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Graphical illustrations of new inequalities involving Caputo Fabrizio integral operators

Maria Tariq[a], Ammara Nosheen[b], Khuram Ali Khan[b], Alhanouf Alburaikan[c,*], Salwa El-Morsy[c,d]

[a] Department of Mathematics and Statistics, University of Lahore, Sargodha Campus, Sargodha, Pakistan.

[b] Department of Mathematics, University of Sargodha, Sargodha, Pakistan.

[c] Department of Mathematics, College of Science, Qassim University, Buraydah, 51452, Saudi Arabia.

[d] Basic Science Department, Nile higher institute for engineering and technology, Mansoura, Egypt.

Abstract

In this research study, we concentrate on Caputo-Fabrizio operators because of their multiple applications. At first, we proceed by providing a new identification for this operator. Subsequently, we use the recently discovered identity to develop a set of integral inequalities via (s, m)-convex function. Moreover, in the context of integral inequalities, we demonstrate how the results enhance and refine a great deal of prior research. Later, in order to provide an improved comprehension of the recently discovered inequalities, we provide particular examples together with the corresponding graphs. Our findings build on preceding research and offer insightful perspectives and strategies for addressing various scientific and mathematical issues.

Keywords: Convex function, fractional derivative, fractional integrals, mathematical operators, probability theory, Hermite-Hadamard inequality.

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1. Introduction

Convex functions, which possess intriguing geometric and analytical properties, are widely used in statistics, graph theory, optimization theory, mathematics and economics. Numerous characterizations exist for convex functions having smoothness features. Readers are refer to [1, 5, 10, 20, 28, 32, 34] for applications of convex functions across various research domains. Convex functions are equally important when investigating integral and discrete inequality. Many classical inequalities can be directly derived from convex functions. Numerous studies have been conducted to show that inequality theory and convex functions are significantly related, see [6, 14, 18, 30]. As a very useful tool, fractional calculus is a vital keystone in mathematics and applied sciences. Caputo operators have exceptional applications in a variety of computer science and mathematics domains. This work aims to provide integral operator inequalities for the class of (s, m)-convex functions. Several forms of bounds for Caputo-Fabrizio operators are proposed. In the year 2014, Eftekhari [13] instigated (s, m)-convex function as following.

*Corresponding author

Email addresses: maria.tariq@math.uol.edu.pk (Maria Tariq), ammara.nosheen@uos.edu.pk (Ammara Nosheen), khuramsms@gmail.com (Khuram Ali Khan), a.albrikan@qu.edu.sa (Alhanouf Alburaikan), s.elmorsy@qu.edu.sa (Salwa El-Morsy)

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Definition 1.1. A function $\chi : [0, j] \longrightarrow \mathbb{R}$, $j > 0$ is (s, m)-convex in the second sense, if

$$\chi\Big(\vartheta\rho + m(1-\rho)\varphi\Big) \leqslant \rho^s\chi(\vartheta) + m(1-\rho)^s\chi(\varphi) \tag{1.1}$$

holds, for $s, m \in (0, 1]$, $\vartheta, \varphi \in [0, j]$ and $\rho \in [0, 1]$.

For $s = m = 1$, (1.1) gives the classical convex function.

The Hermite-Hadamard inequality is the development of the idea of convexity. Since Hermite (1883) and Hadamard (1896) separately found this inequality, other researchers have examined it in great detail (see [3, 29]).

Theorem 1.2 ([29]). *Let $\chi : I \to \mathbb{R}$ be convex function, then*

$$\chi\left(\frac{z+w}{2}\right) \leqslant \frac{1}{w-z}\int_z^w \chi(y)dy \leqslant \frac{\chi(z)+\chi(w)}{2}$$

holds, where $z, w \in I$.

Theorem 1.3 (Hölder's inequality, [24]). *If ζ and χ are real functions on $[\vartheta, \varphi]$ and $|\zeta|^p, |\chi|^q$are integrable on $[\vartheta, \varphi]$, then*

$$\int_\vartheta^\varphi |\zeta(\nu)\chi(\nu)|d\nu \leqslant \Big(\int_\vartheta^\varphi |\zeta(\nu)|^p d\nu\Big)^{\frac{1}{p}}\Big(\int_\vartheta^\varphi |\chi(\nu)|^q d\nu\Big)^{\frac{1}{q}} \tag{1.2}$$

holds, where $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$.

Theorem 1.4 (Power-mean inequality, [24]). *Let ξ, $\chi : [\vartheta, \varphi] \to \mathbb{R}$ and $q > 1$. If $|\xi|, |\xi||\chi|^q$ are integrable on $[\vartheta, \varphi]$, then*

$$\int_\vartheta^\varphi |\xi(\lambda)\chi(\lambda)|d\lambda \leqslant \Big(\int_\vartheta^\varphi |\xi(\lambda)|d\lambda\Big)^{1-\frac{1}{q}}\Big(\int_\vartheta^\varphi |\xi(\lambda)||\chi(\lambda)|^q d\lambda\Big)^{\frac{1}{q}} \tag{1.3}$$

holds.

