



Best proximity point results on $\mathcal{R}$ -metric spaces with applications to fractional differential equation and production-consumption equilibrium



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Abstract

In this article, we introduce the notion of best proximity point in $\mathcal{R}$ -metric space. We prove the best proximity result in $\mathcal{R}$ -metric space and also given some examples to strengthen our obtained results. Finally, an application to fractional differential equation and an application to production-consumption equilibrium are given.

Keywords: Fixed point, fractional differential equation, best proximity point, dynamic market equilibrium problem, $\mathcal{R}$ -contraction, $\mathcal{R}$ -metric space.

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1. Introduction

For the past hundred years, the fixed point has an active and interesting area of research because of its significance and applications see [12, 25, 27]. Similarly, a brief discussion on regarding the concept of proximity pairs between two sets in the theory of functional analysis. Many researchers have analyzed the result of fixed point when the equation does not have an exact solution of $\Gamma(v) = v$. Asadi et al. [3] had generalized the common fixed point on M -metric space. George and Veeramani [10] has given some results in fuzzy metric. Anjum and Aage [2] has established some common fixed point theorem on F -metric space. Latif et al. [15] has generalized the fixed point on multi-valued contractive mapping in metric type space. Mustafa et al. [18] has given some coincidence point results in ordered G -metric space. Likewise Mustafa et al. [19] has proved some common fixed point in ordered partial b -metric space. Menaha et al. [8] have proved fixed point theorem in orthogonal α -admissible on Branciari b -metric spaces. Gunaseelan et al. [16] had established fixed point theorems in C^* -algebra-valued partial b -metric. Notably Fan [9] demonstrated the existence and uniqueness of the best approximation in a

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normed space. Similarly, in 1989, the existence and uniqueness of best approximation in an compact subset of normed space was proved by Segal and Singh [26]. Likewise Prolla [24] have extended the same work to multifunction. In 2010, with the help of Banach contraction principle Basha [5] had found the best proximity point. Similarly, Basha et al. [7] had given the common best proximity points for pairs of non-self-mappings in metric space. Karapınar and Erhan [13] has studied the best proximity point on different types of contractions. In fixed point theory, the most important mathematical finding is Banach contraction principle. In addition to that, some generalized metric spaces such as fuzzy metric, quasi-metric, semi-metric, pseudo-metric, and partial metric spaces had extended to best approximation point. Pragadeeswarar et al. [23] have proved the common best proximity point on partially ordered metric space. In 2012, Basha [6] have proved best proximity point on partially ordered metric space. Abkar and Gabeleh [1] has proved best proximity points of non-self mapping. Satyendra Kumar et al. [11] established best proximity point in partial ordered metric space. Poonguzali et al. [22] has given best proximity point in O-CM space. Khalehghli et al. [14] has introduced fixed point result on $\mathcal{R}$ -Metric space. Gunaseelan et al. [17] have proved fixed point theorems in $\mathcal{R}$ -Metric space. For more details see [20, 21, 28].

Motivated by the results of [11, 22], in this paper, we have extended the best proximity point on non-self mapping in the background of $\mathcal{R}$ -M space. From this result, we have proved best proximity point results. Appropriate examples are also given to support our results. Finally, we give some applications on our presented results.

2. Preliminaries

Throughout this paper, M space means metric space, $\mathcal{R}$ -M space means $\mathcal{R}$ -metric space, CM space means complete metric space, $\mathcal{R}$ -CM space means $\mathcal{R}$ complete metric space, $\mathcal{R}$ -Per means $\mathcal{R}$ -preserving, $\mathcal{R}$ -Con means $\mathcal{R}$ -continuous, and $\mathcal{R}$ -Seq means $\mathcal{R}$ -sequence. The following Definitions 2.1-2.8 have been introduced in [22].

Definition 2.1. Let V and U be any two non-empty subsets of a metric space (X, d_X) and $\Gamma: V \rightarrow U$ be a mapping. An element $v \in V$ is called best proximity point of the mapping Γ , if the following condition is fulfilled

$$d_X(v, \Gamma v) = d_X(V, U). \quad (2.1)$$

Definition 2.2. Let X be a non-empty set and let $\mathcal{R} \subseteq X \times X$ be an binary realtion. Then $(X, \mathcal{R})$ is an $\mathcal{R}$ -set, if there exists $v_0 \in V$ such that $\omega \mathcal{R} v_0$ or $v_0 \mathcal{R} \omega$, for all $\omega \in X$.

Definition 2.3. Let $(X, \mathcal{R})$ be an $\mathcal{R}$ -set and $\{v_n\}$ be any sequence, then $\{v_n\}$ is an $\mathcal{R}$ -Seq, if $v_n \mathcal{R} v_{n+1}$ or $v_{n+1} \mathcal{R} v_n$, for all $n \in \mathbb{N}$.

Definition 2.4. Let $(X, \mathcal{R})$ be an $\mathcal{R}$ -set and V be any subset of X . Then V is $\mathcal{R}$ -closed set, if $\mathcal{R}$ -Seq $\{v_n\} \subset V$, $v_n \rightarrow v$, then $v \in V$.

Definition 2.5. Let (V, U) be a pair of non-empty subset of (X, d_X) . The pair (V, U) satisfies the P-property, if $v_1, v_2 \in V$ and $\omega_1, \omega_2 \in U$ with

$$\left. \begin{array}{l} d_X(v_1, \omega_1) = d_X(V, U) \\ d_X(v_2, \omega_2) = d_X(V, U) \end{array} \right\} \text{ implies } d_X(v_1, v_2) = d_X(\omega_1, \omega_2).$$

Definition 2.6. A binary relation $\mathcal{R}$ on a metric space (X, d_X) is called a $\mathcal{R}$ -metric space and it is denoted by $(X, d_X, \mathcal{R})$.

Definition 2.7. Let $(X, d_X, \mathcal{R})$ be an $\mathcal{R}$ -M space $(X, \mathcal{R})$ is an $\mathcal{R}$ -set and (X, d_X) be a M space. A map $\Gamma: X \rightarrow X$ is called $\mathcal{R}$ -Con at $v \in X$, for each $\mathcal{R}$ -Seq $\{v_n\}_{n \in \mathbb{N}}$ in X with $v_n \rightarrow v$, we have $\Gamma(v_n) \rightarrow \Gamma(v)$. Γ is called $\mathcal{R}$ -Con on X , if Γ is $\mathcal{R}$ -C in each $v \in V$.

