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Strongly bounded variation functions in Krein spaces

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Abstract

In the present paper, we introduce the notion of strongly bounded variation function in Krein spaces, we then show that the definition of bounded variation is independent of the decomposition of the Krein space, and the definition of bounded variation of a function in Hilbert spaces given in [V. V. Chistyakov, J. Dynam. Control Syst., **3** (1997), 261–289], is a particular case of the one introduced in this paper. We show a technique for constructing bounded variation functions on Krein spaces from bounded variation functions on the Hilbert subspaces composing the Krein space.

Keywords: Indefinite metric, Krein space, bounded variation, negative variation.

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1. Introduction

The concept of bounded variation function admits several generalizations. Vitali, Hardy, Arzela, Pierpont, Fréchet, and Tonelli extended in different ways the concept of bounded variation function for real functions of two variables [2, 13, 14, 18–20]. Adams [1, 8] studied the relationship between these concepts. Chistyakov in [5, 10, 11, 15] studied a concept of bounded variation function, for functions $f : [a, b] \to X$, where X is a normed or metric space. In the framework of this type of functions one cannot posit that a bounded variation function is the difference of two increasing functions, since a metric space one does not necessarily order that allows the introduction of the concept of increasing function. However, some alternative Jordan decomposition theorems have been proved in [4, 7, 16].

On the other hand, some spaces which generalize to Hilbert spaces, such as spaces with indefinite inner product had appeared before 1994 in the scientific literature in papers by Dirac and Pauli [9, 17]. But the year 1944 is considered to be the origin of the systematic study of the theory of operators in such spaces. Lev Pontryagin's article "Hermitian Operators in spaces with an indefinite metric" appeared in the Soviet journal"Izvestya Akademii nauk CCCP" (News of the Academy of Sciences of the USSR) that is why 1944 is considered by several authors as the beginning of this theory. Nowadays extending results of Hilbert spaces to spaces with indefinite metric is quite attractive for researchers, an example of them is the extension of frames to such spaces ([10–12]). In this paper we extend the notion of strong bounded variation function to spaces with indefinite metric and prove the most relevant results of such functions in these spaces.

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2. Preliminaries

Definition 2.1 ([15]). Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a normed space, $f : [a, b] \to \mathcal{X}$ is *strongly of bounded variation* (BV) on $[a, b]$, if $\sup\{\sum \|f(t_i) - f(t_{i-1})\|_{\mathcal{X}}\}$ is finite, where the supremum is taken over all partitions of $[a, b]$.

Definition 2.2 ([3, 5]). A space $\mathcal{K}$ with an indefinite inner product $[\cdot,\cdot]$ that admits a fundamental decomposition of the form $\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$, such that $(\mathcal{K}_+, [\cdot,\cdot])$ and $(\mathcal{K}_-, -[\cdot,\cdot])$ are Hilbert spaces, is called *Krein space*, denoted by $(\mathcal{K}, [\cdot,\cdot])$.

Definition 2.3 ([3, 5]). Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space with fundamental decomposition $\mathcal{K} = \mathcal{K}_-[\dot{+}]\mathcal{K}_+$ and define two operators $\mathcal{P}^+ : \mathcal{K} \longrightarrow \mathcal{K}_+$ and $\mathcal{P}^- : \mathcal{K} \longrightarrow \mathcal{K}_-$, by $\mathcal{P}^+(x) = x^+$ and $\mathcal{P}^-(x) = x^-$ for all $x \in \mathcal{K}$, respectively, where $x^+ \in \mathcal{K}_+$, $x^- \in \mathcal{K}_-$ and $x = x^+ + x^-$. The operators $\mathcal{P}^+$ and $\mathcal{P}^-$ are called *fundamental projectors*. The operator $\mathcal{J} : \mathcal{K} \longrightarrow \mathcal{K}$ defined by $\mathcal{J} = \mathcal{P}^+ - \mathcal{P}^-$, this is $\mathcal{J}x = \mathcal{P}^+x - \mathcal{P}^-x = x^+ - x^-$, for all $x \in \mathcal{K}$, is known by the fundamental symmetry of the Krein space $\mathcal{K}$.

Remark 2.4. The Krein space with its fundamental decomposition $\mathcal{K} = \mathcal{K}_-[\dot{+}]\mathcal{K}_+$ and their fundamental symmetry $\mathcal{J}$, from now on will be written as $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot], \mathcal{J})$.

Theorem 2.5 ([3, 5]). *Let $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot], \mathcal{J})$ be a Krein space, then $\mathcal{J}$ is an invertible operator such that $\mathcal{J}^2 = I$, $\mathcal{J}^{-1} = \mathcal{J}$, and additionally satisfies that $\mathcal{J}$ is a symmetric, isometric, and self-adjoint operator in the Krein space and in the associated Hilbert spaces.*

Definition 2.6 ([3, 5]). Let $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot], \mathcal{J})$ be a Krein space and the function $[\cdot,\cdot]_{\mathcal{J}} : \mathcal{K} \times \mathcal{K} \longrightarrow \mathbb{C}$ defined by $[x, y]_{\mathcal{J}} = [\mathcal{J}x, y]$, $x, y \in \mathcal{K}$. This function is called *$\mathcal{J}$-internal product*.

Note that if we have another fundamental decomposition, then we would have another fundamental symmetry and consequently another $\mathcal{J}$-internal product.

Definition 2.7 ([3, 5]). The fundamental symmetry $\mathcal{J}$ associated with the Krein space $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$ induces a norm on $\mathcal{K}$ defined by $\|x\|_{\mathcal{J}} := \sqrt{[x, x]_{\mathcal{J}}}$, for all $x \in \mathcal{K}$, this norm is called *$\mathcal{J}$-norm* of $\mathcal{K}$. Explicitly,

$$\|x\|_{\mathcal{J}} = ([x^+, x^+] - [x^-, x^-])^{1/2}, \text{ for all } x \in \mathcal{K}.$$

Remark 2.8. The following norms are defined

$$\|x^+\|_+ = \sqrt{[x^+, x^+]},\ x^+ \in \mathcal{K}^+ \quad \text{and} \quad \|x^-\|_- = \sqrt{-[x^-, x^-]},\ x^- \in \mathcal{K}^-.$$

From now on, the topology studied on the Krein spaces is directly related to the $\mathcal{J}$-norm of $\mathcal{K}$.

Next, we prove a result that relates the $\mathcal{J}$-norm to positive and negative norms in a Krein space. This result will be quite useful in the remainder of this paper.

Theorem 2.9. *Let $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot], \mathcal{J})$ be a Krein space, then $\|x\|_{\mathcal{J}} \leqslant \|x^+\|_+ + \|x^-\|_-$ for all $x = x^+ + x^- \in \mathcal{K}$.*

Proof. $\|x\|_{\mathcal{J}}^2 = [x, x]_{\mathcal{J}} = [\mathcal{J}x, x] = [x^+ - x^-, x] = [x^+, x] - [x^-, x] = [x^+, x^+] + (-[x^-, x^-]) = ([x^+, x^+]^{1/2})^2 + (-[x^-, x^-]^{1/2})^2 = \|x^+\|_+^2 + \|x^-\|_-^2 \leqslant \|x^+\|_+^2 + 2\|x^+\|_+\|x^-\|_- + \|x^-\|_-^2 = (\|x^+\|_+ + \|x^-\|_-)^2$. Therefore, $\|x\|_{\mathcal{J}} \leqslant \|x^+\|_+ + \|x^-\|_-$ for all $x = x^+ + x^- \in \mathcal{K}$. □

Example 2.10. Let $\mathcal{K} = \mathbb{R}^2$ with the indefinite inner product $[\cdot,\cdot] : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$[(a, b), (c, d)] := ac - bd, \ \ (a, b), (c, d) \in \mathbb{R}^2.$$