The calculus of integral and differential forms was distinctly discovered in the 17th century by Newton and Leibniz. Since then, differentiation and integration have become important concepts in calculus and analysis. A rich history of the Caputo fractional derivative dates back to the seventeenth century, when semi-derivatives were first addressed by Leibniz and the Marquis de l'Hospital. Fused with fractional differentiation and integration, it is a fundamental tool in the discipline of fractional calculus. The first integral operator in the field of fractional calculus is the Riemann-Liouville fractional integral, which was employed to obtain the Riemann-Liouville fractional derivative. Later, Caputo made improvements to the Riemann-Liouville derivatives, which led to the creation of the well-known Caputo fractional derivative formula defined as follows ([7, 31]). Let $W^1(\vartheta, \varphi)$ be the Sobolev space of order one as

$$W^1(\vartheta, \varphi) = \{g \in L^2(\vartheta, \varphi) : g' \in L^2(\vartheta, \varphi)\},$$

where

$$L^2(\vartheta, \varphi) = \{g(z) : \Big(\int_\vartheta^\varphi g^2(z)dz\Big)^{\frac{1}{2}} < \infty\}.$$

Let $\chi \in W^1(\vartheta, \varphi), \vartheta < \varphi$ and $\sigma \in [0, 1]$, then left derivative in the sense of Caputo-Fabrizio is:

$$({}^{CFD}_{\vartheta}D^\sigma\chi)(z) = \frac{T(\sigma)}{1-\sigma}\int_\vartheta^z \chi'(h)e^{\frac{-\sigma(z-h)^\sigma}{1-\sigma}}dh,$$

$z > \sigma$ and the corresponding integral operator is

$$({}_{\vartheta}^{CF}I^{\sigma}\chi)(z) = \frac{1-\sigma}{T(\sigma)}\chi(z) + \frac{\sigma}{T(\sigma)}\int_{\vartheta}^{z}\chi(\nu)d\nu.$$

For $\sigma = 0$ and $\sigma = 1$ the left derivatives are defined as $({}_{\vartheta}^{CFD}D^{0}\chi)(z) = \chi'(z)$ and $({}_{\vartheta}^{CFD}I^{1}\chi)(z) = \chi(z) - \chi(\vartheta)$, respectively. For the right derivative operator

$$({}_{\varphi}^{CFD}D^{\sigma}\chi)(z) = \frac{-T(\sigma)}{1-\sigma}\int_{z}^{\varphi}\chi'(l)e^{\frac{-\sigma(l-z)^{\sigma}}{1-\sigma}}dl,$$

$z < \varphi$ and the corresponding integral operator is

$$({}^{CF}I_{\varphi}^{\sigma}\chi)(z) = \frac{1-\sigma}{T(\sigma)}\chi(z) + \frac{\sigma}{T(\sigma)}\int_{z}^{\varphi}\chi(\nu)d\nu.$$

$T(\sigma) > 0$ is a normalization function, and $T(0) = T(1) = 1$.

Kavurmaci et al. determined the following identity, which is generalized in the form of Caputo-Fabrizio integrals in this paper.

Lemma 1.5 ([19])**.** *Consider the differentiable function $\chi : [\vartheta, \varphi] \to \mathbb{R}$, if χ' is integrable on $[\vartheta, \varphi]$ and $u \in [\vartheta, \varphi]$, then*

$$\begin{aligned}
&\frac{(u-\vartheta)\chi(\vartheta) + (\varphi-u)\chi(\varphi)}{(\varphi-\vartheta)} - \frac{1}{\varphi-\vartheta}\int_{\vartheta}^{\varphi}\chi(\nu)d\nu \\
&= \frac{(u-\vartheta)^2}{\varphi-\vartheta}\int_{0}^{1}(\varpi-1)\chi'(\varpi u + (1-\varpi)\vartheta)d\varpi + \frac{(\varphi-u)^2}{\varphi-\vartheta}\int_{0}^{1}(1-\varpi)\chi'(\varpi u + (1-\varpi)\varphi)d\varpi,
\end{aligned} \tag{1.4}$$

where $0 \leqslant \varpi \leqslant 1$.

Definition 1.6 (Means)**.** Let $0 < \vartheta < \varphi$, for $z \in \mathbb{R} - \{0, -1\}$ and $\vartheta \neq \varphi$, arithmetic mean and Stolarsky mean are formulized as $A(\vartheta, \varphi) = \frac{\vartheta+\varphi}{2}$ and $L_z(\vartheta, \varphi) = \left(\frac{\vartheta^{z+1} - \varphi^{z+1}}{(z+1)(\vartheta-\varphi)}\right)^{\frac{1}{z}}$, respectively.

Mani et al. discussed bicomplex-valued controlled metric spaces with applications to fractional differential equations [22]. They derived some applications of fuzzy b-Metric space to Fredholm integral equations [23]. Hermite-Hadamard type inequalities for uniformly p-convex and uniformly q-convex functions are provided in [4]. Nosheen et al. introduced the Hermite-Hadamard type inequalities for the class of (s, m)-convex functions in [26], which involve Caputo fractional derivatives and Caputo-Fabrizio integrals. For twice differentiable h-convex functions, Vivas-Cortez et al. investigated the Hermite-Hadamard-Type integral inequalities [36]. Xu et al. presented mathematical exploration on control of bifurcation for a dynamical model [41]. In [38], authors gave mechanism for fractional-order three-triangle multi-delayed neural networks. In [9], Chinnamuniyandi et al. investigated the presence of unique solutions and quasi-uniform stability for a class of fractional-order uncertain BAM neural networks utilizing the Banach fixed point concept, the contraction mapping principle, and analysis techniques. Some dynamics of fractional-order dynamical systems are constructed in [11, 21, 37, 39, 40]. For more recent work on fractional inequalities, one can see [8, 12, 15, 17, 25, 27, 33, 35].

Inspired by recently conducted studies on the Caputo-Fabrizio fractional integral operator and generalizations of Hermite-Hadamard-type inequalities for various convex functions, we develop a new identity for the differentiable functions via Caputo-Fabrizio integrals. We are able to deduce numerous new fractional inequalities for (s, m)-convex function by applying this identity to produce several new Hermite-Hadmard-type inequalities. This work is arranged according to the following structure. In Section 2, new Hermite-Hadamard inequalities associated with the fractional operator are proposed. In Section 3, we examine fascinating uses associated with the trapezoidal formula. Section 4 presents bounds for the mean of the probability density function. Difference of the arithmetic mean and Logarithmic mean are estimated with a graphical example in Section 5. Lastly, the conclusion and future possibilities for further research are discussed in Section 6.

