Delta = (\mathbf{i}I_n - \mathrm{Diag}(d_{kk}I[\alpha_k]))\,(\mathbf{i}I_n + \mathrm{Diag}(d_{kk}I[\alpha_k]))^{-1},\ \forall k = 1 : p, 1 \leqslant p \leqslant n.$$

As,

$$\lambda_i\left(\mathrm{Diag}(d_{kk}I[\alpha_k])A + A^T(\mathrm{Diag}(d_{kk}I[\alpha_k]))\right) \neq 0,\ \forall i, \forall k = 1 : p, 1 \leqslant p \leqslant n.$$

This can be written as

$$\lambda_i\left(\mathrm{Diag}(d_{kk}I[\alpha_k])A + A^T(\mathrm{Diag}(d_{kk}I[\alpha_k])) + (\mathrm{Diag}(d_{kk}I[\alpha_k]))\right) \neq 0,\ \forall i, \forall k = 1 : p, 1 \leqslant p \leqslant n,$$

and this is true if and only if

$$\lambda_i\left(\mathrm{Diag}(d_{kk}I[\alpha_k])A + A^T(\mathrm{Diag}(d_{kk}I[\alpha_k])) + \mathbf{i}(\mathbf{i}I_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta)\right) \neq 0,\ \forall i, \forall k = 1 : p, 1 \leqslant p \leqslant n.$$

This further implies that

$$\lambda_i\left((\mathbf{i}I_n + \mathrm{Diag}(d_{kk}I[\alpha_k])A + A^T\mathrm{Diag}(d_{kk}I[\alpha_k])A) - (\mathbf{i}I_n - \mathrm{Diag}(d_{kk}I[\alpha_k])A - A^T\mathrm{Diag}(d_{kk}I[\alpha_k]))\Delta\right) \neq 0,$$

$\forall i, \forall k = 1 : p$, $1 \leqslant p \leqslant n$, $\forall \Delta \in \mathbb{B}$. Thus,

$$\lambda_i\left(I_n - (\mathbf{i}I_n + \mathrm{Diag}(d_{kk}I[\alpha_k])A + A^T\mathrm{Diag}(d_{kk}I[\alpha_k]))^{-1}(\mathbf{i}I_n - \mathrm{Diag}(d_{kk}I[\alpha_k])A - A^T\mathrm{Diag}(d_{kk}I[\alpha_k]))\Delta\right) \neq 0,$$

$\forall i, \forall k = 1 : p$, $1 \leqslant p \leqslant n$, $\forall \Delta \in \mathbb{B}$. The final expression for λ_i, $\forall i$ is equivalent to the fact that $0 \leqslant \mu_{\mathbb{B}}(M) < 1$, see [34]. □

The following theorem provides necessary condition to matrix $A \in \mathbb{C}^{n,n}$ being as $D(\alpha)$-stable.

Theorem 4.2. *Let* $A \in \mathbb{C}^{n,n}$. *For matrix* A *being a* $D(\alpha)$*-stable matrix the necessary condition is that* $A = e^B$ *(the hermitian* $B \in \mathbb{C}^{n,n}$*) be a stable matrix and* $0 \leqslant \mu_{\mathbb{B}}\left((\boldsymbol{i}I_+ e^B)^{-1}(\boldsymbol{i}I_- e^B)\right) < 1$.

Proof. We desire to prove that A is $D(\alpha)$-stable matrix if $\lambda_k(\mathbf{i}I_n + e^B\, \mathrm{Diag}(d_{kk}I[\alpha_k])) \neq 0,\ \forall\, k = 1 : p,\ 1 \leqslant p \leqslant n$. Let $\Delta \in \mathbb{B}$ with a block-diagonal structure, and let

$$\mathrm{Diag}(d_{kk}I[\alpha_k]) = (\mathbf{i}I_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta).$$

Then, we conclude the fact that

$$\lambda_k(\mathbf{i}I + e^B(\mathbf{i}I + \Delta)^{-1}(\mathbf{i}I - \Delta)) \neq 0.$$

This further ensures that

$$\lambda_k((\mathbf{i}I + e^B)^{-1}(\mathbf{i}I - e^B)\Delta) \neq 0,\ k = 1 : p,\ 1 \leqslant p \leqslant n,\ \forall\ \Delta \in \mathbb{B}.$$

Thus the definition of structured singular values allows us to conclude that

$$0 \leqslant \mu_{\mathbb{B}}\left((\mathbf{i}I_n + e^B)^{-1}(\mathbf{i}I_n - e^B)\right) < 1.$$

This complete the proof. □

5. The rank-1 perturbation to D-semi-stable matrices

Next, we aim to establish some novel results to study connection between D-semi-stable matrices, and the structured singular values. Theorem 5.1 shows that $A \in \mathbf{C}^{n,n}$ is D-semi-stable iff it is semi-stable and all eigenvalues of rank-1 perturbation to A, that is $A + \nu\omega^*$, are non-zero.