Let

$$\mathcal{K}_+ := \{(x, 0)|x \in \mathbb{R}\} \quad \text{and} \quad \mathcal{K}_- := \{(0, y)|y \in \mathbb{R}\}.$$

Therefore, $\mathcal{K}$ is a Krein space with fundamental symmetry,

$$\mathcal{J}(x, y) = \mathcal{P}^+(x, y) - \mathcal{P}^-(x, y) = (x, 0) - (0, y) = (x, -y), \text{ for all } (x, y) \in \mathbb{R}^2.$$

Furthermore

$$[(a, b), (c, d)]_{\mathcal{J}} = [\mathcal{J}(a, b), (c, d)] = [(a, -b), (c, d)] = ac + bd.$$

Remark 2.11. By Definition 2.7 we have $\|(1,2)\|_{\mathcal{J}} = \sqrt{1^2+2^2} = \sqrt{5}$. Furthermore, $(1,2) = (1,0)+(0,2)$. However,

$$\|(1,0)\|_+ = \sqrt{[(1,0),(1,0)]} = 1 \text{ and } \|(0,2)\|_- = \sqrt{-[(0,2),(0,2)]} = \sqrt{-(-4)} = 2,$$

from where we have

$$\|(1,2)\|_{\mathcal{J}} < \|(1,0)\|_+ + \|(0,2)\|_-.$$

Thus, we show that the inequality obtained in Theorem 2.9 can be strict.

Theorem 2.12 ([3])**.** *Let $(\mathcal{K},[\cdot,\cdot])$ be a Krein space and let $\mathcal{K} = \mathcal{K}_{+1}[\dot{+}]\mathcal{K}_{-1}$, $\mathcal{K} = \mathcal{K}_{+2}[\dot{+}]\mathcal{K}_{-2}$ be two fundamental decompositions. If $\mathcal{J}_1$ and $\mathcal{J}_2$ are the fundamental symmetries, respectively, then $\|\cdot\|_{\mathcal{J}_1}$ and $\|\cdot\|_{\mathcal{J}_2}$ are equivalent norms.*

Theorem 2.13 ([5])**.** *Let $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-,[\cdot,\cdot],\mathcal{J})$ be a Krein space, then $(\mathcal{K},[\cdot,\cdot]_{\mathcal{J}})$ is a Hilbert space.*

Theorem 2.14 ([10])**.** *Let $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-,[\cdot,\cdot],\mathcal{J})$ be a Krein space, and let $\mathcal{P}$ be an orthogonal projection that commutes with $\mathcal{J}$, then the spaces $\mathcal{P}\mathcal{K}$ and $(I-\mathcal{P})\mathcal{K}$ are Krein spaces, with fundamental symmetries $\mathcal{P}\mathcal{J}$ and $(I-\mathcal{P})\mathcal{J}$, respectively.*

3. Functions of bounded variation in Hilbert spaces associated to a Krein space

In this section we establish the notion of bounded variation functions on spaces with indefinite metric. First we will show an example with which we guarantee that we are working on a non-empty set.

Example 3.1. Let $(\mathbb{R}^2,[\cdot,\cdot])$ with the indefinite inner product $[\cdot,\cdot]:\mathbb{R}^2\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $[(x_1,y_1),(x_2,y_2)] := x_1x_2 - y_1y_2$, $(x_1,y_1),(x_2,y_2)\in\mathbb{R}^2$, $\mathcal{K}_+ := \{(x,0)|x\in\mathbb{R}\}$ and $\mathcal{K}_- := \{(0,y)|y\in\mathbb{R}\}$, $\mathcal{J}(x,y) = (x,-y)$, for all $(x,y)\in\mathbb{R}^2$, $\|(x,y)\|_{\mathcal{J}} = \sqrt{[(x,y),(x,y)]_{\mathcal{J}}} = \sqrt{x^2+y^2}$, $\|(x,0)\|_+ = \sqrt{[(x,0),(x,0)]} = \sqrt{x\cdot x - 0\cdot 0} = \sqrt{x^2} = |x|$, $\|(0,y)\|_- = \sqrt{-[(0,y),(0,y)]} = \sqrt{-(0\cdot 0 - y\cdot y)} = \sqrt{-(-y^2)} = \sqrt{y^2} = |y|$. Let $f:[a,b]\to\mathbb{R}^2$ such that $f(x) = (x,x)$. Consider the partition $P = \{a,t_1,t_2,\dots,t_{i-1},t_1,\dots,t_{n-1},b\}\in\mathcal{P}[a,b]$, then

$$\begin{aligned}\|f(t_i)-f(t_{i-1})\|_{\mathcal{J}} &= \|(t_i,t_i)-(t_{i-1},t_{i-1})\|_{\mathcal{J}}\\ &= \|(t_i-t_{i-1},t_i-t_{i-1})\|_{\mathcal{J}}\\ &= \sqrt{[(t_i-t_{i-1},t_i-t_{i-1}),(t_i-t_{i-1},t_i-t_{i-1})]_{\mathcal{J}}}\\ &= \sqrt{(t_i,t_{i-1})^2+(t_i,t_{i-1})^2}\\ &= \sqrt{2(t_i,t_{i-1})^2} = \sqrt{2}\sqrt{(t_i,t_{i-1})^2} = \sqrt{2}|t_i-t_{i-1}| = \sqrt{2}(t_i-t_{i-1}).\end{aligned}$$

Therefore,

$$\sum_{i=1}^{n}\|f(t_i)-f(t_{i-1})\|_{\mathcal{J}} = \sum_{i=1}^{n}\sqrt{2}(t_i-t_{i-1}) = \sqrt{2}\sum_{i=1}^{n}(t_i-t_{i-1}) = \sqrt{2}(b-a) < \infty.$$

The set $A = \{\sum_{i=1}^{n}\|f(t_i)-f(t_{i-1})\|_{\mathcal{J}} : P\in\mathcal{P}[a,b]\}$ is upper bounded by $\sqrt{2}(b-a)$. Therefore,

$$V_a^b(f,\mathbb{R}^2) = \sup\left\{\sum_{i=1}^{n}\|f(t_i)-f(t_{i-1})\|_{\mathcal{J}} : P\in\mathcal{P}[a,b]\right\} \leqslant \sqrt{2}(b-a) < \infty.$$

This guarantees that f is of bounded variation in Hilbert space $(\mathbb{R}^2,[\cdot,\cdot]_{\mathcal{J}})$ associated with the Krein space $(\mathbb{R}^2,[\cdot,\cdot])$.

With the following result we show that the strongly bounded variation functions in the Hilbert spaces associated with a Krein space are independent of the fundamental decomposition of the Krein space.

Theorem 3.2. *Let* $(\mathcal{K}, [\cdot,\cdot])$ *be a Krein space and let* $\mathcal{K} = \mathcal{K}_{+1}[\dot{+}]\mathcal{K}_{-1}$, $\mathcal{K} = \mathcal{K}_{+2}[\dot{+}]\mathcal{K}_{-2}$ *be two fundamental decompositions. If* $\mathcal{J}_1$ *and* $\mathcal{J}_2$ *are the fundamental symmetries, respectively, and if* $f : [a,b] \to \mathcal{K}$ *is strongly of bounded variation in Hilbert space* $(\mathcal{K}, [\cdot,\cdot]_{\mathcal{J}_1})$, *then* f *is strongly of bounded variation in Hilbert space* $(\mathcal{K}, [\cdot,\cdot]_{\mathcal{J}_2})$.