2. Main results

One generalization of Lemma 1.5 is as the following.

Lemma 2.1. *Consider the function* $\chi : [\vartheta, \varphi] \to \mathbb{R}$ *differentiable on* (ϑ, φ)*. If* χ' *is integrable on* $[m\vartheta, m\varphi]$*,* $k \in [m\vartheta, m\varphi]$*, and* $m \in (0, 1]$*, then next relation holds:*

$$\begin{aligned}
&\frac{(k-m\vartheta)\chi(m\vartheta)+(m\varphi-k)\chi(m\varphi)}{(\varphi-\vartheta)} - \frac{T(\nu)}{\nu(\varphi-\vartheta)}\Big[({}^{CF}_{m\vartheta}I^{\nu}\chi)(k) + ({}^{CF}I^{\nu}_{m\varphi}\chi)(k)\Big] + \frac{2(1-\nu)}{(\varphi-\vartheta)\nu}\chi(k) \\
&= \frac{(k-m\vartheta)^2}{\varphi-\vartheta}\int_0^1 (\varpi-1)\chi'(\varpi k + m(1-\varpi)\vartheta)d\varpi + \frac{(m\varphi-k)^2}{\varphi-\vartheta}\int_0^1 (1-\varpi)\chi'(\varpi k + m(1-\varpi)\varphi)d\varpi,
\end{aligned} \tag{2.1}$$

where $0 \leqslant \varpi \leqslant 1$.

Proof. Let

$$U = \frac{(k-m\vartheta)^2}{\varphi-\vartheta}\int_0^1 (\varpi-1)\chi'(\varpi k + m(1-\varphi)\vartheta)d\varpi + \frac{(m\varphi-k)^2}{\varphi-\vartheta}\int_0^1 (1-\varpi)\chi'(\varpi k + m(1-\varpi)\varphi)d\varpi. \tag{2.2}$$

Applying integration by parts on the right side of (2.2),

$$U = \frac{(k-m\vartheta)\chi(m\vartheta)+(m\varphi-k)\chi(m\varphi)}{(\varphi-\vartheta)} - \frac{1}{\varphi-\vartheta}\int_{m\vartheta}^{m\varphi}\chi(l)dl. \tag{2.3}$$

Multiplying (2.3) with $\frac{\nu}{T(\nu)}$ and subtracting $\frac{2(1-\nu)\chi(k)}{T(\nu)(\varphi-\vartheta)}$ we get

$$\begin{aligned}
&\frac{\nu}{T(\nu)}U - \frac{2(1-\nu)\chi(k)}{T(\nu)(\varphi-\vartheta)} \\
&\quad = \frac{\nu}{T(\nu)}\Big[\frac{(k-m\vartheta)\chi(m\vartheta)+(m\varphi-k)\chi(m\varphi)}{(\varphi-\vartheta)} - \frac{1}{\varphi-\vartheta}\int_{m\vartheta}^{m\varphi}\chi(l)dl\Big] - \frac{2(1-\nu)\chi(k)}{T(\nu)(\varphi-\vartheta)} \\
&\quad = \frac{\nu}{T(\nu)}\frac{(k-m\vartheta)\chi(m\vartheta)+(m\varphi-k)\chi(m\varphi)}{(\varphi-\vartheta)} - \frac{1}{(\varphi-\vartheta)}\Big[({}^{CF}_{m\vartheta}I^{\nu}\chi)(k) + ({}^{CF}I^{\nu}_{m\varphi}\chi)(k)\Big].
\end{aligned} \tag{2.4}$$

Further simplification of (2.4) leads toward (2.1). □

Remark 2.2.

(i) Putting $\nu = m = 1$ in Lemma 2.1 we get (1.4).
(ii) Putting $m = 1$ in Lemma 2.1 gives [15, Lemma 2.2].

Notations: Consider

$$|Y| := \left|\frac{(k-m\vartheta)\chi(m\vartheta)+(m\varphi-k)\chi(m\varphi)}{(\varphi-\vartheta)} - \frac{T(\nu)}{\nu(\varphi-\vartheta)}\Big[\Big(({}^{CF}_{m\vartheta}I^{\nu}\chi)(k)+({}^{CF}I^{\nu}_{m\varphi}\chi)(k)\Big)\Big] + \frac{2(1-\nu)}{\nu(\varphi-\vartheta)}\chi(k)\right|$$

and

$$|K| := \left|\frac{(k-m\vartheta)\chi(m\vartheta)+(m\varphi-k)\chi(m\varphi)}{(\varphi-\vartheta)} - \frac{1}{(\varphi-\vartheta)}\int\limits_{m\vartheta}^{m\varphi}\chi(c)dc\right|,$$

throughout the remaining text.

Theorem 2.3. *Let* $\chi : [m\vartheta, m\varphi] \to \mathbb{R}$ *be a differentiable function on* $(m\vartheta, m\varphi)$. *If* $|\chi'|$ *is* (s, m)*-convex function and integrable on* $[m\vartheta, m\varphi]$, *then*

$$\begin{aligned}&\left|\frac{(k-m\vartheta)\chi(m\vartheta)+(m\varphi-k)\chi(m\varphi)}{(\varphi-\vartheta)} - \frac{T(\nu)}{\nu(\varphi-\vartheta)}\Big[({}^{CF}_{m\vartheta}I^{\nu}\chi)(k)+({}^{CF}I^{\nu}_{m\varphi}\chi)(k)\Big] + \frac{2(1-\nu)}{(\varphi-\vartheta)\nu}\chi(k)\right|\\ &\leqslant \frac{(k-m\vartheta)^2}{\varphi-\vartheta}\Big[\frac{|\chi'(k)|}{(s+1)(s+2)}+m\frac{|\chi'(\vartheta)|}{(s+2)}\Big] + \frac{(m\varphi-k)^2}{\varphi-\vartheta}\Big[\frac{|\chi'(k)|}{(s+1)(s+2)}+m\frac{|\chi'(\varphi)|}{(s+2)}\Big]\end{aligned}$$

holds for $k \in [m\vartheta, m\varphi]$ *and* $s, m \in (0, 1]$.

