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Approximate controllability of a semilinear impulsive fractional stochastic integro-differential inclusions with Poisson jumps

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Abstract

This work explores the approximate controllability of a semilinear impulsive fractional stochastic integro-differential inclusions with Poisson jumps involving the Caputo fractional derivative of order $q \in (0, 1)$. We consider a class of control systems governed by fractional differential inclusions by using Bohnenblust-Karlin's fixed point theorem and stochastic analysis theory to derive a new set of sufficient conditions for the approximate controllability of a semilinear impulsive fractional stochastic integro-differential inclusions with Poisson jumps. Finally, an example is given to illustrate the results obtained.

Keywords: Approximate controllability, semilinear systems, mild solutions, impulsive systems, Poisson jumps.

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1. Introduction

Controllability, a fundamental principle of mathematical control theory, is essential to both deterministic and stochastic control systems. The idea of controllability applies to systems represented by ordinary and partial differential equations in finite and infinite-dimensional spaces. In essence, this property states that a dynamical control system may be controlled by a set of allowed controls from an arbitrary beginning state to an arbitrary end state. However, fractional differential equations may be an effective tool for characterizing a number of diverse situations. For many real-world applications, fractional-order models are preferred over integer-order ones. For more information, refer to books [17, 27, 45] and the research articles [26, 38, 41].

On the other hand, processes that occasionally undergo an abrupt change in state are the focus of impulsive differential equations. Processes having this characteristic frequently include control theory, physics, population dynamics, biology, medicine, and many other fields, as well as processes that naturally arise in the dynamics of various populations. These disturbances are best described as sudden shifts in state or impulses. Impulsive differential equations are used to model these processes. Boudaoui and Lakhel [9] studied the controllability of stochastic impulsive neutral functional differential equations

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with jumps. Lin and Hu [19] established the existence results for impulsive neutral stochastic functional integro-differential inclusions with nonlocal initial conditions. Sakthivel [33] discussed the approximate controllability of impulsive stochastic evolution equations. Rajchakit et al. [29–31] investigated the stability, global exponential stability, and passivity for impulsive neural networks with time-varying delays.

The Poisson jumps have attracted a lot of interest and are commonly employed to replicate a wide range of events that occur in disciplines including economics, biology, physics, medicine, economics, and other sciences. Moreover, many real-life problems (such as those that experience abrupt price changes [jumps] as a result of earthquakes, market collapses, epidemics, and so on) might experience a few stochastic disturbances of the jump kind. Such systems have non-continuous sampling pathways. To better explain such models, stochastic processes involving jumps should be considered. Frequently, Poisson random data is the source of these jump models. Such systems have right-continuous sample routes with left bounds. Recently, the exploration of stochastic differential equations with jumps has become more and more popular. Evidence for the existence and uniqueness of stochastic differential equations with Markovian switching and Poisson jumps was presented by [10, 15, 20, 35, 44].

The concept of the controllability of finite-dimensional deterministic linear control systems was first proposed by Kalman [16]. See [7] and [11], respectively, for details on the core concepts of control theory in finite and infinite-dimensional spaces. However, in several cases, some reasonable randomness can appear in the problem, so that the system should be modeled by a stochastic form. Few writers have examined how deterministic controllability ideas may be applied to stochastic control systems. By utilizing the Banach fixed point method, [12] examined the controllability of semilinear stochastic systems in Dauer and Mahmudov systems. Mahmudov [21, 22, 25] investigated the controllability results for linear, semilinear and nonlinear stochastic systems. Author [23] presented the approximate controllability of semilinear deterministic and stochastic evolution equations. Controllability of nonlinear stochastic systems was discussed by [3, 6, 34]. Mahmudov et al. [24] investigated the approximate controllability results for fractional semilinear integro-differential inclusions in Hilbert spaces. In [39, 42], authors studied the approximate controllability for integro-differential systems.

Sakthivel and others [36, 37] looked at the existence of findings for fractional stochastic differential equations. On the other hand, very few writers have used semigroup theory to look into the neutral functional integro-differential systems in Banach space that are controllable. Balachandran et al. [4, 5] established the controllability results for stochastic integro-differential systems. Additionally, by utilizing Bohnenblust-Karlin's fixed point theorem and multivalued maps, researchers in [40] discussed fractional stochastic semilinear differential inclusion. Using Poisson jumps and nonlocal conditions, [2] Anguraj et al. looked on the approximate controllability of a semilinear impulsive stochastic system.

Inspired by the above-mentioned work, this paper aims to fill this gap. The purpose of the article is to show the approximate controllability of semilinear impulsive fractional stochastic integro-differential inclusions with Poisson jumps of the form:

$$\begin{cases} {}^{C}D^{q}x(t) \in Ax(t) + Bu(t) + f(t,x(t)) + \int_0^t e(t,s,x(s))ds + g(t,x(t))\frac{dw(t)}{dt} \\ \qquad + \int_{\mathfrak{U}} h(t,x(t),u)\tilde{N}(dt,du), \quad t \in J = [0,b],\ 0<q<1,\ t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)), \quad t = t_k,\ k = 1,2,\dots,m,\ x(0) = x_0 + p(x), \end{cases} \tag{1.1}$$

where ${}^{C}D^{q}$ denotes the Caputo fractional derivative of order q, A represents the infinitesimal generator for the bounded linear operator $T(t)$, $t \in [0,b]$ on $\mathbb{H}$. $B : \mathfrak{U} \to \mathbb{H}$ represents a bounded linear operator. The control $u \in L^2_{\mathfrak{F}}([0,b],\mathfrak{U})$, $f : [0,b] \times \mathbb{H} \to \mathbb{H}$, $e : J \times J \times \mathbb{H} \to \mathbb{H}$, $g : [0,b] \times \mathbb{H} \to L^2_0$, $h : [0,b] \times \mathbb{H} \times \mathfrak{U} \to \mathbb{H}$, and $I : \mathbb{H} \to \mathbb{H}$ are suitable functions, and $p : C([0,b],\mathbb{H}) \to \mathbb{H}$. Additionally, the constant moments of time t_k fulfill $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, consequently. $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ denotes the jump in the state x at time t_k with I_k is the size of the jump.

The following are the main findings of our manuscript.

(i) A study on the approximate controllability of a semilinear impulsive fractional stochastic integro-differential inclusions with Poisson jumps is an unexplored topic in the literature and this is an additional motivation for writing this paper.
(ii) In this manuscript, using Bohnenblust-Karlin's fixed point methodology, we establish the approximate controllability of a semilinear impulsive fractional stochastic integro-differential inclusions with Poisson jumps.
(iii) The approximate controllability of (1.1) is illustrated, considering that the corresponding linear system is approximately controllable.
(iv) The main findings are demonstrated using an example.

The format of this article is as follows. In Section 2, it offers some basic definitions, lemmas, and theorems that are useful in demonstrating the main findings. Section 3 examines the adequate condition to show approximate controllability for the system (1.1). In Section 4, the main conclusion is illustrated by an example.

2. Preliminaries

Let (Ω, F, P) be the complete probability space equipped with normal filtration F_t, $t \in J = [0, b]$. The separable Hilbert spaces are $\mathtt{H}$, $\mathcal{U}$, and Y. Consider w denote a Q-Wiener process on (Ω, F, P), and let Q be the covariance operator such that $\mathrm{tr}(Q) \leqslant \infty$. Suppose that Y contains a bounded sequence of nonnegative real numbers λ_n, which is a full orthonormal system e_n, where $\lambda_n > 0$ such that $Qe_n = \lambda_n e_n$, $n = 1, 2, \ldots$, and β_n of independent Brownian motions such that

$$w(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n(t)e_n, \quad t \in J,$$

and $F_t = F_t^w$, where F_t^w is the σ-algebra that w produced and the value should be $L_2^0 = L_2(Q^{\frac{1}{2}}Y, \mathtt{H})$. The space that contains all Hilbert-Schmidt operators is often used, ranging from $Q^{\frac{1}{2}}Y$ to $\mathtt{H}$, additionally the norm is $\|\delta\| = \mathrm{tr}[\delta Q \delta_*]$. Let $J_1 = [-h, b]$ and the Banach space of all piecewise continuous functions $x(t) : J_1 \to L_2(\Omega, F_t, \mathtt{H})$ be specified, which fulfills $\sup_{t \in J_1} E\|x(t)\|^2 < \infty$. Assume that $PC(J_1, L_2(\Omega, F_t, \mathtt{H}))$ has a closed subspace $\mathscr{C} = C([0, b]; \mathtt{H})$ it includes all processes that can be measured and F_t-adapted, $x(\cdot) : t \in [-h, b]$ while using the norm topology by

$$\|\Phi\| = \left(\sup_{t \in [0,b]} E\|\Phi(t)\|_{\mathtt{H}}^2 \right)^{\frac{1}{2}}.$$

Assume $P(t)$, $t \geqslant 0$ is a stationary Poisson point process that values are taken from the measurable space $(\mathcal{U}, B(\mathcal{U}))$ and F_t-adapted. It is known as the Poisson random measure denoted by $P(\cdot)$. It is defined as $N_p((0, t] \times \Lambda) := \sum_{s \in (0,b]} I_\Lambda(P(s))$ for $\Lambda \in B(\mathcal{U})$. After that, provide the measure $\tilde{N}$ by

$$\tilde{N}(dt, du) = N_p(dt, du) - \nu(du)dt,$$

this is known as the compensated Poisson random measure.