Proof. If f is strongly of bounded variation, then there exists $M > 0$ such that

$$V_a^b(f, \mathcal{K}, \mathcal{J}_1) = \sup\left\{\sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\|_{\mathcal{J}_1} : P \in \mathcal{P}[a,b]\right\} \leqslant M.$$

By Theorem 2.12, $\|\cdot\|_{\mathcal{J}_1}$ and $\|\cdot\|_{\mathcal{J}_2}$ are equivalent norms, then there exists $\beta > 0$ such that

$$\|f(t_i) - f(t_{i-1})\|_{\mathcal{J}_2} \leqslant \beta\|f(t_i) - f(t_{i-1})\|_{\mathcal{J}_1}.$$

Therefore,

$$\sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\|_{\mathcal{J}_2} \leqslant \beta \sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\|_{\mathcal{J}_1} \leqslant \beta M.$$

Then,

$$V_a^b(f, \mathcal{K}, \mathcal{J}_2) = \sup\left\{\sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\|_{\mathcal{J}_2} : P \in \mathcal{P}[a,b]\right\} \leqslant \beta M.$$

Thus the function f is strongly of bounded variation in $[a,b]$ with respect to $\mathcal{J}_2$. □

Considering the above result, we conclude that strongly of bounded variation functions are independent of the fundamental decomposition into Krein spaces. Therefore, from now on we will note $V_a^b(f, \mathcal{K}, [\cdot,\cdot]_{\mathcal{J}})$.

With the following result we guarantee the existence of functions of bounded variation in spaces with indefinite metric.

Theorem 3.3. *Let* $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot], \mathcal{J})$ *be a Krein space and* $f_+ : [a,b] \to \mathcal{K}_+$, $f_- : [a,b] \to \mathcal{K}_-$ *strongly of bounded variation in the Hilbert spaces* $(\mathcal{K}_+, [\cdot,\cdot])$ *and* $(\mathcal{K}_-, -[\cdot,\cdot])$, *respectively, then* $f : [a,b] \to \mathcal{K}$ *defined as* $f(t) = f_+(t) + f_-(t)$ *is strongly of bounded variation in the Hilbert space* $(\mathcal{K}, [\cdot,\cdot]_{\mathcal{J}})$.

Proof. Consider the partition $P = \{a, t_1, t_2, \ldots, t_{i-1}, t_i, \ldots, t_{n-1}, b\} \in \mathcal{P}[a,b]$. Let f_+ and f_- strongly of bounded variation in $(\mathcal{K}_+, [\cdot,\cdot])$ and $(\mathcal{K}_-, -[\cdot,\cdot])$, respectively, then there exist $M^+, M^- > 0$ such that

$$\overset{+}{V}{}_a^b(f_+, \mathcal{K}_+) = \sup\left\{\sum_{i=1}^{n} \|f_+(t_i) - f_+(t_{i-1})\|_+ : P \in \mathcal{P}[a,b]\right\} \leqslant M^+$$

and

$$\overset{-}{V}{}_a^b(f_-, \mathcal{K}_-) = \sup\left\{\sum_{i=1}^{n} \|f_-(t_i) - f_-(t_{i-1})\|_- : P \in \mathcal{P}[a,b]\right\} \leqslant M^-.$$

Let $f : [a,b] \to \mathcal{K}$ defined as $f(t) = f_+(t) + f_-(t)$ for any $t \in [a,b]$,

$$\begin{aligned}\|f(t_i) - f(t_{i-1})\|_{\mathcal{J}} &= \|(f_+(t_i) + f_-(t_i)) - (f_+(t_{i-1}) + f_-(t_{i-1}))\|_{\mathcal{J}} \\ &= \|(f_+(t_i) - f_+(t_{i-1})) + (f_-(t_i) - f_-(t_{i-1}))\|_{\mathcal{J}} \\ &\leqslant \|f_+(t_i) - f_+(t_{i-1})\|_+ + \|f_-(t_i) - f_-(t_{i-1})\|_-.\end{aligned}$$

Therefore,

$$\sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\|_{\mathcal{J}} \leqslant \sum_{i=1}^{n} \|f_+(t_i) - f_+(t_{i-1})\|_+ + \sum_{i=1}^{n} \|f_-(t_i) - f_-(t_{i-1})\|_-.$$

From which we obtain

$$V_a^b(f,(\mathcal{K},[\cdot,\cdot])) \leqslant V_a^b(f_+,\mathcal{K}_+) + V_a^b(f_-,\mathcal{K}_-) \leqslant M^+ + M^- = M < \infty,$$

where $M^+ + M^- = M$. Then,

$$V_a^b(f,\mathcal{K}) = \sup\left\{\sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\|_{\mathcal{J}} : P \in \mathcal{P}[a,b]\right\} \leqslant M.$$

□

4. Functions of bounded variation in Krein spaces

In this section we introduce the notion of strongly bounded variation functions in Krein spaces (4.3), which generalizes the notion of a bounded variation function given in [15], furthermore prove some properties of such functions.

Remark 4.1. Let $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot], \mathcal{J})$ be a Krein space, and $f : [a,b] \to \mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$. Considering that for any t in $[a,b]$, $f(t)$ belongs to $\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$, we will henceforth write the image of t under f as $f(t) = f^+(t) + f^-(t)$, for any t in $[a,b]$.

Remark 4.2. Let $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot], \mathcal{J})$ be a Krein space, and $f : [a,b] \to \mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$. Considering that for any t in $[a,b]$, $f(t) \in \mathcal{K}$, therefore there exist $k^+ \in \mathcal{K}_+$ and $k^- \in \mathcal{K}_-$ such that $f(t) = k^+ + k^-$, we will write $k^+ = f^+(t)$ and $k^- = f^-(t)$. Therefore,

$$(\mathcal{J}f)(t) = \mathcal{J}(f(t)) = \mathcal{J}(k^+ + k^-) = k^+ - k^- = f^+(t) - f^-(t).$$

Definition 4.3. Let $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$ be a Krein space and let f defined in $[a,b]$, we will say that f is strongly of bounded variation in $[a,b]$ on $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-)$ if

$$V_a^b(f,(\mathcal{K},[\cdot,\cdot])) = \sup\left\{\sum_{i=1}^{n} \left(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-\right)\right\}$$

for all P in $\mathcal{P}[a,b]$ is finite.

Remark 4.4. The set of all functions strongly of bounded variation in $[a,b]$ on $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$ is denoted as $BV([a,b],\mathcal{K},[\cdot,\cdot])$.

Next, we show an example of a strongly of bounded variation function in a Krein space.

Example 4.5. Consider $(\mathbb{C}^2, [\cdot,\cdot])$ with indefinite inner product $[\cdot,\cdot] : \mathbb{C}^2 \times \mathbb{C}^2 \longrightarrow \mathbb{C}$ given by

$$[(x,y),(w,z)] := x\overline{w} - y\overline{z}, \quad (x,y),(w,z) \in \mathbb{C}^2,$$

Is a Krein space, with $\mathcal{K}_+ := \{(x,0) | x \in \mathbb{C}\}$ and $\mathcal{K}_- := \{(0,y) | y \in \mathbb{C}\}$, $\mathcal{J}(x,y) = (x,-y)$, for all $(x,y) \in \mathbb{C}^2$,

$$\begin{aligned}
\|(x,y)\|_{\mathcal{J}} &= \sqrt{[(x,y),(x,y)]_{\mathcal{J}}} = \sqrt{[\mathcal{J}(x,y),(x,y)]} = \sqrt{[(x,-y),(x,y)]} = \sqrt{|x|^2 + |y|^2},\\
\|(x,0)\|_+ &= \sqrt{[(x,0),(x,0)]} = \sqrt{|x|^2} = |x|,\\
\|(0,y)\|_- &= \sqrt{[(0,y),(0,y)]} = \sqrt{-y(-\overline{y})} = \sqrt{|y|^2} = |y|.
\end{aligned}$$