Proof. Utilizing Lemma 2.1 and properties of absolute value function,

$$|Y| \leqslant \frac{(k-m\vartheta)^2}{\varphi-\vartheta}\int\limits_0^1(1-\varpi)|\chi'(\varpi k+m(1-\varpi)\vartheta)|d\varpi + \frac{(m\varphi-k)^2}{\varphi-\vartheta}\int\limits_0^1(1-\varpi)|\chi'(\varpi k+m(1-\varpi)\varphi)|d\varpi.$$

Since $|\chi'(\cdot)|$ is (s, m)-convex, therefore

$$\begin{aligned}|Y| &\leqslant \frac{(k-m\vartheta)^2}{\varphi-\vartheta}\int\limits_0^1(1-\varpi)[\varpi^s|\chi'(k)|+m(1-\varpi)^s|\chi'(\vartheta)|]d\varpi\\ &\quad+\frac{(m\varphi-k)^2}{\varphi-\vartheta}\int\limits_0^1(1-\varpi)[\varpi^s|\chi'(k)|+m(1-\varpi)^s|\chi'(\varphi)|]d\varpi\\ &\leqslant \frac{(k-m\vartheta)^2}{\varphi-\vartheta}\Big[\frac{|\chi'(k)|}{(s+1)(s+2)}+m\frac{|\chi'(\vartheta)|}{(s+2)}\Big] + \frac{(m\varphi-k)^2}{\varphi-\vartheta}\Big[\frac{|\chi'(k)|}{(s+1)(s+2)}+m\frac{|\chi'(\varphi)|}{(s+2)}\Big].\end{aligned}$$

□

The repercussions of Theorem 2.3 are summed up in the following remark.

Remark 2.4. Substitution of $\nu = 1$ in Theorem 2.3 gives

$$\begin{aligned}|K| &= \left|\frac{(k-m\vartheta)\chi(m\vartheta)+(m\varphi-k)\chi(m\varphi)}{(\varphi-\vartheta)} - \frac{1}{(\varphi-\vartheta)}\int\limits_{m\vartheta}^{m\varphi}\chi(j)dj\right|\\ &\leqslant \frac{(k-m\vartheta)^2}{\varphi-\vartheta}\Big[\frac{|\chi'(k)|}{(s+1)(s+2)}+m\frac{|\chi'(\vartheta)|}{(s+2)}\Big] + \frac{(m\varphi-k)^2}{\varphi-\vartheta}\Big[\frac{|\chi'(k)|}{(s+1)(s+2)}+m\frac{|\chi'(\varphi)|}{(s+2)}\Big].\end{aligned} \tag{2.5}$$

(a) Substituting $m = 1$ and $m = s = 1$ in (2.5), we obtain [2, Theorem 5] and [19, Theorem 4].
(b) Substituting $m = s = 1$ and choosing $k = \frac{\vartheta+\varphi}{2}$ in (2.5), we get [19, Corollary 2].

Theorem 2.5. *Let* $\chi : [m\vartheta, m\varphi] \to \mathbb{R}$ *be a differentiable function on interval* $(m\vartheta, m\varphi)$. *If* $|\chi'|^q$, $q > 1$, *is* (s, m)*-convex function and integrable on* $[m\vartheta, m\varphi]$, *then*

$$|Y| \leqslant \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\frac{(k-m\vartheta)^2(m|\chi'(\vartheta)|^q + |\chi'(k)|^q)^{\frac{1}{q}} + (m\varphi-k)^2(m|\chi'(\varphi)|^q + |\chi'(k)|^q)^{\frac{1}{q}}}{(\varphi-\vartheta)}\right]$$

holds, for $s, m \in (0, 1]$, $k \in [m\vartheta, m\varphi]$ *and* $p^{-1} = 1 - q^{-1}$.

Proof. Use Lemma 2.1 and (1.2),

$$\begin{aligned}|Y| \leqslant{}& \frac{(k-m\vartheta)^2}{\varphi-\vartheta}\left(\int_0^1 (1-\varpi)^p d\varpi\right)^{\frac{1}{p}}\left(\int_0^1 |\chi'(\varpi k + m(1-\varpi)\vartheta)|^q d\varpi\right)^{\frac{1}{q}}\\ &+ \frac{(m\varphi-k)^2}{\varphi-\vartheta}\left(\int_0^1 (1-\varpi)^p d\varpi\right)^{\frac{1}{p}}\left(\int_0^1 |\chi'(\varpi k + m(1-\varpi)\varphi)|^q d\varpi\right)^{\frac{1}{q}}.\end{aligned}$$

(s, m)-convexity of $|\chi'|^q$ gives

$$\left|Y\right| \leqslant \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{(s+1)}\right)^{\frac{1}{q}}\left[\frac{(k-m\vartheta)^2(m|\chi'(\vartheta)|^q + |\chi'(k)|^q)^{\frac{1}{q}} + (m\varphi-k)^2(m|\chi'(\varphi)|^q + |\chi'(k)|^q)^{\frac{1}{q}}}{(\varphi-\vartheta)}\right].$$

□

Remark 2.6. Substituting $\nu = 1$ in Theorem 2.5 we get

$$|K| \leqslant \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\frac{(k-m\vartheta)^2(m|\chi'(\vartheta)|^q + |\chi'(k)|^q)^{\frac{1}{q}} + (m\varphi-k)^2(m|\chi'(\varphi)|^q + |\chi'(k)|^q)^{\frac{1}{q}}}{(\varphi-\vartheta)}\right]. \tag{2.6}$$

(a) Choosing $m = 1$ and $m = s = 1$ in (2.6) we obtain [2, Theorem 6] and [19, Theorem 5], respectively.
(b) Substituting $m = 1$ and choosing $k = \frac{\vartheta+\varphi}{2}$ in (2.6) we get [2, Corollary 4].
(c) Putting $m = s = 1$ and choosing $k = \frac{\vartheta+\varphi}{2}$ in (2.6) we obtain [19, Corollary 3].