Theorem 5.1. *Let* $A \in \mathbb{C}^{n,n}$, *then* A *is* D*-semi-stable iff* A *is a semi-stable matrix and* $\lambda_k(A + \nu\omega^*) \neq 0, \forall k$, *with* $\nu\omega^*$, *a rank-1 matrix.*

Proof. Consider that matrix A is D-semi-stable, meaning that $\mathrm{Re}(\lambda_k(AD)) \geqslant 0$, for all k and for all $D \in \Omega$. We require to prove that matrix A is semi-stable, means that $\mathrm{Re}(\lambda_k(A)) \geqslant 0$ for all k and $\lambda_k(A + \nu\omega^*) \neq 0$, $\forall k,\ \nu\omega \in \mathbb{C}^{n,1}$. Since, $\mathrm{Re}(\lambda_k(AD)) \geqslant 0, \forall k, \forall D \in \Omega$. This implies that $\mathrm{Re}(\lambda_k(AI_n)) \geqslant 0, \forall k$ or $\mathrm{Re}(\lambda_k(A)) \geqslant 0$. This implies that A is semi-stable matrix. Furthermore, for a rank-1 perturbation to A, that is, $(A + \nu\omega^*)$, we aim that

$$\mathrm{Re}(\lambda_k(A + \nu\omega^*)) \neq 0,\ \ \forall k.$$

We assume that for $\nu, \omega \in \mathbb{C}^{n,1}$, $z_l^*\nu \neq 0$ and $\omega^* z_r \neq 0$. Here, z_l and z_r are the left hand side and sight hand side eigenvectors corresponding to a simple eigenvalue of A. Additionally, suppose that $y \in \mathrm{Ker}(A + \nu\omega^*)$. This allows to have homogeneous system of linear equations of the form

$$z_l^*\nu\omega^* y = z_l^*(A + \nu\omega^*)y = 0.$$

This implies that $\omega^* y = 0$ because $z_l^*\nu \neq 0$. On the other hand, $(A + \nu\omega^*)y = 0$, so that $y = \alpha z_r$ for some $\alpha \in \mathbb{C}$. This, in turn implies that $\omega^* y = \alpha\omega^* z_r = 0 \Rightarrow \alpha = 0, y = 0$. Conversely, suppose that if $\omega^* z_r = 0$, then $(A + \nu\omega^*)z_r = 0$. Also, $z_l^*(A + \nu\omega^*) = 0$ if $z_e^*\nu = 0$. □

The following given Theorem 5.2 shows that a square complex valued matrix is D-semi-stable if only matrix A is semi-stable and the structured singular value is bounded from above by 1.

Theorem 5.2. *Let* $A \in \mathbb{C}^{n,n}$, *then* A *is* D*-semi-stable iff it is a semi-stable matrix while* $0 \leqslant \mu_{\mathbb{B}}(M) < 1$. *Here*

$$M = (\boldsymbol{i}I_n + \widehat{A})^{-1}(\boldsymbol{i}I_n - \widehat{A}),$$

with $\widehat{A} = A + \nu\omega^*$ *for* $\nu, \omega \in \mathbb{C}^{n,1}$.