Consider $f : [1,2] \to \mathbb{C}^2$ defined by $f(t) = (t,-t)$. Let us show that f is strongly of bounded variation. In fact, let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[1,2]$. Note that $f(t) = (t,-t) = (t,0) + (0,-t)$, where $(t,0) \in \mathcal{K}_+$ and $(0,-t) \in \mathcal{K}_-$, also $f^+(t) = (t,0)$ and $f^-(t) = (0,-t)$. Therefore,

$$f^+(t_i) = (t_i,0), \quad f^-(t) = (0,-t_i), \quad f^+(t_{i-1}) = (t_{i-1},0), \quad \text{and} \quad f^-(t_{i-1}) = (0,-t_{i-1}).$$

Then,

$$\|f^+(t_i) - f^+(t_{i-1})\|_+ = \|(t_i - t_{i-1}, 0)\|_+ = |t_i - t_{i-1}| = t_i - t_{i-1},$$
$$\|f^-(t_i) - f^-(t_{i-1})\|_- = \|(0, t_{i-1} - t_i)\|_- = |t_{i-1} - t_i| = t_i - t_{i-1}.$$

Thus,

$$\sum_{i=1}^{n} (\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-) = \sum_{i=1}^{n} (2(t_i - t_{i-1})) = 2(t_n - t_0) = 2.$$

Therefore, we have to $V_1^2(f, (\mathbb{C}^2, [\cdot, \cdot]))$ is bounded, thus it follows that f is strongly of bounded variation.

Definition 4.6 (Positive and negative variations of functions in Krein spaces)**.** Let $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot, \cdot], \mathcal{J})$ be a Krein space, and $f : [a, b] \to \mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$, with $f(t) = f^+(t) + f^-(t)$, for all t in the interval $[a, b]$. The positive and negative variation of f on $[a, b]$ with respect to $(\mathcal{K}_+, [\cdot, \cdot])$ and $(\mathcal{K}_-, -[\cdot, \cdot])$, respectively, are defined by

$$\overset{+}{V}{}_a^b(f, (\mathcal{K}_+, [\cdot, \cdot])) = \sup \left\{ \sum_{i=1}^{n} \|f^+(t_i) - f^+(t_{i-1})\|_+ : P \in \mathcal{P}[a, b] \right\}$$

and

$$\overset{-}{V}{}_a^b(f, (\mathcal{K}_-, -[\cdot, \cdot])) = \sup \left\{ \sum_{i=1}^{n} \|f^-(t_i) - f^-(t_{i-1})\|_- : P \in \mathcal{P}[a, b] \right\}.$$

Remark 4.7. Note that $\overset{+}{V}{}_a^b(f, (\mathcal{K}_+, [\cdot, \cdot])) \geqslant 0$, $\overset{-}{V}{}_a^b(f, (\mathcal{K}_-, -[\cdot, \cdot])) \geqslant 0$, and $V_a^b(f, (\mathcal{K}, [\cdot, \cdot])) \geqslant 0$.

Theorem 4.8. *Let $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot, \cdot])$ be a Krein space and $f : [a, b] \to \mathcal{K}$ be strongly of bounded variation on $[a, b]$ in $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-)$, then f is strongly of bounded variation in the Hilbert spaces $(\mathcal{K}^+, [\cdot, \cdot])$ and $(\mathcal{K}^-, -[\cdot, \cdot])$.*

Proof. Consider the partition $P = \{a, t_1, t_2, \ldots, t_{i-1}, t_i, \ldots, t_{n-1}, b\} \in \mathcal{P}[a, b]$. The proof is a consequence of

$$\|f^-(t_i) - f^-(t_{i-1})\|_-, \|f^+(t_i) - f^+(t_{i-1})\|_+ \leqslant \|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-.$$

□

The following result shows that every strongly of bounded variation function on a Krein space is strongly of bounded variation function in the Hilbert space associated.

Theorem 4.9. *Let $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot, \cdot])$ be a Krein space and $f : [a, b] \to \mathcal{K}$ strongly of bounded variation function in $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot, \cdot])$, then f is strongly of bounded variation function in the Hilbert space $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{J}})$.*

Proof. If f is strongly of bounded variation in $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot, \cdot])$, then, there exists $M > 0$ such that

$$V_a^b(f, (\mathcal{K}, [\cdot, \cdot])) = \sup \left\{ \sum_{i=1}^{n} (\|f^+(t_i) - f^+(t_{i-1})\| + \|f^-(t_i) - f^-(t_{i-1})\|) : P \in \mathcal{P}[a, b] \right\} \leqslant M.$$

Since,

$$\begin{aligned} \|f(t_i) - f(t_{i-1})\|_{\mathcal{J}} &= \|(f_+(t_i) + f_-(t_i)) - (f_+(t_{i-1}) + f_-(t_{i-1}))\|_{\mathcal{J}} \\ &= \|(f_+(t_i) - f_+(t_{i-1})) + (f_-(t_i) - f_-(t_{i-1}))\|_{\mathcal{J}} \\ &\leqslant \|f_+(t_i) - f_+(t_{i-1})\|_+ + \|f_-(t_i) - f_-(t_{i-1})\|_-, \end{aligned}$$

therefore, it follows that

$$\sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\|_{\mathcal{J}} \leqslant \sum_{i=1}^{n} (\|f^+(t_i) - f^+(t_{i-1})\|_+ \|f^-(t_i) - f^-(t_{i-1})\|_-).$$

Then,

$$V_a^b(f,\mathcal{K},\mathcal{J}) = \sup\left\{\sum_{i=1}^{n} \|f(t_i)-f(t_{i-1})\|_{\mathcal{J}} : P\in\mathcal{P}[a,b]\right\}$$
$$\leqslant \sup\left\{\sum_{i=1}^{n} \|f^+(t_i)-f^+(t_{i-1})\|_+ + \|f^-(t_i)-f^-(t_{i-1})\|_- : P\in\mathcal{P}[a,b]\right\} \leqslant M<\infty.$$

Therefore, f is strongly of bounded variation in the Hilbert space $(\mathcal{K},[\cdot,\cdot]_{\mathcal{J}})$. □

Remark 4.10. Note that this result is obtained using Theorem 2.9 and taking into account Remark 2.11, it follows that in general the reciprocal of Theorem 4.9 is not true.

Theorem 4.11. *Let* $(\mathcal{K}=\mathcal{K}_+[\dot{+}]\mathcal{K}_-,[\cdot,\cdot])$ *be a Krein space and* $f:[a,b]\to\mathcal{K}=\mathcal{K}_+[\dot{+}]\mathcal{K}_-$ *a function,* $f\in BV([a,b],\mathcal{K}=\mathcal{K}_+[\dot{+}]\mathcal{K}_-)$*, then* $\overset{+}{V}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot]))=0=\overset{-}{V}{}_a^b(f,(\mathcal{K}_-,-[\cdot,\cdot]))$ *if and only if* f *is constant in* $[a,b]$ *with respect to* $(\mathcal{K}=\mathcal{K}_+[\dot{+}]\mathcal{K}_-)$.

Proof.