Theorem 2.7. *Let* $\chi : [m\vartheta, m\varphi] \to \mathbb{R}$ *be a differentiable function on* $(m\vartheta, m\varphi)$. *If* $|\chi'|^q$, $q > 1$, *is an* (s, m)*-convex function and integrable on* $[m\vartheta, m\varphi]$, *then*

$$|Y| \leqslant \frac{(k-m\vartheta)^2}{\varphi-\vartheta}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{|\chi'(k)|^q}{(s+1)(s+2)} + \frac{m|\chi'(\vartheta)|^q}{(s+2)}\right]^{\frac{1}{q}} + \frac{(m\varphi-k)^2}{\varphi-\vartheta}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{|\chi'(k)|^q}{(s+1)(s+2)} + \frac{m|\chi'(\varphi)|^q}{s+2}\right]^{\frac{1}{q}}$$

holds for $s, m \in (0, 1]$ *and* $k \in [m\vartheta, m\varphi]$.

Proof. Utilizing Lemma 2.1 and (1.3) we obtain

$$\begin{aligned}|Y| \leqslant{}& \frac{(k-m\vartheta)^2}{\varphi-\vartheta}\left(\int_0^1 (1-\varpi)d\varpi\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-\varpi)|\chi'(\varpi k + m(1-\varpi)\vartheta)|^q d\varpi\right)^{\frac{1}{q}}\\ &+ \frac{(m\varphi-k)^2}{\varphi-\vartheta}\left(\int_0^1 (1-\varpi)d\varpi\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-\varpi)|\chi'(\varpi k + m(1-\varpi)\varphi)|^q d\varpi\right)^{\frac{1}{q}}.\end{aligned}$$

Since $|\chi'|^q$ is (s,m)-convex, therefore

$$
\begin{aligned}
|Y| &\leqslant \frac{(k-m\vartheta)^2}{\varphi-\vartheta}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-\varpi)[\varpi^s|\chi'(k)|^q + m(1-\varpi)^s|\chi'(\vartheta)|^q]d\varpi\right)^{\frac{1}{q}} \\
&\quad + \frac{(m\varphi-k)^2}{\varphi-\vartheta}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-\varpi)[\varpi^s|\chi'(k)|^q + m(1-\varpi)^s|\chi'(\varphi)|^q]d\varpi\right)^{\frac{1}{q}} \\
&\leqslant \frac{(k-m\vartheta)^2}{\varphi-m\vartheta}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{|\chi'(k)|^q}{(s+1)(s+2)} + \frac{m|\chi'(\vartheta)|^q}{(s+2)}\right]^{\frac{1}{q}} + \frac{(m\varphi-k)^2}{\varphi-\vartheta}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{|\chi'(k)|^q}{(s+1)(s+2)} + \frac{m|\chi'(\varphi)|^q}{s+2}\right]^{\frac{1}{q}}.
\end{aligned}
$$

□

Remark 2.8. Substituting $\nu = 1$ in Theorem 2.7, we get

$$
\begin{aligned}
|K| \leqslant & \frac{(k-m\vartheta)^2}{\varphi-m\vartheta}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{|\chi'(k)|^q}{(s+1)(s+2)} + \frac{m|\chi'(\vartheta)|^q}{(s+2)}\right]^{\frac{1}{q}} \\
& + \frac{(m\varphi-k)^2}{\varphi-\vartheta}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{|\chi'(k)|^q}{(s+1)(s+2)} + \frac{m|\chi'(\varphi)|^q}{s+2}\right]^{\frac{1}{q}}.
\end{aligned}
\tag{2.7}
$$

(a) In (2.7), substituting $m = 1$ and $m = s = 1$ provides [2, Theorem 7] and [19, Theorem 7], respectively.
(b) Putting $m = s = 1$ and choosing $k = \frac{\vartheta+\varphi}{2}$ in (2.7), we get [19, Corollary 4].

Example 2.9. Substituting $\nu = 1$, $k = \frac{\vartheta+\varphi}{2}$, and $\chi(x) = \frac{x^u}{u}$ in Theorem 2.7, where u=s+m and $0 < s+m < 1$, we get

$$
\begin{aligned}
&\left|A(\vartheta^u,\varphi^u) - L_u^u(\vartheta,\varphi)\right| \\
&\leqslant \frac{(\varphi-\vartheta)(u)}{4m^{u-1}}\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\left[\frac{|(\frac{\vartheta+\varphi}{2})^{u-1}|^q}{(s+1)(s+2)} + \frac{m|(\vartheta)^{u-1}|^q}{(s+2)}\right]^{\frac{1}{q}} + \left[\frac{|(\frac{\vartheta+\varphi}{2})^{u-1}|^q}{(s+1)(s+2)} + \frac{m|(\varphi)^{u-1}|^q}{(s+2)}\right]^{\frac{1}{q}}\right).
\end{aligned}
\tag{2.8}
$$

Through Theorem 2.7, we are enable to find bound for difference of means in this example.

3. Applications to error estimation formulae

In this part, some numerical analysis applications of our results are explored.

