J. Math. Computer Sci., 37 (2025), 274–286

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Results on fixed points and common fixed points on bipolar $\mathfrak{b}$-metric space with applications

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Abstract

In this paper, we present the concept of bipolar $\mathfrak{b}$-metric spaces and establish both fixed point and common fixed point theorems within this framework. Our results broaden and extend several well-known findings in the existing literature. To illustrate the applicability of our theorems, we provide a detailed example and a relevant application.

Keywords: Bipolar $\mathfrak{b}$-metric space, fixed point, covariant map, contravariant map.

2020 MSC: 47H10, 54H25.

1. Introduction

Fixed point theory is a foundational tool across numerous mathematical disciplines, with a key area of research focusing on the identification of fixed points in generalized contraction mappings. Recent studies have produced a plethora of papers addressing fixed point theory and its applications in various contexts. In 1989, Bakhtin [1] introduced the concept of $\mathfrak{b}$-metric spaces, which was subsequently expanded by Czerwik [4] in 1993. These developments have led to numerous generalizations of the Banach fixed point theorem in $\mathfrak{b}$-metric spaces, with significant contributions from Czerwik [5], Pacurar [22], Boriceanu [2], Bota [3], Mehmet [18]. The convergence of measurable functions with respect to measure has also been extensively studied, as seen in [10, 24, 25], the works of Czerwik and others. Czerwik [4] first presented a generalization of banach fixed point theorem in $\mathfrak{b}$-metric spaces. Recently, the study of fixed points of contraction mappings in bipolar metric spaces, a generalization of the Banach contraction principle, has become a vibrant area of research. In 2016, Mutlu and Gürdal [19] introduced bipolar metric spaces and explored basic fixed point and coupled fixed point theorems for co-variant and contra-variant maps under contractive conditions; see [19, 20]. In bipolar metric spaces, a lot of significant work has been done (see [8, 9, 14–17, 19–21, 23]). Noteworthy advancements have been made in this field, including the work

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doi: 10.22436/jmcs.037.03.02

Received: 2024-06-21 *Revised:* 2024-07-23 *Accepted:* 2024-09-03

of Gaba et al. [8] in 2021, who established fixed point theorems in bipolar metric spaces. The authors highlighted particular ordered vector spaces that are derived from the bipolar $\mathfrak{b}$-metric space and explored their significance in economic research. Balaji Ramalingam [26] had explored fixed point theorem on fuzzy bipolar $\mathfrak{b}$-metric space. Deep and Kazemi [7] had expolred non-linear fractional integral equations by Petryshyn's fixed point theorem. Kazemi et al. [13] had given application of fixed point theorem on fractional stochastic functional integral equations. Das et al. [6] had presented existence of solutions of fractional hybrid differential equations. In this paper, we extend this line of inquiry by introducing the notion of bipolar $\mathfrak{b}$-metric spaces and proving fixed point and common fixed point theorems within this novel framework.

2. Preliminaries

In this section, we present some basic definitions.

Definition 2.1. Let Ξ and Λ be nonempty sets and $\mathfrak{b} \geqslant 1$ be a given real number and $\varphi : \Xi \times \Lambda \to \mathbb{R}^+$ be a mapping. Consider the following properties.

1. If $\varphi(\sigma,\eta) = 0$ iff $\sigma = \eta$, for all $(\sigma,\eta) \in \Xi \times \Lambda$;
2. $\varphi(\sigma,\eta) = \varphi(\eta,\sigma)$, for all $(\sigma,\eta) \in \Xi \cap \Lambda$;
3. $\varphi(\sigma,\eta) \leqslant \mathfrak{b}(\varphi(\sigma,\omega) + \varphi(\alpha,\omega) + \varphi(\alpha,\eta))$.

The triplet (Ξ,Λ,φ) is called a bipolar $\mathfrak{b}$-metric spaces (B$\mathfrak{b}$MS).

Definition 2.2.

1. Let (Ξ,Λ,φ) be a B$\mathfrak{b}$MSs with $\mathfrak{b} \geqslant 1$. Then the points of the sets Ξ, Λ, and $\Xi \cap \Lambda$ are named as left, right, and central points, respectively, and any sequence, that is consisted of only left (or right, or central) points is called a left (or right, or central) sequence on (Ξ,Λ,φ).
2. Let $(\Xi_1,\Lambda_1,\varphi_1)$ and $(\Xi_2,\Lambda_2,\varphi_2)$ be B$\mathfrak{b}$MSs with $\mathfrak{b} \geqslant 1$ and $\Omega : \Xi_1 \cup \Lambda_1 \to \Xi_2 \cup \Lambda_2$ be a function. If $\Omega(\Xi_1) \subseteq \Xi_2$ and $\Omega(\Lambda_1) \subseteq \Lambda_2$, then Ω is called a covariant map, or a map from $(\Xi_1,\Lambda_1,\varphi_1)$ to $(\Xi_2,\Lambda_2,\varphi_2)$ and this is written as $\Omega : (\Xi_1,\Lambda_1,\varphi_1) \rightrightarrows (\Xi_2,\Lambda_2,\varphi_2)$. If $\Omega : (\Xi_1,\Lambda_1,\varphi_1) \rightrightarrows (\Lambda_2,\Xi_2,\overline{\varphi}_2)$ is a map, then Ω is called a contravariant map from $(\Xi_1,\Lambda_1,\varphi_1)$ to $(\Xi_2,\Lambda_2,\varphi_2)$ and this is denoted as $\Omega : (\Xi_1,\Lambda_1,\varphi_1) \rightleftarrows (\Xi_2,\Lambda_2,\varphi_2)$.

Definition 2.3. Let (Ξ,Λ,φ) be a B$\mathfrak{b}$MSs with $\mathfrak{b} \geqslant 1$. A left sequence $\{\sigma_\vartheta\}$ converges to a right point η if and only if for every $\varepsilon > 0$ we can find an $\vartheta_0 \in \mathbb{N}$ satisfying $\varphi(\sigma_\vartheta,\eta) < \varepsilon$ for all $\vartheta \geqslant \vartheta_0$. Similarly, a right seqence $\{\eta_\vartheta\}$ converges to a left point σ if and only if, for every $\varepsilon > 0$ we can find an $\vartheta_0 \in \mathbb{N}$ satisfying $\varphi(\sigma,\eta_\vartheta) < \varepsilon$ for all $\vartheta \geqslant \vartheta_0$.

Definition 2.4. Let $(\Xi_1,\Lambda_1,\varphi_1)$ and $(\Xi_2,\Lambda_2,\varphi_2)$ be B$\mathfrak{b}$MSs with $\mathfrak{b} \geqslant 1$.

1. A map $\Omega : (\Xi_1,\Lambda_1,\varphi_1) \rightrightarrows (\Xi_2,\Lambda_2,\varphi_2)$ is said to be continuous at a point $\sigma_0 \in \Xi$, if for every $\varepsilon > 0$, we can find a $\delta > 0$ such that whenever $\eta \in \Lambda_1$ and $\varphi_1(\sigma_0,\eta) < \delta$, $\varphi_2(\Omega(\sigma_0), \Omega(\eta)) < \varepsilon$. It is continuous at a point $\eta_0 \in \Lambda_1$ if for every $\varepsilon > 0$, we can find a $\delta > 0$ satisfying whenever $\sigma \in \Xi_1$ and $\varphi_1(\sigma,\eta_0) < \Xi$, $\varphi_2(\Omega(\sigma),\Omega(\eta_0)) < \varepsilon$. If f is continuous at each point $\sigma \in \Xi_1$ and $\eta \in \Lambda_1$, then it is called continuous on a set.
2. A contravariant map $\Omega : (\Xi_1,\Lambda_1,\varphi_1) \leftrightarrows (\Xi_2,\Lambda_2,\varphi_2)$ is continuous if and only if it is continuous as a covariant map $\Omega : (\Xi_1,\Lambda_1,\varphi_1) \rightrightarrows (\Xi_2,\Lambda_2,\bar{\varphi}_2)$.

Definition 2.5. Let $(\Xi_1,\Lambda_1,\varphi_1)$ and $(\Xi_2,\Lambda_2,\varphi_2)$ be B$\mathfrak{b}$MSs with $\mathfrak{b} \geqslant 1$ and $\wp \in (0,1)$. A covariant map $\Omega : (\Xi_1,\Lambda_1,\varphi_1)\rightrightarrows(\Xi_2,\Lambda_2,\varphi_2)$ such that

$$\varphi(\Omega(\sigma),\Omega(\eta)) \leqslant \wp\varphi(\sigma,\eta) \text{ for all } \sigma \in \Xi_1, \eta \in \Lambda_1$$

or a contravariant map $\Omega : (\Xi_1, \Lambda_1, \varphi_1) \leftrightarrows (\Xi_2, \Lambda_2, \varphi_2)$ such that

$$\varphi(\Omega(\sigma), \Omega(\eta)) \leqslant \wp\varphi(\sigma, \eta) \text{ for all } \sigma \in \Xi_1, \eta \in \Lambda_1$$

is said to be contraction.

Definition 2.6. Let (Ξ, Λ, φ) be a B$\mathfrak{b}$MSs with $\mathfrak{b} \geqslant 1$.

1. A sequence $(\{\sigma_n\}, \{\eta_n\})$ on the set $\Xi \times \Lambda$ is called a bisequence on (Ξ, Λ, φ).
2. If both $\{\sigma_n\}$ and $\{\eta_n\}$ converge, then the bisequence (σ_n, η_n) is said to be convergent. If $\{\sigma_n\} \to u$ and $\{\eta_n\} \to u$, where $u \in \Xi \cap \Lambda$, then it is said to be biconvergent.
3. A bisequence $(\{\sigma_n\}, \{\eta_n\})$ on (Ξ, Λ, φ) is said to be a Cauchy bisequence, if for each $\varepsilon > 0$, we can find a number $\vartheta_0 \in \mathbb{N}$, satisfying for all $\vartheta, m \in \mathbb{N}$, $\vartheta, m \geqslant \vartheta_0$, $\varphi(\sigma_\vartheta, \eta_m) < \varepsilon$.