(→) Suppose that $\overset{+}{V}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot]))=0=\overset{-}{V}{}_a^b(f,(\mathcal{K}_-,-[\cdot,\cdot]))$, that is,

$$\overset{+}{V}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot])) = \sup\left\{\sum_{i=1}^{n} \|f^+(t_i)-f^+(t_{i-1})\|_+ : P\in\mathcal{P}[a,b]\right\}=0$$

and

$$\overset{-}{V}{}_a^b(f,(\mathcal{K}_-,-[\cdot,\cdot])) = \sup\left\{\sum_{i=1}^{n} \|f^-(t_i)-f^-(t_{i-1})\|_- : P\in\mathcal{P}[a,b]\right\}=0.$$

Then,

$$\sum_{i=1}^{n} \|f^+(t_i)-f^+(t_{i-1})\|_+ = 0 \quad \text{and} \quad \sum_{i=1}^{n} \|f^-(t_i)-f^-(t_{i-1})\|_- = 0.$$

Let $x\in[a,b]$. Then, in particular, for the partition $P=\{a,x,b\}$, we have that

$$\|f^+(x)-f^+(a)\|_+ + \|f^+(b)-f^+(x)\|_+ = 0 \quad \text{and} \quad \|f^-(x)-f^-(a)\|_- + \|f^-(b)-f^-(x)\|_- = 0.$$

Thus, $\big(\|f^+(x)-f^+(a)\|_+=0, \|f^+(b)-f^+(x)\|_+=0\big)$ and $\big(\|f^-(x)-f^-(a)\|_-=0, \|f^-(b)-f^-(x)\|_-=0\big)$. Therefore, $\big(f^+(x)-f^+(a)=\mathbf{0},\ f^+(b)-f^+(x)=\mathbf{0}\big)$ and $\big(f^-(x)-f^-(a)=\mathbf{0}, f^-(b)-f^-(x)=\mathbf{0}\big)$. Next, $\big(f^+(x)=f^+(a), f^+(b)=f^+(x)\big)$ and $\big(f^-(x)=f^-(a), f^-(b)=f^-(x)\big)$ follows $f^+(x)=f^+(a)=f^+(b)$ and $f^-(x)=f^-(a)=f^-(b)$. Therefore, $f(x)=f^+(x)+f^-(x)=f^+(a)+f^-(a)=f^+(b)+f^-(b)$. Thus, f is constant in the interval $[a,b]$.

(←) Suppose that f is constant in $[a,b]$, then there exists $c=(c^++c^-)\in(\mathcal{K}=\mathcal{K}_+[\dot{+}]\mathcal{K}_-)$ such that for any $x\in[a,b]$, $f(x)=f^+(x)+f^-(x)=c^++c^-$, then $f^+(x)-c^+=c^--f^-(x)$. Thus, $f^+(x)-c^+=\mathbf{0}$ and $c^--f^-(x)=\mathbf{0}$, therefore, $f^+(x)=c^+$ and $c^-=f^-(x)$. Thus,

$$\overset{+}{V}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot])) = \sup\left\{\sum_{i=1}^{n} \|f^+(t_i)-f^+(t_{i-1})\|_+ : P\in\mathcal{P}[a,b]\right\}$$
$$= \sup\left\{\sum_{i=1}^{n} \|c^+-c^+\|_+ : P\in\mathcal{P}[a,b]\right\} = \sup\{0\}=0$$

and

$$\bar{V}_a^b(f,(\mathcal{K}_-,-[\cdot,\cdot])) = \sup\left\{\sum_{i=1}^{n} \|f^-(t_i)-f^-(t_{i-1})\|_- : P\in\mathcal{P}[a,b]\right\}$$
$$= \sup\left\{\sum_{i=1}^{n} \|c^- - c^-\|_- : P\in\mathcal{P}[a,b]\right\} = \sup\{0\} = 0.$$

□

Theorem 4.12. *Let* $(\mathcal{K}=\mathcal{K}_+[\dot{+}]\mathcal{K}_-,[\cdot,\cdot])$ *be a Krein space, if* $f:[a,b]\to\mathcal{K}$ *is strongly of bounded variation in* $(\mathcal{K}=\mathcal{K}_+[\dot{+}]\mathcal{K}_-,[\cdot,\cdot])$*, then* $\mathcal{J}f$ *is strongly of bounded variation in* $(\mathcal{K}=\mathcal{K}_+[\dot{+}]\mathcal{K}_-,[\cdot,\cdot])$ *and* $(\mathcal{K},[\cdot,\cdot]_{\mathcal{J}})$*.*

Proof. If f is strongly of bounded variation in $(\mathcal{K}=\mathcal{K}_+[\dot{+}]\mathcal{K}_-,[\cdot,\cdot])$, then, there exists $M>0$ such that

$$V_a^b(f,(\mathcal{K},[\cdot,\cdot])) = \sup\left\{\sum_{i=1}^{n}(\|f^+(t_i)-f^+(t_{i-1}\|+\|f^-(t_i)-f^-(t_{i-1}\|) : P\in\mathcal{P}[a,b]\right\} \leqslant M.$$

Let $P=\{a,t_1,t_2,\dots,t_{n-1},b\}$ be a partition of $[a,b]$, then

$$\|(\mathcal{J}f)^+(t_i)-(\mathcal{J}f)^+(t_{i-1})\|_+ = \|\mathcal{J}f^+(t_i)-\mathcal{J}f^+(t_{i-1})\|_+ = \|f^+(t_i)-f^+(t_{i-1})\|_+$$

and

$$\begin{aligned}\|(\mathcal{J}f)^-(t_i)-(\mathcal{J}f)^-(t_{i-1})\|_- &= \|\mathcal{J}f^-(t_i)-(\mathcal{J}f^-(t_{i-1}))\|_- \\ &= \|-f^-(t_i)-(-f^-(t_{i-1}))\|_- \\ &= \|-(f^-(t_i)-f^-(t_{i-1}))\|_- = \|f^-(t_i)-f^-(t_{i-1}))\|_-.\end{aligned}$$

Then,

$$\|f^+(t_i)-f^+(t_{i-1}\|_- + \|f^-(t_i)-f^-(t_{i-1}\|_- = \|(\mathcal{J}f)^+(t_i)-(\mathcal{J}f)^+(t_{i-1})\|_+ + \|(\mathcal{J}f)^-(t_i)-(\mathcal{J}f)^-(t_{i-1})\|_-.$$

Thus,

$$V_a^b(\mathcal{J}f,\mathcal{K}) = \sup\left\{\sum_{i=1}^{n}(\|\mathcal{J}f^+(t_i)-\mathcal{J}f^+(t_{i-1}\|+\|\mathcal{J}f^-(t_i)-\mathcal{J}f^-(t_{i-1}\|) : P\in\mathcal{P}[a,b]\right\} \leqslant M.$$

Therefore, $\mathcal{J}f$ is strongly of bounded variation in $[a,b]$ on $(\mathcal{K}=\mathcal{K}_+[\dot{+}]\mathcal{K}_-,[\cdot,\cdot])$ and, consequently, by Theorem [4.9] is strongly of bounded variation in the Hilbert space $(\mathcal{K},[\cdot,\cdot]_{\mathcal{J}})$. □

Remark 4.13. The reciprocal of the previous result is true, since if $\mathcal{J}f$ is strongly of bounded variation in $[a,b]$ on $(\mathcal{K}=\mathcal{K}_+[\dot{+}]\mathcal{K}_-,[\cdot,\cdot])$, then $\mathcal{J}\mathcal{J}f=\mathcal{I}f=f$ is strongly of bounded variation in $[a,b]$ on $(\mathcal{K}=\mathcal{K}_+[\dot{+}]\mathcal{K}_-,[\cdot,\cdot])$.

The following result shows that in Krein spaces, the set of strongly of bounded variation functions is a subset of the bounded functions.