Definition 2.1 ([1]). System (1.1) is approximately controllable on J if $\overline{R(b)} = L_2(\Omega, F_b, \mathtt{H})$, where $R(b) = \{x(b; u) : u \in L_F^2([0, b], \mathcal{U})\}$.

For each $0 \leqslant t \leqslant b$, $\delta(\delta I + \psi_0^b)^{-1} \to 0$ is the strong operator topology as $\delta \to 0^+$, the controllability Gramian is of the form

$$\psi_0^b = \int_0^b S_q(t - s)BB^*S_q^*(t - s)ds.$$

Keep in mind the linear deterministic system associated with (1.1),

$$^{C}D^{q}x(t) \in Ax(t)+Bu(t),\ t\in[0,b],\ x(0)=x_0+p(x),$$

is approximately controllable on J iff $\delta(\delta I+\psi_0^b)^{-1}\to 0$ strongly as $\delta\to 0^+$.

Definition 2.2 ([28])**.** The Caputo derivative of order q for $f:[0,\infty)\to\mathbb{R}$ can be expressed as

$$^{C}D^{q}f(t) = {}^{L}D^{q}\left(f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^k(0)\right),\quad t>0,\ n-1<q<n.$$

If $f(t)\in C^n[0,\infty)$, then

$$^{L}D^{q}f(t)=\frac{1}{\Gamma(n-q)}\frac{d^n}{dt^n}\int_0^t\frac{f(s)}{(t-s)^{q+1-n}}ds = I^{n-q}f^n(s),\quad t>0,\ n-1<q<n.$$

Now, we will provide some basic definitions of multivalued maps. Refer Deimling [14] for further information about multivalued maps and the basic definitions of upper semicontinuous, relatively compact, the readers can refer to [43].

Definition 2.3 ([43])**.** g is called completely continuous if $g(C)$ is relatively compact for every bounded subset C of H.

The multivalued map g is completely continuous with nonempty values, then g is upper semicontinuous if g has a closed graph, i.e., $x_n\to x_*,\ y_n\to y_*,\ y_n\in g(x_n)$ imply $y_*\in g(x_*)$. g has a fixed point if there is $x\in H$ such that $x\in g(x)$.

In the preceding $P_{bd,cl,cv}(H)$ stands for family of all nonempty bounded, closed and convex subset of H.

A multivalued map $g: J\to P_{bd,cl,cv}(H)$ is called measurable if $\forall x\in H$ the mean square distance between x and $g(t)$ is measurable function on J.

Define a collection of choices of g, $\forall x\in L_2^0$,

$$\sigma\in S_{g,x}=\left\{\sigma\in L_2^0:\sigma(t)\in g(t,x(t)) \text{ for a.e. } t\in J\right\}.$$

Definition 2.4 ([36])**.** A stochastic processes $x\in\mathscr{C}$ is a mild solution of (1.1) if, $\forall u\in L_F^2(J,\mathcal{U})$, it represents to the integral equation

$$\begin{aligned}x(t)=&T_q(t)(x_0+p(x))+\int_0^t S_q(t-s)[Bu(s)+f(s,x(s))]ds\\&+\int_0^t S_q(t-s)\left[\int_0^s e(s,r,x(r))dr\right]ds+\int_0^t S_q(t-s)g(s,x(s))dw(s)\\&+\int_0^t S_q(t-s)h(s,x(s),u)\tilde{N}(ds,du)+\sum_{0<t_k<t_1}T_q(t-t_k)I_k(x(t_k)),\end{aligned}$$

where

$$T_q(t)=E_{q,1}(t^qA)=\frac{1}{2\pi i}\int_{B_r}e^{\lambda t}\frac{\lambda^{q-1}}{\lambda^Q-A}d\lambda,\quad S_q(t)=t^{q-1}E_{q,q(t^qA)}=\frac{1}{2\pi i}\int_{B_r}e^{\lambda t}\frac{1}{\lambda^Q-A}d\lambda,$$

B_r denotes the Bromwich path. If $q\in(0,1)$ and $A\in A^q(\theta_0,\omega_0)$, then for each $x\in C$ and $t>0$, we obtain $\|T_q(t)\|\leqslant Me^{\omega t}$ and $\|S_q(t)\|\leqslant Ce^{\omega t}(1+t^{q-1}),\ t>0,\ \omega>\omega_0$. Let

$$M_T=\sup_{0\leqslant t\leqslant b}\|T_q(t)\|,\quad M_S=\sup_{0\leqslant t\leqslant b}Ce^{\omega t}(1+t^{1-q}).$$

Hence, we have $\|T_q(t)\|\leqslant M_T$ and $\|S_q(t)\|\leqslant t^{q-1}M_S$.

Definition 2.5 ([32]). The multi-valued map $g : J \times H \to P_{bd,cl,cv}(H)$ is called L_2-Caratheodory if

(i) $t \to g(t,\nu)$ is measurable $\forall \nu \in H$;
(ii) $\nu \to g(t,\nu)$ is u.s.c., for almost all $t \in J$;
(iii) $\forall r > 0$, $\exists\, L_{\sigma,r} \in L_1(J,\mathbb{R}^+)$ such that

$$\|g(t,\nu)\|^2 := \sup_{\sigma\in g(t,\nu)} E\|\sigma\|^2 \leqslant L_{\sigma,r}(t),\ \forall\ \|\nu\|^2 \leqslant r \text{ and for a.e. } t \in J.$$

Lemma 2.6 ([13]). *Let $\sigma : J \times \Omega \to L_2^0$ be the strongly measurable function such that $\int_0^b E\|\sigma(t)\|^p_{L_2^0} < \infty$. Then,*

$$E\left\| \int_0^b \sigma(s)dw(s)\right\|^p \leqslant L_\eta \int_0^b E\|\sigma(s)\|^p_{L_2^0} ds,$$

$\forall t \in J$ and $p \geqslant 2$, where L_η is a constant involving p and b.

Lemma 2.7 ([18]). *Consider J as compact real interval, $P_{bd,cl,cv}(H)$ be the collection of all nonempty, closed, bounded and convex subsets of H and g be a multi-valued maps fulfilling $g : J \times H \to P_{bd,cl,cv}(H)$ is measurable to $t\ \forall\, x \in H$ is u.s.c., to $x\ \forall\, t \in J$, and $\forall x \in C(J,H)$ the set*

$$S_{g,x} = \{\sigma \in L^1(J,H) : \sigma(t) \in g(t,x(t)), t \in J\}$$

is nonempty. Let $\zeta : L^1(J,H) \times C$ be a continuous function, then

$$\zeta \circ S_g : C(J,H) \to P_{bd,cl,cv}(C(J,H)), \quad x \to (\zeta \circ S_g)(x) = \zeta(S_{g,x}).$$

is a closed operator in $\mathscr{C} \times \mathscr{C}$.

Lemma 2.8 ([8]). *Let $\mathcal{D}$ be a nonempty subset of H, which is closed, bounded and convex. Suppose $g : \mathcal{D} \to 2^{\mathcal{D}}\backslash\{\emptyset\}$ is u.s.c., with closed, convex values and such that $g(\mathcal{D}) \subseteq \mathcal{D}$ and $G(\mathcal{D})$ is compact. Then g has a fixed point.*

3. Main results

The requirements for the approximate controllability of a semilinear impulsive fractional stochastic integro-differential inclusions with Poisson jumps are initially developed in this section using the Bohnenblust-Karlin's fixed point approach. To support our claim, we require the following assumptions.

(A1) The operators $T_q(t)$ and $S_q(t)$ are compact.
(A2) The function $f : J \times H \to H$ is continuous and there exists a constant $C_1 > 0$ such that

$$\|f(t,x)\|^2 \leqslant C_1(1 + \|x\|^2),\ \forall\, x \in H,\ t \in J.$$

(A3) $g : J \times H \to P_{bd,cl,cv}(H)$ is a L_2-Caratheodory and satisfies the following criteria.
  (i) For every $t \in J$, $g(t,\cdot) : H \to P_{bd,cl,cv}(H)$ is u.s.c., and for all $x \in H$, $g(\cdot,x)$ is measurable. Furthermore, the collection of all $x \in \mathscr{C}$,

$$S_{g,x} = \{\sigma \in L_2^0 : \sigma(t) \in g(t,x(t)) \text{ for a.e. } t \in J\},$$

  is nonempty.
  (ii) For each $r > 0$ and $x \in \mathscr{C}$ with $\|x\|_{\mathscr{C}} \leqslant r$, there exists a constant $\beta \in (0,q)$ and $L_{\sigma,r}(\cdot) \in L^{\frac{1}{\beta}}(J,\mathbb{R}^+)$ such that $\|g(t,x(t))\|^2 = \sup\{\|\sigma\|^2_{L_2^0} : \sigma \in g(t,x(t))\} \leqslant L_{\sigma,r},\ t \in J$.