In this paper, we introduce the notion of bipolar $\mathfrak{b}$-metric space and prove fixed point and common fixed point theorems on bipolar $\mathfrak{b}$-metric space.

3. Main results

In this section, we present our first result.

Theorem 3.1. *Let (Ξ, Λ, φ) be a complete B$\mathfrak{b}$MSs with $\mathfrak{b} \geqslant 1$ and given a contraction $\Omega : (\Xi, \Lambda, \varphi) \rightrightarrows (\Xi, \Lambda, \varphi)$. Then the function $\Omega : \Xi \cup \Lambda \to \Xi \cup \Lambda$ has a UFP (unique fixed point).*

Proof. Since Ω is a contraction, there exists a $\wp \in (0, 1)$ such that

$$\varphi(\Omega(\sigma), \Omega(\eta)) \leqslant \wp\varphi(\sigma, \eta) \text{ for all } (\sigma, \eta) \in \Xi \times \Lambda.$$

Let $\sigma_0 \in \Xi$ and $\eta_0 \in \Lambda$. For each $\vartheta \in \mathbb{N}$, define $\Omega(\sigma_\vartheta) = \sigma_{\vartheta+1}$ and $\Omega(\eta_\vartheta) = \eta_{\vartheta+1}$. Then $(\{\sigma_\vartheta\}, \{\eta_\vartheta\})$ is a bisequence on (Ξ, Λ, φ). Say $\mathcal{M} := \varphi(\sigma_0, \eta_0) + \varphi(\sigma_0, \eta_1)$. Then, for all $\vartheta, \mathfrak{p} \in \mathbb{N}$, we have

$$\varphi(\sigma_\vartheta, \eta_\vartheta) = \varphi(\Omega(\sigma_{\vartheta-1}), \Omega(\eta_{\vartheta-1})) \leqslant \wp\varphi(\sigma_{\vartheta-1}, \eta_{\vartheta-1}) \leqslant \cdots \leqslant \wp^\vartheta \varphi(\sigma_0, \eta_0),$$

and also

$$\begin{aligned}
\varphi(\sigma_\vartheta, \eta_{\vartheta+1}) &= \varphi(\Omega(\sigma_{\vartheta-1}), \Omega(\eta_\vartheta)) \leqslant \wp\varphi(\sigma_{\vartheta-1}, \eta_\vartheta) \leqslant \cdots \leqslant \wp^\vartheta \varphi(\sigma_0, \eta_1), \\
\varphi(\sigma_{\vartheta+\mathfrak{p}}, \eta_\vartheta) &\leqslant \mathfrak{b}(\varphi(\sigma_{\vartheta+\mathfrak{p}}, \eta_{\vartheta+1}) + \varphi(\sigma_\vartheta, \eta_{\vartheta+1}) + \varphi(\sigma_\vartheta, \eta_\vartheta)) \\
&\leqslant \mathfrak{b}\varphi(\sigma_{\vartheta+\mathfrak{p}}, \eta_{\vartheta+1}) + \mathfrak{b}\wp^\vartheta\mathcal{M} \\
&\leqslant \mathfrak{b}^2(\varphi(\sigma_{\vartheta+\mathfrak{p}}, \eta_{\vartheta+2}) + \varphi(\sigma_{\vartheta+1}, \eta_{\vartheta+2}) + \varphi(\sigma_{\vartheta+1}, \eta_{\vartheta+1})) + \mathfrak{b}\wp^\vartheta\mathcal{M} \\
&\leqslant \mathfrak{b}^2\varphi(\sigma_{\vartheta+\mathfrak{p}}, \eta_{\vartheta+2}) + (\mathfrak{b}^2\wp^{\vartheta+1} + \mathfrak{b}\wp^\vartheta)\mathcal{M} \\
&\;\;\vdots \\
&\leqslant \mathfrak{b}^\mathfrak{p}\varphi(\sigma_{\vartheta+\mathfrak{p}}, \eta_{\vartheta+\mathfrak{p}}) + (\mathfrak{b}^\mathfrak{p}\wp^{\vartheta+\mathfrak{p}-1} + \cdots + \mathfrak{b}^2\wp^{\vartheta+1} + \mathfrak{b}\wp^\vartheta)\mathcal{M} \\
&\leqslant (\mathfrak{b}^\mathfrak{p}\wp^{\vartheta+\mathfrak{p}} + \cdots + \mathfrak{b}^2\wp^{\vartheta+1} + \mathfrak{b}\wp^\vartheta)\mathcal{M} \leqslant \frac{\mathfrak{b}\wp^\vartheta\mathcal{M}}{1 - \mathfrak{b}\wp} = \mathcal{K}_\vartheta,
\end{aligned}$$

and similarly $\varphi(\sigma_\vartheta, \eta_{\vartheta+\mathfrak{p}}) \leqslant \mathcal{K}_\vartheta$. Let $\varepsilon > 0$. Since $\wp \in (0, 1)$, there exists an $\vartheta_0 \in \mathbb{N}$ such that $\mathcal{K}_{\vartheta_0} = \frac{\mathfrak{b}\wp_0^\vartheta}{1-\mathfrak{b}\wp} < \frac{\varepsilon}{3}$. Then

$$\varphi(\sigma_\vartheta, \eta_m) \leqslant \varphi(\sigma_\vartheta, \eta_{\vartheta_0}) + \varphi(\sigma_{\vartheta_0}, \eta_{\vartheta_0}) + \varphi(\sigma_{\vartheta_0}, \eta_m) \leqslant 3K_{\vartheta_0} < \varepsilon$$

and $(\{\sigma_\vartheta\},\{\eta_\vartheta\})$ is a Cauchy bisequence. Since (Ξ,Λ,φ) is complete, $\{\sigma_\vartheta\}\to\varpi$, $\{\eta_\vartheta\}\to\varpi$, where $\varpi\in\Xi\cap\Lambda$ and

$$\{\Omega(\eta_\vartheta)\}=\{\eta_{\vartheta+1}\}\to\varpi\in\Xi\cap\Lambda.$$

Since Ω is continuous $\Omega(\eta_\vartheta)\to\Omega(\varpi)$, so $\Omega(\varpi)=\varpi$. Let ρ be a another fixed point of Ω, then

$$\varphi(\varpi,\rho)=\varphi(\Omega(\varpi),\Omega(\rho))\leqslant\varphi(\varpi,\rho),$$

where $0<\wp<1$, which implies $\varphi(\varpi,\rho)=0$, and so $\varpi=\rho$. □

Example 3.1. Let $\Xi=\{\mathcal{U}_\vartheta(\mathbb{R}):\mathcal{U}_\vartheta(\mathbb{R})$ be an upper triangular matrice over $\mathbb{R}\}$, $\Lambda=\{\mathcal{L}_\vartheta(\mathbb{R}):\mathcal{L}_\vartheta(\mathbb{R})$ is an upper triangular matrices over $\mathbb{R}\}$, and the map $\mathfrak{d}:\Xi\times\Lambda\to[1,\infty)$ is defined by

$$\mathfrak{d}(\mathcal{P},\mathcal{Q})=\sum_{i,j=1}^{\vartheta}|\sigma_{ij}-\mathfrak{q}_{ij}|^2$$

for all $\mathcal{P}=(\sigma_{ij})_{\vartheta\times\vartheta}\in\Xi$ and $\mathcal{Q}=(\mathfrak{q}_{ij})_{\vartheta\times\vartheta}\in\Lambda$. Then $(\Xi,\Lambda,\mathfrak{d})$ is a complete B𝔟MSs. Define $\Omega:(\Xi,\Lambda,\mathfrak{d})\rightrightarrows(\Xi,\Lambda,\mathfrak{d})$ by

$$\Omega(\mathcal{P})=\left(\frac{\sigma_{ij}}{3}\right)_{\vartheta\times\vartheta}$$

for all $\mathcal{P}=(\sigma_{ij})_{\vartheta\times\vartheta}\in\mathcal{U}_\vartheta(\mathbb{R})\cup\mathcal{L}_\vartheta(\mathbb{R})$. Now,

$$\mathfrak{d}(\Omega(\mathcal{P}),\Omega(\mathcal{Q}))=\frac{1}{9}\sum_{i,j=1}^{\vartheta}|\sigma_{ij}-\mathfrak{q}_{ij}|^2\leqslant\frac{1}{2}\sum_{i,j=1}^{\vartheta}|\sigma_{ij}-\mathfrak{q}_{ij}|^2=\wp\mathfrak{d}(\mathcal{P},\mathcal{Q})$$

for all $\mathcal{P}=(\sigma_{ij})_{\vartheta\times\vartheta}\in\Xi$ and $\mathcal{Q}=(\mathfrak{q}_{ij})_{\vartheta\times\vartheta}\in\Lambda$. All the axioms of Theorem 3.1 are verified with $\wp=\frac{1}{2}$ and Ω has a UFP $(0_{\vartheta\times\vartheta},0_{\vartheta\times\vartheta})\in\mathcal{U}_\vartheta(\mathbb{R})\cup\mathcal{L}_\vartheta(\mathbb{R})$, where $0_{\vartheta\times\vartheta}$ is the null matrix.

Example 3.2. Let $\Xi=[0,1]$ and $\Lambda=\{0\}\cup\mathbb{N}-\{1\}$ be equipped with $\varphi(\sigma,\eta)=|\sigma-\eta|^2$ for all $\sigma\in\Xi$, $\eta\in\Lambda$. Then, (Ξ,Λ,φ) is a complete B𝔟MSs. Define $\Omega:\Xi\cup\Lambda\rightrightarrows\Xi\cup\Lambda$ by

$$\Omega(\sigma)=\begin{cases}\frac{\sigma}{5}, & \text{if } \sigma\in(0,1],\\ 0, & \text{if } \sigma\in\{0\}\cup\mathbb{N}-\{1\},\end{cases}$$

$\forall\ \sigma\in\Xi\cup\Lambda$. Let $\sigma\in\Xi$ and $\eta\in\Lambda$, then

$$\varphi(\Omega\sigma,\Omega\eta)=|\frac{\sigma}{5}-0|^2\leqslant\frac{1}{2}|\sigma-\eta|^2.$$

Therefore, axioms of Theorem 3.1 are fulfilled. Hence Ω has a UFP $\sigma=0$.