Definition 2.8. Let $(X, d_X, \mathcal{R})$ be an $\mathcal{R}$ -M space and $0 < \rho < 1$. A map $\Gamma: X \rightarrow X$ is called $\mathcal{R}$ -contraction with Lipschitz constant ρ , if for all $v, \omega \in X$ with $v\mathcal{R}\omega$, $d_X(\Gamma v, \Gamma \omega) \leq \rho d_X(v, \omega)$.

Example 2.9. Let $X = [0, 0.95]$ and X be Euclidean metric. Let $v\mathcal{R}\omega$ if $v\omega \in \{v, \omega\}, \forall v, \omega \in X$. Let the map $\Gamma: X \rightarrow X$ is defined by

$$\Gamma(v) = \begin{cases} v^3, & \text{if } v \in \mathbb{Q} \cap X, \\ 0, & \text{if } v \in \mathbb{Q}^c \cap X. \end{cases}$$

Suppose, $v = 0.5, \omega = \frac{1}{\sqrt{3}}$, and $0 < \rho < 1$, then $|\Gamma(v) - \Gamma(\omega)| = 0.125 \not\leq \rho|0.5 - \frac{1}{\sqrt{3}}|$. Hence, Γ is not a contraction. Now, let $v\mathcal{R}\omega$, therefore $v = 0$ or $\omega = 0$. Suppose $v = 0$, thus

$$\Gamma(\omega) = \begin{cases} \omega^3, & \text{if } \omega \in \mathbb{Q} \cap X, \\ 0, & \text{if } \omega \in \mathbb{Q}^c \cap X. \end{cases}$$

Hence by choosing $\rho = 0.90$, then $|\Gamma(v) - \Gamma(\omega)| \leq \omega^3 \leq \rho|0 - \omega| = \rho\omega$. Hence Γ is $\mathcal{R}$ -contraction.

Definition 2.10. Let $\Gamma: X \rightarrow X$ be a mapping, Γ is called $\mathcal{R}$ -Per if $v\mathcal{R}\omega$, then $\Gamma(v)\mathcal{R}\Gamma(\omega)$ for all $v, \omega \in X$.

Definition 2.11. An $\mathcal{R}$ -sequence $\{v_n\}$ in X is said to be an $\mathcal{R}$ -Cauchy sequence, if for every $\epsilon > 0$ there exists an integer N such that $d_X(v_n, v_m) < \epsilon$ if $n \geq N$ and $m \geq N$. It is clear that $v_n\mathcal{R}v_m$ or $v_m\mathcal{R}v_n$.

3. Main results

Next, we will prove the theorems will be used to demonstrate the existence of best proximity point in $\mathcal{R}$ -metric space.

Lemma 3.1. Let V be an $\mathcal{R}$ -closed subset of an $\mathcal{R}$ -complete metric space X , then V is an $\mathcal{R}$ -complete metric space.

Proof. Let $\{v_n\}$ be any $\mathcal{R}$ -Cauchy sequence in V . Then, $\{v_n\} \subseteq X$. Since X is an $\mathcal{R}$ -complete metric space, there exists $v \in X$ such that $v_n \rightarrow v$. Additionally, $\{v_n\}$ is an $\mathcal{R}$ -sequence, which converges to $v \in X$. Hence, $v \in V$. $\square$

Definition 3.2. Let V and U be any two subsets of a metric space (X, d_X) . A map $\Gamma: V \rightarrow U$ is called proximally $\mathcal{R}$ -Per if

$$\left. \begin{aligned} d_X(v_1, \Gamma \omega_1) &= d_X(V, U) \\ d_X(v_2, \Gamma \omega_2) &= d_X(V, U) \end{aligned} \right\} \text{ implies } v_1\mathcal{R}v_2, \text{ if } \omega_1\mathcal{R}\omega_2.$$

Theorem 3.3. Let V and U be two non-empty $\mathcal{R}$ -closed subsets of an $\mathcal{R}$ -CM space $(X, d_X, \mathcal{R})$ such that V_0 be a non-empty subset. If (V, U) has the P -property and also $\Gamma: V \rightarrow U$ such that:

1. Γ is $\mathcal{R}$ -Con and a $\mathcal{R}$ -contraction mapping;
2. $\Gamma(V_0) \subseteq U_0$;
3. Γ is proximally $\mathcal{R}$ -Per;
4. V_0 is an $\mathcal{R}$ -set,

then, $d_X(v, \Gamma v) = d_X(V, U)$ for some $v \in V$.

Proof. Since V_0 is an $\mathcal{R}$ -set, there exists $\phi \in V_0$, then there is some v such that $v\mathcal{R}\phi$ (or) $\phi\mathcal{R}v$ for all $v \in V_0$. Let us consider that $v\mathcal{R}\phi$. From above assumption 2, we have $\Gamma\phi \in U_0$. Similarly, $v_1 \in V_0$ such that $d_X(v_1, \Gamma\phi) = d_X(V, U)$. Let $\Gamma v_1 \in U_0$, and hence $d_X(v_2, \Gamma v_1) = d_X(V, U)$. By the proximally, $\mathcal{R}$ -Per property of Γ is given by $v_1\mathcal{R}v_2$. Like this manner, we construct an $\mathcal{R}$ -Seq, $v_1\mathcal{R}v_2\mathcal{R}v_3\mathcal{R}\dots\mathcal{R}v_n\mathcal{R}\dots$ with $d_X(v_{n+1}, \Gamma v_n) = d_X(V, U), \forall n \in \mathbb{N}$. By P -property of (V, U) , we have

$$d_X(v_n, v_{n+1}) = d_X(\Gamma v_{n-1}, \Gamma v_n).$$

We have,

$$d_X(v_n, v_{n+1}) = d_X(\Gamma v_{n-1}, \Gamma v_n) \leq \rho d_X(v_{n-1}, v_n) \leq \cdots \leq \rho^n d_X(v_0, v_1).$$