(A4) The function $s \to L_{\sigma,r}(s) \in L^1([0,b],\mathbb{R}^+)$ and there exists a $\gamma > 0$ such that

$$\lim_{r\to\infty} \inf \frac{\int_0^b (t-s)^{2(q-1)} L_{\sigma,r}(s)ds}{r} = \gamma < +\infty.$$

(A5) The function $e : J \times J \times H \to H$ fulfills the Lipschitz and linear growth condition, i.e., there exist a constant $N_1 > 0$ such that
$$\left\| \int_0^t e(t,s,x)ds \right\|^2 \leqslant N_1(1+\|x\|^2).$$

(A6) The function $h : [0,b] \times H \times \mathcal{U} \to H$ fulfills that there exists a constants $K_1 > 0,\ L_1 > 0$ such that
$$\int_{\mathcal{U}} \|h(t,x,u)\|^2 \nu(du) \leqslant K_1(1+\|x\|^2), \quad \int_{\mathcal{U}} \|h(t,x,u)\|^4 \nu(du) \leqslant L_1(1+\|x\|^4).$$

(A7) $I_k : H \times H$ are continuous and there exist constants d_k satisfying
$$\|I_k(x)\|^2 \leqslant d_k(1+\|x\|^2),\ k = 1,2,\dots,m.$$

(A8) The function p is continuous and there exists a constant $M_p > 0$ such that
$$\|p(x)\|^2 \leqslant M_p(1+\|x\|^2),\ \forall x \in \mathscr{C}.$$

Lemma 3.1. *For any* $\tilde{x}_b \in L^2(F_b, H)$, $\exists \tilde{\Phi} \in L^2_F(\Omega; L^2(J, L^0_2))$ *such that* $\tilde{x}_b = E\tilde{x}_b + \int_0^b \tilde{\Phi}(s)dw(s)$. *Now,* $\forall \delta > 0$ *and* $\tilde{x}_b \in L^2(F_b, H)$, *the control function is defined as*
$$\begin{aligned}
u_x^\delta(t) = {} & B^* S_q^*(b-t)\left[(\delta I + \psi_0^b)^{-1}[E\tilde{x}_b - T_q(b)(x_0 + p(x))] + \int_0^b (\delta I + \psi_0^b)^{-1}\tilde{\Phi}(s)dw(s)\right] \\
& - B^* S_q^*(b-t)\int_0^b (\delta I + \psi_0^b)^{-1} S_q(b-s) f(s,x(s))ds \\
& - B^* S_q^*(b-t)\int_0^b (\delta I + \psi_0^b)^{-1} S_q(b-s)\left[\int_0^s e(s,r,x(r))dr\right]ds \\
& - B^* S_q^*(b-t)\int_0^b (\delta I + \psi_0^b)^{-1} S_q(b-s)\sigma(s)dw(s) \\
& - B^* S_q^*(b-t)\int_0^b \int_{\mathcal{U}} (\delta I + \psi_0^b)^{-1} S_q(b-s) h(s,x(s),u)\tilde{N}(ds,du) \\
& - B^* S_q^*(b-t)(\delta I + \psi_0^b)^{-1} \sum_{k=1}^m T(b-t_k) I_k(x(t_k)).
\end{aligned}$$

Theorem 3.2. *Assume* ($\mathbb{A}$1)-($\mathbb{A}$8) *are fulfilled. Then the fractional control system* (1.1) *has a mild solution on* J *if*
$$\left(14M_T^2 M_P + 7M_T^2 d_k + 7M_S^2 \frac{b^{2q-1}}{2q-1}(bC_1 + bN_1 + K_1 + \sqrt{L_1}) + 7L_\eta M_S^2 \gamma\right)\left(1 + \frac{7}{\delta^2} b M_B^4 M_S^4 \frac{b^{4q-3}}{4q-3}\right) < 1. \tag{3.1}$$

Proof. For $\delta > 0$, we define the multivalued operator $\Gamma : \mathscr{C} \to 2^{\mathscr{C}}$, specified by
$$\begin{aligned}
\Gamma(x) = \Bigg\{ \Phi \in \mathscr{C} : \Phi(t) = {} & T_q(t)(x_0 + p(x)) + \int_0^t S_q(t-s)[Bu_x^\delta(s) + f(s,x(s))]ds \\
& + \int_0^t S_q(t-s)\left[\int_0^s e(s,r,x(r))dr\right]ds + \int_0^t S_q(t-s)\sigma(s)dw(s) \\
& + \int_0^t \int_{\mathcal{U}} S_q(t-s) h(s,x(s),u)\tilde{N}(ds,du) + \sum_{k=1}^m T(t-t_k) I_k(x(t_k)),\ \sigma \in S_{\sigma,x} \Bigg\},
\end{aligned}$$
it contains a fixed point x, which provides a mild solution for the system (1.1).

Now, we prove that Γ satisfies all required conditions of Lemma 2.8. For the sake of convenience, we subdivide the proof into in several steps.

Step 1: Γ is convex $\forall x \in \mathscr{C}$. In actuality, if $\Phi_1,\ \Phi_2 \in \Gamma(x)$, $\exists \sigma_1, \sigma_2 \in S_{g,x}$ such that for each $t \in J$, we have

$$\begin{aligned}
\Phi_i(t) = {} & T_q(t)(x_0 + p(x)) + \int_0^t S_q(t-s)f(s,x(s))ds + \int_0^t S_q(t-s)\left[\int_0^s e(s,r,x(r))dr\right]ds \\
& + \int_0^t S_q(t-s)\sigma_i(s)dw(s) + \int_0^t\int_u S_q(t-s)h(s,x(s),u)\tilde{N}(ds,du) \\
& + \sum_{k=1}^m T(t-t_k)I_k(x(t_k)) + \int_0^t S_q(t-\xi)BB^*S_q^*(b-\xi) \\
& \times \Bigg\{(\delta I + \psi_0^b)^{-1}[E\tilde{x}_b - T_q(b)(x_0+p(x))] + \int_0^b (\delta I + \psi_0^b)^{-1}\tilde{\Phi}(s)dw(s) \\
& - \int_0^b (\delta I + \psi_0^b)^{-1}S_q(b-s)f(s,x(s))ds - \int_0^b (\delta I + \psi_0^b)^{-1}S_q(b-s)\left[\int_0^s e(s,r,x(r))dr\right]ds \\
& - \int_0^b (\delta I + \psi_0^b)^{-1}S_q(b-s)\sigma_i(s)dw(s) - \int_0^b\int_u (\delta I + \psi_0^b)^{-1}S_q(b-s)h(s,x(s),u)\tilde{N}(ds,du) \\
& - (\delta I + \psi_0^b)^{-1}\sum_{k=1}^m T(b-t_k)I_k(x(t_k))\Bigg\}d\xi,\ i = 1,2.
\end{aligned}$$

Let $\lambda \in [0,1]$, $\forall t \in J$,

$$\begin{aligned}
& \lambda\Phi_1(t) + (1-\lambda)\Phi_2(t) \\
& = T_q(t)(x_0 + p(x)) + \int_0^t S_q(t-s)f(s,x(s))ds + \int_0^t S_q(t-s)\left[\int_0^s e(s,r,x(r))dr\right]ds \\
& \quad + \int_0^t S_q(t-s)[\lambda\sigma_1(s) + (1-\lambda)\sigma_2(s)]dw(s) + \int_0^t\int_u S_q(t-s)h(s,x(s),u)\tilde{N}(ds,du) \\
& \quad + \sum_{k=1}^m T(t-t_k)I_k(x(t_k)) + \int_0^t S_q(t-\xi)BB^*S_q^*(b-\xi) \\
& \quad \times \Bigg\{(\delta I + \psi_0^b)^{-1}[E\tilde{x}_b - T_q(b)(x_0+p(x))] + \int_0^b (\delta I + \psi_0^b)^{-1}\tilde{\Phi}(s)dw(s) \\
& \quad - \int_0^b (\delta I + \psi_0^b)^{-1}S_q(b-s)f(s,x(s))ds - \int_0^b (\delta I + \psi_0^b)^{-1}S_q(b-s)\left[\int_0^s e(s,r,x(r))dr\right]ds \\
& \quad - \int_0^b (\delta I + \psi_0^b)^{-1}S_q(b-s)[\lambda\sigma_1(s) + (1-\lambda)\sigma_2(s)]dw(s) \\
& \quad - \int_0^b\int_u (\delta I + \psi_0^b)^{-1}S_q(b-s)h(s,x(s),u)\tilde{N}(ds,du) - (\delta I + \psi_0^b)^{-1}\sum_{k=1}^m T(b-t_k)I_k(x(t_k))\Bigg\}d\xi.
\end{aligned}$$