We present our second result.

Theorem 3.2. *Let (Ξ,Λ,φ) be a complete B𝔟MSs with $\mathfrak{b}\geqslant 1$ and given a contravariant contraction $\Omega:(\Xi,\Lambda,\varphi)\leftrightarrows(\Xi,\Lambda,\varphi)$. Then the function $\Omega:\Xi\cup\Lambda\to\Xi\cup\Lambda$ has a UFP.*

Proof. Since Ω is a contravariant contraction, we can find a $\wp\in(0,1)$ satisfying $\varphi(\Omega(\eta),\Omega(\sigma))\leqslant\wp\varphi(\sigma,\eta)$ and $(\sigma,\eta)\in\Xi\times\Lambda$. Then for all $\vartheta,\mathfrak{p}\in\mathbb{Z}^+$,

$$\begin{aligned}\varphi(\sigma_\vartheta,\eta_\vartheta)&=\varphi(\Omega(\eta_{\vartheta-1}),\Omega(\sigma_\vartheta))\\&\leqslant\varphi(\sigma_\vartheta,\eta_{\vartheta-1})\end{aligned}$$

$$
\begin{aligned}
&= \wp\varphi(\Omega(\eta_{\vartheta-1}),\Omega(\sigma_{\vartheta-1}))\\
&\leqslant \wp^2\varphi(\sigma_{\vartheta-1},\eta_{\vartheta-1})\\
&\ \vdots\\
&\leqslant \wp^{2\vartheta}\varphi(\sigma_0,\eta_0),\\
\varphi(\sigma_{\vartheta+1},\eta_\vartheta) &= \varphi(\Omega(\eta_\vartheta),\Omega(\sigma_\vartheta)) \leqslant \wp\varphi(\sigma_\vartheta,\eta_\vartheta) \leqslant \wp^{2\vartheta+1}\varphi(\sigma_0,\eta_0),\\
\varphi(\sigma_{\vartheta+\mathfrak{p}},\eta_\vartheta) &\leqslant \mathfrak{b}(\varphi(\sigma_{\vartheta+\mathfrak{p}},\eta_{\vartheta+1}) + \varphi(\sigma_{\vartheta+1},\eta_{\vartheta+1}) + \varphi(\sigma_{\vartheta+1},\eta_\vartheta))\\
&\leqslant \mathfrak{b}\varphi(\sigma_{\vartheta+\mathfrak{p}},\eta_{\vartheta+1}) + (\mathfrak{b}\wp^{2\vartheta+2}+\mathfrak{b}\wp^{2\vartheta+1})\varphi(\sigma_0,\eta_0)\\
&\leqslant \mathfrak{b}^2(\varphi(\sigma_{\vartheta+\mathfrak{p}},\eta_{\vartheta+2}) + \varphi(\sigma_{\vartheta+2},\eta_{\vartheta+2}) + \varphi(\sigma_{\vartheta+2},\eta_{\vartheta+1}) + (\mathfrak{b}\wp^{2\vartheta+2}+\mathfrak{b}\wp^{2\vartheta+1})\varphi(\sigma_0,\eta_0)\\
&\leqslant \varphi(\sigma_{\vartheta+\mathfrak{p}},\eta_{\vartheta+2}) + (\mathfrak{b}^2\wp^{2\vartheta+4}+\mathfrak{b}^2\wp^{2\vartheta+3}+\mathfrak{b}\wp^{2\vartheta+2}+\mathfrak{b}\wp^{2\vartheta+1})\varphi(\sigma_0,\eta_0)\\
&\ \vdots\\
&\leqslant \mathfrak{b}^{\mathfrak{p}}\varphi(\sigma_{\vartheta+\mathfrak{p}},\eta_{\vartheta+\mathfrak{p}-1}) + (\mathfrak{b}^{\mathfrak{p}-1}\wp^{2\vartheta+2\mathfrak{p}-2}+\cdots.+\mathfrak{b}\wp^{2\vartheta+1})\varphi(\sigma_0,\eta_0)\\
&\leqslant (\mathfrak{b}^{\mathfrak{p}}\wp^{2\vartheta+2\vartheta-1}+\mathfrak{b}^{\mathfrak{p}-1}\wp^{2\vartheta+2\mathfrak{p}-2}+\mathfrak{b}^{\mathfrak{p}-1}\wp^{2\vartheta+2\mathfrak{p}-3}+\cdots.+\mathfrak{b}\wp^{2\vartheta+1})\varphi(\sigma_0,\eta_0)\\
&\leqslant (\mathfrak{b}^{\mathfrak{p}}\wp^{2\vartheta+2\vartheta-1}+\cdots+\mathfrak{b}\wp^{2\vartheta+1})\varphi(\sigma_0,\eta_0) + (\mathfrak{b}^{\mathfrak{p}-1}\wp^{2\vartheta+2\mathfrak{p}-2}+\cdots+\mathfrak{b}\wp^{2\vartheta+2})\varphi(\sigma_0,\eta_0)\\
&\leqslant \frac{\mathfrak{b}\wp^{2\vartheta+1}}{1-\mathfrak{b}\wp^2}\varphi(\sigma_0,\eta_0) + \frac{\mathfrak{b}\wp^{2\vartheta+2}}{1-\mathfrak{b}\wp^2}\varphi(\sigma_0,\eta_0).
\end{aligned}
$$

As $\vartheta \to \infty$, we get

$$
\begin{aligned}
\varphi(\sigma_{\vartheta+\mathfrak{p}},\eta_\vartheta) &\to 0,\\
\varphi(\sigma_\vartheta,\eta_{\vartheta+\mathfrak{p}}) &\leqslant \mathfrak{b}(\varphi(\sigma_\vartheta,\eta_\vartheta) + \varphi(\sigma_{\vartheta+1},\eta_\vartheta) + \varphi(\sigma_{\vartheta+1},\eta_{\vartheta+\mathfrak{p}}))\\
&\leqslant (\mathfrak{b}\wp^{2\vartheta}+\mathfrak{b}\wp^{2\vartheta+1})\varphi(\sigma_0,\eta_0) + \mathfrak{b}\varphi(\sigma_{\vartheta+1},\eta_{\vartheta+\mathfrak{p}})\\
&\leqslant (\mathfrak{b}\wp^{2\vartheta}+\mathfrak{b}\wp^{2\vartheta+1})\varphi(\sigma_0,\eta_0) + \mathfrak{b}^2\varphi(\sigma_{\vartheta+1},\eta_{\vartheta+1}) + \mathfrak{b}^2\varphi(\sigma_{\vartheta+2},\eta_{\vartheta+1}) + \mathfrak{b}^2\varphi(\sigma_{\vartheta+2},\eta_{\vartheta+\mathfrak{p}})\\
&\leqslant (\mathfrak{b}\wp^{2\vartheta}+\mathfrak{b}\wp^{2\vartheta+1}+\mathfrak{b}^2\wp^{2\vartheta+2}+\mathfrak{b}^2\wp^{2\vartheta+3})\varphi(\sigma_0,\eta_0) + \mathfrak{b}^2\varphi(\sigma_{\vartheta+2},\eta_{\vartheta+\mathfrak{p}})\\
&\ \vdots\\
&\leqslant (\mathfrak{b}\wp^{2\vartheta}+\mathfrak{b}\wp^{2\vartheta+1}+\cdots+\mathfrak{b}^{\mathfrak{p}}\wp^{2\vartheta+2\mathfrak{p}-1})\varphi(\sigma_0\eta_0) + \mathfrak{b}^{\mathfrak{p}}\varphi(\sigma_{\vartheta+\mathfrak{p}},\eta_{\vartheta+\mathfrak{p}})\\
&\leqslant (\mathfrak{b}\wp^{2\vartheta}+\mathfrak{b}\wp^{2\vartheta+1}+\cdots+\mathfrak{b}^{\mathfrak{p}}\wp^{2\vartheta+2\mathfrak{p}-1}+\mathfrak{b}^{\mathfrak{p}}\wp^{2\vartheta+2\mathfrak{p}})\varphi(\sigma_0,\eta_0)\\
&= (\mathfrak{b}\wp^{2\vartheta}+\cdots+\mathfrak{b}^{\mathfrak{p}}\wp^{2\vartheta+2\mathfrak{p}})\varphi(\sigma_0,\eta_0) + (\mathfrak{b}\wp^{2\vartheta+1}+\cdots+\mathfrak{b}^{\mathfrak{p}}\wp^{2\vartheta+2\mathfrak{p}-1})\varphi(\sigma_0,\eta_0)\\
&\leqslant \frac{\mathfrak{b}\wp^{2\vartheta}}{1-\mathfrak{b}\wp^2}\varphi(\sigma_0,\eta_0) + \frac{\mathfrak{b}\wp^{2\vartheta+1}}{1-\mathfrak{b}\wp^2}\varphi(\sigma_0,\eta_0).
\end{aligned}
$$

As $\vartheta \to \infty$, we get $\varphi(\sigma_\vartheta,\eta_{\vartheta+\mathfrak{p}}) \to 0$. Therefore, $(\{\sigma_\vartheta\},\{\eta_\vartheta\})$ is a Cauchy bisequence. Since (Ξ,Λ,φ) is complete, $(\{\sigma_\vartheta\},\{\eta_\vartheta\})$ converges, then $\{\sigma_\vartheta\} \to \varpi, \{\eta_\vartheta\} \to \varpi$, where $\varpi \in \Xi \cap \Lambda$. Since Ω is a continuous, $\{\sigma_n\} \to \varpi$, which means that

$$\{\eta_\vartheta\} = \{\Omega(\sigma_\vartheta)\} \to \Omega(\varpi)$$

and combining this with $\{\eta_\vartheta\} \to \varpi$ gives $\Omega(\varpi) = \varpi$. Let ρ be a another fixed point of Ω, then

$$\varphi(\varpi,\rho) = \varphi(\Omega(\varpi),\Omega(\rho)) \leqslant \wp\varphi(\varpi,\rho),$$

which gives $\varphi(\varpi,\rho,\mathfrak{w}) = 0$. Hence $\varpi = \rho$. □

Example 3.3. Let $\Xi = \{0,1,2,7\}$ and $\Lambda = \{0,\frac{1}{4},\frac{1}{2},3\}$ be equipped with $\varphi(\sigma,\eta) = |\sigma-\eta|^2$ for all $\sigma \in \Xi, \eta \in \Lambda$.