3.1. *Error bound for the quadrature formula*

Let χ have a bounded first derivative over $[\vartheta,\varphi]$. Mean value theorem for χ gives [16]:

$$
(k-\vartheta)\chi'(\xi) = \chi(k) - \chi(\vartheta) \tag{3.1}
$$

for some $\xi \in (\vartheta, k]$. Integrating (3.1) and taking absolute value:

$$
\left|\int_\vartheta^\varphi \chi(k)dk - (\varphi-\vartheta)\chi(\vartheta)\right| = \left|\int_\vartheta^\varphi (k-\vartheta)\chi'(\xi)dk\right|,
$$

we can obtain a more accurate approximation of the integral on the right side by incorporating the absolute value into the integrand and substituting the term χ' with an upper bound $Z = \sup|\chi'(\xi)|$ as

$$
\left|\int_\vartheta^\varphi \chi(k)dk - (\varphi-\vartheta)\chi(\vartheta)\right| \leqslant \frac{(\varphi-\vartheta)^2 Z}{2}. \tag{3.2}
$$

Putting $k = \varphi$ and $s = m = 1$ in (2.5), then using the fact that $|\chi'(\vartheta)|, |\chi'(\varphi)| \leqslant Z$, we get (3.2), which is error bound for the quadrature formula.

Next, trapezoidal error bounds are estimated using the main findings.

3.2. *The trapezoidal formula* [19]

Let b belong to set of natural numbers, $B : \vartheta = \delta_0 < \delta_1 < \delta_2 < \cdots < \delta_b = \varphi$ is a partition of interval $[\vartheta, \varphi]$, and consider the quadrature formula

$$\int_{\vartheta}^{\varphi} \chi(h)dh = \mathfrak{Y}(\chi, B) + \mathfrak{S}(\chi, B), \tag{3.3}$$

where

$$\mathfrak{Y}(\chi, B) = \sum_{a=0}^{b-1} \left(\frac{\chi(\delta_a) + \chi(\delta_{a+1})}{2} \right) (\delta_{a+1} - \delta_a)$$

for the area of trapeziums and $\mathfrak{S}(\chi, B)$ represents the corresponding approximation error.

Proposition 3.1. *Considering the premises of Theorem 2.3, the trapezoidal error estimate for each partition B of $[\vartheta, \varphi]$ satisfies:*

$$|\mathfrak{S}(\chi, B)| \leqslant \sum_{a=0}^{b-1} \frac{(\delta_{a+1} - \delta_a)^2}{4} \left[\frac{|\chi'(\delta_{a+1})| + |\chi'(\delta_a)|}{s+2} + \frac{2|\chi'(\frac{\delta_{a+1}+\delta_a}{2})|}{(s+1)(s+2)} \right].$$

Proof. In (2.5), choosing $m = 1$ and $k = \frac{\vartheta+\varphi}{2}$ we get

$$\begin{aligned} \left| \frac{\chi(\vartheta) + \chi(\varphi)}{2} - \frac{1}{(\varphi - \vartheta)} \int_{\vartheta}^{\varphi} \chi(c)dc \right| &\leqslant \frac{\varphi - \vartheta}{4} \left[\frac{|\chi'(\frac{\vartheta+\varphi}{2})|}{(s+1)(s+2)} + \frac{|\chi'(\vartheta)|}{(s+2)} \right] \\ &\quad + \frac{\varphi - \vartheta}{4} \left[\frac{|\chi'(\frac{\vartheta+\varphi}{2})|}{(s+1)(s+2)} + \frac{|\chi'(\varphi)|}{(s+2)} \right]. \end{aligned} \tag{3.4}$$

Considering (3.4) for $[\delta_a, \delta_{a+1}]$ $(a = 0, 1, \ldots, b-1)$ we get

$$\left| \frac{\chi(\delta_a) + \chi(\delta_{a+1})}{2} - \frac{1}{(\delta_{a+1} - \delta_a)} \int_{\delta_a}^{\delta_{a+1}} \chi(c)dc \right| \leqslant \frac{(\delta_{a+1} - \delta_a)}{4} \left[\frac{|\chi'(\delta_{a+1})| + |\chi'(\delta_a)|}{s+2} + \frac{2|\chi'(\frac{\delta_{a+1}+\delta_a}{2})|}{(s+1)(s+2)} \right].$$

Hence, in (3.3) we have

$$\begin{aligned} \left| \int_{\vartheta}^{\varphi} \chi(c)dc - \mathfrak{Y}(\chi, B) \right| &\leqslant \sum_{a=0}^{b-1} \left| \int_{\delta_a}^{\delta_{a+1}} \chi(c)dc - \frac{\chi(\delta_a) + \chi(\delta_{a+1})}{2} (\delta_{a+1} - \delta_a) \right| \\ &\leqslant \sum_{a=0}^{b-1} \frac{(\delta_{a+1} - \delta_a)^2}{4} \left[\frac{|\chi'(\delta_{a+1})| + |\chi'(\delta_a)|}{s+2} + \frac{2|\chi'(\frac{\delta_{a+1}+\delta_a}{2})|}{(s+1)(s+2)} \right]. \end{aligned}$$

□

Remark 3.2. The next set of error estimates meets the requirements for every partition B of $[\vartheta, \varphi]$.

The next two inequalities, result, respectively, if we adhere to the steps in the proof of Proposition 3.1 for (2.7) and (2.6).

$$\begin{aligned}|\mathfrak{S}(\chi, B)| \leqslant \sum_{a=0}^{b-1} \frac{(\delta_{a+1}-\delta_a)^2}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\Bigg(&\left[\frac{|\chi'(\frac{\delta_{a+1}+\delta_a}{2})|^q}{(s+1)(s+2)}+\frac{|\chi'(\delta_a)|^q}{s+2}\right]^{\frac{1}{q}}\\&+\left[\frac{|\chi'(\frac{\delta_{a+1}+\delta_a}{2})|^q}{(s+1)(s+2)}+\frac{|\chi'(\delta_{a+1})|^q}{s+2}\right]^{\frac{1}{q}}\Bigg).\end{aligned} \tag{3.5}$$

$$\begin{aligned}|\mathfrak{S}(\chi, B)| \leqslant \left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \sum_{a=0}^{b-1} \frac{(r_{a+1}-r_a)^2}{4}\Bigg[&\left(|\chi'(r_a)|^q+|\chi'(\frac{r_{a+1}+r_a}{2})|^q\right)^{\frac{1}{q}}\\&+\left(|\chi'(r_{a+1})|^q+|\chi'(\frac{r_{a+1}+r_a}{2})|^q\right)^{\frac{1}{q}}\Bigg].\end{aligned} \tag{3.6}$$