Due to the convex values of g, it is clear that $S_{\sigma,x}$ is convex. So, that if $\sigma_1, \sigma_2 \in S_{g,x}$ then $\lambda\sigma_1(s) + (1-\lambda)\sigma_2(s) \in S_{g,x}$. Thus,

$$\lambda\sigma_1(s) + (1-\lambda)\sigma_2(s) \in \Gamma(x).$$

Step 2: Γ maps bounded sets into bounded sets in $\mathscr{C}$. For each positive integer $r > 0$, let $\mathcal{B}_r = \{x \in \mathscr{C} : E\|x(t)\|^2_{\mathscr{C}} \leqslant r\}$. The bounded, closed, and convex set $\mathscr{C}$ is $\mathcal{B}_r$. According to our argument, $r \ni \Gamma(\mathcal{B}_r) \subseteq \mathcal{B}_r$. If this is false, then there exists a function $x^r \in \mathcal{B}_r$ for each positive number r, but $\Gamma(x^r)$ does not belong to $\mathcal{B}_r$. For all $x \in \mathscr{C},\ t \in J$. By the hypotheses, we have

$$E\|u_x^\delta(t)\|^2 \leqslant \frac{7}{\delta^2}M_B^2M_S^2(b-t)^{2q-2}\Bigg\{\left\|E\tilde{x}_b + \textstyle\int_0^b \tilde{\Phi}(s)dw(s)\right\|^2 + E\left\|T_q(b)(x_0+p(x))\right\|^2$$

$$
\begin{aligned}
&+ E\left\| \int_0^b S_q(b-s) f(s,x(s))ds \right\|^2 + E\left\| \int_0^b S_q(b-s)\left[\int_0^s e(s,r,x(r))dr\right]ds \right\|^2 \\
&+ E\left\| \int_0^b S_q(b-s)\sigma(s)dw(s) \right\|^2 + E\left\| \int_0^b \int_u S_q(b-s)h(s,x(s),u)\tilde{N}(ds,du) \right\|^2 \\
&+ E\left\| \sum_{k=1}^m T(b-t_k)I_k(x(t_k)) \right\|^2 \Bigg\} \\
\leqslant\ & \frac{7}{\delta^2} M_B^2 M_S^2 (b-t)^{2q-2}\Bigg\{ 2\|E\tilde{x}_b\|^2 + 2\int_0^b E\|\tilde{\Phi}(s)\|^2_{L_2^0} ds + 2M_T^2(E\|x_0\|^2 + M_p(1+\|x\|^2)) \\
&+ bM_S^2 \frac{b^{2q-1}}{2q-1} C_1(1+\|x\|^2) + bM_S^2\frac{b^{2q-1}}{2q-1} N_1(1+\|x\|^2) + L_\eta M_S^2 \int_0^b (t-s)^{2(q-1)} L_{\sigma,r}(s)ds \\
&+ M_S^2 \frac{b^{2q-1}}{2q-1} K_1(1+\|x\|^2) + M_S^2 \frac{b^{2q-1}}{2q-1}\sqrt{L_1}(1+\|x\|^2) + M_T^2 d_k(1+\|x\|^2) \Bigg\} \leqslant (b-t)^{2q-2} M_U,
\end{aligned}
$$

where

$$
\begin{aligned}
M_U = & \frac{7}{\delta^2} M_B^2 M_S^2 \Bigg\{ 2\|E\tilde{x}_b\|^2 + 2\int_0^b E\|\tilde{\Phi}(s)\|^2_{L_2^0} ds + 2M_T^2(E\|x_0\|^2 + M_p(1+r)) + M_T^2 d_k(1+r) \\
&+ M_S^2 \frac{b^{2q-1}}{2q-1}\left(bC_1(1+r) + bN_1(1+r) + K_1(1+r) + \sqrt{L_1}(1+r) \right) \\
&+ M_S^2 L_\eta \int_0^b (t-s)^{2(q-1)} L_{\sigma,r}(s)ds \Bigg\}.
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
r \leqslant & E\|\Phi(t)\|^2 \\
\leqslant & 7E\left\| T_q(t)(x_0+p(x)) \right\|^2 + 7E\left\| \int_0^t S_q(t-s)Bu_x^\delta(s)ds \right\|^2 + 7E\left\| \int_0^t S_q(t-s)f(s,x(s))ds \right\|^2 \\
&+ 7E\left\| \int_0^t S_q(t-s)\left[\int_0^s e(s,r,x(r))dr\right]ds \right\|^2 + 7E\left\| \int_0^t S_q(t-s)\sigma(s)dw(s) \right\|^2 \\
&+ 7E\left\| \int_0^t \int_u S_q(t-s)h(s,x(s),u)\tilde{N}(ds,du) \right\|^2 + 7E\left\| \sum_{k=1}^m T(t-t_k)I_k(x(t_k)) \right\|^2 \\
\leqslant & 14M_T^2 E\|x_0\|^2 + 14M_T^2 M_P(1+r) + 7bM_B^2M_S^2\frac{b^{4q-3}}{4q-3}M_U + 7bM_S^2\frac{b^{2q-1}}{2q-1}C_1(1+r) \\
&+ 7bM_S^2\frac{b^{2q-1}}{2q-1}N_1(1+r) + 7L_\eta M_S^2\int_0^t (t-s)^{2(q-1)}L_{\sigma,r}(s)ds + 7bM_S^2\frac{b^{2q-1}}{2q-1}K_1(1+r) \\
&+ 7bM_S^2\frac{b^{2q-1}}{2q-1}\sqrt{L_1}(1+r) + 7M_T^2 d_k(1+r) \\
\leqslant & 14M_T^2 E\|x_0\|^2 + 14M_T^2 M_P(1+r) + 7bM_B^2M_S^2\frac{b^{4q-3}}{4q-3}\Bigg(\frac{7}{\delta^2}M_B^2M_S^2\Bigg\{ 2\|E\tilde{x}_b\|^2 + 2\int_0^b E\|\tilde{\Phi}(s)\|^2_{L_2^0}ds \\
&+ 2M_T^2(E\|x_0\|^2 + M_p(1+r)) + M_T^2 d_k(1+r) + M_S^2\frac{b^{2q-1}}{2q-1}\Big(bC_1(1+r) + bN_1(1+r) \\
&+ K_1(1+r) + \sqrt{L_1}(1+r)\Big) + M_S^2 L_\eta \int_0^b (t-s)^{2(q-1)}L_{\sigma,r}(s)ds \Bigg\}\Bigg) + 7bM_S^2\frac{b^{2q-1}}{2q-1}C_1(1+r) \\
&+ 7bM_S^2\frac{b^{2q-1}}{2q-1}N_1(1+r) + 7L_\eta M_S^2\int_0^t (t-s)^{2(q-1)}L_{\sigma,r}(s)ds + 7bM_S^2\frac{b^{2q-1}}{2q-1}K_1(1+r)
\end{aligned}
$$

$$+7bM_S^2\frac{b^{2q-1}}{2q-1}\sqrt{L_1}(1+r)+7M_T^2d_k(1+r)$$
$$\leqslant 14M_T^2E\|x_0\|^2\left(1+\frac{7}{\delta^2}bM_B^4M_S^4\frac{b^{4q-3}}{4q-3}\right)+14bM_B^4M_S^4\frac{b^{4q-3}}{4q-3}\frac{7}{\delta^2}\left(\|E\tilde{x}_b\|^2+\int_0^b E\|\tilde{\Phi}(s)\|^2_{L_2^0}ds\right)$$
$$+14M_T^2M_P(1+r)\left(1+\frac{7}{\delta^2}bM_B^4M_S^4\frac{b^{4q-3}}{4q-3}\right)+7M_T^2d_k(1+r)\left(1+\frac{7}{\delta^2}bM_B^4M_S^4\frac{b^{4q-3}}{4q-3}\right)$$
$$+7M_S^2\frac{b^{2q-1}}{2q-1}(1+r)(bC_1+bN_1+K_1+\sqrt{L_1})\left(1+\frac{7}{\delta^2}bM_B^4M_S^4\frac{b^{4q-3}}{4q-3}\right)$$
$$+7L_\eta M_S^2\int_0^t(t-s)^{2(q-1)}L_{\sigma,r}(s)ds\left(1+\frac{7}{\delta^2}bM_B^4M_S^4\frac{b^{4q-3}}{4q-3}\right).$$

Now, by dividing both side by r and taking the $\lim r\to\infty$, we get

$$\left(14M_T^2M_P+7M_T^2d_k+7M_S^2\frac{b^{2q-1}}{2q-1}(bC_1+bN_1+K_1+\sqrt{L_1})+7L_\eta M_S^2\gamma\right)\left(1+\frac{7}{\delta^2}bM_B^4M_S^4\frac{b^{4q-3}}{4q-3}\right)>1.$$

This is contradiction to condition (3.1). Consequently, $\forall r>0,\ \Gamma(\mathcal{B}_r)\subseteq\mathcal{B}_r$.