Then, (Ξ, Λ, φ) is a complete B$\mathfrak{b}$MSs. Define $\Omega : \Xi \cup \Lambda \rightleftarrows \Xi \cup \Lambda$ by

$$\Omega(\sigma) = \begin{cases} \frac{1}{4}, & \text{if } \sigma \in \{2, 7\}, \\ 0, & \text{if } \sigma \in \{0, \frac{1}{4}, \frac{1}{2}, 1, 3\}, \end{cases}$$

$\forall\ \sigma \in \Xi \cup \Lambda$. Let $\sigma \in \Xi$ and $\eta \in \Lambda$, then we can easily get

$$\varphi(\Omega\sigma, \Omega\eta) \leqslant \frac{1}{2}\varphi(\sigma, \eta).$$

Therefore, axioms of Theorem 3.2 are fulfilled. Hence Ω has a UFP $\sigma = 0$.

Next, we present a result based on Kannan's type contraction [12].

Theorem 3.3. *Let $\Omega : (\Xi, \Lambda, \varphi) \leftrightarrows (\Xi, \Lambda, \varphi)$, where (Ξ, Λ, φ) is a complete B$\mathfrak{b}$MSs with $\mathfrak{b} \geqslant 1$ and let $\alpha \in (0, \frac{1}{2})$ such that the inequality*

$$\varphi(\Omega\eta, \Omega\sigma) \leqslant \alpha(\varphi(\sigma, \Omega\sigma) + \varphi(\Omega\eta, \eta)),$$

holds for all $\sigma \in \Xi$ and $\eta \in \Lambda$. Then the function $\Omega : \Xi \cup \Lambda \to \Xi \cup \Lambda$ has a unique fixed point.

Proof. Let $\sigma_0 \in \Xi$, for each $\vartheta \geqslant 0$, we define $\eta_\vartheta = \Omega\sigma_\vartheta$ and $\sigma_{\vartheta+1} = \Omega\eta_\vartheta$. Then

$$\varphi(\sigma_\vartheta, \eta_\vartheta) = \varphi(\Omega\eta_{\vartheta-1}, \Omega\sigma_\vartheta) \leqslant \alpha(\varphi(\sigma_\vartheta, \Omega\sigma_\vartheta) + \varphi(\Omega\eta_{\vartheta-1}, \eta_{\vartheta-1})) = \alpha(\varphi(\sigma_\vartheta, \eta_\vartheta) + \varphi(\sigma_\vartheta, \eta_{\vartheta-1}))$$

for all integers $\vartheta \geqslant 1$. Then,

$$\varphi(\sigma_\vartheta, \eta_\vartheta) \leqslant \frac{\alpha}{1-\alpha}\varphi(\sigma_\vartheta, \eta_{\vartheta-1}),$$

and

$$\begin{aligned} \varphi(\sigma_\vartheta, \eta_{\vartheta-1}) = \varphi(\Omega\eta_{\vartheta-1}, \Omega\sigma_{\vartheta-1}) &\leqslant \alpha(\varphi(\sigma_{\vartheta-1}, \Omega\sigma_{\vartheta-1}) + \varphi(\Omega\eta_{\vartheta-1}, \eta_{\vartheta-1})) \\ &= \alpha(\varphi(\sigma_{\vartheta-1}, \eta_{\vartheta-1}) + \varphi(\sigma_\vartheta, \eta_{\vartheta-1})), \end{aligned}$$

so that

$$\varphi(\sigma_\vartheta, \eta_{\vartheta-1}) \leqslant \frac{\alpha}{1-\alpha}\varphi(\sigma_{\vartheta-1}, \eta_{\vartheta-1}).$$

If we say $\wp := \frac{\alpha}{1-\alpha}$, then we have $\wp \in (0, 1)$ since $\alpha \in (0, \frac{1}{2})$. Now

$$\varphi(\sigma_\vartheta, \eta_\vartheta) \leqslant \wp^{2\vartheta}\varphi(\sigma_0, \eta_0), \qquad \varphi(\sigma_\vartheta, \eta_{\vartheta-1}) \leqslant \wp^{2\vartheta-1}\varphi(\sigma_0, \eta_0).$$

For each $\mathfrak{m} > \vartheta$,

$$\begin{aligned} \varphi(\sigma_\vartheta, \eta_\mathfrak{m}) &\leqslant \mathfrak{b}(\varphi(\sigma_\vartheta, \eta_\vartheta) + \varphi(\sigma_{\vartheta+1}, \eta_\vartheta) + \varphi(\sigma_{\vartheta+1}, \eta_\mathfrak{m})) \\ &\leqslant (\mathfrak{b}\wp^{2\vartheta} + \mathfrak{b}\wp^{2\vartheta+1})\varphi(\sigma_0, \eta_0) + \mathfrak{b}\varphi(\sigma_{\vartheta+1}, \eta_\mathfrak{m}) \\ &\ \vdots \\ &\leqslant (\mathfrak{b}\wp^{2\vartheta} + \mathfrak{b}\wp^{2\vartheta+1} + \cdots + \mathfrak{b}^\mathfrak{m}\wp^{2\mathfrak{m}-1} + \mathfrak{b}^\mathfrak{m}\wp^{2\mathfrak{m}})\varphi(\sigma_0, \eta_0) \\ &= (\mathfrak{b}\wp^{2\vartheta} + \cdots + \mathfrak{b}^\mathfrak{m}\wp^{2\mathfrak{m}})\varphi(\sigma_0, \eta_0) + (\mathfrak{b}\wp^{2\vartheta+1} + \cdots + \mathfrak{b}^\mathfrak{m}\wp^{2\mathfrak{m}-1})\varphi(\sigma_0, \eta_0) \\ &\leqslant \frac{\mathfrak{b}\wp^{2\vartheta}}{1-\mathfrak{b}\wp^2}\varphi(\sigma_0, \eta_0) + \frac{\mathfrak{b}\wp^{2\vartheta+1}}{1-\mathfrak{b}\wp^2}\varphi(\sigma_0, \eta_0). \end{aligned}$$

As $\vartheta, \mathfrak{m} \to \infty$, we get

$$\varphi(\sigma_\vartheta, \eta_\mathfrak{m}) \to 0.$$

Consequently, for each $\mathfrak{m} < \vartheta$,

$$\begin{aligned}
\varphi(\sigma_\vartheta, \eta_\mathfrak{m}) &\leqslant \mathfrak{b}(\varphi(\sigma_{\mathfrak{m}+1}, \eta_\mathfrak{m}) + \varphi(\sigma_{\mathfrak{m}+1}, \eta_{\mathfrak{m}+1}) + \varphi(\sigma_\vartheta, \eta_{\mathfrak{m}+1}))\\
&\leqslant (\mathfrak{b}\wp^{2\mathfrak{m}+1} + \mathfrak{b}\wp^{2\mathfrak{m}+2})\varphi(\sigma_0, \eta_0) + \mathfrak{b}\varphi(\sigma_\vartheta, \eta_{\mathfrak{m}+1})\\
&\vdots\\
&\leqslant (\mathfrak{b}\wp^{2\mathfrak{m}+1} + \mathfrak{b}\wp^{2\mathfrak{m}+2} + \cdots + \mathfrak{b}^\vartheta\wp^{2\vartheta})\varphi(\sigma_0, \eta_0) + \mathfrak{b}^\vartheta\varphi(\sigma_\vartheta, \eta_\vartheta)\\
&\leqslant (\mathfrak{b}\wp^{2\mathfrak{m}+1} + \mathfrak{b}\wp^{2\mathfrak{m}+2} + \cdots + \mathfrak{b}^\vartheta\wp^{2\vartheta+1})\varphi(\sigma_0, \eta_0)\\
&= (\mathfrak{b}\wp^{2\mathfrak{m}+1} + \cdots + \mathfrak{b}^\vartheta\wp^{2\vartheta+1})\varphi(\sigma_0, \eta_0) + (\mathfrak{b}\wp^{2\mathfrak{m}+2} + \cdots + \mathfrak{b}^\vartheta\wp^{2\vartheta})\varphi(\sigma_0, \eta_0)\\
&\leqslant \frac{\mathfrak{b}\wp^{2\mathfrak{m}+1}}{1 - \mathfrak{b}\wp^2}\varphi(\sigma_0, \eta_0) + \frac{\mathfrak{b}\wp^{2\mathfrak{m}+2}}{1 - \mathfrak{b}\wp^2}\varphi(\sigma_0, \eta_0).
\end{aligned}$$

As $\vartheta, \mathfrak{m} \to \infty$, we get $\varphi(\sigma_\vartheta, \eta_\mathfrak{m}) \to 0$. Therefore, $(\{\sigma_\vartheta\}, \{\eta_\mathfrak{m}\})$ is a Cauchy bisequence. Since (Ξ, Λ, φ) is complete, $\{\sigma_\vartheta\} \to \varpi, \{\eta_\mathfrak{m}\} \to \varpi$, where $\varpi \in \Xi \cup \Lambda$. Since $\{\Omega\sigma_\vartheta\} = \{\eta_\vartheta\} \to \varpi$. On the other hand,

$$\varphi(\Omega\varpi, \Omega\sigma_\vartheta) \leqslant \alpha(\varphi(\sigma_\vartheta, \Omega\sigma_\vartheta) + \varphi(\Omega\varpi, \varpi)) = \alpha(\varphi(\sigma_\vartheta, \eta_\vartheta) + \varphi(\Omega\varpi, \varpi)),$$

which in turn implies that $\varphi(\Omega\varpi, \varpi) \leqslant \alpha\varphi(\Omega\varpi, \varpi)$. Hence $\Omega\varpi = \varpi$. Let ρ be a another fixed point of Ω, then

$$\varphi(\varpi, \rho) = \varphi(\Omega\varpi, \Omega\rho) \leqslant \alpha(\varphi(\varpi, \Omega\varpi) + \varphi(\Omega\rho, \rho)) = \alpha(\varphi(\varpi, \varpi) + \varphi(\nu, \rho)) = 0.$$

Consequently $\varpi = \rho$. □

Now, we prove our first CFP (common fixed point) result.