Proof. Given matrix A be a D-semi-stable iff A be a semi-stable while $\lambda_k(A+\nu\omega^*+\mathbf{i}P)$ not equal to zero for all k, and for all $P \in \Omega$. To prove $\lambda_k(A+\nu\omega^*+\mathbf{i}P) \neq 0$ for all k and for all $P \in \Omega$, consider that $\Delta = (\mathbf{i}I - P)(\mathbf{i}I + P)^{-1}$ having a block diagonal structure, and $\Delta \in \mathbb{B}$. The diagonal structured matrix P possesses all positive entries on its principal diagonal takes following form

$$P = (\mathbf{i}I_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta).$$

Since, $\lambda_k(A+\nu\omega^*+\mathbf{i}P) \neq 0$, which ensure that

$$\lambda_k\left(A+\nu\omega^*+\mathbf{i}(\mathbf{i}I+\Delta)^{-1}(\mathbf{i}I-\Delta)\right) \neq 0, \quad \forall \Delta \in \mathbb{B}.$$

Furthermore,

$$\operatorname{rank}\left(A+\nu\omega^*+\mathbf{i}(\mathbf{i}I_n+\Delta)^{-1}(\mathbf{i}I_n-\Delta)\right) = \operatorname{rank}\left((\mathbf{i}I_n+A+\nu\omega^*)-(\mathbf{i}I_n-A+\nu\omega^*)\Delta\right), \quad \forall \Delta \in \mathbb{B}.$$

In turn, this yields

$$\left(A+\nu\omega^*+\mathbf{i}(\mathbf{i}I_n+\Delta)^{-1}(\mathbf{i}I_n-\Delta)\right) \sim \left((\mathbf{i}I_n+A+\nu\omega^*)-(\mathbf{i}I_n-A+\nu\omega^*)\Delta\right), \quad \forall \Delta \in \mathbb{B}.$$

Also,

$$\left((\mathbf{i}I_n+A+\nu\omega^*)-(\mathbf{i}I_n-A+\nu\omega^*)-(\mathbf{i}I_n-A+\nu\omega^*)\Delta\right), \quad \forall \Delta \in \mathbb{B}.$$

Finally, We conclude that

$$\lambda_k\left(I_n-(\mathbf{i}I_n+A+\nu\omega^*)^{-1}(\mathbf{i}I_n-A+\nu\omega^*)\Delta)\right) \neq 0, \quad \forall \Delta \in \mathbb{B}.$$

This is necessary condition that $0 \leqslant \mu_{\mathbb{B}}(A) < 1$. This complete the proof. □

5.1. Pseudo-spectrum

The pseudo-spectrum of a matrix M is the set of which contains the spectrum (all the eigenvalues), that is, the spectrum of matrix M. The important question one can raise is about the singularity of M which does not appear as a robust in the sense that a small perturbation ϵ may vary the answer from yes to no in a dramatic way. This helps to think that either $\|M^{-1}\|$ is large enough or not? For λ, an eigenvalue of M, a much better question is to ask: is $\|(\lambda I_n - M)^{-1}\|$ large or not? such a pattern allows following definitions and results [40] of pseudo-spectrum.

Definition 5.3. Let M be a given n-dimensional matrix, $\epsilon > 0$, a small perturbation. The ϵ-pseudospectrum $\sigma_\epsilon(M)$ is the set of eigenvalues $\lambda \in \mathbb{C}$ such that

$$\|(\lambda I_n - M)^{-1}\| > \frac{1}{\epsilon}.$$

Remark 5.4. For $\lambda \in \sigma(M)$, $\sigma(M)$ being as the set of eigenvalues of M, $\|(\lambda I_n - M)^{-1}\| = \infty$. The second definition of pseudo-spectrum is given as follows.

Definition 5.5. Let M be a given n-dimensional matrix, $\epsilon > 0$, a small perturbation. The ϵ-pseudospectrum $\sigma_\epsilon(M)$ is the set of eigenvalues $\lambda \in \mathbb{C}$ such that $\lambda \in \sigma(M+E)$, for some E with $\|E\| < \epsilon$.

The third characterization of pseudo-spectrum is given as bellow.

Definition 5.6. Let M be a given n-dimensional matrix, $\epsilon > 0$, a small perturbation. The ϵ-pseudospectrum $\sigma_\epsilon(M)$ is the set of eigenvalues $\lambda \in \mathbb{C}$ such that $\|(\lambda I_n - M)\nu\| < \epsilon$, for some $\nu \in \mathbb{C}^{n,1}$, $\|\nu\| = 1$.

The following Theorem gives an equivalence of all above definitions of pseudo-spectrum.