Step 3: Γ maps bounded sets into equicontinuous sets of $\mathscr{C}$, $\forall x\in\mathcal{B}_r,\ \Phi\in\Gamma(x),\ \exists\ \sigma\in S_{g,x}$ such that

$$\begin{aligned}\Phi(t)=T_q(t)(x_0+p(x))&+\int_0^t S_q(t-s)[Bu_x^\delta(s)+f(s,x(s))]ds\\&+\int_0^t S_q(t-s)\left[\int_0^s e(s,r,x(r))dr\right]ds+\int_0^t S_q(t-s)\sigma(s)dw(s)\\&+\int_0^t\int_u S_q(t-s)h(s,x(s),u)\tilde{N}(ds,du)+\sum_{k=1}^m T(t-t_k)I_k(x(t_k)).\end{aligned}$$

Let $0<t_1<t_2\leqslant b$, then

$$\begin{aligned}&E\|\Phi(t_2)-\Phi(t_1)\|^2\\&\leqslant 13\Bigg[[2(E\|(T_q(t_2)-T_q(t_1))x_0\|^2+E\|(T_q(t_2)-T_q(t_1))p(x)\|^2)]\\&\quad+t_1\int_0^{t_1}E\left\|(S_q(t_2-s)-S_q(t_1-s))f(s,x(s))\right\|^2ds+M_S^2\frac{(t_2-t_1)^{2q-1}}{2q-1}\int_{t_1}^{t_2}E\|f(s,x(s))\|^2ds\\&\quad+t_1\int_0^{t_1}E\left\|(S_q(t_2-s)-S_q(t_1-s))Bu_x^\delta(s)\right\|^2ds+\|B\|^2M_S^2\frac{(t_2-t_1)^{2q-1}}{2q-1}\int_{t_1}^{t_2}E\|u_x^\delta\|^2ds\\&\quad+t_1\int_0^{t_1}E\left\|(S_q(t_2-s)-S_q(t_1-s))\left[\int_0^s e(s,r,x(r))dr\right]\right\|^2ds\\&\quad+M_S^2\frac{(t_2-t_1)^{2q-1}}{2q-1}\int_{t_1}^{t_2}E\left\|\int_0^s e(s,r,x(r))dr\right\|^2ds+L_\eta\int_0^{t_1}E\left\|(S_q(t_2-s)-S_q(t_1-s))\sigma(s)\right\|^2dw(s)\\&\quad+M_S^2L_\eta\frac{(t_2-t_1)^{2q-1}}{2q-1}\int_{t_1}^{t_2}E\|\sigma(s)\|^2dw(s)+\left(\int_0^{t_1}\int_u E\|(S_q(t_2-s)-S_q(t_1-s))h(s,x(s),u)\|^2\nu(du)ds\right.\\&\quad\left.+\int_{t_1}^{t_2}\int_u E\|S_q(t_2-s))h(s,x(s),u)\|^2\nu(du)ds\right)\\&\quad+\left(\left(\int_0^{t_1}\int_u E\|(S_q(t_2-s)-S_q(t_1-s))h(s,x(s),u)\|^4\nu(du)ds\right)^{\frac{1}{2}}\right.\\&\quad\left.+\left(\int_{t_1}^{t_2}\int_u E\|S_q(t_2-s))h(s,x(s),u)\|^4\nu(du)ds\right)^{\frac{1}{2}}\right)\end{aligned}$$

$$+\sum_{0<t_k<t_1} E(\|T(t_2-t_k)-T(t_1-t_k)\|\|I_k(x(t_k))\|)^2+\sum_{t_1<t_k<t_2} E(\|T(t_2-t_k)\|\|I_k(x(t_k))\|)^2\Big].$$

As $t_1 \to t_2$, the RHS of the aforementioned inequality approaches to 0. $T_q(t)$ and $S_q(t)$ are chosen because of their compactness, which ensures continuity in the uniform operator topology. As a result, Φ is equicontinuous.

Step 4: We show that $\Pi(t)=\{\Phi(t):\Phi\in\Gamma(\mathcal{B}_r)\}$ is relatively compact for $t\in J$. Let $t\in(0,b]$ be fixed and η a real number satisfying $0<\eta<t$. We define $x\in\mathcal{B}_r$ as

$$\begin{aligned}\Phi_\eta(t)&=T_q(t)(x_0+p(x))+S_q(\eta)\int_0^{t-\eta} S_q(t-\eta-s)[Bu_x^\delta(s)+f(s,x(s))]ds\\&+S_q(\eta)\int_0^{t-\eta} S_q(t-\eta-s)\left[\int_0^s e(s,r,x(r))dr\right]ds+S_q(\eta)\int_0^{t-\eta} S_q(t-\eta-s)\sigma(s)dw(s)\\&+S_q(\eta)\int_0^{t-\eta}\int_{\mathcal{U}} S_q(t-\eta-s)h(s,x(s),u)\tilde{N}(ds,du)+\sum_{0<t_k<t-\eta} T(t-t_k)I_k(x(t_k)).\end{aligned}$$

Since $S_q(t)$ is compact operator, the set $\Pi_\eta(t)=\{\Phi_\eta(t):\Phi_\eta\in\Gamma(\mathcal{B}_r)\}$ is relatively compact in $\mathtt{H}$ $\forall\eta$, $0<\eta<t$. However, we also have

$$\begin{aligned}E\|\Phi(t)-\Phi_\eta(t))\|^2&\leqslant 6E\left\|\int_{t-\eta}^t S_q(t-s)Bu_x^\delta(s)ds\right\|^2+6E\left\|\int_{t-\eta}^t S_q(t-s)f(s,x(s))ds\right\|^2\\&+6E\left\|\int_{t-\eta}^t S_q(t-s)\left[\int_0^s e(s,r,x(r))dr\right]ds\right\|^2+6E\left\|\int_{t-\eta}^t S_q(t-s)\sigma(s)ds\right\|^2\\&+6E\left\|\int_{t-\eta}^t\int_{\mathcal{U}} S_q(t-s)h(s,x(s),u)\tilde{N}(ds,du)\right\|^2+6\sum_{t-\eta<t_k<t} E\|T(t-t_k)I_k(x(t_k))\|^2\\&\leqslant 6M_B^2M_S^2 b\frac{\eta^{4q-3}}{4q-3}M_{\mathcal{U}}+6M_S^2 b\frac{\eta^{2q-1}}{2q-1}C_1(1+\|x\|^2)+6M_S^2 b\frac{\eta^{2q-1}}{2q-1}N_1(1+\|x\|^2)\\&+6M_S^2L_\eta\int_{t-\eta}^t (t-s)^{2(q-1)}L_{\sigma,r}(s)ds+6M_S^2 b\frac{\eta^{2q-1}}{2q-1}(bK_1+\sqrt{L_1}\sqrt{b})(1+\|x\|^2)\\&+6M_T^2\sum_{0<t_k<t-\eta} d_k(1+\|x\|^2).\end{aligned}$$

As a result, there are sets that are arbitrarily near to the collection $\Pi(t)=\{\Phi(t):\Phi\in\Gamma(\mathcal{B}_r)\}$ and the collection $\Phi(t)$ is relatively compact in $\mathtt{H}$ $\forall t\in J$. $\Pi(t)$ is relatively compact $\mathtt{H}$, for all $t\in J$, since it is compact at $t=0$.

Step 5: The closed graph of Γ. Let $x_n\to x_*$ and $\Phi_n\to\Phi_*$, $n\to\infty$. We demonstrate that $\Phi_*\in\Gamma(x_*)$. Since $\Phi_n\in\Gamma(x_n)$, $\exists$ $\sigma_n\in S_{g,x_n}$, such that $\forall t\in J$,

$$\begin{aligned}\Phi_n(t)&=T_q(t)(x_0+p(x_n))+\int_0^t S_q(t-s)f(s,x_n(s))ds+\int_0^t S_q(t-s)\left[\int_0^s e(s,r,x_n(r))dr\right]ds\\&+\int_0^t S_q(t-s)\sigma_n(s)dw(s)+\int_0^t\int_{\mathcal{U}} S_q(t-s)h(s,x_n(s),u)\tilde{N}(ds,du)\\&+\sum_{k=1}^m T(t-t_k)I_k(x_n(t_k))+\int_0^t S_q(t-\xi)B\Big\{B^*S_q^*(b-\xi)\\&\times\left[(\delta I+\psi_0^b)^{-1}[E\tilde{x}_b-T_q(b)(x_0+p(x_n))]+\int_0^b(\delta I+\psi_0^b)^{-1}\tilde{\Phi}(s)dw(s)\right]\end{aligned}$$

$$
\begin{aligned}
&- B^*S_q^*(b-\xi)\int_0^b (\delta I+\psi_0^b)^{-1}S_q(b-s)f(s,x_n(s))ds\\
&- B^*S_q^*(b-\xi)\int_0^b (\delta I+\psi_0^b)^{-1}S_q(b-s)\left[\int_0^s e(s,r,x_n(r))dr\right]ds\\
&- B^*S_q^*(b-\xi)\int_0^b (\delta I+\psi_0^b)^{-1}S_q(b-s)\sigma_n(s)dw(s)\\
&- B^*S_q^*(b-\xi)\int_0^b \int_u (\delta I+\psi_0^b)^{-1}S_q(b-s)h(s,x_n(s),u)\tilde{N}(ds,du)\\
&- B^*S_q^*(b-\xi)(\delta I+\psi_0^b)^{-1}\sum_{k=1}^m T(b-t_k)I_k(x_n(t_k))\Bigg\}d\xi.
\end{aligned}
$$