Theorem 3.4. *Let (Ξ, Λ, φ) be a complete B$\mathfrak{b}$MSs with $\mathfrak{b} \geqslant 1$ and let $\Omega, \mathcal{S} : (\Xi, \Lambda, \varphi) \rightleftarrows (\Xi, \Lambda, \varphi)$ be a contravariant mapping satisfying*

$$\varphi(\mathcal{S}\eta, \Omega\sigma) \leqslant \alpha\frac{\varphi(\sigma, \Omega\sigma)\varphi(\mathcal{S}\eta, \eta)}{\varphi(\sigma, \eta)} + \gamma\varphi(\sigma, \eta) + \imath[\varphi(\sigma, \Omega\sigma) + \varphi(\mathcal{S}\eta, \eta)], \tag{3.1}$$

for all $(\sigma, \eta) \in \Xi \times \Lambda$, with $\sigma \neq \eta$ and $0 \leqslant \alpha + \gamma + 2\imath < 1$. Then $\Omega, \mathcal{S} : \Xi \cup \Lambda \to \Xi \cup \Lambda$ have a unique CFP.

Proof. Let $\sigma_0 \in \Xi$ and $\eta_0 \in \Lambda$, then for each $\vartheta \in \mathbb{N} \cup \{0\}$, we define

$$\mathcal{S}\sigma_{2\vartheta} = \eta_{2\vartheta}, \Omega\sigma_{2\vartheta+1} = \eta_{2\vartheta+1}, \mathcal{S}\eta_{2\vartheta} = \sigma_{2\vartheta+1}, \Omega\eta_{2\vartheta+1} = \sigma_{2\vartheta+2}.$$

Now by (3.1), we get

$$\begin{aligned}
\varphi(\sigma_{2\vartheta+1}, \eta_{2\vartheta+1}) &= \varphi(\mathcal{S}\eta_{2\vartheta}, \Omega\sigma_{2\vartheta+1})\\
&\leqslant \alpha\frac{\varphi(\sigma_{2\vartheta+1}, \Omega\sigma_{2\vartheta+1})\varphi(\mathcal{S}\eta_{2\vartheta+1}, \eta_{2\vartheta})}{\varphi(\sigma_{2\vartheta+1}, \eta_{2\vartheta})}\\
&\quad + \gamma\varphi(\sigma_{2\vartheta+1}, \eta_{2\vartheta}) + \imath[\varphi(\sigma_{2\vartheta+1}, \Omega\sigma_{2\vartheta+1}) + \varphi(\mathcal{S}\eta_{2\vartheta}, \eta_{2\vartheta})]\\
&= \alpha\frac{\varphi(\sigma_{2\vartheta+1}, \eta_{2\vartheta+1})\varphi(\sigma_{2\vartheta+1}, \eta_{2\vartheta})}{\varphi(\sigma_{2\vartheta+1}, \eta_{2\vartheta})} + \gamma\varphi(\sigma_{2\vartheta+1}, \eta_{2\vartheta}) + \imath[\varphi(\sigma_{2\vartheta+1}, \eta_{2\vartheta+1}) + \varphi(\sigma_{2\vartheta+1}, \eta_{2\vartheta})]\\
&= \alpha\varphi(\sigma_{2\vartheta+1}, \eta_{2\vartheta+1}) + \gamma\varphi(\sigma_{2\vartheta+1}, \eta_{2\vartheta}) + \imath\varphi(\sigma_{2\vartheta+1}, \eta_{2\vartheta+1}) + \imath\varphi(\sigma_{2\vartheta+1}, \eta_{2\vartheta}),
\end{aligned}$$

which implies that

$$\varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta+1}) \leqslant \frac{\gamma+\iota}{1-\alpha-\iota}\varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta}). \tag{3.2}$$

Also, we have

$$\begin{aligned}
\varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta}) &= \varphi(\mathcal{S}\eta_{2\vartheta},\Omega\sigma_{2\vartheta})\\
&\leqslant \alpha\frac{\varphi(\sigma_{2\vartheta},\Omega\sigma_{2\vartheta})\varphi(\mathcal{S}\eta_{2\vartheta},\eta_{2\vartheta})}{\varphi(\sigma_{2\vartheta},\eta_{2\vartheta})}+\gamma\varphi(\sigma_{2\vartheta},\eta_{2\vartheta})+\iota[\varphi(\sigma_{2\vartheta},\Omega\sigma_{2\vartheta})+\varphi(\mathcal{S}\eta_{2\vartheta},\eta_{2\vartheta})]\\
&= \alpha\frac{\varphi(\sigma_{2\vartheta},\eta_{2\vartheta})\varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta})}{\varphi(\sigma_{2\vartheta},\eta_{2\vartheta})}+\gamma\varphi(\sigma_{2\vartheta},\eta_{2\vartheta})+\iota[\varphi(\sigma_{2\vartheta},\eta_{2\vartheta})+\varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta})]\\
&= \alpha\varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta})+\gamma\varphi(\sigma_{2\vartheta},\eta_{2\vartheta})+\iota\varphi(\sigma_{2\vartheta},\eta_{2\vartheta})+\iota\varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta}),
\end{aligned}$$

which implies that

$$\varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta}) \leqslant \frac{\gamma+\iota}{1-\alpha-\iota}\varphi(\sigma_{2\vartheta},\eta_{2\vartheta}). \tag{3.3}$$

Since $\alpha+\gamma+2\iota \in [0,1)$ and $\frac{\gamma+\iota}{1-\alpha-\iota} = \wp$ (say), $\wp \in [0,1)$. Hence, from (3.2) and (3.3), we get

$$\varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta+1}) \leqslant \wp^{4\vartheta+2}\varphi(\sigma_0,\eta_0) \quad \text{and} \quad \varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta}) \leqslant \wp^{4\vartheta+1}\varphi(\sigma_0,\eta_0).$$

Now, we can get that for any $\vartheta \in \mathbb{N}$,

$$\varphi(\sigma_{\vartheta+1},\eta_{\vartheta+1}) \leqslant \wp^{2\vartheta+2}\varphi(\sigma_0,\eta_0), \quad \varphi(\sigma_{\vartheta+1},\eta_{\vartheta}) \leqslant \wp^{2\vartheta+1}\varphi(\sigma_0,\eta_0), \quad \text{and} \quad \varphi(\sigma_{\vartheta},\eta_{\vartheta}) \leqslant \wp^{2\vartheta}\varphi(\sigma_0,\eta_0).$$

For all $\mathfrak{m},\vartheta \in \mathbb{N}$, we have two cases.

Case 1. If $\mathfrak{m} > \vartheta$,

$$\begin{aligned}
\varphi(\sigma_{\vartheta},\eta_{\mathfrak{m}}) &\leqslant \mathfrak{b}(\varphi(\sigma_{\vartheta},\eta_{\vartheta})+\varphi(\sigma_{\vartheta+1},\eta_{\vartheta})+\varphi(\sigma_{\vartheta+1},\eta_{\mathfrak{m}}))\\
&\leqslant \mathfrak{b}\wp^{2\vartheta}\varphi(\sigma_0,\eta_0)+\mathfrak{b}\wp^{2\vartheta+1}\varphi(\sigma_0,\eta_0)+\mathfrak{b}\varphi(\sigma_{\vartheta+1},\eta_{\mathfrak{m}})\\
&\leqslant (\mathfrak{b}\wp^{2\vartheta}+\mathfrak{b}\wp^{2\vartheta+1})\varphi(\sigma_0,\eta_0)+\mathfrak{b}^2(\varphi(\sigma_{\vartheta+1},\eta_{\vartheta+1})+\varphi(\sigma_{\vartheta+2},\eta_{\vartheta+1})+\varphi(\sigma_{\vartheta+2},\eta_{\mathfrak{m}}))\\
&\leqslant (\mathfrak{b}\wp^{2\vartheta}+\mathfrak{b}\wp^{2\vartheta+1})\varphi(\sigma_0,\eta_0)+\mathfrak{b}^2\wp^{2\vartheta+2}\varphi(\sigma_0,\eta_0)+\mathfrak{b}^2\wp^{2\vartheta+3}\varphi(\sigma_0,\eta_0)+\mathfrak{b}^2\varphi(\sigma_{\vartheta+2},\eta_{\mathfrak{m}})\\
&\ \ \vdots\\
&\leqslant \mathfrak{b}\wp^{2\vartheta}(1+\mathfrak{b}\wp^2+\mathfrak{b}^2\wp^4+\mathfrak{b}^3\wp^6+\cdots)\varphi(\sigma_0,\eta_0)+\mathfrak{b}\wp^{2\vartheta+1}(1+\mathfrak{b}\wp^2+\mathfrak{b}^2\wp^4+\mathfrak{b}^3\wp^6+\cdots)\varphi(\sigma_0,\eta_0)\\
&= \mathfrak{b}\wp^{2\vartheta}\left(\frac{1}{1-\mathfrak{b}\wp^2}\right)\varphi(\sigma_0,\eta_0)+\mathfrak{b}\wp^{2\vartheta+1}\left(\frac{1}{1-\mathfrak{b}\wp^2}\right)\varphi(\sigma_0,\eta_0).
\end{aligned}$$

Since $\wp < 1$, $\lim_{\vartheta,\mathfrak{m}\to\infty}\varphi(\sigma_{\vartheta},\eta_{\mathfrak{m}}) = 0$.