Theorem 4.14. *Let* $(\mathcal{K}=\mathcal{K}_+[\dot{+}]\mathcal{K}_-,[\cdot,\cdot])$ *be a Krein space and* $f:[a,b]\to\mathcal{K}$ *strongly of bounded variation in* $[a,b]$ *on* $(\mathcal{K}=\mathcal{K}_+[\dot{+}]\mathcal{K}_-)$*, then* f *is bounded.*

Proof. If f is strongly of bounded variation in $(\mathcal{K}=\mathcal{K}_+[\dot{+}]\mathcal{K}_-,[\cdot,\cdot])$, then there exists $M>0$ such that

$$V_a^b(f,(\mathcal{K},[\cdot,\cdot])) \leqslant M.$$

Therefore, $\overset{+}{V}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot])) \leqslant V_a^b(f,(\mathcal{K},[\cdot,\cdot])) \leqslant M$ and $\overset{-}{V}{}_a^b(f,(\mathcal{K}_-,-[\cdot,\cdot])) \leqslant V_a^b(f,(\mathcal{K},[\cdot,\cdot])) \leqslant M$. Let $x \in [a,b]$, consider the partition $P = \{a,x,b\} \in \mathcal{P}[a,b]$. In particular,

$$\sum_{i=1}^{3} \|f^+(t_i) - f^+(t_{i-1})\|_+ \leqslant M \quad \text{and} \quad \sum_{i=1}^{3} \|f^-(t_i) - f^-(t_{i-1})\|_- \leqslant M,$$

therefore,

$$\|f^+(x) - f^+(a)\|_+ + \|f^+(b) - f^+(x)\|_+ \leqslant M \quad \text{and} \quad \|f^-(x) - f^-(a)\|_- + \|f^-(b) - f^-(x)\|_- \leqslant M.$$

Then,

$$\begin{aligned}
\|f(x)\|_{\mathcal{J}} &= \|f^+(x) + f^-(x)\|_{\mathcal{J}} \\
&\leqslant \|f^+(x)\|_+ + \|f^-(x)\|_- \\
&= \|(f^+(x) - f^+(a)) + f^+(a)\|_+ + \|(f^-(x) - f^-(b)) + f^-(b)\|_- \\
&\leqslant \|f^+(x) - f^+(a)\|_+ + \|f^+(a)\|_+ + \|f^-(x) - f^-(b)\|_- + \|f^-(b)\|_- \\
&\leqslant M + \|f^+(a)\|_+ + M + \|f^-(b)\|_- \leqslant 2M + \|f^+(a)\|_+ + \|f^-(b)\|_-.
\end{aligned}$$

Therefore, f is bounded in $[a,b]$. □

Remark 4.15. We see that if $f : [a,b] \to \mathcal{K}$ is strongly of bounded variation in $[a,b]$ on $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-)$, then f is bounded. Reciprocal is not always true.

Example 4.16. Consider $(\mathbb{R}^2,[\cdot,\cdot])$ with indefinite inner product $[\cdot,\cdot] : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$[(x_1,y_1),(x_2,y_2)] := x_1x_2 - y_1y_2, \quad (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$$

is a Krein space [5], with $\mathcal{K}_+ := \{(x,0)|x \in \mathbb{R}\}$ and $\mathcal{K}_- := \{(0,y)|y \in \mathbb{R}\}$, $\mathcal{J}(x,y) = (x,-y)$, for all $(x,y) \in \mathbb{R}^2$, $\|(x,y)\|_{\mathcal{J}} = \sqrt{[(x,y),(x,y)]_{\mathcal{J}}} = \sqrt{x^2+y^2}$,

$$\|(x,0)\|_+ = \sqrt{[(x,0),(x,0)]} = \sqrt{x^2} = |x|, \quad \|(0,y)\|_- = \sqrt{[(0,y),(0,y)]} = \sqrt{-y(-y)} = \sqrt{y^2} = |y|.$$

Consider $f : [\sqrt{2},3] \to \mathbb{R}^2$ defined by

$$f(t) = \begin{cases} (1,1), & \text{if } t \text{ is rational, } t \in [\sqrt{2},3], \\ (0,0), & \text{if } t \text{ is irrational, } t \in [\sqrt{2},3]. \end{cases}$$

We will prove that f is bounded.

Proof. Let's see that f is bounded on $[\sqrt{2},3]$. In fact,

$$\|f(t)\|_{\mathcal{J}} = \begin{cases} \sqrt{2}, & \text{if } t \text{ is rational, } t \in [\sqrt{2},3], \\ 0, & \text{if } t \text{ is irrational, } t \in [\sqrt{2},3]. \end{cases}$$

Therefore, $\|f(t)\|_{\mathcal{J}} \leqslant \sqrt{2}$ for all $t \in [\sqrt{2},3]$. Thus f is bounded. Now, let's see that f is not of bounded variation. Let $t_0 = \sqrt{2}$, as between any two reals there is a rational number and an irrational number, we can choose t_1 as a rational number between $\sqrt{2}$ and 3, t_2 as an irrational number between t_1 and 3, t_3 as a rational number between t_2 and 3, and so on t_{2i} would be an irrational number between t_{2i-1} and 3, and t_{2i+1} would be a rational number between t_{2i} and 3, finally we choose $t_n = 3$. Then,

$$V_{\sqrt{2}}^3(f,(\mathbb{R}^2,[\cdot,\cdot])) \geqslant \sum_{i=1}^{n} \left(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-\right)$$

$$
\begin{aligned}
&= (\|-(1,1)\|_+ + \|(1,1)\|_-) + \cdots + (\|-(1,1)\|_+ + \|(1,1)\|_-) \\
&= (\|(1,1)\|_+ + \|(1,1)\|_-) + \cdots + (\|(1,1)\|_+ + \|(1,1)\|_-) \\
&\geqslant \|(1,1)\|_{\mathcal{J}} + \cdots + \|(1,1)\|_{\mathcal{J}} = \sqrt{2} + \cdots + \sqrt{2} = \sqrt{2} \cdot n.
\end{aligned}
$$

□

Note that a partition of the interval $[\sqrt{2}, 3]$ was constructed, starting at $\sqrt{2}$, then alternating between rational and irrational numbers until it ends at 3, for which $V_{\sqrt{2}}^{3}(f, (\mathbb{R}^2, [\cdot,\cdot]))$ is not finite, thus f is not strongly of bounded variation in $(\mathbb{R}^2, [\cdot,\cdot])$.

4.1. Algebra of strongly of bounded variation functions in Krein spaces

Theorem 4.17. *Let* $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$ *be a Krein space, if* $f, g : [a, b] \to \mathcal{K}$ *be strongly of bounded variation functions in* $[a, b]$ *on* $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-)$ *and* α *be scalar, then* αf *and* $f + g$ *are also strongly of bounded variation functions in* $[a, b]$ *on* $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-)$*.*

Proof.

(i) If f is strongly of bounded variation function in $[a, b]$ on $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-)$, then there exists $M > 0$ such that $V_a^b(f, (\mathcal{K}, [\cdot,\cdot])) \leqslant M$. Then,

$$
\begin{aligned}
&\sup\left\{\sum_{i=1}^{n} \|\alpha f^+(t_i) - \alpha f^+(t_{i-1})\|_+ + \|\alpha f^-(t_i) - \alpha f^-(t_{i-1})\|_- : P \in \mathcal{P}[a,b]\right\} \\
&\quad = \sup\left\{\sum_{i=1}^{n} \|\alpha\big(f^+(t_i) - f^+(t_{i-1})\big)\|_+ + \|\alpha\big(f^-(t_i) - f^-(t_{i-1})\big)\|_- : P \in \mathcal{P}[a,b]\right\} \\
&\quad = \sup\left\{\sum_{i=1}^{n} |\alpha|\|f^+(t_i) - f^+(t_{i-1})\|_+ + |\alpha|\|f^-(t_i) - f^-(t_{i-1})\|_- : P \in \mathcal{P}[a,b]\right\} \\
&\quad = \sup\left\{|\alpha|\sum_{i=1}^{n} \|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_- : P \in \mathcal{P}[a,b]\right\} \\
&\quad = |\alpha| \sup\left\{\sum_{i=1}^{n} \|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_- : P \in \mathcal{P}[a,b]\right\} \\
&\quad = |\alpha| V_a^b(f, (\mathcal{K}, [\cdot,\cdot])) \leqslant |\alpha| M = L.
\end{aligned}
$$

Thus, $V_a^b(\alpha f, (\mathcal{K}, [\cdot,\cdot])) \leqslant L$. Therefore, αf is strongly of bounded variation function in $[a, b]$ on $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$.