Since $0 < \rho < 1$, we obtain $\lim_{n \rightarrow \infty} \rho^n = 0$. Hence, $\lim_{n \rightarrow \infty} d_X(v_n, v_{n+1}) = 0$. If $n, v \in \mathbb{N}$ and $v < n$, then,

$$\begin{aligned} d_X(v_v, v_n) &\leq d_X(v_v, v_{v+1}) + d_X(v_{v+1}, v_{v+2}) + \cdots + d_X(v_{n-1}, v_n) \\ &\leq \rho^v d_X(v_0, v_1) + \rho^{v+1} d_X(v_0, v_1) + \cdots + \rho^{n-1} d_X(v_0, v_1) \\ &\leq \rho^v [1 + \rho + \cdots + \rho^{n-v-1}] d_X(v_0, v_1) \leq \frac{\rho^v}{1-\rho} d_X(v_0, v_1). \end{aligned}$$

As $v, n \rightarrow \infty$, $d_X(v_v, v_n) \rightarrow 0$, such that v_n is an $\mathcal{R}$ -Cauchy sequence. We know that, V is an $\mathcal{R}$ -closed subset of an $\mathcal{R}$ -CM space. From Lemma 3.1, V is an $\mathcal{R}$ -CM space $(X, d_X, \mathcal{R})$. Thus, we can find $v^* \in V$, such that $\lim_{n \rightarrow \infty} v_n = v^*$. Since Γ is $\mathcal{R}$ -C, $\lim_{n \rightarrow \infty} \Gamma v_{n-1} = \Gamma v^*$, such that $d_X(v_n, \Gamma v_n) \rightarrow d_X(v^*, \Gamma v^*)$ as $n \rightarrow \infty$. Hence, $d_X(v^*, \Gamma v^*) = d_X(V, U)$. $\square$

Corollary 3.4. Let V be a non-empty closed subsets of $(X, d_X, \mathcal{R})$ and let $(X, d_X, \mathcal{R})$ be a $\mathcal{R}$ -CM space such that V_0 be a non-empty subset. Define $\Gamma: V \rightarrow V$ be the mapping such that:

1. Γ is $\mathcal{R}$ -Per;
2. there exists $\Gamma(V_0) \subseteq V_0$ such that $d_X(\Gamma v_1, \Gamma v_2) \leq d_X(v_1, v_2)$, with $v_1 \mathcal{R} v_2$ and $v_1 \neq v_2$;
3. Γ is $\mathcal{R}$ -Con;
4. V_0 is an $\mathcal{R}$ -set.

Then, Γ has a unique fixed point.

Proof. If we substitute $V = U$ into the proof of Theorem 3.3, we obtain the following results respectively. $\square$

Theorem 3.5. Let $(X, d_X, \mathcal{R})$ be any $\mathcal{R}$ -CM space. Let V and U be two non empty subsets of X . Let $\Gamma: V \rightarrow U$ be the mapping such that:

1. Γ is $\mathcal{R}$ -Con and a $\mathcal{R}$ -contraction;
2. by P -property holds that $\Gamma(V_0) \subseteq U_0$ and (V, U) ;
3. Γ is proximally $\mathcal{R}$ -Per;
4. we can find $v_0, v_1 \in V_0$ such that $d_X(v_1, \Gamma v_0) = d_X(V, U)$ and $v_0 \mathcal{R} v_1$.

Then, there exists an element $v \in X$ such that $d_X(v, \Gamma v) = d_X(V, U)$.

Proof. There exist v_0 and v_1 in V_0 such that $d_X(v_1, \Gamma v_0) = d_X(V, U)$ and $v_0 \mathcal{R} v_1$. Since $v_1 \in V_0$, this implies $\Gamma v_1 \in U_0$, and hence, there exists $v_2 \in V_0$ such that $d_X(v_2, \Gamma v_1) = d_X(V, U)$, by the proximally, $\mathcal{R}$ -Per condition of Γ is given by $v_1 \mathcal{R} v_2$. Like this manner, $v_1 \mathcal{R} v_2 \mathcal{R} \cdots \mathcal{R} v_n \mathcal{R} v_{n+1} \mathcal{R} \cdots$. Then $\{v_n\}$ is an $\mathcal{R}$ -Seq with $d_X(v_{n+1}, \Gamma v_n) = d_X(V, U)$, for all $n \in \mathbb{N}$. By P -property of (V, U) , we define,

$$d_X(v_n, v_{n+1}) = d_X(\Gamma v_{n-1}, \Gamma v_n) \leq \rho d_X(v_{n-1}, v_n) \leq \rho^r d_X(v_0, v_1).$$

Since $\rho < 1$, $\rho^r \rightarrow 0$, $\lim_{n \rightarrow \infty} d_X(v_n, v_{n+1}) = 0$, v_n is an $\mathcal{R}$ -Cauchy sequence. If $v, n \in \mathbb{N}$ and $n < v$, we have

$$\begin{aligned} d_X(v_n, v_v) &\leq [d_X(v_n, v_{n+1}) + \cdots + d_X(v_{v-1}, v_v)] \\ &\leq \rho^n d_X(v_0, v_1) + \cdots + \rho^{v-1} d_X(v_0, v_1) \leq \frac{\rho^n}{1-\rho} d_X(v_0, v_1). \end{aligned}$$

Therefore, $d_X(v, v_n) \rightarrow 0$ as $v, n \rightarrow \infty$. Therefore, $\{v_n\}$ is an $\mathcal{R}$ -Cauchy sequence. Hence, $\lim_{n \rightarrow \infty} v_n = v^*$. Since Γ is $\mathcal{R}$ -Con, $\lim_{n \rightarrow \infty} \Gamma v_{n-1} = \Gamma v^*$, which implies $d_X(v_n, \Gamma v_n) \rightarrow d_X(v^*, \Gamma v^*)$. Therefore, v^* is a best proximity point. $\square$

Theorem 3.6. Let $(X, d_X, \mathcal{R})$ be an $\mathcal{R}$ -CM space. Let V and U be two non-empty $\mathcal{R}$ -closed subsets of X such that X_0 be a non-empty set. Moreover, assume that (V, U) has the P -property. Let $\Gamma: V \rightarrow U$ be the mapping such that:

1. Γ is a $\mathcal{R}$ -contraction mapping and proximally $\mathcal{R}$ -Per;
2. $\Gamma(V_0) \subseteq U_0$;
3. if $\{v_n\}$ is any $\mathcal{R}$ -Seq with $v_n \rightarrow v$, then $v_n \mathcal{R} v$ for all $n \in \mathbb{N}$;
4. V_0 is an $\mathcal{R}$ -set.

