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Exploring lock-down effects in a fractional order Covid-19 model with crossover behavior

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Abstract

This research article delves into the intricate dynamics of a COVID-19 model, uniquely characterized by the integration of lock-down measures through a piecewise operator that encompasses both classical and Caputo operators. The article not only examines the model's behavior but also rigorously establishes the existence and uniqueness of solutions for this complex piecewise system. To tackle the numerical approximation of solutions, the study employs Newton's polynomial interpolation scheme, shedding light on the model's behavior under different conditions. Through meticulous graphical representations, the article effectively communicates the results and numerical solutions across various classes of the model, each defined by distinct fractional orders. This comprehensive approach provides valuable insights into the pandemic's multifaceted dynamics, serving as a basis for understanding its progression and evaluating potential control strategies.

Keywords: COVID-19, fractional operator, existence and uniqueness, stability analysis, numerical simulations.

2020 MSC: 26A33, 34A08, 92D30.

1. Introduction

Mathematical models play a pivotal role in the analysis and prediction of the spread of infectious diseases. These models are integral to gaining insights into the efficient control of viruses and the implementation of preventive efforts [11, 24, 30]. Within the realm of infectious diseases, a variety of compartmental models are employed, ranging from the fundamental SIR model to more intricate variations [8]. These models provide researchers and policymakers with valuable tools to comprehend the dynamics of disease transmission, assess the impact of interventions like lock-downs, and guide public health strategies.

Numerous diseases, such as Dengue fever, Hanta fever, and Leptospirosis [27], have a global prevalence. Infectious diseases are characterized by their ability to spread from person to person, in contrast to noninfectious diseases, which stem from inherited or environmental factors. Across history, infectious viral diseases have tragically claimed a significant number of human lives, underscoring their impact on public health and society. Amidst a spectrum of viral diseases, the COVID-19 pandemic has unleashed a global catastrophe, impacting over eight million individuals and displaying a mortality-to-recovery ratio

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that appears to be relatively balanced. Despite this, the absence or existence of the formerly infected individual may be detected using Pcr Method, and the rate of recovery looks to be hopeful in the lack of a curative vaccine. Each quarter, health care specialists, the World Health Organization, and the centers for disease control and prevention grappled with the question about whether a COVID-19 patient may re-infected after being clinically cured. The disease's mild character has gotten the interest of various academics and medical experts, prompting them to embark on a large study effort to properly counterattack and halt the disease's progress.

Four of the seven humans coronaviruses that have been identified are prevalent human influenza infections. MERS-CoV, SARS-CoV, and 2019-nCoV are all viruses that cause severe respiratory problems [2]. Despite the substantial length of time that medical professionals and researchers have dedicated to studying COVID-19, a significant portion of the population remains unaware of the intricacies of this condition. Vaccinations and antiviral medications tailored specifically to prevent or treat COVID-19 are regrettably still unavailable [13, 14, 22, 39]. Reflecting back on China's previous major epidemic, the SARS-CoV outbreak in 2003, one recalls a severe respiratory infection with a distressing mortality rate. However, China managed to control the SARS outbreak by implementing a combination of restrictive measures and effective prevention strategies. In stark contrast, the COVID-19 virus exhibits a considerable and lengthy incubation period, setting it apart from the SARS outbreak. This unique feature has posed unprecedented challenges for managing its spread and impact.

Because of the announced features of the COVID-19 outbreak, each nation's authorities were forced to implement some laws in order to stop the virus's huge spread. Both developed, developing, and underdeveloped countries agree to use lock-down measures to limit people's movement. In certain nations, the lock-down technique may achieve the intended effects, whereas in others, poor management and a poor infrastructure, short term solutions, and incentive may exacerbate the disease' spread. Extensive research has been undertaken by numerous researchers in the realm of mathematical models for infectious diseases, incorporating the utilization of fractional and fractal fractional operators [15, 17–19, 25, 28, 35, 38]. It has been observed that fractal fractional calculus provides better dynamics than classical derivatives [21]. Factor-order dynamical models are useful in many scientific and technical domains because they are superior to integer-order models in capturing intricate behaviors, memory effects, and nonlinear dynamics [1, 20, 34, 36, 37].