We must demonstrate $\exists\sigma_* \in S_{g,x_*}$, such that $\forall t \in J$,

$$
\begin{aligned}
\Phi_*(t) = {} & T_q(t)(x_0+p(x_*)) + \int_0^t S_q(t-s)f(s,x_*(s))ds + \int_0^t S_q(t-s)\left[\int_0^s e(s,r,x_*(r))dr\right]ds\\
&+ \int_0^t S_q(t-s)\sigma_*(s)dw(s) + \int_0^t\int_u S_q(t-s)h(s,x_*(s),u)\tilde{N}(ds,du)\\
&+ \sum_{k=1}^m T(t-t_k)I_k(x_*(t_k)) + \int_0^t S_q(t-\xi)B\Bigg\{B^*S_q^*(b-\xi)\\
&\times\left[(\delta I+\psi_0^b)^{-1}[E\tilde{x}_b - T_q(b)(x_0+p(x_*))] + \int_0^b (\delta I+\psi_0^b)^{-1}\tilde{\Phi}(s)dw(s)\right]\\
&- B^*S_q^*(b-\xi)\int_0^b (\delta I+\psi_0^b)^{-1}S_q(b-s)f(s,x_*(s))ds\\
&- B^*S_q^*(b-\xi)\int_0^b (\delta I+\psi_0^b)^{-1}S_q(b-s)\left[\int_0^s e(s,r,x_*(r))dr\right]ds\\
&- B^*S_q^*(b-\xi)\int_0^b (\delta I+\psi_0^b)^{-1}S_q(b-s)\sigma_*(s)dw(s)\\
&- B^*S_q^*(b-\xi)\int_0^b \int_u (\delta I+\psi_0^b)^{-1}S_q(b-s)h(s,x_*(s),u)\tilde{N}(ds,du)\\
&- B^*S_q^*(b-\xi)(\delta I+\psi_0^b)^{-1}\sum_{k=1}^m T(b-t_k)I_k(x_*(t_k))\Bigg\}d\xi.
\end{aligned}
$$

Since p is continuous, we obtain that

$$
\begin{aligned}
&\Bigg\|\Bigg(\Phi_n(t) - T_q(t)(x_0+p(x_n)) - \int_0^t S_q(t-s)f(s,x_n(s))ds - \int_0^t S_q(t-s)\left[\int_0^s e(s,r,x_n(r))dr\right]ds\\
&\quad - \int_0^t\int_u S_q(t-s)h(s,x_n(s),u)\tilde{N}(ds,du) - \sum_{k=1}^m T(t-t_k)I_k(x_n(t_k))\\
&\quad - \int_0^t S_q(t-\xi)B\times B^*S_q^*(b-\xi)\Bigg\{(\delta I+\psi_0^b)^{-1}[E\tilde{x}_b - T_q(b)(x_0+p(x_n))]\\
&\quad + \int_0^b (\delta I+\psi_0^b)^{-1}\tilde{\Phi}(s)dw(s) - \int_0^b (\delta I+\psi_0^b)^{-1}S_q(b-s)f(s,x_n(s))ds\\
&\quad - \int_0^b (\delta I+\psi_0^b)^{-1}S_q(b-s)\left[\int_0^s e(s,r,x_n(r))dr\right]ds\\
&\quad - \int_0^b\int_u (\delta I+\psi_0^b)^{-1}S_q(b-s)h(s,x_n(s),u)\tilde{N}(ds,du) - (\delta I+\psi_0^b)^{-1}\sum_{k=1}^m T(b-t_k)I_k(x_n(t_k))\Bigg\}d\xi\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&-\Big(\Phi_*(t)-T_q(t)(x_0+p(x_*))-\int_0^t S_q(t-s)f(s,x_*(s))ds-\int_0^t S_q(t-s)\Big[\int_0^s e(s,r,x_*(r))dr\Big]ds\\
&-\int_0^t\int_u S_q(t-s)h(s,x_*(s),u)\tilde{N}(ds,du)-\sum_{k=1}^m T(t-t_k)I_k(x_*(t_k))\\
&-\int_0^t S_q(t-\xi)B\times B^*S_q^*(b-\xi)\Big\{(\delta I+\psi_0^b)^{-1}[E\tilde{x}_b-T_q(b)(x_0+p(x_*))]\\
&+\int_0^b(\delta I+\psi_0^b)^{-1}\tilde{\Phi}(s)dw(s)-\int_0^b(\delta I+\psi_0^b)^{-1}S_q(b-s)f(s,x_*(s))ds\\
&-\int_0^b(\delta I+\psi_0^b)^{-1}S_q(b-s)\Big[\int_0^s e(s,r,x_*(r))dr\Big]ds-\int_0^b\int_u(\delta I+\psi_0^b)^{-1}S_q(b-s)h(s,x_*(s),u)\tilde{N}(ds,du)\\
&-(\delta I+\psi_0^b)^{-1}\sum_{k=1}^m T(b-t_k)I_k(x_*(t_k))\Big\}d\xi\Big)\Big\|^2\to 0,\ \text{as } n\to\infty.
\end{aligned}
$$

Let $\psi: L^2(L(\Omega,H))\to C(J,H)$ is a linear continuous operator,

$$\sigma\to\psi(\sigma)(t)=\int_0^t S_q(t-s)\Big[\sigma(s)+BB^*S_q^*(b-s)\Big(\int_0^b(\delta I+\psi_0^b)^{-1}S_q(b-\tau)\sigma(\tau)d\tau\Big)\Big]dw(s).$$

Lemma 2.7 clearly proves that $\psi\circ S_{g,x}$ is a closed graph operator, where $S_{g,x}=\{\sigma\in g(t,x(t))\}$. Also from the definition of ψ, we have

$$
\begin{aligned}
&\Big(\Phi_n(t)-T_q(t)(x_0+p(x_n))-\int_0^t S_q(t-s)f(s,x_n(s))ds-\int_0^t S_q(t-s)\Big[\int_0^s e(s,r,x_n(r))dr\Big]ds\\
&-\int_0^t\int_u S_q(t-s)h(s,x_n(s),u)\tilde{N}(ds,du)-\sum_{k=1}^m T(t-t_k)I_k(x_n(t_k))\\
&-\int_0^t S_q(t-\xi)B\times B^*S_q^*(b-\xi)\Big\{(\delta I+\psi_0^b)^{-1}[E\tilde{x}_b-T_q(b)(x_0+p(x_n))]\\
&+\int_0^b(\delta I+\psi_0^b)^{-1}\tilde{\Phi}(s)dw(s)-\int_0^b(\delta I+\psi_0^b)^{-1}S_q(b-s)f(s,x_n(s))ds\\
&-\int_0^b(\delta I+\psi_0^b)^{-1}S_q(b-s)\Big[\int_0^s e(s,r,x_n(r))dr\Big]ds-\int_0^b\int_u(\delta I+\psi_0^b)^{-1}S_q(b-s)h(s,x_n(s),u)\tilde{N}(ds,du)\\
&-(\delta I+\psi_0^b)^{-1}\sum_{k=1}^m T(b-t_k)I_k(x_n(t_k))\Big\}d\xi\Big)\in\psi(S_{g,x_n}).
\end{aligned}
$$

As $x_n\to x_*$, Lemma 2.7 implies that

$$
\begin{aligned}
&\Big(\Phi_*(t)-T_q(t)(x_0+p(x_*))-\int_0^t S_q(t-s)f(s,x_*(s))ds-\int_0^t S_q(t-s)\Big[\int_0^s e(s,r,x_*(r))dr\Big]ds\\
&-\int_0^t\int_u S_q(t-s)h(s,x_*(s),u)\tilde{N}(ds,du)-\sum_{k=1}^m T(t-t_k)I_k(x_*(t_k))\\
&-\int_0^t S_q(t-\xi)B\times B^*S_q^*(b-\xi)\Big\{(\delta I+\psi_0^b)^{-1}[E\tilde{x}_b-T_q(b)(x_0+p(x_*))]\\
&+\int_0^b(\delta I+\psi_0^b)^{-1}\tilde{\Phi}(s)dw(s)-\int_0^t(\delta I+\psi_0^b)^{-1}S_q(b-s)f(s,x_*(s))ds\\
&-\int_0^b(\delta I+\psi_0^b)^{-1}S_q(b-s)\Big[\int_0^s e(s,r,x_*(r))dr\Big]ds-\int_0^b\int_u(\delta I+\psi_0^b)^{-1}S_q(b-s)h(s,x_*(s),u)\tilde{N}(ds,du)
\end{aligned}
$$

$$- (\delta I+\psi_0^b)^{-1}\sum_{k=1}^{m} T(b-t_k)I_k(x_*(t_k))\Big\}d\xi\Big) \in \psi(S_{g,x_*}).$$

As a result, Φ has a closed graph.