Then, there exists $v \in V$ such that $d_X(v, \Gamma v) = d_X(V, U)$.

Proof. By Theorem 3.5, we can give an $\mathcal{R}$ -Cauchy sequence $\{v_n\}$ with $d_X(v_{n+1}, \Gamma v_n) = d_X(X, \Sigma)$ and there exists $v \in V$, such that $v_n \rightarrow v$. Thus, for any $\frac{\epsilon}{2} > 0$, there exists $\Pi_1 \in \mathbb{N}$ such that $d_X(v_n, v) \leq \frac{\epsilon}{2}$, $\forall n \geq \Pi_1$. Similarly, for any $\frac{\epsilon}{2\rho} > 0$, there exists $\Pi_2 \in \mathbb{N}$ such that $d_X(v_\nu, v) \leq \frac{\epsilon}{2\rho}$, where ρ is the contraction condition of Γ and for all $\nu \geq \Pi_2$. Choosing $\Pi = \max\{\Pi_1, \Pi_2\}$ is given by the following

$$\begin{aligned} d_X(v, \Gamma v) &\leq d_X(v, v_\Pi) + d_X(v_\Pi, \Gamma v_\Pi) + d_X(\Gamma v_\Pi, \Gamma v) \\ &\leq \frac{\epsilon}{2} + d_X(X, \Sigma) + \rho d_X(v_\Pi, v) \leq \frac{\epsilon}{2} + d_X(X, \Sigma) + \frac{\epsilon}{2} \leq d_X(X, \Sigma) + \epsilon. \end{aligned}$$

Since ϵ is arbitrary, then $d_X(v, \Gamma v) = d_X(X, \Sigma)$. □

Let us denote the new notion called weakly proximally $\mathcal{R}$ -preserving as follows.

Definition 3.7. The mappings $\Gamma, \Omega: V \rightarrow U$ are said to be weakly proximally $\mathcal{R}$ -Per if

1. for all $v \in V$, we can find that $\kappa_1, \kappa_2 \in V$ with $d_X(\kappa_1, \Gamma v) = d_X(X, \Sigma)$, $d_X(\kappa_2, \Omega \kappa_1) = d_X(X, \Sigma)$ and $\kappa_1 \mathcal{R} \kappa_2$;
2. for all $v \in V$, there exist $\vartheta_1, \vartheta_2 \in V$ with $d_X(\vartheta_1, \Gamma v) = d_X(X, \Sigma)$, $d_X(\vartheta_2, \Omega \vartheta_1) = d_X(X, \Sigma)$ and $\vartheta_1 \mathcal{R} \vartheta_2$.

Theorem 3.8. Let V and U be two non-empty subsets of $\mathcal{R}$ -closed with an $\mathcal{R}$ -CM space $(X, d_X, \mathcal{R})$ with V_0 be a non-empty subset. Assume that (X, Σ) has the P -property. Let $\Gamma, \Omega: V \rightarrow U$ be two non self-mappings satisfying

1. (Γ, Ω) is weakly proximally $\mathcal{R}$ -Per;
2. Γ or Ω is $\mathcal{R}$ -Con;
3. for all v, κ with $v \mathcal{R} \kappa$, $d_X(\Gamma v, \Omega \kappa) \leq \rho d_X(v, \kappa)$, for some $\rho \in [0, 1)$;
4. if any $\mathcal{R}$ -Seq $\{v_n\}$ converges, then $v_n \mathcal{R} v$, for all $n \in \mathbb{N}$, where $v = \lim_{n \rightarrow \infty} v_n$.

Then, there exists $v \in V$ such that $d_X(v, \Gamma v) = d_X(v, \Omega v) = d_X(V, U)$.

Proof. Since V_0 be an non-empty subsets, choose any $v_0 \in V_0$. Applying Γ on v_0 , then $\Gamma v_0 \in U_0$. As (Γ, Ω) is weakly proximally, we have

$$d_X(v, \Gamma v_0) = d_X(V, U), d_X(\Gamma v_2, \Omega v_1) = d_X(V, U)$$

and $v_1 \mathcal{R} v_2$. Continuing this manner, we get the weakly proximally $\mathcal{R}$ -Per condition of (Γ, Ω) . Let $\{v_n\}$ of $\mathcal{R}$ -Seq with

$$d_X(v_{2n+1}, \Gamma v_{2n}) = d_X(X, \Sigma), d_X(v_{2n+2}, \Omega v_{2n+1}) = d_X(V, U)$$

and $v_{2n+1} \mathcal{R} v_{n+2}$. Let us prove $\{v_n\}$ to be Cauchy sequence. We have

$$\begin{aligned} d_X(v_{2n+1}, v_{2n+2}) &= d_X(\Gamma v_{2n}, \Omega v_{2n+1}) \\ &\leq \rho d_X(v_{2n}, v_{2n+1}) \\ &= \rho d_X(\Gamma v_{2n-1}, \Omega v_{2n}) \leq \rho^2 d_X(v_{2n-1}, v_{2n}) \leq \dots \leq \rho^{2n+1} d_X(v_0, v_1). \end{aligned}$$

Since $0 < \rho < 1$, we obtain that $\rho^{2n+1} \rightarrow 0$, this implies

$$\lim_{n \rightarrow \infty} d_X(v_{2n+1}, v_{2n+2}) = 0.$$

For $n, v \in \mathbb{N}$ with $v > n$, we have

$$\begin{aligned} d_X(v_n, v_v) &\leq d_X(v_n, v_{n+1}) + d_X(v_{n+1}, v_{n+2}) + \cdots + d_X(v_{v-1}, v_v) \\ &\leq \rho^n d_X(v_0, v_1) + \rho^{n+1} d_X(v_0, v_1) + \cdots + \rho^{v-1} d_X(v_0, v_1) \\ &\leq \rho^n [1 + \rho + \rho^2 + \cdots + \rho^{v-n-1}] d_X(v_0, v_1). \end{aligned}$$