In this section, we bolster the content by introducing an innovative and comprehensive model formulation that delves into the intricate dynamics of infectious disease spread. Hence, we consider the model [3] as bellow:

$$\begin{aligned} \dot{S}(t) &= \Lambda - (\beta I + \lambda_1 L + d)S + \gamma_1 I + \gamma_2 I_L + \theta_1 S_L, \\ \dot{S_L}(t) &= \lambda_1 SL - (d+\theta_1)S_L, \\ \dot{I}(t) &= \beta SI - \gamma_1 I - \alpha_1 I - dI - \gamma_2 IL + \theta_2 I_L, \\ \dot{I_L}(t) &= \gamma_2 IL - dI_L - \theta_2 I_L - \gamma_2 I_L - \alpha_2 I_L, \\ \dot{L}(t) &= \mu I - \phi L. \end{aligned} \tag{1.1}$$

In model (1.1) S and I represent susceptible and infected individuals, respectively, S_L and I_L are susceptible and infected individuals, respectively on which lock down is imposed. The system state variable L shows the cumulative density of lock down. The parameters' description appeared in model (1.1) are given in Table 1.

Atangana et al. have presented a new class of operators called as piecewise derivatives as well as integrals [5]. These novel operators offer significant utility as they enable the examination of a mathematical model using both classical and fractional operators within a single interval, further divided into two distinct sub-intervals [6, 26, 29]. This approach not only allows for the application of traditional methodologies but also incorporates more intricate fractional behaviors, enhancing the model's versatility and accuracy. These operators have proven to be particularly valuable to researchers, providing a novel avenue for investigating cross-over behaviors within the context of mathematical modeling [4, 12, 23] and many other useful mathematical model on neural network along with other techniques [9, 10, 16, 32]. These cross-over behaviors often underlie complex phenomena in infectious disease dynamics, crucial for understanding scenarios where disease transmission patterns abruptly change due to interventions, population dynamics, or external factors. Motivated by these advantageous features, we embark on an exploration of the proposed COVID-19 model. We delve into its dynamics through both the lens of classical operators and the utilization of the Caputo piecewise operator. By employing this dual approach, our intention is to gain a comprehensive understanding of how the model responds to different mathematical frameworks, thereby shedding light on the potential influence of cross-over behaviors in disease transmission dynamics.

Table 1: Detail of the used parameters for model (1.1).

Variables	Description
Λ	Recruitment rate
β	Contact rate of infection
λ_1	Lock down imposition on susceptible individuals
λ_2	Lock down imposition on infected individuals
d	Natural death rate
γ_1	Recovery rate in I
γ_2	Recovery rate in I_L
θ_1	Transfer rate from S_L to S
α_1	Rate of death due to infections in I
θ_2	Transfer rate from I_L to I
α_2	Rate of death due to infections in I_L
μ	Lock down implementation rate
ϕ	Lock down depletion rate

The equation (1.1) can be written in piecewise fractional form

$$\begin{aligned}
{}^{PCC}_{0}\mathbf{D}_0^{\delta}S(t) &= \Lambda - (\beta I + \lambda_1 L + d)S + \gamma_1 I + \gamma_2 I_L + \theta_1 S_L,\\
{}^{PCC}_{0}\mathbf{D}_0^{\delta}S_L(t) &= \lambda_1 SL - (d + \theta_1)S_L,\\
{}^{PCC}_{0}\mathbf{D}_0^{\delta}I(t) &= \beta SI - \gamma_1 I - \alpha_1 I - dI - \gamma_2 IL + \theta_2 I_L,\\
{}^{PCC}_{0}\mathbf{D}_0^{\delta}I_L(t) &= \gamma_2 IL - dI_L - \theta_2 I_L - \gamma_2 I_L - \alpha_2 I_L,\\
{}^{PCC}_{0}\mathbf{D}_0^{\delta}L(t) &= \mu I - \phi L,
\end{aligned} \tag{1.2}$$