Theorem 5.7. *For an n-dimensional complex valued M, all three definitions of pseudo-spectrum are equivalent.*

6. Numerical experimentation

In this section of the article, we a give a comparison on the numerical approximation of lower bounds of μ-values. The numerical algorithms under consideration for approximation of lower bounds of μ-values are: the well-known Matlab routine **mussv**, the power algorithm (PA) [32], Gain Based Algorithm (GBA) [37], Poles Migration Algorithm (PMA) [25], Non-linear optimization Algorithm (NLA) [11], and the Low-rank ordinary differential equations based algorithm (LRA) given by first author [10]. The matrices are taken from various models of economy and finance. Furthermore, we use EigTool [39] for the computation of the pseudo-spectrum of each structured matrix.

The Matlab routine **mussv** which is freely available in the Matlab Control ToolBox providing the best results on the numerical computation of both lower and upper bounds of μ-values. The proposed methodology provides results on the numerical computation of lower bounds of μ-values of structured matrices from various economic models. The reasons for the sharpness of new results as compared to existing techniques are following.

1. The proposed methodology uses the computation of singular values instead of eigenvalues of structured matrices.
2. The computation of singular values is obtained with singular value decomposition technique. The Matlab command **svd** is being used to compute the singular values.
3. The computation of largest singular value of structured matrices in the sense of low-rank yields the sharper lower bounds of structured singular values, and also computational cost is not high once compared with exiting techniques.

We give a number of numerical examples for economy models on the numerical computation of lower bounds of μ-values or structured singular values. The obtained results on the computation of lower bounds of μ-values demonstrate the effectiveness of proposed methodology and existing techniques.

The following Examples 6.1 and 6.2 are take from [29], where authors aim to determine the capital demands which is a complicated procedure in economy and finance. Almost every factory, company faces such problems and then seek their optimal solutions. From mathematical prospectus, this demands the study of the **cost matrix** obtain by calculating capital demands.

Example 6.1. We consider a 2-dimensional real valued cost matrix, which is taken from [29],

$$A = \begin{bmatrix} 4 & 6 \\ 5 & 7 \end{bmatrix}.$$

We present a comparison on numerical computation of the lower bounds to structured singular values in Table 1.

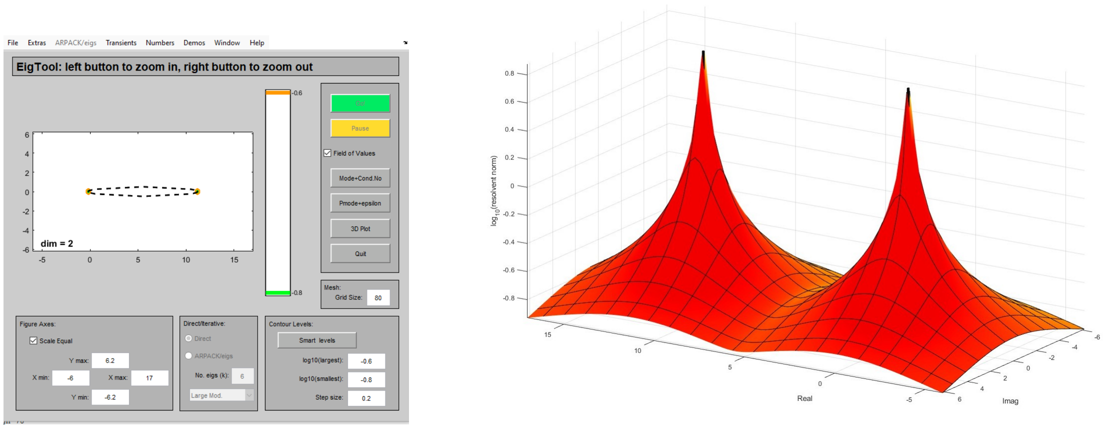

Figure 1: The pseudo-spectrum of the structured matrix given in Example 6.1.

Table 1: The μ-values lower bounds.

mussv	PA	GBA	PMA	NLA	LRA
11.2236	11.2236	11.4496	11.4501	11.5512	11.2236

Example 6.2. We take a 3-dimensional real valued matrix that denotes the quantity of product in each material in a economy model as

$$A = \begin{bmatrix} 10 & 30 & 20 \\ 80 & 70 & 90 \\ 40 & 50 & 60 \end{bmatrix}.$$

We present the comparison on numerical approximation of the lower bounds of structured singular values in Table 2.