By the above inequality, then $\{v_n\}$ is an $\mathcal{R}$ -Cauchy sequence. Since, we have $\mathcal{R}$ -CM space $\{v_n\}$ converges which implies $v_n \mathcal{R} v$, for all $n \in \mathbb{N}$. Assume that Γ is $\mathcal{R}$ -Con, then we can easily say that $d_X(v_{2n+1}, \Gamma v_{2n}) \rightarrow d_X(v, \Gamma v)$; proceed with this, we get $d_X(v, \Gamma v) = d_X(X, \Sigma)$. Thus, v is the best proximity point of Γ . Next, we prove κ be the best proximity point on Ω . By convergence of $\{v_n\}$, for $\frac{\epsilon}{2} > 0$, we can find $\Pi_1 \in \mathbb{N}$ such that $d_X(v_n, v) \leq \frac{\epsilon}{2}$, for all $n \geq \Pi_1$, similarly, for $\frac{\epsilon}{2\rho} > 0$, we can find $\Pi_2 \in \mathbb{N}$, such that $d_X(v_n, v) \leq \frac{\epsilon}{2}, \forall n \geq \Pi_2$. Let us choose $\Pi = \max\{\Pi_1, \Pi_2\}$, we obtain

$$\begin{aligned} d_X(v, \Omega v) &\leq d_X(v, v_{2\Pi+1}) + d_X(v_{2\Pi+1}, \Gamma v_{2\Pi}) + d_X(\Gamma v_{2\Pi}, \Omega v) \\ &\leq \frac{\epsilon}{2} + d_X(v_{2\Pi+1}, \Gamma v_{2\Pi}) + \rho d_X(v_{2\Pi}, v) \\ &\leq \frac{\epsilon}{2} + d_X(v_{2\Pi+1}, \Gamma v_{2\Pi}) + \frac{\epsilon}{2} \leq d_X(v_{2\Pi+1}, \Gamma v_{2\Pi}) + \epsilon. \end{aligned}$$

Now, $d_X(v, \Omega v) \leq d_X(V, U) + \epsilon$. Since ϵ is arbitrary, we get the conclusion that $d_X(v, \Omega v) = d_X(X, \Sigma)$. Thus, $d_X(v, \Gamma v) = d_X(v, \Omega v) = d_X(V, U)$ and v is the best proximity point of Γ and Ω . $\square$

Theorem 3.9. Let V and U be two-empty closed subsets of an $\mathcal{R}$ -CM space $(X, d_X, \mathcal{R})$ with V_0 be non-empty subset. Assume that (V, U) has the P -property. Let the mappings $\Gamma, \Omega: V \rightarrow U$ be two non-self mappings such that:

1. (Γ, Ω) is weakly proximally $\mathcal{R}$ -Per;
2. Γ or Ω is $\mathcal{R}$ -Con;
3. for all v, κ with $v \mathcal{R} \kappa$, $d_X(\Gamma v, \Omega \kappa) \leq \rho d_X(v, \kappa)$, for some $\rho \in [0, 1)$;
4. if v is a best proximity point of either Γ or Ω , then $v \mathcal{R} v$.

Then, there exists $v \in V$ such that $d_X(v, \Gamma v) = d_X(v, \Omega v) = d_X(V, U)$.

Proof. From Theorem 3.8, we can see that the $\mathcal{R}$ -Cauchy sequence $\{v_n\}$ such that $d_X(v_{2n+1}, \Gamma v_{2n}) = d_X(V, U)$ and $d_X(v_{2n+1}, \Omega v_{2n+2}) = d_X(V, U)$. $\mathcal{R}$ -completeness provides the convergence of $\{v_n\}$, that is, we can find $v \in V$ such that $v_n \rightarrow v$. By the condition 2, Ω is $\mathcal{R}$ -Con, then $d_X(v_{2n+1}, \Omega v_{2n+2}) \rightarrow d_X(v, \Omega v)$. Therefore $d_X(v, \Omega v) = d_X(V, U)$. Hence, v is the best proximity point for Ω , thus $v \mathcal{R} v$. By utilizing triangular property,

$$d_X(v, \Gamma v) \leq d_X(v, \Omega v) + d_X(\Omega v, \Gamma v) \leq d_X(v, \Omega v) + \rho d_X(v, v) \leq d_X(v, \Omega v).$$

Similarly,

$$d_X(v, \Omega v) \leq d_X(v, \Gamma v) + d_X(\Gamma v, \Omega v) \leq d_X(v, \Gamma v) + \rho d_X(v, v) \leq d_X(v, \Gamma v).$$

Hence, $d_X(v, \Gamma v) = d_X(v, \Omega v)$, which means that $d_X(v, \Gamma v) = d_X(v, \Omega v) = d_X(V, U)$. $\square$

Example 3.10. Let $X = \mathbb{R}^2$ and define $\Gamma: \{0\} \times \mathbb{R} \rightarrow \{1\} \times \mathbb{R}$ with $\mathcal{R}$ defined as $v \mathcal{R} \omega$ and $(v, \omega) \in \mathcal{R}$ if $v, \omega \geq 0$ by

$$\Gamma(0, v) = \begin{cases} (1, \frac{v}{4}), & v \in \mathbb{Q} \cap \mathbb{R}, \\ (1, 0), & v \in \mathbb{Q}^c \cap \mathbb{R}. \end{cases}$$

Since, Γ is $\mathcal{R}$ -contraction, we observe that $V_0 = V$ and $U_0 = U$; therefore $\Gamma(V_0) \subseteq U_0$. Clearly, (V, U) has the P -property. Therefore $(0, 0)$ is the best proximity point of Γ . Hence, Γ satisfies all conditions of Theorem 3.5.