where PCC represents piecewise derivative with classical and Caputo operators with two subintervals in $[0, T]$. In more concise form we can express equation (1.2) as

$$\begin{aligned}
&{}^{PCC}_{0}\mathbf{D}_t^{\delta}(S(t)) = \begin{cases} {}^{C}_{0}\mathbf{D}_t(S(t)) = \frac{d}{dt}G_1(S,t), \\ {}^{C}_{0}\mathbf{D}_t^{\delta}(S(t)) = G_1(S,t), \end{cases} \quad {}^{PCC}_{0}\mathbf{D}_t^{\delta}(S_L(t)) = \begin{cases} {}^{C}_{0}\mathbf{D}_t(S_L(t)) = \frac{d}{dt}G_2(S_L,t), \\ {}^{C}_{0}\mathbf{D}_t^{\delta}(S_L(t)) = G_2(S_L,t), \end{cases}\\
&{}^{PCC}_{0}\mathbf{D}_t^{\delta}(I(t)) = \begin{cases} {}^{C}_{0}\mathbf{D}_t(I(t)) = \frac{d}{dt}G_3(I,t), \\ {}^{C}_{0}\mathbf{D}_t^{\delta}(I(t)) = G_3(I,t), \end{cases} \quad {}^{PCC}_{0}\mathbf{D}_t^{\delta}(I_L(t)) = \begin{cases} {}^{C}_{0}\mathbf{D}_t(I_L(t)) = \frac{d}{dt}G_4(I_L,t), \\ {}^{C}_{0}\mathbf{D}_t^{\delta}(I_L(t)) = G_4(I_L,t), \end{cases}\\
&{}^{PCC}_{0}\mathbf{D}_t^{\delta}(L(t)) = \begin{cases} {}^{C}_{0}\mathbf{D}_t(L(t)) = \frac{d}{dt}G_5(L,t), \\ {}^{C}_{0}\mathbf{D}_t^{\delta}(L(t)) = G_5(L,t), \end{cases}
\end{aligned} \tag{1.3}$$

here $0 < t \leqslant t_1$, $t_1 < t \leqslant t_2$, ${}^{C}_{0}\mathbf{D}_t$, and ${}^{C}_{0}\mathbf{D}_t^{\delta}$ represents classical and the Caputo derivative, respectively, and G_i, where $i = 1, 2, 3, 4, 5$ are the left hand side of equation (1.2).

Motivated by the above kinds of literature the concerned paper considered a novel mathematical model for differential equations in the sense of piecewise Caputo and classical operators, which represents the crossover behaviors of the considered model. In the framework of piecewise Caputo and classical operators, the existence of a solution as well as its uniqueness is also developed. For each quantity, the piecewise subinterval approximation solution is developed using the Newton polynomial interpolation method. The available data on various periods and fractional orders convergent to integer orders are compared to the numerical simulation.

The structure of the paper as is follows. In Section 2, basic definition are presented and Section 3 investigates the existence and uniqueness solution for the proposed model by using the fixed point theory. In Section 4, the approximate solution of the model is studied with the aid of Newton interpolation formula and the required solution is obtained. Simulations are presented along with figures and mentioned the behavior of the obtained results in Section 5. Finally, we conclude our results in Section 6.

2. Basic results

In this part, we present some definitions.