Example 3.3. Substituting $\chi(x)=\frac{q}{4+q}x^{\frac{4}{q}+1}$ in (3.5) and (3.6) we get the following inequalities, respectively,

$$\begin{aligned}R(q) &= \left|\frac{q(1+2^{\frac{4}{q}+2}+3^{\frac{4}{q}+1})}{8+2q}+\frac{q^2(3^{\frac{4}{q}+1}-1)}{(4+q)(4+2q)}\right|\\&\leqslant \frac{1}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{7}{4}\right)^{\frac{1}{q}}+\left(\frac{31}{4}\right)^{\frac{1}{q}}+\left(\frac{1009}{60}\right)^{\frac{1}{q}}+\left(\frac{2569}{60}\right)^{\frac{1}{q}}\right]=Y(q),\\R(q) &\leqslant \frac{1}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{97}{16}\right)^{\frac{1}{q}}+\left(\frac{337}{16}\right)^{\frac{1}{q}}+\left(\frac{881}{16}\right)^{\frac{1}{q}}+\left(\frac{1921}{16}\right)^{\frac{1}{q}}\right]=U(q),\end{aligned} \tag{3.7}$$

where $R(q)$ represents the trapezoidal error for $\chi(x)=\frac{q}{4+q}x^{\frac{4}{q}+1}$, and $Y(q)$ (purple) and $U(q)$ (yellow) are

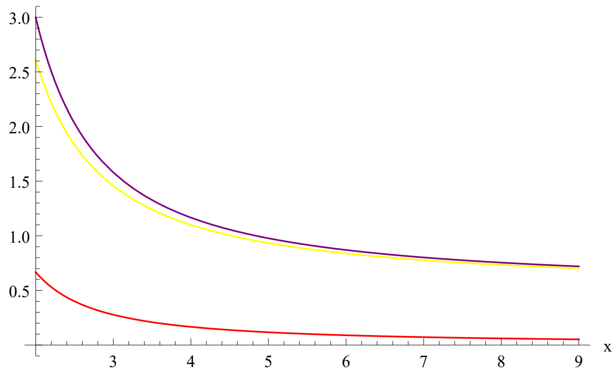

Figure 1: An illustration of the inequalities shown in Example 3.3, where red color represents $R(q)$, purple is $Y(q)$, and yellow shows $U(q)$.

bounds of trapezoidal error. It is clear from Figure 1 that yellow curve is more closer to red curve, which shows that (3.7) gives the best estimation.

4. Bounds for mean of probability density function

For a continuous random variable S, let $\chi:[m\omega, m\zeta]\to[0,1]$ be probability density function and the cumulative distribution function is $N(s)=P(S\leqslant s)=\int_{m\vartheta}^{s}\chi(h)dh$, where $s\in[\vartheta,\varphi]$, $N(m\vartheta)=0$, and

$N(m\varphi) = 1$. The mean of χ is given by the following formula:

$$E(S) = \int_{m\vartheta}^{m\varphi} r\chi(r)dr = m\varphi - \int_{m\vartheta}^{m\varphi} N(r)dr. \tag{4.1}$$

Theorem 4.1. *Taking into consideration Theorem 2.3's presumptions,*

$$|E(S) - k| \leqslant (k - m\vartheta)^2\Big[\frac{|\chi(k)|}{(s+1)(s+2)} + m\frac{|\chi(\vartheta)|}{(s+2)}\Big] + (m\varphi - k)^2\Big[\frac{|\chi(k)|}{(s+1)(s+2)} + m\frac{|\chi(\varphi)|}{(s+2)}\Big]. \tag{4.2}$$

Proof. Putting $\chi = N$ in (2.5) and applying (4.1) we obtain (4.2). □

Theorem 4.2. *Taking into consideration Theorem 2.5's presumptions we get*

$$|E(S) - k| \leqslant (\frac{1}{p+1})^{\frac{1}{p}}(\frac{1}{s+1})^{\frac{1}{q}}\Big[(k - m\vartheta)^2(m|\chi(\vartheta)|^q + |\chi(k)|^q)^{\frac{1}{q}} + (m\varphi - k)^2(m|\chi(\varphi)|^q + |\chi(k)|^q)^{\frac{1}{q}}\Big]. \tag{4.3}$$

Proof. Choosing $\chi = N$ in (2.6), and then applying (4.1), one obtains (4.3). □

Theorem 4.3. *Taking into consideration Theorem 2.7's presumptions,*

$$\begin{aligned} |E(S) - k| \leqslant& (k - m\vartheta)^2(\frac{1}{2})^{1-\frac{1}{q}}[\frac{|\chi(k)|^q}{(s+1)(s+2)} + \frac{m|\chi(\vartheta)|^q}{(s+2)}]^{\frac{1}{q}} \\ &+ (m\varphi - k)^2(\frac{1}{2})^{1-\frac{1}{q}}[\frac{|\chi(k)|^q}{(s+1)(s+2)} + \frac{m|\chi(\varphi)|^q}{s+2}]^{\frac{1}{q}}. \end{aligned} \tag{4.4}$$

Proof. Choosing $\chi = N$ in (2.7), and then applying (4.1), one obtains (4.4). □