Table 2: The μ-values lower bounds.

mussv	PA	GBA	PMA	NLA	LRA
167.8493	167.8493	167.9103	167.1001	167.9814	167.8413

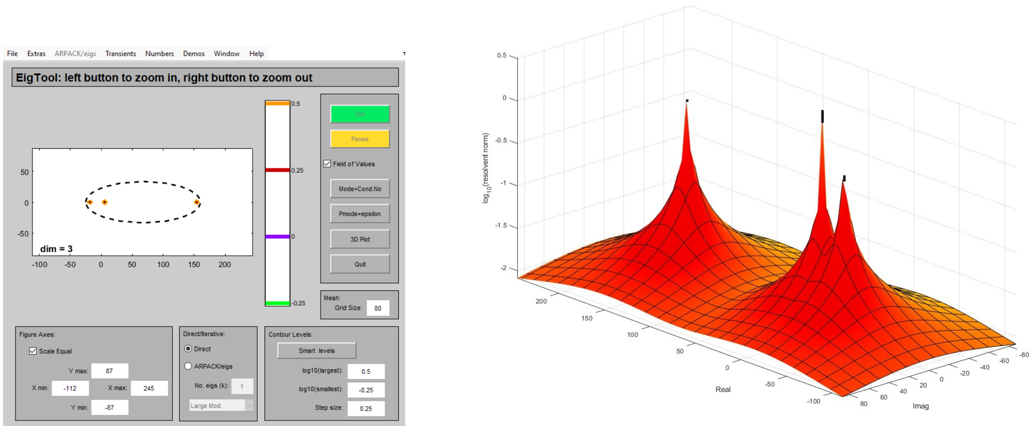

Figure 2: The pseudo-spectrum of the structured matrix given in Example 6.2.

The dynamic stochastic models are used in macro economy. The coefficients of the linear state-space system is represented by structured D-stable matrices. Example 6.3 is a 2-dimensional structured D-stable matrix corresponding to an economy model.

Example 6.3. Consider following 2-dimensional D-stable matrix given as

$$A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}.$$

We present the comparison on numerical computation of the lower bounds of μ-values in Table 3.

Table 3: The μ-values lower bounds.

mussv	PA	GBA	PMA	NLA	LRA
1.0096	1.0096	1.0118	1.0198	1.0213	1.0090

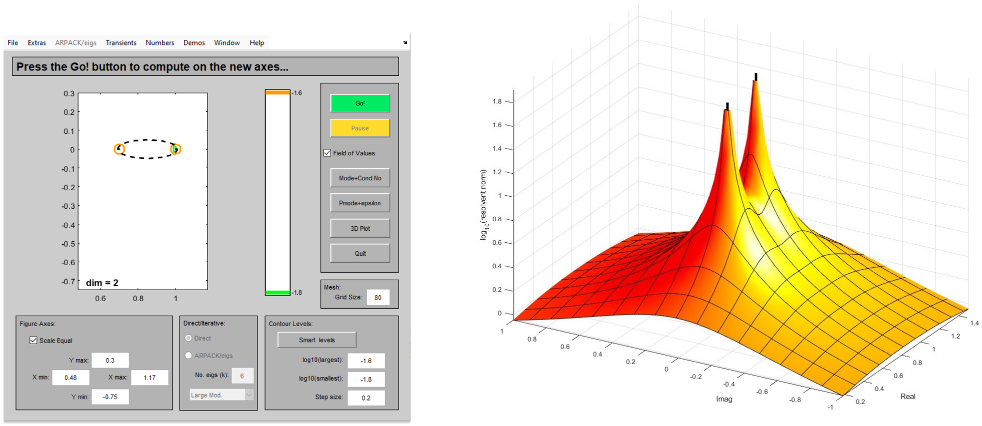

Figure 3: The pseudo-spectrum of the structured matrix given in Example 6.3.

Example 6.4. Consider following 5-dimensional matrix taken from [35],

$$A = \begin{bmatrix} 0 & 0 & 0.71 & 0 & 0 \\ 0.18 & 0 & 0 & 0.71 & 0 \\ 0.71 & 0.59 & 0.71 & 0.71 & 0.40 \\ 0.70 & 0.71 & 0.59 & 0.71 & 0.40 \\ 0 & 0 & 0 & 0.59 & 40 \end{bmatrix}.$$

We present the comparison on numerical computation of the lower bounds of μ-values in Table 4.