Example 3.11. Consider $X = \mathbb{R}^2$ with $\mathcal{R}$ defined as $(v_1, v_2)\mathcal{R}(\omega_1, \omega_2)$, if $v_1 \leq \omega_1$ and $v_2 \leq \omega_2$ and $(v, \omega) \in \mathcal{R}$ if $v, \omega \geq 0$. Let us choose $d_X(v, \omega) = |v_1 - v_2| + |\omega_1 - \omega_2|$. Then $(X, d_X, \mathcal{R})$ is an $\mathcal{R}$ -CM space. Define

$$V = \{(0, \sigma) : \sigma \in \mathbb{R}\} \text{ and } U = \{(1, \eta) : \eta \in \mathbb{R}\}.$$

Then $d_X(V, U) = 1$. Define $\Gamma: V \rightarrow U$ by

$$\Gamma(0, \sigma) = \begin{cases} (1, \frac{-\sigma}{4}), & \sigma \in \mathbb{Q} \cap \mathbb{R}, \\ (1, \frac{-\sigma}{8}), & \sigma \in \mathbb{Q}^c \cap \mathbb{R}, \end{cases}$$

and $\Omega: V \rightarrow U$ by $\Omega(0, \eta) = (1, \frac{-\eta}{8})$. Then we have following cases.

1. (Γ, Ω) is weakly proximally $\mathcal{R}$ -Per. Let $v \in X$, then $v = (0, v_1)$, where $v_1 \in \mathbb{R}$.

1. If $v_1 \in \mathbb{Q} \cap \mathbb{R}$, then $\Gamma v = (1, \frac{-v_1}{4})$ and let us take $\kappa = (0, \frac{-v_1}{4})$ and $w = (0, \frac{-v_1}{16})$, then $d_X(v, \Gamma \kappa) = d_X(V, U) = d_X(\kappa, \Omega w)$ and $\kappa \mathcal{R} w$.
2. If $v_1 \in \mathbb{Q}^c \cap \mathbb{R}$, then $\Gamma v = (1, \frac{-v_1}{8})$ and let us take $\kappa = (0, \frac{-v_1}{4})$ and $w = (0, \frac{-v_1}{32})$, then $d_X(v, \Gamma \kappa) = d_X(V, U) = d_X(\kappa, \Omega w)$ and $\kappa \mathcal{R} w$.

Similarly, for all $v \in V$, thus, $w, w' \in V$ with

$$d_X(w, \Omega v) = d_X(V, U), d_X(w', \Gamma w) = d_X(V, U),$$

which implies $w \mathcal{R} w'$.

2. Γ or Ω is $\mathcal{R}$ -Con. Hence, Ω is a continuous function. Thus Ω is $\mathcal{R}$ -Con. We assume that Γ is not $\mathcal{R}$ -Con, since $\mathcal{R}$ -Seq $v_n = (0, -1 - \frac{\sqrt{2}}{n})$ converges to $v = (0, -1)$. But, $\Gamma(v_n) = (1, \frac{-(1, \frac{-\sqrt{2}}{n})}{8})$ converges to $(1, \frac{1}{8})$, which is not equal to $\Gamma v = (1, \frac{1}{4})$.

3. If $v \mathcal{R} \kappa$, then $d_X(\Gamma v, \Omega \kappa) \leq \rho d_X(v, \kappa)$ for some $\rho \in [0, 1)$. Let $v = (0, v_1), \kappa = (0, \kappa_1) \in V$.

1. If $v_1 \in \mathbb{Q}$, then

$$d_X(\Gamma v, \Omega \kappa) = d_X((1, \frac{-v_1}{4}), (1, \frac{-\kappa_1}{8})) = |\frac{-v_1}{4} + \frac{\kappa_1}{8}| \leq |\frac{-v_1}{4} + \frac{\kappa_1}{4}| \text{ (since } v_1 \leq \kappa_1) \leq \frac{1}{4} d_X(v, \kappa).$$

2. If $v_1 \in \mathbb{Q}^c$, then

$$d_X(\Gamma v, \Omega \kappa) = d_X((1, \frac{-v_1}{8}), (1, \frac{-\kappa_1}{8})) = |\frac{-v_1}{8} + \frac{\kappa_1}{8}| \leq \frac{1}{8} d_X(v, \kappa) \leq \frac{1}{4} d_X(v, \kappa).$$

By choosing $\rho = \frac{1}{4}$, then $v \mathcal{R} \omega, d_X(\Gamma v, \Omega \kappa) \leq \rho d_X(v, \kappa)$.

4. If $\{v_n\}$ is an $\mathcal{R}$ -Seq with $v_n \rightarrow v$, then $v_n \mathcal{R} v, \forall n \in \mathbb{N}$. Since $\{v_n\}$ is an $\mathcal{R}$ -Seq, we have $v_n = (0, \sigma_n) \leq v_{n+1} = (0, \sigma_{n+1})$, such that $\sigma_n \leq \sigma_{n+1}$. Thus $\{v_n\}$ has monotonically increasing sequence and converges to supremum, say $v: = (0, \sigma)$. We say that $v_n \mathcal{R} v, \forall n \in \mathbb{N}$. We observe that (V, U) has the P-property. Hence $v^* = (0, 0)$ satisfies $d_X(v^*, \Gamma v^*) = d_X(v^*, \Omega v^*) = d_X(V, U)$.

4. An application on fractional differential equation

Ayari et al. [4] has first presented fractional differential equation in best proximity point. Let $\mathcal{C}[0, 1]$ be the set of all continuous function on $[0, 1]$ and the mapping $d_X: \mathcal{C}([0, 1]) \times \mathcal{C}([0, 1]) \rightarrow \mathbb{R}$ defined by

$$d_X(v, \omega) = \|v - \omega\|_\infty = \sup_{\mathfrak{x} \in [0, 1]} |v(\mathfrak{x}) - \omega(\mathfrak{x})|.$$

Define $(\nu, \omega) \in \mathcal{R}$ iff $\nu, \omega \geq 0$. Let the map $\mathfrak{h}: [0, +\infty) \rightarrow \mathbb{R}$ be a Caputo derivative of fractional order β of a continuous function defined as

$${}^c\mathcal{D}^\beta(\mathfrak{h}(\mathfrak{N})) = \frac{1}{\Gamma(\nu - \beta)} \int_0^\mathfrak{N} (\mathfrak{N} - \ell)^{\nu - \beta - 1} \mathfrak{g}^{(\nu)}(\ell) d\ell, \quad (\nu - 1 < \beta < \mathfrak{n}, \nu = [\beta] + 1),$$

where Γ be gamma function. Let us see, the existence result of a non-linear fractional order differential equation,

$${}^c\mathcal{D}^\beta(\kappa(\mathfrak{N})) + \mathfrak{f}(\mathfrak{N}, \kappa(\mathfrak{N})) = 0, \quad (0 \leq \mathfrak{N} \leq 1, \beta < 1), \quad (4.1)$$

with $\kappa(0) = \kappa(1) = 0$ and continuous function of $\mathfrak{f}: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ and Green's function of a problem (4.1) is given by

$$\mathcal{G}(\mathfrak{N}, \ell) = \begin{cases} \mathfrak{N}(1 - \ell)^{\sigma-1} - (\mathfrak{N} - \ell)^{\sigma-1}, & \text{if } 0 \leq \mathfrak{N} \leq \ell \leq 1, \\ \frac{\mathfrak{N}(1 - \ell)^{\sigma-1}}{\Gamma(\sigma)}, & \text{if } 0 \leq \ell \leq \mathfrak{N} \leq 1. \end{cases}$$

Assume that the following conditions hold.