Case 2. If $\mathfrak{m} < \vartheta$, we have

$$\begin{aligned}
\varphi(\sigma_{\vartheta},\eta_{\mathfrak{m}}) &\leqslant \mathfrak{b}(\varphi(\sigma_{\mathfrak{m}+1},\eta_{\mathfrak{m}})+\varphi(\sigma_{\mathfrak{m}+1},\eta_{\mathfrak{m}+1})+\varphi(\sigma_{\vartheta},\eta_{\mathfrak{m}+1}))\\
&\leqslant \mathfrak{b}\wp^{2\mathfrak{m}+1}\varphi(\sigma_0,\eta_0)+\mathfrak{b}\wp^{2\mathfrak{m}+2}\varphi(\sigma_0,\eta_0)+\mathfrak{b}\varphi(\sigma_{\vartheta},\eta_{\mathfrak{m}+1})\\
&\leqslant (\mathfrak{b}\wp^{2\mathfrak{m}+1}+\mathfrak{b}\wp^{2\mathfrak{m}+2})\varphi(\sigma_0,\eta_0)+\mathfrak{b}^2(\varphi(\sigma_{\mathfrak{m}+2},\eta_{\mathfrak{m}+1})+\varphi(\sigma_{\mathfrak{m}+2},\eta_{\mathfrak{m}+2})+\varphi(\sigma_{\vartheta},\eta_{\mathfrak{m}+2}))\\
&\ \ \vdots\\
&\leqslant (\mathfrak{b}\wp^{2\mathfrak{m}+1}+\mathfrak{b}\wp^{2\mathfrak{m}+2}+\mathfrak{b}^2\wp^{2\mathfrak{m}+3}+\mathfrak{b}^2\wp^{2\mathfrak{m}+4}+\cdots)\varphi(\sigma_0,\eta_0)
\end{aligned}$$

$$= \mathfrak{b}\wp^{2m+1}(\frac{1}{1-\mathfrak{b}\wp^2})\varphi(\sigma_0,\eta_0) + \mathfrak{b}\wp^{2m+2}(\frac{1}{1-\mathfrak{b}\wp^2})\varphi(\sigma_0,\eta_0).$$

Again, since $\wp < 1$, $\lim_{\vartheta,m\to\infty} \varphi(\sigma_\vartheta,\eta_m) = 0$.

Therefore, $(\{\sigma_\vartheta\},\{\eta_m\})$ is a Cauchy bisequence. Since (Ξ,Λ,φ) is complete, $\{\sigma_\vartheta\} \to \sigma^*, \{\eta_m\} \to \sigma^*$, where $\sigma^* \in \Xi\cup\Lambda$. Also, $\{\mathcal{S}(\sigma_{2\vartheta})\} = \{\eta_{2\vartheta}\} \to \sigma^* \in \Xi\cap\Lambda$ implies that $\mathcal{S}(\sigma_{2\vartheta})$ has a unique limit σ^*, and $\{\sigma_\vartheta\} \to \sigma^*$ implies that $\{\sigma_{2\vartheta}\} \to \sigma^*$. Since $\mathcal{S}$ is a continuous, $\mathcal{S}(\sigma_{2\vartheta}) \to \mathcal{S}\sigma^*$. Therefore, $\mathcal{S}\sigma^* = \sigma^*$. Similarly, $\{\Omega(\eta_{2\vartheta+1})\} = \{\sigma_{2\vartheta+2}\} \to \sigma^* \in \Xi\cap\Lambda$ implies that $\Omega(\eta_{2\vartheta+1})$ has a unique limit σ^*, and $\{\eta_\vartheta\} \to \sigma^*$ implies that $\{\eta_{2\vartheta+1}\} \to \sigma^*$. Now, the continuity of Ω implies that $\{\Omega(\eta_{2\vartheta+1})\} \to \Omega\sigma^*$. Therefore, $\Omega\sigma^* = \sigma^*$. Let $\eta^* \in \Xi\cap\Lambda$ such that $\mathcal{S}\eta^* = \Omega\eta^* = \eta^* \in \Xi\cap\Lambda$. Then, we get

$$\begin{aligned}\varphi(\eta^*,\sigma^*) = \varphi(\mathcal{S}\eta^*,\Omega\sigma^*) &\leqslant \alpha\frac{\varphi(\sigma^*,\Omega\sigma^*)\varphi(\mathcal{S}\eta^*,\eta^*)}{\varphi(\sigma^*,\eta^*)} + \gamma\varphi(\sigma^*,\eta^*) + \imath[\varphi(\sigma^*,\Omega\sigma^*) + \varphi(\mathcal{S}\eta^*,\eta^*)]\\ &= \alpha\frac{\varphi(\sigma^*,\sigma^*)\varphi(\eta^*,\eta^*)}{\varphi(\sigma^*,\eta^*)} + \gamma\varphi(\sigma^*,\eta^*) + \imath[\varphi(\sigma^*,\sigma^*) + \varphi(\eta^*,\eta^*)].\end{aligned}$$

Therefore, $\varphi(\eta^*,\sigma^*) \leqslant \gamma\varphi(\sigma^*,\eta^*)$, which is a contradiction and hence, $\sigma^* = \eta^*$. □

Now, we present our second CFP result.

Theorem 3.5. *Let (Ξ,Λ,φ) be a complete B$\mathfrak{b}$MSs with $\mathfrak{b} \geqslant 1$ and $\Omega,\mathcal{S} : (\Xi,\Lambda,\varphi) \rightleftarrows (\Xi,\Lambda,\varphi)$ be a contravariant mapping satisfying*

$$\varphi(\mathcal{S}\eta,\Omega\sigma) \leqslant \alpha\frac{\varphi(\sigma,\Omega\sigma)\varphi(\sigma,\mathcal{S}\eta) + \varphi(\mathcal{S}\eta,\eta)\varphi(\eta,\Omega\sigma)}{\varphi(\sigma,\mathcal{S}\eta) + \varphi(\eta,\Omega\sigma)}, \tag{3.4}$$

for all $(\sigma,\eta) \in \Xi\times\Lambda$, with $\sigma \neq \eta$ and $0 < \alpha < 1$. Then $\Omega,\mathcal{S} : \Xi\cup\Lambda \to \Xi\cup\Lambda$ have a unique CFP.

Proof. Let $\sigma_0 \in \Xi$ and $\eta_0 \in \Lambda$, then for each $\vartheta \in \mathbb{N}\cup\{0\}$, we define

$$\mathcal{S}\sigma_{2\vartheta} = \eta_{2\vartheta}, \quad \Omega\sigma_{2\vartheta+1} = \eta_{2\vartheta+1}, \quad \mathcal{S}\eta_{2\vartheta} = \sigma_{2\vartheta+1}, \quad \Omega\eta_{2\vartheta+1} = \sigma_{2\vartheta+2}.$$

Now by (3.4), we get

$$\begin{aligned}\varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta+1}) &= \varphi(\mathcal{S}\eta_{2\vartheta},\Omega\sigma_{2\vartheta+1})\\ &\leqslant \alpha\frac{\varphi(\sigma_{2\vartheta+1},\Omega\sigma_{2\vartheta+1})\varphi(\sigma_{2\vartheta+1},\mathcal{S}\eta_{2\vartheta}) + \varphi(\mathcal{S}\eta_{2\vartheta},\eta_{2\vartheta})\varphi(\eta_{2\vartheta},\Omega\sigma_{2\vartheta+1})}{\varphi(\sigma_{2\vartheta+1},\mathcal{S}\eta_{2\vartheta}) + \varphi(\eta_{2\vartheta},\Omega\sigma_{2\vartheta+1})}\\ &= \alpha\frac{\varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta+1})\varphi(\sigma_{2\vartheta+1},\sigma_{2\vartheta+1}) + \varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta})\varphi(\eta_{2\vartheta},\eta_{2\vartheta+1})}{\varphi(\sigma_{2\vartheta+1},\sigma_{2\vartheta+1}) + \varphi(\eta_{2\vartheta},\eta_{2\vartheta+1})} = \alpha\varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta}),\end{aligned}$$

which implies that

$$\varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta+1}) \leqslant \alpha\varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta}). \tag{3.5}$$

Also, we have

$$\begin{aligned}\varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta}) &= \varphi(\mathcal{S}\eta_{2\vartheta},\Omega\sigma_{2\vartheta})\\ &\leqslant \alpha\frac{\varphi(\sigma_{2\vartheta},\Omega\sigma_{2\vartheta})\varphi(\sigma_{2\vartheta},\mathcal{S}\eta_{2\vartheta}) + \varphi(\mathcal{S}\eta_{2\vartheta},\eta_{2\vartheta})\varphi(\eta_{2\vartheta},\Omega\sigma_{2\vartheta})}{\varphi(\sigma_{2\vartheta},\mathcal{S}\eta_{2\vartheta}) + \varphi(\eta_{2\vartheta},\mathcal{S}\sigma_{2\vartheta})}\\ &= \alpha\frac{\varphi(\sigma_{2\vartheta},\eta_{2\vartheta})\varphi(\sigma_{2\vartheta},\sigma_{2\vartheta+1}) + \varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta})\varphi(\eta_{2\vartheta},\eta_{2\vartheta})}{\varphi(\sigma_{2\vartheta},\sigma_{2\vartheta+1}) + \varphi(\eta_{2\vartheta},\eta_{2\vartheta})} = \alpha\varphi(\sigma_{2\vartheta},\eta_{2\vartheta}),\end{aligned}$$

which implies that

$$\varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta}) \leqslant \alpha\varphi(\sigma_{2\vartheta},\eta_{2\vartheta}). \tag{3.6}$$

Hence, from the previous two inequalities (3.5) and (3.6), we get

$$\varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta+1}) \leqslant \alpha^{4\vartheta+2}\varphi(\sigma_0,\eta_0) \quad \text{and} \quad \varphi(\sigma_{2\vartheta+1},\eta_{2\vartheta}) \leqslant \alpha^{4\vartheta+1}\varphi(\sigma_0,\eta_0).$$

Now, we can get that for any $\vartheta \in \mathbb{N}$,

$$\varphi(\sigma_{\vartheta+1},\eta_{\vartheta+1}) \leqslant \alpha^{2\vartheta+2}\varphi(\sigma_0,\eta_0), \quad \varphi(\sigma_{\vartheta+1},\eta_{\vartheta}) \leqslant \alpha^{2\vartheta+1}\varphi(\sigma_0,\eta_0), \quad \text{and} \quad \varphi(\sigma_{\vartheta},\eta_{\vartheta}) \leqslant \alpha^{2\vartheta}\varphi(\sigma_0,\eta_0).$$

For all $\mathfrak{m},\vartheta \in \mathbb{N}$, we have two cases.