Definition 2.1 ([25])**.** Let a function be $\mathbb{F}(t)$, with order $n-1<\delta<n$, then the fractional order derivative in the sense of Caputo can be written as

$$^{C}\mathbf{D}^{\delta}\mathbb{F}(t)=\frac{1}{\Gamma(n-\delta)}\int_0^t (t-\wp)^{n-\delta-1}[\mathbb{F}'(\wp)]d\wp$$

and the integration is

$$^{C}\mathbf{I}^{\delta}(\mathbb{F}(t))=\frac{1}{\Gamma(\delta)}\int_0^t (t-\wp)^{\delta-1}d\wp,\ \delta>0.$$

Definition 2.2 ([5])**.** Consider a differentiable function $\mathbb{F}(t)$ and an increasing mapping, then classical piecewise derivative is

$$^{PF}\mathbf{D}\mathbb{F}(t)=\begin{cases}\mathbb{F}'(t), & 0<t\leqslant t_1,\\ \frac{\mathbb{F}'(t)}{f'(t)}, & t_1<t\leqslant t_2=T,\end{cases}$$

and the integration is

$$^{PF}\mathbf{I}(\mathbb{F}(t))=\begin{cases}\int_0^t \mathbb{F}(\tau)d\tau, & 0<t\leqslant t_1,\\ \int_{t_1}^t \mathbb{F}(\tau)f'(\tau)d(\tau), & t_1<t\leqslant t_2,\end{cases}$$

here $^{PF}\mathbf{D}\mathbb{F}(t)$ and $^{PF}\mathbf{I}\mathbb{F}(t)$ are used for classical derivative and integration with $0<t\leqslant t_1$, global derivative and integration for $t_1<t\leqslant t_2$.

Definition 2.3 ([5])**.** Let $\mathbb{F}(t)$ be differentiable then classical and fractional piecewise derivative is defined as

$$^{PF}\mathbf{D}^{\delta}\mathbb{F}(t)=\begin{cases}\mathbb{F}'(t), & 0<t\leqslant t_1,\\ {}_0^{C}D_t^{\delta}\mathbb{F}(t), & t_1<t\leqslant t_2,\end{cases}$$

while for integration

$$^{PF}\mathbf{I}(\mathbb{F}(t))=\begin{cases}\int_0^t \mathbb{F}(\tau)d\tau, & 0<t\leqslant t_1,\\ \frac{1}{\Gamma\delta}\int_{t_1}^t (t-\wp)^{\delta-1}U(\wp)d(\wp), & t_1<t\leqslant t_2,\end{cases}$$

here $^{PF}\mathbf{D}^{\delta}\mathbb{F}(t)$ and ${}_0^{PF}\mathbf{I}_t\mathbb{F}(t)$ represented the classical Caputo derivative and integration for $0<t\leqslant t_1$, fractional Caputo derivative and integration for $t_1<t\leqslant t_2$.

3. Existence and uniqueness results

We will study the existence of a solution as well as the uniqueness of the suggested piecewise model in this section. For this, we express the system (1.2) as

$$^{PCC}\mathbf{D}^{\delta}\mathbb{F}(t) = G(t, \mathbb{F}(t)), \quad 0 < \delta \leqslant 1$$

is

$$\mathbb{F}(t) = \begin{cases} \mathbb{F}_0 + \int_0^t G(\wp, \mathbb{F}(\wp))d\wp, & 0 < t \leqslant t_1 \\ \mathbb{F}(t_1) + \frac{1}{\Gamma\delta}\int_{t_1}^{t_2}(t-\wp)^{\sigma-1}G(\wp, \mathbb{F}(\wp))d(\wp), & t_1 < t \leqslant t_2, \end{cases} \tag{3.1}$$