Using the Arzela-Ascoli theorem and Steps 1-5, we get to the conclusion that Γ is a compact multi-valued map, u.s.c., with closed convex values. It can be deduced from Lemma 2.8 that Γ has a fixed point x, which is a mild solution of (1.1). □

Theorem 3.3. *If* f, g, e *and* h *are uniformly bounded and it is assumed that* ($\mathbb{A}$1)-($\mathbb{A}$8) *hold, then equation* (1.1) *is approximately controllable on* $[0, b]$.

Proof. Applying the stochastic Fubini's theorem and x_β has a fixed point on Γ, it is easy to see that

$$\begin{aligned}
x_\beta(b) = \tilde{x}_b &- \delta(\delta I+\psi_0^b)^{-1}\Big(E\tilde{x}_b - T_q(b)(x_0+p(x_\beta))\Big) + \delta\int_0^b (\delta I+\psi_0^b)^{-1}S_q(b-s)f(s,x_\beta(s))ds \\
&+ \delta\int_0^b (\delta I+\psi_0^b)^{-1}S_q(b-s)\Big[\int_0^s e(s,r,x_\beta(r))dr\Big]ds \\
&+ \delta\int_0^b (\delta I+\psi_0^b)^{-1}S_q(b-s)[g(s,x_\beta(s))-\Phi(s)]dw(s) \\
&+ \delta\int_0^b (\delta I+\psi_0^b)^{-1}S_q(b-s)h(s,x_\beta(s),u)\tilde{N}(ds,du) + \delta(\delta I+\psi_0^b)^{-1}\sum_{0<t_k<t} T_q(b-t_k)I_k(x_\beta(t_k)).
\end{aligned}$$

Given that f, e, g, and h are uniformly bounded according to our assumptions, then $\mathcal{D} > 0$ exists such that

$$\|f(s,x_\beta(s))\|^2 + \|\int_0^s e(s,r,x_\beta(r))ds\|^2 + \|g(s,x_\beta(s))\|^2 + \|h(s,x_\beta(s),u)\|^2 \leqslant \mathcal{D}$$

in $[0, b]\times\Omega$. Now, there are consequences, as shown by $\{f(s,x_\beta(s))\}$, $\{\int_0^s e(s,r,x_\beta(r))ds\}$, $\{g(s,x_\beta(s))\}$, and $\{h(s,x_\beta(s),u)\}$, which converge weakly to $f(s)$, $g(s)$ in $\mathbb{H}\times L_2^0$ and $\int_0^s e(s,r,x)$, $h(s,u)$ in $\mathbb{H}\times\mathbb{H}\times L_2^0$, respectively. Therefore, the compactness of $S_q(t)$ implies that, $S_q(t-s)f(s,x_\beta(s)) \to S_q(t-s)f(s)$, $S_q(t-s)e(s,r,x_\beta(s)) \to S_q(t-s)e(s,r,x)$, $S_q(t-s)g(s,x_\beta(s)) \to S_q(t-s)g(s)$, and $S_q(t-s)h(s,x_\beta(s),u) \to S_q(t-s)h(s,u)$ in $[0, b]\times\Omega$. From the aforementioned equation, we obtained

$$\begin{aligned}
E\|x_\beta(b)-\tilde{x}_b\|^2 \leqslant{}& \Big\|\delta(\delta I+\psi_0^b)^{-1}\Big[E\tilde{x}_b - T_q(b)(x_0+p(x_\beta))\Big]\Big\|^2 + E\Big(\int_0^b \Big\|\delta(\delta I+\psi_0^b)^{-1}\Phi(s)\Big\|_{L_2^0}^2 ds\Big) \\
&+ E\Big(\int_0^b \Big\|\delta(\delta I+\psi_0^b)^{-1}S_q(t-s)[f(s,x_\beta(s))-f(s)]\Big\|ds\Big)^2 \\
&+ E\Big(\int_0^b \Big\|\delta(\delta I+\psi_0^b)^{-1}S_q(t-s)f(s)\Big\|ds\Big)^2 \\
&+ E\Big(\int_0^b \Big\|\delta(\delta I+\psi_0^b)^{-1}S_q(t-s)\int_0^s\Big[e(s,r,x_\beta(r))-e(s,r)\Big]dr\Big\|ds\Big)^2 \\
&+ E\Big(\int_0^b \Big\|\delta(\delta I+\psi_0^b)^{-1}S_q(t-s)\Big[\int_0^s e(s,r)\Big]dr\Big\|ds\Big)^2 \\
&+ E\Big(\int_0^b \Big\|\delta(\delta I+\psi_0^b)^{-1}S_q(t-s)[g(s,x_\beta(s))-g(s)]\Big\|_{L_2^0}^2 ds\Big) \\
&+ E\Big(\int_0^b \Big\|\delta(\delta I+\psi_0^b)^{-1}S_q(t-s)g(s)\Big\|_{L_2^0}^2 ds\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ E\left(\int_0^b \int_{\mathfrak{u}} \left\| \delta(\delta I + \psi_0^b)^{-1} S_q(t-s)[h(s, x_\beta(s), u) - h(s, u)]\tilde{N}(ds, du) \right\| \right)^2 \\
&+ E\left(\int_0^b \int_{\mathfrak{u}} \left\| \delta(\delta I + \psi_0^b)^{-1} S_q(t-s) h(s, u)\tilde{N}(ds, du) \right\| \right)^2 \\
&+ E\left(\left\| \delta(\delta I + \psi_0^b)^{-1} \sum_{0<t_k<t} T_q(t-t_k) I_k(x_\beta(t_k)) \right\| \right)^2 .
\end{aligned}
$$

However, in accordance with Definition 2.1, $\forall 0 \leqslant s \leqslant b$, $\delta(\delta I + \psi_0^b)^{-1} \to 0$ strongly as $\delta \to 0^+$, and furthermore, $\|\delta(\delta I + \psi_0^b)^{-1}\| \leqslant 1$. As a result, we use the Lebesgue dominated convergence approach to produce $E\|x_\beta(b) - \tilde{x}_b\|^2 \to 0$ as $\delta \to 0^+$. This gives the approximate controllability of system (1.1). □

4. Example

Let us consider the impulsive fractional stochastic system

$$
\begin{cases}
{}^C D^q x(t,\nu) = x_{\nu\nu} + Bu(t,\nu) + \tilde{f}(t, x(t,\nu)) + \int_0^t \tilde{e}(t,s,x(s,\nu))ds + \tilde{g}(t, x(t,\nu))\frac{dw(t)}{dt} \\
\qquad\qquad + \int_{\mathfrak{u}} \tilde{h}(t, x(t,\nu), u)\tilde{N}(ds, du), \quad t \in J,\ t \neq t_k, \\
\Delta x(t_k, \nu) = \tilde{I}_k(x(t_k), \nu),\ t = t_k,\ k = 1, 2, \dots, n, \\
x(t, 0) = x(t, \pi) = 0,\ 0 < \nu < \pi, \\
x(0, \nu) + \sum_{i=1}^m \delta_i x(t_i, \nu) = x_0(\nu),\ \nu \in [0, \pi],
\end{cases}
\tag{4.1}
$$

where $B : \mathcal{U} \to H$ represents a bounded linear operator, and $\tilde{f} : J \times H \to H$, $\tilde{e} : J \times J \times H \to H$, $\tilde{g} : J \times H \to L_2^0$, and $\tilde{h} : J \times H \times \mathcal{U} \to H$ are all uniformly bounded and continuous, $u(t)$ is a feedback control and $w(t)$ is a Q-Wiener process. Let $H = L_2[0, \pi]$ and $A : D(A) \subset H \to H$ be an operator specified by $Ax = x_{\nu\nu}$ with

$$D(A) = \{x \in H : x, x_\nu \text{ are absolutely continuous},\ x_{\nu\nu} \in H,\ x(0) = x(\pi) = 0\}.$$

Additionally, A has a discrete spectrum, and its eigenvalues are $-n^2$, $n = 1, 2, \dots$, respectively, with the accompanying normalized characteristic vectors $e_n(s) = \sqrt{\frac{2}{\pi}} \sin ns$, then

$$Ax = \sum_{n=1}^\infty -n^2 \langle x, e_n \rangle e_n, \quad x \in H.$$

In this case, A creates a compact semigroup $T(t)$, where $t > 0$ and is provided by

$$T(t)x = \sum_{n=1}^\infty e^{-n^2} \langle x, e_n \rangle e_n(\nu), \quad x \in H.$$