Case 1. If $\mathfrak{m} > \vartheta$,

$$\begin{aligned}
\varphi(\sigma_\vartheta,\eta_\mathfrak{m}) &\leqslant \mathfrak{b}(\varphi(\sigma_\vartheta,\eta_\vartheta)+\varphi(\sigma_{\vartheta+1},\eta_\vartheta)+\varphi(\sigma_{\vartheta+1},\eta_\mathfrak{m}))\\
&\leqslant \mathfrak{b}\alpha^{2\vartheta}\varphi(\sigma_0,\eta_0)+\mathfrak{b}\alpha^{2\vartheta+1}\varphi(\sigma_0,\eta_0)+\mathfrak{b}\varphi(\sigma_{\vartheta+1},\eta_\mathfrak{m})\\
&\leqslant (\mathfrak{b}\alpha^{2\vartheta}+\mathfrak{b}\alpha^{2\vartheta+1})\varphi(\sigma_0,\eta_0)+\mathfrak{b}^2(\varphi(\sigma_{\vartheta+1},\eta_{\vartheta+1})+\varphi(\sigma_{\vartheta+2},\eta_{\vartheta+1})+\varphi(\sigma_{\vartheta+2},\eta_\mathfrak{m}))\\
&\leqslant (\mathfrak{b}\alpha^{2\vartheta}+\mathfrak{b}\alpha^{2\vartheta+1})\varphi(\sigma_0,\eta_0)+\mathfrak{b}^2\alpha^{2\vartheta+2}\varphi(\sigma_0,\eta_0)+\mathfrak{b}^2\alpha^{2\vartheta+3}\varphi(\sigma_0,\eta_0)+\mathfrak{b}^2\varphi(\sigma_{\vartheta+2},\eta_\mathfrak{m})\\
&\ \ \vdots\\
&\leqslant (\mathfrak{b}\alpha^{2\vartheta}+\mathfrak{b}\alpha^{2\vartheta+1}+\mathfrak{b}^2\alpha^{2\vartheta+2}+\mathfrak{b}^2\alpha^{2\vartheta+3}+\cdots)\varphi(\sigma_0,\eta_0)\\
&= \mathfrak{b}\alpha^{2\vartheta}(\frac{1}{1-\mathfrak{b}\alpha^2})\varphi(\sigma_0,\eta_0)+\mathfrak{b}\alpha^{2\vartheta+1}(\frac{1}{1-\mathfrak{b}\alpha^2})\varphi(\sigma_0,\eta_0).
\end{aligned}$$

Since $\alpha < 1$, $\lim_{\vartheta,\mathfrak{m}\to\infty}\varphi(\sigma_\vartheta,\eta_\mathfrak{m}) = 0$.

Case 2. If $\mathfrak{m} < \vartheta$,

$$\begin{aligned}
\varphi(\sigma_\vartheta,\eta_\mathfrak{m}) &\leqslant \mathfrak{b}(\varphi(\sigma_{\mathfrak{m}+1},\eta_\mathfrak{m})+\varphi(\sigma_{\mathfrak{m}+1},\eta_{\mathfrak{m}+1})+\varphi(\sigma_\vartheta,\eta_{\mathfrak{m}+1}))\\
&\leqslant \mathfrak{b}\alpha^{2\mathfrak{m}+1}\varphi(\sigma_0,\eta_0)+\mathfrak{b}\alpha^{2\mathfrak{m}+2}\varphi(\sigma_0,\eta_0)+\mathfrak{b}\varphi(\sigma_\vartheta,\eta_{\mathfrak{m}+1})\\
&\leqslant (\mathfrak{b}\alpha^{2\mathfrak{m}+1}+\mathfrak{b}\alpha^{2\mathfrak{m}+2})\varphi(\sigma_0,\eta_0)+\mathfrak{b}^2(\varphi(\sigma_{\mathfrak{m}+2},\eta_{\mathfrak{m}+1})+\varphi(\sigma_{\mathfrak{m}+2},\eta_{\mathfrak{m}+2})+\varphi(\sigma_\vartheta,\eta_{\mathfrak{m}+2}))\\
&\ \ \vdots\\
&\leqslant (\mathfrak{b}\alpha^{2\mathfrak{m}+1}+\mathfrak{b}\alpha^{2\mathfrak{m}+2}+\mathfrak{b}^2\alpha^{2\mathfrak{m}+3}+\mathfrak{b}^2\alpha^{2\mathfrak{m}+4}+\cdots)\varphi(\sigma_0,\eta_0)\\
&= \mathfrak{b}\wp^{2\mathfrak{m}+1}(\frac{1}{1-\mathfrak{b}\wp^2})\varphi(\sigma_0,\eta_0)+\mathfrak{b}\wp^{2\mathfrak{m}+2}(\frac{1}{1-\mathfrak{b}\wp^2})\varphi(\sigma_0,\eta_0).
\end{aligned}$$

Again, since $\alpha < 1$, $\lim_{\vartheta,\mathfrak{m}\to\infty}\varphi(\sigma_\vartheta,\eta_\mathfrak{m}) = 0$.

Therefore, $(\{\sigma_\vartheta\},\{\eta_\mathfrak{m}\})$ is a Cauchy bisequence. Since (Ξ,Λ,φ) is complete, $\{\sigma_\vartheta\} \to \sigma^*, \{\eta_\mathfrak{m}\} \to \sigma^*$, where $\sigma^* \in \Xi\cup\Lambda$. Also, $\{\mathcal{S}(\sigma_{2\vartheta})\} = \{\eta_{2\vartheta}\} \to \sigma^* \in \Xi\cap\Lambda$ implies that $\mathcal{S}(\sigma_{2\vartheta})$ has a unique limit σ^*, and $\{\sigma_\vartheta\} \to \sigma^*$ implies that $\{\sigma_{2\vartheta}\} \to \sigma^*$. Since $\mathcal{S}$ is a continuous, $\{\mathcal{S}(\sigma_{2\vartheta})\} \to \mathcal{S}\sigma^*$. Therefore, $\mathcal{S}\sigma^* = \sigma^*$. Similarly, $\{\Omega(\eta_{2\vartheta+1})\} = \{\sigma_{2\vartheta+2}\} \to \sigma^* \in \Xi\cap\Lambda$ implies that $\Omega(\eta_{2\vartheta+1})$ has a unique limit σ^*, and $\{\eta_\vartheta\} \to \sigma^*$ implies that $\{\eta_{2\vartheta+1}\} \to \sigma^*$. Now, the continuity of Ω implies that $\{\Omega(\eta_{2\vartheta+1})\} \to \Omega\sigma^*$. Therefore, $\Omega\sigma^* = \sigma^*$. Hence, Ω and $\mathcal{S}$ have a common fixed point. Let $\eta^* \in \Xi\cap\Lambda$ such that $\mathcal{S}\eta^* = \Omega\eta^* = \eta^* \in \Xi\cap\Lambda$. Then, we get

$$\begin{aligned}
\varphi(\eta^*,\sigma^*) = \varphi(\mathcal{S}\eta^*,\Omega\sigma^*) &\leqslant \alpha\frac{\varphi(\sigma^*,\Omega\sigma^*)\varphi(\sigma^*,\mathcal{S}\eta^*)+\varphi(\mathcal{S}\eta^*,\eta^*)\varphi(\eta^*,\Omega\sigma^*)}{\varphi(\sigma^*,\mathcal{S}\eta^*)+\varphi(\eta^*,\Omega\sigma^*)}\\
&= \alpha\frac{\varphi(\sigma^*,\sigma^*)\varphi(\sigma^*,\eta^*)+\varphi(\eta^*,\eta^*)\varphi(\eta^*,\sigma^*)}{\varphi(\sigma^*,\eta^*)+\varphi(\eta^*,\sigma^*)} = 0.
\end{aligned}$$

Therefore, $\sigma^* = \eta^*$. □

4. Applications

4.1. Application I

In this section, we present our supporting result of Theorem 3.1.