where

$$\mathbb{F}(t) = \begin{cases} S(t), \\ S_L(t), \\ I(t), \\ I_L(t), \\ L(t), \end{cases} \quad \mathbb{F}_0 = \begin{cases} S(0), \\ S_L(0), \\ I(0), \\ I_L(0), \\ L(0), \end{cases} \quad \mathbb{F}_{t_1} = \begin{cases} S_{t_1}, \\ S_{L(t_1)}, \\ I_{t_1}, \\ I_{L(t_1)}, \\ L_{t_1}, \end{cases} \quad G(t, \mathbb{F}(t)) = \begin{cases} G_1 = \begin{cases} \frac{d}{dt}G_1(S, S_L, I, I_L, L, t), \\ {}^{C}G_1(S, S_L, I, I_L, L, t), \end{cases} \\ G_2 = \begin{cases} \frac{d}{dt}G_2(S, S_L, I, I_L, L, t), \\ {}^{C}G_2(S, S_L, I, I_L, L, t), \end{cases} \\ G_3 = \begin{cases} \frac{d}{dt}G_3(S, S_L, I, I_L, L, t), \\ {}^{C}G_3(S, S_L, I, I_L, L, t), \end{cases} \\ G_4 = \begin{cases} \frac{d}{dt}G_4(S, S_L, I, I_L, L, t), \\ {}^{C}G_4(S, S_L, I, I_L, L, t), \end{cases} \\ G_5 = \begin{cases} \frac{d}{dt}G_5(S, S_L, I, I_L, L, t), \\ {}^{C}G_5(S, S_L, I, I_L, L, t). \end{cases} \end{cases}$$

Let $\infty > t_2 \geqslant t > t_1 > 0$ having Banach space $E_1 = C[0, T]$ having

$$\|\mathbb{F}\| = \max_{t \in [0,T]} |\mathbb{F}(t)|.$$

For obtaining the results, we consider growth condition, on non-linear operator in the form:

(C1) if there exist a constant $L_{\mathbf{P}} > 0$ such that for all $\mathbb{F}, \bar{\mathbb{F}} \in E_1$, we have

$$|\mathbf{P}(t, \mathbb{F}) - \mathbf{P}(t, \bar{\mathbb{F}})| \leqslant L_{\mathbf{P}}|\mathbb{F} - \bar{\mathbb{F}}|;$$

(C2) if there exist a constant $C_{\mathbf{P}} > 0$ & $M_{\mathbf{P}} > 0$,

$$|\mathbf{P}(t, \mathbb{F}(t))| \leqslant C_{\mathbf{P}}|\mathbb{F}| + M_{\mathbf{P}}.$$

Theorem 3.1. *Suppose* $\mathbf{P}$ *be continuous (piece-wise) on sub-interval* $0 < t \leqslant t_1$ *and* $t_1 < t \leqslant t_2$ *on* $[0, T]$*, also satisfy* (C2)*, then Eq.* (1.3) *has one or more solution the sub-intervals.*

Proof. From Schauder theorem, consider the closed subset in the sub-intervals $0, T$ as B of E in the form

$$B = \{\mathbb{F} \in E : \ \|\mathbb{F}\| \leqslant R_{1,2}, \ R > 0\}.$$

Now, suppose the operator $\mathbb{V} : B \to B$ and apply system (3.1) as

$$\mathbb{V}(\mathbb{F}) = \begin{cases} \mathbb{F}_0 + \int_0^t \mathbf{P}(\wp, \mathbb{F}(\wp))d\wp, & 0 < t \leqslant t_1, \\ \mathbb{F}(t_1) + \frac{1}{\Gamma(\delta)}\int_{t_1}^{t_2}(t-\wp)^{\sigma-1}\mathbf{P}(\wp, \mathbb{F}(\wp))d(\wp), & t_1 < t \leqslant t_2. \end{cases}$$

For $\mathbb{F} \in B$, we have

$$|\mathbb{V}(\mathbb{F})(t)| \leqslant \begin{cases} |\mathbb{F}_0| + \int_0^{t_1} |\mathbf{P}(\wp, \mathbb{F}(\wp))|d\wp, \\ |\mathbb{F}_{(t_1)}| + \frac{1}{\Gamma(\delta)}\int_{t_1}^{t_2}(t-\wp)^{\delta-1}|\mathbf{P}(\wp\mathbb{F}(\wp))|d(\wp), \end{cases}$$