Table 4: The μ-values lower bounds.

mussv	PA	GBA	PMA	NLA	LRA
2.1279	2.1277	2.1283	2.1378	2.1301	2.1276

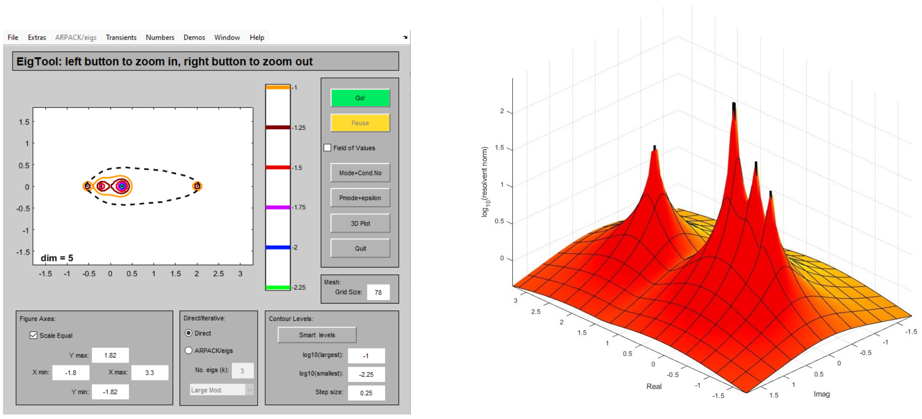

Figure 4: The pseudo-spectrum of the structured matrix given in Example 6.4.

Example 6.5. Consider following 3-dimensional matrix obtained from a dynamic forward-looking model taken from [38]:

$$A = \begin{bmatrix} 0.50 & 0.15 & 0.10 \\ 0.10 & 0.40 & 0.05 \\ 0 & 0.20 & 0.80 \end{bmatrix}.$$

We present the comparison on numerical computation of the lower bounds of μ-values in Table 5.

Table 5: The μ-values lower bounds.

mussv	PA	GBA	PMA	NLA	LRA
0.8637	0.8731	0.8810	0.8710	0.8793	0.8738

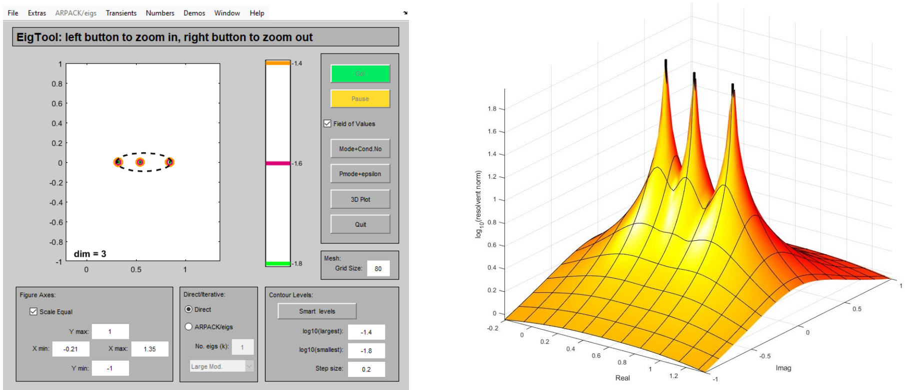

Figure 5: The pseudo-spectrum of the structured matrix given in Example 6.5.

Finally, we give pseudo-spectrum of Frank matrix obtained by Matlab command F = gallery('frank',N).

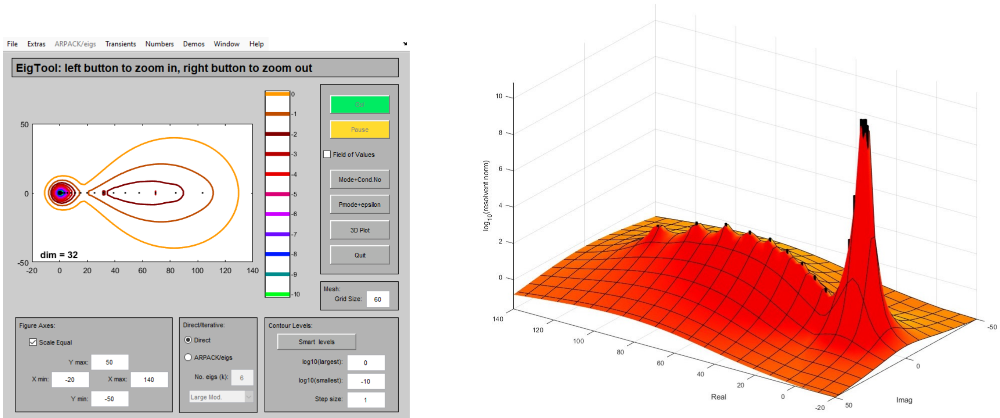

Figure 6: The pseudo-spectrum of Frank matrix.