(ii) If f and g are strongly of bounded variation functions in $[a, b]$ on $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-)$, then there are $M_1, M_2 > 0$ such that $V_a^b(f, (\mathcal{K}, [\cdot,\cdot])) \leqslant M_1$ and $V_a^b(g, (\mathcal{K}, [\cdot,\cdot])) \leqslant M_2$. The result is obtained by considering that $(f + g)^+(x) = f^+(x) + g^+(x)$, $(f + g)^-(x) = f^-(x) + g^-(x)$, the triangular inequality, and that the supremum of a sum of sets is the sum of the supremums whenever they exist. Thus we obtain that

$$V_a^b((f + g), (\mathcal{K}, [\cdot,\cdot])) \leqslant V_a^b(f, (\mathcal{K}, [\cdot,\cdot])) + V_a^b(g, (\mathcal{K}, [\cdot,\cdot])) \leqslant M_1 + M_2 = M.$$

Therefore, $f + g$ is strongly of bounded variation in $[a, b]$ on the space $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$. □

Theorem 4.18. *Let* $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$ *be a Krein space and let* $f : [a, b] \to \mathcal{K}$ *a strongly of bounded variation function in* $[a, b]$ *on* $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-)$*, suppose that* $c \in (a, b)$*, then f is strongly of bounded variation in* $[a, c]$ *and in* $[c, b]$ *on* $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-)$*. In this case we have* $V_a^b(f, (\mathcal{K}, [\cdot,\cdot])) = V_a^c(f, \mathcal{K}) + V_c^b(f, \mathcal{K})$*.*

Proof. If f is strongly of bounded variation in $[a, b]$ on $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-)$, then, there exists $M > 0$ such that

$$\overset{+}{V}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot])) \leqslant V_a^b(f,(\mathcal{K},[\cdot,\cdot])) \leqslant M \quad \text{and} \quad \overset{-}{V}{}_a^b(f,(\mathcal{K}_-,-[\cdot,\cdot])) \leqslant V_a^b(f,(\mathcal{K},[\cdot,\cdot])) \leqslant M.$$

Consider the partition $P = \{a, t_1, t_2, t_3, \ldots, t_{n-1}, b\}$. Since $c \in (a, b)$, then

$$P^* = \{a, t_1, t_2, t_3, \ldots, t_{m-1}, c = t_m, t_{m+1}, \ldots, t_{n-1}, b\}$$

is also a partition for $[a, b]$. Then,

$$\begin{aligned} M &\geqslant \sum_{i=1}^{n} \left(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-\right) \\ &= \sum_{i=1}^{m} \left(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-\right) \\ &\quad + \sum_{i=m}^{n} \left(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-\right) \\ &\geqslant \sum_{i=1}^{m} \left(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-\right). \end{aligned}$$

Therefore,

$$\sup\left\{\sum_{i=1}^{m} \left(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-\right) : P \in \mathcal{P}[a, c]\right\} \leqslant M,$$

which implies that $V_a^c(f,(\mathcal{K},[\cdot,\cdot])) \leqslant M$. Furthermore, it is similarly obtained that

$$\sup\left\{\sum_{i=m}^{n} \left(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-\right) : P \in \mathcal{P}[c, b]\right\} \leqslant M,$$

which implies that $V_c^b(f,(\mathcal{K},[\cdot,\cdot])) \leqslant M$. Therefore, f is strongly of bounded variation in $[a, c]$ and in $[c, b]$. Furthermore,

$$\begin{aligned} &\sup\left\{\sum_{i=1}^{m} \left(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-\right) : P \in \mathcal{P}[a, c]\right\} \\ &\quad + \sup\left\{\sum_{i=m}^{n} \left(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-\right) : P \in \mathcal{P}[c, b]\right\} \\ &= \sup\left\{\sum_{i=1}^{m} \left(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-\right) : P \in \mathcal{P}[a, c]\right. \\ &\quad \left. + \sum_{i=m}^{n} \left(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-\right) : P \in \mathcal{P}[c, b]\right\} \\ &= \sup\left\{\sum_{i=1}^{n} \left(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-\right) : P \in \mathcal{P}[a, b]\right\} V_a^b(f,(\mathcal{K},[\cdot,\cdot])). \end{aligned}$$

Therefore, $V_a^c(f,(\mathcal{K},[\cdot,\cdot])) + V_c^b(f,(\mathcal{K},[\cdot,\cdot])) = V_a^b(f,(\mathcal{K},[\cdot,\cdot]))$. □

5. Construction of increasing functions by variations in Krein spaces

In this section we wish to show how to construct increasing functions from the variation of a function in Krein spaces.

Theorem 5.1. *Let* $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot], \mathcal{J})$ *be a Krein space, and* $f : [a,b] \to \mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$ *be a strongly of bounded variation, then* V *defined by* $V : [a,b] \to \mathbb{R}$ *as* $V(x) = \overset{+}{V}{}_a^x(f,(\mathcal{K}^+,[\cdot,\cdot])) + \overset{-}{V}{}_a^x(f,(\mathcal{K}^-,-[\cdot,\cdot]))$ *if* $a < x \leqslant b$, $V(a) = 0$, *is increasing in* $[a,b]$.

Proof. Let $x_1, x_2 \in (a,b)$ such that $x_1 \leqslant x_2$, then

$$\overset{+}{V}{}_a^{x_1}(f,(\mathcal{K}_+,[\cdot,\cdot])) = \sup\left\{\sum_{i=1}^{n}\|f^+(t_i) - f^+(t_{i-1})\|_+ : P \in \mathcal{P}[a,x_1]\right\}$$

and

$$\overset{+}{V}{}_a^{x_2}(f,(\mathcal{K}_+,[\cdot,\cdot])) = \sup\left\{\sum_{i=1}^{n}\|f^+(t_i) - f^+(t_{i-1})\|_+ : P \in \mathcal{P}[a,x_2]\right\}.$$

Since $x_1 \leqslant x_2$, then $[a,x_1] \subseteq [a,x_2]$. Let $P_1 = \{a,t_1,t_2,\dots,t_{n-1},x_1\}$ be a partition for $[a,x_1]$. Therefore, $P_2 = \{a,t_1,t_2,\dots,t_{n-1},x_1,x_2\}$ is a partition for $[a,x_2]$. Then,

$$\begin{aligned}&\|f^+(t_1) - f^+(a)\|_+ + \|f^+(t_2) - f^+(t_1)\|_+ + \cdots + \|f^+(x_1) - f^+(t_{n-1})\|_+ \\ &\quad\leqslant \|f^+(t_1) - f^+(a)\|_+ + \|f^+(t_2) - f^+(t_1)\|_+ + \cdots + \|f^+(x_1) - f^+(t_{n-1})\|_+ + \|f^+(x_2) - f^+(x_1)\|_+.\end{aligned}$$

Therefore,

$$\sup\left\{\sum_{i=1}^{n}\|f^+(t_i) - f^+(t_{i-1})\|_+ : P_1 \in \mathcal{P}[a,x_1]\right\} \leqslant \sup\left\{\sum_{i=1}^{n}\|f^+(t_i) - f^+(t_{i-1})\|_+ : P_2 \in \mathcal{P}[a,x_2]\right\}.$$

Thus, $\overset{+}{V}{}_a^{x_1}(f,(\mathcal{K}_+,[\cdot,\cdot])) \leqslant \overset{+}{V}{}_a^{x_2}(f,(\mathcal{K}_+,[\cdot,\cdot]))$. For negative variations, the procedure is analogous and it is obtained that $\overset{-}{V}{}_a^{x_1}(f,(\mathcal{K}_-,-[\cdot,\cdot])) \leqslant \overset{-}{V}{}_a^{x_2}(f,(\mathcal{K}_-,-[\cdot,\cdot]))$. Then,

$$V(x_1) = \overset{+}{V}{}_a^{x_1}(f,(\mathcal{K}_+,[\cdot,\cdot])) + \overset{-}{V}{}_a^{x_1}(f,(\mathcal{K}_-,-[\cdot,\cdot])) \leqslant \overset{+}{V}{}_a^{x_2}(f,(\mathcal{K}_+,[\cdot,\cdot])) + \overset{-}{V}{}_a^{x_2}(f,(\mathcal{K}_-,-[\cdot,\cdot])) = V(x_2).$$

Therefore, $V : [a,b] \to \mathbb{R}$ is increasing function in $[a,b]$. □

Next we endow the set $(BV([a,b],(\mathcal{K},[\cdot,\cdot])))$ of strongly of bounded variation functions in Krein space with a norm.