$$\leqslant \begin{cases} |\mathbb{F}_0| + \int_0^{t_1}[C_{\mathbf{P}}|\mathbb{F}| + M_{\mathbf{P}}]d\wp, \\ |\mathbb{F}_{(t_1)}| + \frac{1}{\Gamma(\delta)}\int_{t_1}^{t_2}(t-\wp)^{\delta-1}[C_{\mathbf{P}}|\mathbb{F}| + M_{\mathbf{P}}]d(\wp), \end{cases}$$

$$\leqslant \begin{cases} |\mathbb{F}_0| + \mathbf{t_1}[C_{\mathbf{P}}\|\mathbb{F}\| + M_{\mathbf{P}}] =\leqslant R_{1,2}, & 0 < t \leqslant t_1, \\ |\mathbb{F}_{(t_1)}| + \frac{T^\delta}{\Gamma(\delta+1)}[C_{\mathbf{P}}\|\mathbb{F}\| + M_{\mathbf{P}}] \leqslant R_{1,2}, & t_1 < t \leqslant t_2, \end{cases}$$

for $t_1 < t \leqslant T$, using $|(t_1-\wp)^\delta - (T-\wp)^\delta| \leqslant T^\delta$, one has

$$R_{1,2} \geqslant \max \begin{cases} \frac{|\mathbb{F}_0| + \mathbf{t_1} M_{\mathbf{P}}}{1-\mathbf{t_1} C_{\mathbf{P}}}, & 0 < t \leqslant t_1, \\ \frac{|\mathbb{F}_{(t_1)}|\Gamma(\delta+1) + T^\delta M_{\mathbf{P}}}{\Gamma(\delta+1) - T^\delta C_{\mathbf{P}}}, & t_1 < t \leqslant t_2. \end{cases}$$

The previous equation shows that $\|\mathbb{V}(\mathbb{F})\| \leqslant R_{1,2}$ implies that $\mathbb{V}(B) \subset B$. So we got that $\mathbb{V}$ is complete as well as bounded. Now to show the complete continuity, we advance as consider $t_n > t_m \in [0, t_1]$ in first interval, assume

$$\begin{aligned} |\mathbb{V}(\mathbb{F})(t_n) - \mathbb{V}(\mathbb{F})(t_m)| &= \left| \int_0^{t_n} \mathbf{P}(\wp, \mathbb{F}(\wp))d\wp - \int_0^{t_m} \mathbf{P}(\wp, \mathbb{F}(\wp))d\wp \right| \\ &\leqslant \int_0^{t_n} |\mathbf{P}(\wp, \mathbb{F}(\wp))|d\wp - \int_0^{t_m} |\mathbf{P}(\wp, \mathbb{F}(\wp))|d\wp \\ &\leqslant \left[\int_0^{t_n} (C_{\mathbf{P}}|\mathbb{F}| + M_{\mathbf{P}}) - \int_0^{t_m} (C_{\mathbf{P}}|\mathbb{F}| + M_{\mathbf{P}}) \leqslant (C_{\mathbf{P}}\mathbb{F} + M_{\mathbf{P}})[t_n - t_m]. \right. \end{aligned} \tag{3.2}$$

Now from Eq. (3.2), when $t_m \to t_n$, then

$$|\mathbb{V}(\mathbb{F})(t_n) - \mathbb{V}(\mathbb{F})(t_m)| \to 0 \text{ as } t_m \to t_n.$$

Hence $\mathbb{V}$ holds the condition of equi-continuity in $[0, t_1]$. Further consider $t_i, t_j \in [t_1, T]$ in the Caputo sense as