7. Merits and demerits of proposed methodology

The proposed methodology to construct novel results on the interconnections amongst structured D-stability and its variants, and structured singular values (μ-values) for economy models is based on the collection of various tools from linear algebra, matrix analysis and system theory. The results on D-stability and structured singular values (μ-values) provide an approach to understand the stability and robustness for economy models subject to constraints in terms of structured and unstructured uncertainties.

Merits: The proposed method to study the interconnection between D-stability theory and μ-theory has the following merits.

1. The proposed methodology is applicable to n-dimensional complex valued matrices instead of n-dimensional real valued matrices.
2. The collection of various tools from linear algebra, matrix analysis and system theory gives an accessible approach to build a new methodology rather developing a complex geometric approach to study the interconnection between D-stability and μ-theory.

3. The proposed methodology ensure that equilibrium points of an economy models will remains stable under structured or unstructured perturbations.
4. A large number of economic models are sensitive to different uncertainties, for instance, the shocks to supply chains, the sudden change in market policy. The proposed methodology allows to analyze the robust analysis of performance of economy models under structured or unstructured uncertainties.
5. The economic models particularly those based on linear assumptions, can benefit from the proposed methodology to study D-stability and its connection with structured singular values to ensure that a number of economic indicators will remain within acceptable range.

Demerits: The proposed method to study the interconnection between D-stability theory and μ-theory has the following merits.

1. The proposed methodology faces problems while dealing with such economy models where the coefficient matrices are of higher dimensions.
2. As most of the real world economy models are highly nonlinear in their nature, the proposed methodology suits well to linear economy models and when applying it non-linear economy models then it may produce inaccurate results.
3. The economy models can be of much large scale and complex in their nature and the proposed methodology requires a significant computation power. It is specially very difficult to deal with high-dimensional economy models with the help of proposed methodology.
4. The proposed methodology does provide insights to study and discuss the stability of economy models, but it might not work well for micro-level dynamics of economy mode4ls, for instance, for the behavior of individual firms or markets.
5. The proposed methodology lacks the estimation to a number of an exact parameters and structured perturbations to an economy model which can be very challenging. This may happen due to difficulty level while obtaining the precise data on economy variables.

8. Conclusion

In this article, we have considered the problem related to the interconnection among D-stability theory, and that of μ-theory. We have developed some new mathematical results linking the bridge for H-stability, $D(\alpha)$-stability, a rank-1 perturbation to D-semi-stable matrices, and structured singular values (μ-values). The numerical experimentation show the comparison of bounds from below to the μ-values approximated by various numerical techniques. The Matlab EigTool is used for the computation of pseud-spectrum of structured matrices appearing across different economy models. The advantages and limitations of the proposed work are listed as following.

Advantages of the proposed work: The proposed work in this article provides an interconnections between structured singular values and a larger class of D-stable matrices. The work presented in this paper on D-stability is applicable to analyze the nonlinear systems with help of Lyapunov functions. The mathematical results on D-stability helps to obtain stable equilibrium points under different conditions, for instance, market shocks and policy changes. Most of the economic models involve structured uncertainties, for instance, the structured and unstructured uncertainties among the tax policies, the rates of interest. The study on μ-analysis helps to identifying how such structured or unstructured uncertainties can affect the overall economy while allowing policymakers to design more robust policies.

Limitations of the proposed work: The study of D-stability and structured singular values has limitations in the sense that the model under consideration are often linear or linear time varying. The application of D-stability to nonlinear systems can be overly simplistic. The obtained results from D-stability may not capture the true dynamics of the economy models. The μ-analysis is computationally expensive. Unfortunately, an exact computation of μ-values is a hard problem, infact an NP-hard problem. The economy models mostly does involve a large number of parameters and variables and hence it demands very high computational requirements for robust analysis. Unfortunately, this leads practical difficulties to scale μ-analysis to large scale macroeconomic models.

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