5. Estimates for difference of arithmetic mean and Stolarsky mean

Proposition 5.1. *Consider the presumptions in Theorem 2.5 with* $i \in (-\infty, 0] \bigcup [1, \infty) \backslash \{-2q, q\}$, *then the following bounds exist:*

$$\begin{aligned} &\Big|\frac{q}{i+q}\Big(A(\vartheta^{\frac{i+q}{q}}, \varphi^{\frac{i+q}{q}}) - L^{\frac{i+q}{q}}_{\frac{i+q}{q}}(\vartheta, \varphi)\Big)\Big| \\ &\quad \leqslant (\frac{1}{p+1})^{\frac{1}{p}}(\frac{1}{s+1})^{\frac{1}{q}}\frac{(\varphi - \vartheta)}{4}\Big[\Big((\vartheta)^i + (\frac{\vartheta+\varphi}{2})^i\Big)^{\frac{1}{q}} + \Big((\varphi)^i + (\frac{\vartheta+\varphi}{2})^i\Big)^{\frac{1}{q}}\Big], \end{aligned} \tag{5.1}$$

$$\begin{aligned} &\Big|\frac{q}{i+q}\Big(A(\vartheta^{\frac{i+q}{q}}, \varphi^{\frac{i+q}{q}}) - L^{\frac{i+q}{q}}_{\frac{i+q}{q}}(\vartheta, \varphi)\Big)\Big| \\ &\quad \leqslant (\frac{1}{2})^{1-\frac{1}{q}}\frac{(\varphi - \vartheta)}{4}\Big[\Big(\frac{(\vartheta)^i}{s+2} + \frac{(\frac{\vartheta+\varphi}{2})^i}{(s+1)(s+2)}\Big)^{\frac{1}{q}} + \Big(\frac{(\varphi)^i}{s+2} + \frac{(\frac{\vartheta+\varphi}{2})^i}{(s+1)(s+2)}\Big)^{\frac{1}{q}}\Big]. \end{aligned} \tag{5.2}$$

Proof. $\chi(x) = \frac{q}{i+q}x^{\frac{i}{q}+1}$, $m = 1$, and $k = \frac{\vartheta+\varphi}{2}$ in Theorems 2.5 and 2.7 provide the desired outcomes. □

Example 5.2. In inequalities (5.1) and (5.2), substituting $i = 4, q = 2$ we get the following inequalities, respectively,

$$M(\vartheta, \varphi) = \Big|\frac{1}{3}(\frac{\vartheta^3 + \varphi^3}{2} - \frac{\vartheta^4 - \varphi^4}{4(\vartheta - \varphi)})\Big| \leqslant (\frac{1}{3})^{\frac{1}{2}}(\frac{2}{3})^{\frac{1}{2}}\frac{\varphi - \vartheta}{4}\Big[(\vartheta^4 + (\frac{\vartheta+\varphi}{2})^4)^{\frac{1}{2}} + (\varphi^4 + (\frac{\vartheta+\varphi}{2})^4)^{\frac{1}{2}}\Big] = h(\vartheta, \varphi),$$

$$M(\vartheta, \varphi) \leqslant (\frac{1}{2})^{\frac{1}{2}}\frac{\varphi - \vartheta}{4}\Big[(\frac{2}{5}\vartheta^4 + \frac{4}{15}(\frac{\vartheta+\varphi}{2})^4)^{\frac{1}{2}} + (\frac{2}{5}\varphi^4 + \frac{4}{15}(\frac{\vartheta+\varphi}{2})^4)^{\frac{1}{2}}\Big] = l(\vartheta, \varphi).$$

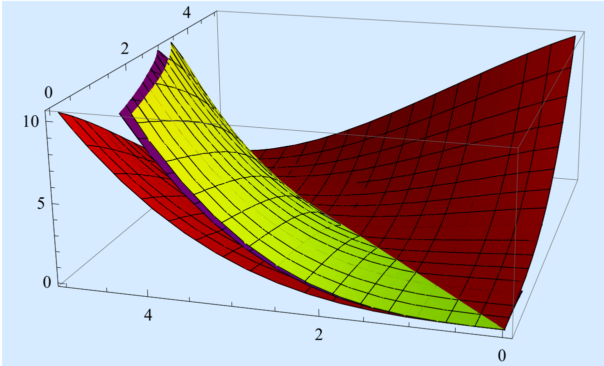

Figure 2: A visual representation of the inequalities shown in Example 5.2, where red color represents $M(\vartheta, \varphi)$, purple is $l(\vartheta, \varphi)$, and yellow shows $h(\vartheta, \varphi)$.

For $\vartheta, \varphi \in \mathbb{R}^+$ with $\vartheta < \varphi$, $M(\vartheta, \varphi)$ shows difference of Arithmetic and Stolarsky mean. $h(\vartheta, \varphi)$ and $l(\vartheta, \varphi)$ are bounds of this difference. Figure 2 shows that surface with purple color is more closer to red surface, clearly (5.2) gives better bound.

6. Conclusion

In view of current developments in the field of fractional analysis, numerous studies have been conducted to optimize the bounds with the assistance of various fractional integral operators. The Caputo-Fabrizio operator counts among them. Since this topic is significant and has many repercussions in simulating real-world natural events, the primary objective of this study is to obtain novel and widespread inequalities that make a relationship between inequality theory and fractional analysis by using this operator. Furthermore, we highlight how the findings, when examined within the context of integral inequality, build upon and improve a significant body of previous research. After that, in order to help people better grasp the recently discovered inequalities, we offer particular examples along with the associated graphs. It is anticipated that these theoretical investigations will open up new directions for research into fresh ways for the Caputo-Fabrizio operator and in numerous additional domains of application. With the use of this operator, future research can examine a variety of inequality categories, including Simpson-type, Grüss-type, and Chebyshev-type inequality. Additionally, one can focus on the practical applications of these inequalities.

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Use of AI tools

No AI tool is used in this study.

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