Define $f : J \times H \to H$, $e : J \times J \times H \to H$, $g : J \times H \to L_2^0$, $h : J \times H \times \mathcal{U} \to H$ and $I : H \times H$ by

$$
\begin{aligned}
&f(t, x(t))(\nu) = \tilde{f}(t, x(t,\nu)), && e(t,s,x(s))(\nu) = \tilde{e}(t,s,x(s,\nu)), \\
&g(t, x(t))(\nu) = \tilde{g}(t, x(t,\nu)), && h(t, x(t), u)(\nu) = \tilde{h}(t, x(t,\nu), u), \\
&I_k(x(t_k))(\nu) = \tilde{I}_k(x(t_k, \nu)),\ (t, x(t)) \in J \times H,\ \nu \in [0, \pi].
\end{aligned}
$$

The function $g : C([0, b]; H) \to H$ is defined as

$$g(x)(\nu) = \sum_{i=1}^n \delta_i x(t_i, \nu), \quad 0 < t_i < b,\ \nu \in [0, \pi],$$

with the choices of A, B, e, f, g, h, I, and g, system (1.1) the abstract form of system (4.1) such that the conditions in ($\mathbb{A}$1)-($\mathbb{A}$8) are satisfied. An infinite dimensional space is defined by

$$\mathcal{U} = \left(u : u = \sum_{n=2}^{\infty} u_n e_n(\nu) \middle| \sum_{n=2}^{\infty} u_n^2 < 2 \right),$$

using the norm as described by

$$\|u\|_{\mathcal{U}} = \left(\sum_{n=2}^{\infty} u_n^2 \right)^{\frac{1}{2}},$$

and $B : \mathcal{U} \to \mathtt{H}$ is linear continuous mapping follows

$$Bu = 2u_2(t)e_1(\nu) + \sum_{n=2}^{\infty} u_n(t)e_n(\nu).$$

It is evident that, for $u(t, \nu, \omega) = \sum_{n=2}^{\infty} u_n(t, \omega)e_n(\nu) \in L_2^F([0, b]; \mathcal{U})$,

$$Bu = 2u_2(t)e_1(\nu) + \sum_{n=2}^{\infty} u_n(t)e_n(\nu) \in L_2^F([0, b]; \mathcal{U}).$$

Moreover,

$$B^*\nu = (2\nu_1 + \nu_2)e_2(\nu) + \sum_{n=3}^{\infty} \nu_n e_n(\nu), \quad B^*S^*(t)x = (2x_1e^{-t} + x_2e^{-4t})e_2(\nu) + \sum_{m=3}^{\infty} x_n e^{-n^2 t} e_n(\nu),$$

for $\nu = \sum_{n=1}^{\infty} \nu_n e_n(\nu)$ and $x = \sum_{n=1}^{\infty} x_n e_n(\nu)$. Let $\|B^*S^*(t)x\| = 0,\ t \in [0, b]$, the conclusion is that

$$\|2x_1e^{-t} + x_2e^{-4t}\|^2 + \sum_{n=3}^{\infty} \|x_n e^{-n^2 t}\|^2 = 0,\ t \in [0, b], \quad \Rightarrow x_n = 0,\ n = 1, 2, \ldots, \quad \Rightarrow x = 0.$$

According to Theorem 4.1.7 of [16], the deterministic linear system associated with (4.1) is thus approximately controllable on $[0, b]$. As a result, system (4.1) is approximately controllable if $\tilde{f}, \tilde{e}, \tilde{g}, \tilde{h}, \tilde{I}$, and g fulfill the following hypotheses ($\mathbb{A}$1)-($\mathbb{A}$8).

Next, we verify that the hypothesis ($\mathbb{A}$1)-($\mathbb{A}$8) for the above system (4.1) one by one.

Verification of ($\mathbb{A}$1): The operators $T_q(t)$ and $S_q(t)$ are compact. Therefore, ($\mathbb{A}$1) is verified.

Verification of ($\mathbb{A}$2): Assume that $f(t, x(t))(\nu) = \tilde{f}(t, x(t, \nu))$. The function $\tilde{f} : J \times \mathtt{H} \to \mathtt{H}$ is continuous and there exist a constant $C_1 > 0$ such that

$$\|\tilde{f}(t, x(t, \nu)\|^2 \leqslant C_1(1 + \|x(t, \nu)\|^2),\ \forall\ x \in \mathtt{H},\ t \in J.$$

Hence, ($\mathbb{A}$2) is verified.

Verification of ($\mathbb{A}$3): Assume that $g(t, x(t))(\nu) = \tilde{g}(t, x(t, \nu))$. $\tilde{g} : J \times \mathtt{H} \to P_{bd,cl,cv}(\mathtt{H})$ is a L_2-Caratheodory and which satisfy the following criteria.

(i) For every $(t, \nu) \in J \times \mathtt{H}$, $\tilde{g}(t, \cdot) : \mathtt{H} \to P_{bd,cl,cv}(\mathtt{H})$ is u.s.c., and for all $x \in \mathtt{H}$, $\tilde{g}(\cdot, x)$ is measurable. Furthermore, the collection of all $x \in \mathscr{C}$,

$$S_{\tilde{g},x} = \{\sigma \in L_2^0 : \sigma(t) \in \tilde{g}(t, x(t, \nu)) \text{ for a.e. } (t, \nu) \in J \times \mathtt{H}\},$$

is nonempty.

(ii) For each $r > 0$ and $x \in \mathscr{C}$ with $\|x\|_{\mathscr{C}} \leqslant r$, there exists a constant $\beta \in (0, q)$ and $L_{\sigma,r}(\cdot) \in L^{\frac{1}{\beta}}(J, \mathbb{R}^+)$ such that

$$\|\tilde{g}(t, x(t,\nu))\|^2 = \sup\{\|\sigma\|^2_{L_2^0} : \sigma \in \tilde{g}(t, x(t,\nu))\} \leqslant L_{\sigma,r}.$$

Therefore, assumption ($\mathbb{A}3$) is verified.

Verification of ($\mathbb{A}4$): The function $s \to L_{\sigma,r}(s) \in L^1([0,t], \mathbb{R}^+)$ and there exists a constant $\gamma > 0$ such that

$$\lim_{r\to\infty} \inf \frac{\int_0^b (t-s)^{2(q-1)} L_{\sigma,r}(s)ds}{r} = \gamma < +\infty.$$

Hence, ($\mathbb{A}4$) is verified.

Verification of ($\mathbb{A}5$): Assume that $e(t,s,x(s))(\nu) = \tilde{e}(t,s,x(s,\nu))$. The function $\tilde{e} : J \times J \times H \to H$ fulfills the Lipschitz and linear growth condition, i.e., there exist a constant $N_1 > 0$ such that

$$\left\|\int_0^t \tilde{e}(t,s,x(s,\nu))ds\right\|^2 \leqslant N_1(1 + \|x(s,\nu)\|^2).$$

Therefore, assumption ($\mathbb{A}5$) is satisfied.

Verification of ($\mathbb{A}6$): Assume that $h(t,x(t),u)(\nu) = \tilde{h}(t,x(t,\nu),u)$. The function $\tilde{h} : [0,b] \times H \times \mathcal{U} \to H$ fulfills that there exists $K_1 > 0, L_1 > 0$ such that

$$\int_{\mathcal{U}} \|\tilde{h}(t,x(t,\nu),u)\|^2 \nu(du) \leqslant K_1(1 + \|x(t,\nu)\|^2), \quad \int_{\mathcal{U}} \|\tilde{h}(t,x(t,\nu),u)\|^4 \nu(du) \leqslant L_1(1 + \|x(t,\nu)\|^4).$$

Hence, ($\mathbb{A}6$) is verified.

Verification of ($\mathbb{A}7$): Assume that $I_k(x(t))(\nu) = \tilde{I}_k(x(t,\nu))$. $\tilde{I}_k : H \times H$ are continuous and there exists constants d_k satisfying

$$\|\tilde{I}_k(x(t_k,\nu))\| \leqslant d_k(1 + \|x(t_k,\nu)\|^2), \; k = 1, 2, \ldots, m.$$

Therefore, assumption ($\mathbb{A}7$) is verified.

Verification of ($\mathbb{A}8$): Assume that $p(x(t))(\nu) = g(x(t,\nu))$. The function g is continuous and there exists a constant $M_g > 0$ such that

$$\|g(x(t,\nu))\|^2 \leqslant M_g(1 + \|x(t,\nu)\|^2),$$

$\forall x \in \mathscr{C}$. Hence, ($\mathbb{A}8$) is verified.

Clearly, all the assumptions of the Theorems 3.2 and 3.3 are satisfied, then we conclude that system (4.1) is approximately controllable on $[0, b]$.

5. Conclusion

In this work, we have demonstrated the approximate controllability of a semilinear impulsive fractional stochastic integro-differential inclusion with Poisson jumps. By using fractional calculus, semigroup theories, and the fixed point technique, a new set of sufficient conditions is formulated that guarantees the approximate controllability of a semilinear impulsive fractional stochastic integro-differential inclusion with Poisson jumps. Additionally, we provided an example to illustrate the main results. In the future, we will focus our study on the existence and approximate controllability of Sobolev-type impulsive fractional neutral stochastic differential inclusions with Poisson jumps by applying the fixed point method.

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