Theorem 5.2. *Let* $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$ *be a Krein space, the function* $\|\cdot\| : BV([a,b],\mathcal{K},[\cdot,\cdot]) \to \mathbb{R}$ *defined by*

$$\|f\|_{BV([a,b],\mathcal{K})} = \|f^+(x)\|_+ + \|f^-(x)\|_- + \overset{+}{V}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot])) + \overset{-}{V}{}_a^b(f,(\mathcal{K}_-,-[\cdot,\cdot])),$$

is a norm in $BV([a,b],\mathcal{K},[\cdot,\cdot])$.

Proof. Let $f, g \in BV([a,b],\mathcal{K},[\cdot,\cdot])$, $\alpha \in \mathbb{R}$. If $x \in [a,b]$, then we have following.

(i) $\|f^+(x)\|_+ \geqslant 0, \|f^-(x)\|_- \geqslant 0$. Furthermore, $\overset{+}{V}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot])) \geqslant 0$ and $\overset{-}{V}{}_a^b(f,(\mathcal{K}_-,-[\cdot,\cdot])) \geqslant 0$. Therefore, $\|f\|_{BV([a,b],\mathcal{K})} \geqslant 0$.

(ii) If $\|f\|_{BV([a,b]),\mathcal{K}} = 0$, then

$$\overset{+}{V}{}_a^b(f, (\mathcal{K}_+, [\cdot,\cdot])) + \overset{-}{V}{}_a^b(f, (\mathcal{K}_-, -[\cdot,\cdot])) + \|f^+(x)\|_+ + \|f^-\|_- = 0.$$

Since $\overset{+}{V}{}_a^b(f, (\mathcal{K}_+, [\cdot,\cdot]))$, $\overset{-}{V}{}_a^b(f, (\mathcal{K}_-, -[\cdot,\cdot]))$, $\|f^+(x)\|_+$, $\|f^-\|_- \geqslant 0$, then for the Theorem 4.11 it is satisfied that

$$\overset{+}{V}{}_a^b(f, (\mathcal{K}_+, [\cdot,\cdot])) = \overset{-}{V}{}_a^b(f, (\mathcal{K}_-, -[\cdot,\cdot])) = \|f^+(x)\|_+ = \|f^-\|_- = 0.$$

Then, there exists $c = c^+ + c^- \in \mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$ such that $f(x) = c = c^+ + c^- = f^+(x) + f^-(x)$. Therefore, $c^+ - f^+(x) = 0$ and $c^- - f^-(x) = 0$, it implies that $f^+(x) = c^+$ and $f^-(x) = c^-$, respectively. Now,

$$\|f^+(x)\|_+ = \|c^+\|_+ = 0 \quad \text{and} \quad \|f^-(x)\|_- = \|c^-\|_- = 0,$$

it implies that $c^+ = 0$ and $c^- = 0$. Therefore, $f(x) = 0 + 0 = 0$ for all $x \in [a, b]$, then $f = \mathbf{0}$.

(iii) Since $f \in BV([a, b], \mathcal{K})$ and $\lambda \in BV([a, b], \mathcal{K})$, then

$$\begin{aligned}
\|\lambda f\|_{BV([a,b],\mathcal{K})} &= \|\lambda f^+(x)\|_+ + \|\lambda f^-(x)\|_- + \overset{+}{V}{}_a^b(\lambda f, (\mathcal{K}^+, [\cdot,\cdot])) + \overset{-}{V}{}_a^b(\lambda f, (\mathcal{K}^-, -[\cdot,\cdot])) \\
&= |\lambda|\|f^+(x)\|_+ + |\lambda|\|f^-(x)\|_- + |\lambda|\overset{+}{V}{}_a^b(f, (\mathcal{K}^+, [\cdot,\cdot])) + |\lambda|\overset{-}{V}{}_a^b(f, (\mathcal{K}^-, -[\cdot,\cdot])) \\
&= |\lambda|(\|f^+(x)\|_+ + \|f^-(x)\|_- + \overset{+}{V}{}_a^b(f, (\mathcal{K}^+, [\cdot,\cdot])) + \overset{-}{V}{}_a^b(f, (\mathcal{K}^-, -[\cdot,\cdot]))) = |\lambda|\|f\|_{BV([a,b],\mathcal{K})}.
\end{aligned}$$

(iv) Triangular inequality. Let $f, g \in BV([a, b], \mathcal{K})$, this implies that

$$\begin{aligned}
\|f+g\|_{BV([a,b],\mathcal{K})} &= \|(f+g)^+(x)\|_+ + \|(f+g)^-(x)\|_- + \overset{+}{V}{}_a^b(f+g, (\mathcal{K}_+, [\cdot,\cdot])) + \overset{-}{V}{}_a^b(f+g, (\mathcal{K}_-, -[\cdot,\cdot])) \\
&\leqslant \|f^+(x)\|_+ + \|g^+(x)\|_+ + \|f^-(x)\|_- + \|g^-(x)\|_- \\
&\quad + \overset{+}{V}{}_a^b(f, (\mathcal{K}_+, [\cdot,\cdot])) + \overset{+}{V}{}_a^b(g, (\mathcal{K}_+, [\cdot,\cdot])) + \overset{-}{V}{}_a^b(f, (\mathcal{K}_-, -[\cdot,\cdot])) + \overset{-}{V}{}_a^b(g, (\mathcal{K}_-, -[\cdot,\cdot])) \\
&= \|f^+(x)\|_+ + \|f^-(x)\|_- + \overset{+}{V}{}_a^b(f, (\mathcal{K}_+, [\cdot,\cdot])) \\
&\quad + \overset{-}{V}{}_a^b(f, (\mathcal{K}_-, -[\cdot,\cdot])) + \|g^+(x)\|_+ + \|g^-(x)\|_- + \overset{+}{V}{}_a^b(g, (\mathcal{K}_+, [\cdot,\cdot])) + \overset{-}{V}{}_a^b(g, (\mathcal{K}_-, -[\cdot,\cdot])) \\
&= \|f\|_{BV([a,b],\mathcal{K})} + \|g\|_{BV([a,b],\mathcal{K})}.
\end{aligned}$$

Thus,

$$\|f\|_{BV([a,b],\mathcal{K})} = \|f^+(x)\|_+ + \|f^-(x)\|_- + \overset{+}{V}{}_a^b(f, (\mathcal{K}_+, [\cdot,\cdot])) + \overset{-}{V}{}_a^b(f, (\mathcal{K}_-, -[\cdot,\cdot]))$$

is a norm in $BV([a, b], \mathcal{K}, [\cdot,\cdot])$. □

6. Conclusion

Every strongly of bounded variation functions in a Krein space preserves the property in the associated Hilbert spaces. The constant functions on a Krein space are precisely those that have zero variation in the associated Hilbert spaces. The set of strongly of bounded variation functions is a subspace of the space of bounded functions. The variation of a function in the Krein spaces can be decomposed as a sum of the variations in the associated Hilbert spaces. The set of strongly bounded functions in Krein space is endowed with a norm.

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