1. There exists $\rho \in (0, 1)$ such that $|\mathfrak{f}(\mathfrak{N}, \nu) - \mathfrak{f}(\mathfrak{N}, \omega)| \leq \rho|\nu - \omega|$, for all $\mathfrak{N} \in [0, 1]$, $\nu, \omega \in \mathbb{R}$;
2. $\sup_{\mathfrak{N} \in [0, 1]} \left(\int_0^1 \mathcal{G}(\mathfrak{N}, \ell) d\ell \right) \leq 1$.

Next, we prove the existence result of a fractional differential equation (4.1).

Theorem 4.1. *Let us assume the conditions (1)-(2) hold, then (4.1) has a unique solution.*

Proof. Let the mapping $\Gamma: \mathcal{C}[0, 1] \rightarrow \mathcal{C}[0, 1]$ be defined by

$$\Gamma(\nu(\mathfrak{N})) = \int_0^1 \mathcal{G}(\mathfrak{N}, \ell) \mathfrak{f}(\ell, \nu(\ell)) d\ell.$$

As our assumption ν is a solution of (4.1) and equivalently, $\nu \in V$ is a solution of integral equation.

$$\kappa(\mathfrak{N}) = \int_0^1 \mathcal{G}(\mathfrak{N}, \ell) \mathfrak{f}(\ell, \kappa(\ell)) d\ell, \quad \forall \mathfrak{N} \in [0, 1].$$

Consider

$$\begin{aligned} |\Gamma\nu(\mathfrak{N}) - \Gamma\omega(\mathfrak{N})| &= \left| \int_0^1 \mathcal{G}(\mathfrak{N}, \ell) \mathfrak{f}(\ell, \nu(\ell)) d\ell - \int_0^1 \mathcal{G}(\mathfrak{N}, \ell) \mathfrak{f}(\ell, \omega(\ell)) d\ell \right| \\ &\leq \int_0^1 |\mathcal{G}(\mathfrak{N}, \ell) (\mathfrak{f}(\ell, \nu(\ell)) - \mathfrak{f}(\ell, \omega(\ell)))| d\ell \\ &\leq \int_0^1 \mathcal{G}(\mathfrak{N}, \ell) |\mathfrak{f}(\ell, \nu(\ell)) - \mathfrak{f}(\ell, \omega(\ell))| d\ell \leq \int_0^1 \mathcal{G}(\mathfrak{N}, \ell) \rho |\nu(\ell) - \omega(\ell)| d\ell. \end{aligned}$$

Now we take supremum, we have $d_X(\Gamma\nu, \Gamma\omega) \leq \rho d_X(\nu, \omega)$. Therefore, all the hypothesis of Corollary 3.4 are satisfied. Hence, Γ has a unique solution in $\mathcal{C}[0, 1]$. $\square$

5. An application in production-consumption equilibrium

For production ν_p and consumption ν_τ , whether prices are rising or decreasing daily pricing patterns and prices have a significant influence on markets. As a result, the economists are interested about the present cost $\mathcal{U}(\mathfrak{N})$. Now, assume

$$\nu_p = \sigma_1 + \eta_1 \mathcal{U}(\mathfrak{N}) + \tau_1 \frac{d\mathcal{U}(\mathfrak{N})}{d\mathfrak{N}} + \zeta_1 \frac{d^2\mathcal{U}(\mathfrak{N})}{d\mathfrak{N}^2}, \quad \nu_\tau = \sigma_2 + \eta_2 \mathcal{U}(\mathfrak{N}) + \tau_2 \frac{d\mathcal{U}(\mathfrak{N})}{d\mathfrak{N}} + \zeta_2 \frac{d^2\mathcal{U}(\mathfrak{N})}{d\mathfrak{N}^2},$$

initially $\mathcal{U}(0) = 0, \frac{d\mathcal{U}}{d\aleph}(0) = 0$, where $\sigma_1, \sigma_2, \eta_1, \eta_2, \tau_1, \tau_2, \zeta_1$, and ζ_2 are constants. A state of dynamic economic equilibrium occurs when market forces are in balance, indicating that the current gap between production and consumption stabilises, that is, $v_p = v_\tau$. Thus,

$$\begin{aligned} \sigma_1 + \eta_1 \mathcal{U}(\aleph) + \tau_1 \frac{d\mathcal{U}(\aleph)}{d\aleph} + \zeta_1 \frac{d^2 \mathcal{U}(\aleph)}{d\aleph^2} &= \sigma_2 + \eta_2 \mathcal{U}(\aleph) + \tau_2 \frac{d\mathcal{U}(\aleph)}{d\aleph} + \zeta_2 \frac{d^2 \mathcal{U}(\aleph)}{d\aleph^2}, \\ (\sigma_1 - \sigma_2) + (\eta_1 - \eta_2) \mathcal{U}(\aleph) + (\tau_1 - \tau_2) \frac{d\mathcal{U}(\aleph)}{d\aleph} + (\zeta_1 - \zeta_2) \frac{d^2 \mathcal{U}(\aleph)}{d\aleph^2} &= 0, \\ \zeta \frac{d^2 \mathcal{U}(\aleph)}{d\aleph^2} + \tau \frac{d\mathcal{U}(\aleph)}{d\aleph} + \eta \mathcal{U}(\aleph) &= -\sigma, \\ \frac{d^2 \mathcal{U}(\aleph)}{d\aleph^2} + \frac{\tau}{\zeta} \frac{d\mathcal{U}(\aleph)}{d\aleph} + \frac{\eta}{\zeta} \mathcal{U}(\aleph) &= \frac{-\sigma}{\zeta}, \end{aligned}$$

where $\sigma = \sigma_1 - \sigma_2, \eta = \eta_1 - \eta_2, \tau = \tau_1 - \tau_2, \zeta = \zeta_1 - \zeta_2$. Now, our initial value problem is modeled as

$$\mathcal{U}''(\aleph) + \frac{\tau}{\zeta} \mathcal{U}'(\aleph) + \frac{\eta}{\zeta} \mathcal{U}(\aleph) = \frac{-\sigma}{\zeta}, \quad \text{with } \mathcal{U}(0) = 0 \text{ and } \mathcal{U}'(0) = 0. \quad (5.1)$$