Theorem 4.1. *Let*

$$\sigma(\varpi) = \mathfrak{b}(\varpi) + \int_{\Psi_1\Psi_2} \mathcal{G}(\varpi, \rho, \sigma(\rho))d\rho, \ \varpi \in \Psi_1 \cup \Psi_2, \tag{4.1}$$

where $\Psi_1 \cup \Psi_2$ is a Lebesgue measurable set. Suppose

1. $\mathcal{G} : (\Psi_1^2 \cup \Psi_2^2) \times [0, \infty) \to [0, \infty)$ *and* $\mathfrak{b} \in L^\infty(\Psi_1) \cup L^\infty(\Psi_2)$;
2. *there is a continuous function* $\theta : \Psi_1^2 \cup \Psi_2^2 \to [0, \infty)$ *and* $\wp \in (0, 1)$ *such that*
$$|\mathcal{G}(\varpi, \rho, \sigma(\rho)) - \mathcal{G}(\varpi, \rho, \eta(\rho)| \leqslant \sqrt{\wp\theta(\varpi, \rho)}(|\sigma(\rho) - \eta(\rho)|,$$
for $\varpi, \rho \in \Psi_1^2 \cup \Psi_2^2$;
3. $\sup_{\varpi\in\Psi_1\cup\Psi_2} \int_{\Psi_1\cup\Psi_2} \theta(\varpi, \rho)d\rho \leqslant 1$.

Then the equation (4.1) has a unique solution in $L^\infty(\Psi_1) \cup L^\infty(\Psi_2)$.

Proof. Let $\Xi = L^\infty(\Psi_1)$ and $\Lambda = L^\infty(\Psi_2)$ be two normed linear spaces, where Ψ_1, Ψ_2 are Lebesgue measurable sets and $m(\Psi_1 \cup \Psi_2) < \infty$. Consider $\varphi : \Xi \times \Lambda \to \mathbb{R}^+$ by $\varphi(\sigma, \eta) = \sup_{\varpi\in\Psi_1\cup\Psi_2} |\sigma - \eta|^2$ for all $(\sigma, \eta) \in \Xi \times \Lambda$. Then (Ξ, Λ, φ) is a complete B$\mathfrak{b}$MSs with $\mathfrak{b} \geqslant 1$. Define $\Omega : L^\infty(\Psi_1) \cup L^\infty(\Psi_2) \rightrightarrows L^\infty(\Psi_1) \cup L^\infty(\Psi_2)$ by

$$\Omega(\sigma(\varpi)) = \mathfrak{b}(\varpi) + \int_{\Psi_1\cup\Psi_2} \mathcal{G}(\varpi, \rho, \sigma(\rho))d\rho, \ \varpi \in \Psi_1 \cup \Psi_2.$$

Now,

$$\begin{aligned}
\varphi(\Omega\sigma(\varpi), \Omega\eta(\varpi)) &= \sup_{\varpi\in\Psi_1\cup\Psi_2} |\Omega\sigma(\varpi) - \Omega\eta(\varpi)|^2 \\
&= \sup_{\varpi\in\Psi_1\cup\Psi_2} \left|\mathfrak{b}(\varpi) + \int_{\Psi_1\cup\Psi_2} \mathcal{G}(\varpi, \rho, \sigma(\rho))d\rho - \left(\mathfrak{b}(\varpi) + \int_{\Psi_1\cup\Psi_2} \mathcal{G}(\varpi, \rho, \sigma(\rho))d\rho\right)\right|^2 \\
&\leqslant \sup_{\varpi\in\Psi_1\cup\Psi_2} \int_{\Psi_1\cup\Psi_2} |\mathcal{G}(\varpi, \rho, \sigma(\rho)) - \mathcal{G}(\varpi, \rho, \eta(\rho))|^2 d\rho \\
&\leqslant \sup_{\varpi\in\Psi_1\cup\Psi_2} \int_{\Psi_1\cup\Psi_2} \wp\theta(\varpi, \rho)(|\sigma(\rho) - \eta(\rho)|^2)d\rho \\
&\leqslant \wp(\sup_{\varpi\in\Psi_1\cup\Psi_2} |\sigma(\rho) - \eta(\rho)|^2) \sup_{\varpi\in\Psi_1\cup\Psi_2} \int_{\Psi_1\cup\Psi_2} \theta(\varpi, \rho)d\rho = \wp\varphi(\sigma, \eta).
\end{aligned}$$

Hence, all the axioms of a Theorem 3.1 are verified and consequently, the integral equation has a unique solution. □

4.2. Application II

We recall many important definitions from fractional calculus theory. For a function $\rho \in \mathcal{C}[0, 1]$, the Reiman-Liouville fractional derivative of order $\delta > 0$ is given by

$$\frac{1}{\Gamma(\vartheta - \delta)} \frac{d^\vartheta}{d\aleph^\vartheta} \int_0^\aleph \frac{\rho(\mathfrak{e})d\mathfrak{e}}{(\aleph - \mathfrak{e})^{\delta-\vartheta+1}} = \mathcal{D}^\delta \rho(\aleph),$$

provided that the right hand side is pointwise defined on $[0,1]$, where $[\delta]$ is the integer part of the number δ, and Γ is the Euler gamma function. Consider the following FDE (fractional differential equation)

$$ {}^{\mathfrak{c}}\mathcal{D}^{\sigma}\rho(\aleph)+\ell(\aleph,\rho(\aleph))=0, \quad 1\leqslant\aleph\leqslant 0, \quad 2\leqslant\sigma>1, \quad \rho(0)=\rho(1)=0, \tag{4.2}$$

where $\ell:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function and ${}^{\mathfrak{c}}\mathcal{D}^{\sigma}$ represents the Caputo fractional derivative of order σ and it is defined by

$$ {}^{\mathfrak{c}}\mathcal{D}^{\sigma}=\frac{1}{\Gamma(\vartheta-\sigma)}\int_0^{\zeta}\frac{\rho^{\vartheta}(\mathfrak{e})d\mathfrak{e}}{(\aleph-\mathfrak{e})^{\sigma-\vartheta+1}}.$$

Let $\Xi=(\mathcal{C}[0,1],[0,\infty))=\{\mathfrak{g}:[0,1]\to[0,\infty):\mathfrak{g}$ is a continuous function$\}$ and $\Lambda=(\mathcal{C}[0,1],(-\infty,0])=\{\mathfrak{g}:[0,1]\to(-\infty,0]:\mathfrak{g}$ is a continuous function$\}$. Consider $\varphi:\Xi\times\Lambda\to\mathbb{R}^+$ by

$$\varphi(\rho,\rho')=\sup_{\aleph\in[0,1]}|\rho(\aleph)-\rho'(\aleph)|^2,$$

for all $(\rho,\rho')\in\Xi\times\Lambda$. Then (Ξ,Λ,φ) is a complete B$\mathfrak{b}$MSs with $\mathfrak{b}\geqslant 1$.

Theorem 4.2. *Assume the nonlinear FDE (4.2). Suppose that the following conditions are satisfied.*

1. *We can find* $\aleph\in[0,1]$, $\wp\in(0,1)$ *and* $(\rho,\rho')\in\Xi\times\Lambda$ *such that* $|\ell(\aleph,\rho)-\ell(\aleph,\rho')|\leqslant\sqrt{\wp}|\rho(\aleph)-\rho'(\aleph)|$.
2. $\sup_{\aleph\in[0,1]}\int_0^1|\mathcal{G}(\aleph,\mathfrak{e})|d\mho\leqslant 1$.

Then the fractional differential equation (4.2) has a unique solution in $\Xi\cup\Lambda$.

Proof. The given fractional differential equation (4.2) is equivalent to the succeeding integral equation

$$\rho(\aleph)=\int_0^1\mathcal{G}(\aleph,\mathfrak{e})\ell(\mho,\rho(\mathfrak{e}))d\mathfrak{e},$$

where

$$\mathcal{G}(\aleph,\mathfrak{e})=\begin{cases}\frac{[\aleph(1-\mathfrak{e})]^{\sigma-1}-(\aleph-\mathfrak{e})^{\sigma-1}}{\Gamma(\sigma)}, & 0\leqslant\mathfrak{e}\leqslant\aleph\leqslant 1,\\ \frac{[\aleph(1-\mathfrak{e})]^{\sigma-1}}{\Gamma(\sigma)}, & 0\leqslant\aleph\leqslant\mathfrak{e}\leqslant 1.\end{cases}$$

Define $\Omega\colon\Xi\cup\Lambda\to\Xi\cup\Lambda$ defined by

$$\Omega\rho(\aleph)=\int_0^1\mathcal{G}(\aleph,\mathfrak{e})\ell(\mho,\rho(\mathfrak{e}))d\mathfrak{e}.$$

Now

$$\begin{aligned}|\Omega\rho(\aleph)-\Omega\rho'(\aleph)|^2&=\left|\int_0^1\mathcal{G}(\aleph,\mathfrak{e})\ell(\mho,\rho(\mathfrak{e}))d\mathfrak{e}-\int_o^1\mathcal{G}(\aleph,\mathfrak{e})\ell(\mho,\rho'(\mathfrak{e}))d\mathfrak{e}\right|^2\\&\leqslant\int_0^1|\mathcal{G}(\aleph,\mathfrak{e})|^2d\mathfrak{e}\cdot\int_0^1\Big|\ell(\mho,\rho(\mathfrak{e}))-\ell(\mho,\rho'(\mathfrak{e}))\Big|^2d\mathfrak{e}\leqslant\wp\Big|\rho(\aleph)-\rho'(\aleph)\Big|^2.\end{aligned}$$

Taking the supremum on both sides, we get $\varphi(\Omega\rho,\Omega\rho')\leqslant\wp\varphi(\rho,\rho')$. Hence, all the axioms of a Theorem 3.1 are verified and consequently, the fractional differential equation (4.2) has a unique solution. □

5. Conclusion

In this article, we given idea on bipolar $\mathfrak{b}$-metric space and proved fixed point and common fixed point theorems. An illustrative example is provided that show the validity of the hypothesis and the degree of usefulness of our findings. In 2017, Kamran et al. [11] introduced the concept of extended $\mathfrak{b}$-metric space and proved fixed point theorems. It is an interesting open problem to introduce the notion of extended bipolar $\mathfrak{b}$-metric space and prove fixed point theorems.

Acknowledgment

The authors S. Aljohani and N. Mlaiki would like to thank Prince Sultan University for paying the publication fees for this work through TAS LAB.

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