The study of production and consumption of the duration time Γ is equivalent to

$$\mathcal{U}(\aleph) = \int_0^\Gamma \mathcal{G}(\aleph, \aleph^*) \mathcal{K}(\aleph^*, \aleph, \mathcal{U}(\aleph)) d\aleph,$$

where Green function $\mathcal{G}(\aleph, \aleph^*)$ is,

$$\mathcal{G}(\aleph, \aleph^*) = \begin{cases} \aleph e^{\frac{\eta}{2\tau}(\aleph^* - \aleph)}, & 0 \leq \aleph \leq \ell \leq \Gamma, \\ \ell e^{\frac{\eta}{2\tau}(\aleph^* - \aleph)}, & 0 \leq \ell \leq \aleph \leq \Gamma, \end{cases}$$

and $\mathcal{K}: [0, \Gamma] \times \mathfrak{D}^2 \rightarrow \mathbb{R}$ is a continuous function. Let an operator $\Delta: \mathfrak{D} \rightarrow \mathfrak{D}$ be described as

$$\Delta \mathcal{U}(\aleph) = \int_0^\Gamma \mathcal{G}(\aleph, \aleph^*) \mathcal{K}(\aleph^*, \aleph, \mathcal{U}(\aleph)) d\aleph.$$

Then, the solution of dynamic market equilibrium problem expressed as (5.1) is a fixed point of Δ . Equation (5.1) controls the current price $\mathcal{U}(\aleph)$. Let $\mathcal{C}[0, \Gamma]$ symbolizes the family of real continuous functions on $[0, \Gamma]$ and assume $\mathfrak{D} = \mathcal{C}[0, \Gamma]$. Define $d_X: \mathfrak{D} \times \mathfrak{D} \rightarrow \mathbb{R}^+$ as $d_X(v, \omega) = \sup_{\aleph \in [0, \Gamma]} |v(\aleph) - \omega(\aleph)|, v, \omega \in \mathfrak{D}$. Then $(\mathfrak{D}, d_X)$ is a CM space.

Theorem 5.1. *Let us assume the map $\Delta: \mathfrak{D} \rightarrow \mathfrak{D}$ is a CM space $(\mathfrak{D}, d_X)$, such that*

1. *a continuous function $\mathcal{K}: [0, \Gamma] \times \mathfrak{D}^2 \rightarrow \mathbb{R}$ and $\rho \in (0, 1)$ such that*

$$|\mathcal{K}(\aleph^*, \aleph, \mathcal{U}_1(\aleph)) - \mathcal{K}(\aleph^*, \aleph, \mathcal{U}_2(\aleph))| \leq \rho |\mathcal{U}_1(\aleph) - \mathcal{U}_2(\aleph)|;$$

2. $\sup_{\aleph \in [0, \Gamma]} \int_0^\Gamma \mathcal{G}(\aleph, \aleph^*) d\aleph \leq 1$.

Then, the dynamic market equilibrium problem (5.1) has a unique solution.

Proof.

$$\begin{aligned} |\Delta \mathcal{U}_1(\aleph) - \Delta \mathcal{U}_2(\aleph)| &= \left| \int_0^\Gamma \mathcal{G}(\aleph, \aleph^*) \mathcal{K}(\aleph^*, \aleph, \mathcal{U}_1(\aleph)) d\aleph - \int_0^\Gamma \mathcal{G}(\aleph, \aleph^*) \mathcal{K}(\aleph^*, \aleph, \mathcal{U}_2(\aleph)) d\aleph \right| \\ &\leq \int_0^\Gamma \left| \mathcal{G}(\aleph, \aleph^*) \mathcal{K}(\aleph^*, \aleph, \mathcal{U}_1(\aleph)) - \mathcal{G}(\aleph, \aleph^*) \mathcal{K}(\aleph^*, \aleph, \mathcal{U}_2(\aleph)) \right| d\aleph \end{aligned}$$

$$\begin{aligned}
&\leq \int_0^\Gamma \left| \mathcal{G}(\aleph, \aleph^*)(\mathcal{K}(\aleph^*, \aleph, \mathcal{U}_1(\aleph)) - \mathcal{K}(\aleph^*, \aleph, \mathcal{U}_2(\aleph))) d\aleph \right| \\
&\leq \int_0^\Gamma \mathcal{G}(\aleph, \aleph^*) |(\mathcal{K}(\aleph^*, \aleph, \mathcal{U}_1(\aleph)) - \mathcal{K}(\aleph^*, \aleph, \mathcal{U}_2(\aleph)))| d\aleph \\
&\leq \int_0^\Gamma \mathcal{G}(\aleph, \aleph^*) |\mathcal{U}_1(\aleph) - \mathcal{U}_2(\aleph)| d\aleph.
\end{aligned}$$

Taking supremum on both sides, we have $d_X(\Gamma v, \Gamma \omega) \leq \rho d_X(v, \omega)$. Thus, the mapping Δ has a unique fixed point. From Corollary 3.4, the equation (5.1) has a unique solution. $\square$

6. Conclusion

In this paper, we introduced the concepts of $\mathcal{R}$ -M space in best proximity point. And also proved the theorems based on best proximity results in $\mathcal{R}$ -M space. Some examples are given related to our obtained results. Finally, we applied our obtained results in the solution of fractional calculus. We concluded that new result are more effective in finding the solutions for further dealing with science and engineering.

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