$$\begin{aligned} &|\mathbb{V}(\mathbb{F})(t_n) - \mathbb{V}(\mathbb{F})(t_m)| \\ &= \left| \frac{1}{\Gamma(\delta)} \int_0^{t_n} (t_n-\wp)^{\delta-1}\mathbf{P}(\wp, \mathbb{F}(\wp))d\wp - \frac{1}{\Gamma(\delta)} \int_0^{t_m} (t_m-\wp)^{\delta-1}\mathbf{P}(\wp, \mathbb{F}(\wp))d\wp \right| \\ &\leqslant \frac{1}{\Gamma(\delta)} \int_0^{t_m} [(t_m-\wp)^{\delta-1} - (t_n-\wp)^{\delta-1}]|\mathbf{P}(\wp, \mathbb{F}(\wp))|d\wp + \frac{1}{\Gamma(\delta)} \int_{t_m}^{t_n} (t_n-\wp)^{\delta-1}|\mathbf{P}(\wp, \mathbb{F}(\wp))|d\wp \\ &\leqslant \frac{1}{\Gamma(\delta)} \left[\int_0^{t_m} [(t_m-\wp)^{\delta-1} - (t_n-\wp)^{\delta-1}]d\wp + \int_{t_m}^{t_n} (t_n-\wp)^{\delta-1}d\wp \right] (C_{\mathbf{P}}\|\mathbb{F}\| + M_{\mathbf{P}}) \\ &\leqslant \frac{(C_{\mathbf{P}}R_{1,2} + M_{\mathbf{P}})}{\Gamma(\delta+1)}[t_n^\delta - t_m^\delta + 2(t_n - t_m)^\delta]. \end{aligned} \tag{3.3}$$

Next from (3.3), we have $t_m \to t_n$, then

$$|\mathbb{V}(\mathbb{F})(t_n) - \mathbb{V}(\mathbb{F})(t_m)| \to 0 \text{ as } t_m \to t_n,$$

and $\mathbb{V}$ is also bounded, therefore

$$\|\mathbb{V}(\mathbb{F})(t_n) - \mathbb{V}(\mathbb{F})(t_m)\| \to 0 \text{ as } t_m \to t_n.$$

As a result, $\mathbb{V}$ is equicontinuous in the $[t_n, t_m]$ interval. As a result, $\mathbb{V}$ is an equicontinuous mapping. Using the Arzela'-Ascoli theorem, the operator $\mathbb{V}$ is continuous (completely), uniformly continuous, and bounded. Therefore, the piece-wise derivable problem (1.3) has at minimum one solution upon every sub interval, according to Schauder theorem. □

Theorem 3.2. *If the following condition holds, the system under consideration has a unique solution if satisfy the following equation*

$$\max\left\{\mathbf{t_1}L_{\mathbf{P}}, \frac{T^{\delta}}{\Gamma(\delta+1)}L_{\mathbf{P}} = \Delta < 1\right\},$$

with the condition (C1).

Proof. Suppose $\mathbb{V}: A \to A$ to be a piece-wise continuous mapping, let $\mathbb{F}$ and $\bar{\mathbb{F}} \in A$ on $[0, t_1]$ in the classical form as

$$\|\mathbb{V}(\mathbb{F}) - \mathbb{V}(\bar{\mathbb{F}})\| = \max_{t\in[0,t_1]}\left|\int_0^{t_1}\mathbf{P}(\wp, \mathbb{F}(\wp))d\wp - \int_0^{t_1}\mathbf{P}(\wp, \bar{\mathbb{F}}(\wp))d\wp\right| \leqslant \mathbf{t_1}L_{\mathbf{P}}\|\mathbb{F} - \bar{\mathbb{F}}\|. \tag{3.4}$$

From the Eq. (3.4), we can write

$$\|\mathbb{V}(\mathbb{F}) - \mathbb{V}(\bar{\mathbb{F}})\| \leqslant \mathbf{t_1}L_{\mathbf{P}}\|\mathbb{F} - \bar{\mathbb{F}}\|. \tag{3.5}$$

Hence, $\mathbb{V}$ is the contraction map. As a result of the Banach contraction principle, the considered model has a single solution in the provided sub-interval. Furthermore, for sub-interval $t \in [t_1, t_2]$, we consider Caputo derivative as

$$\begin{aligned}\|\mathbb{V}(\mathbb{F}) - \mathbb{V}(\bar{\mathbb{F}})\| &= \max_{t\in[t_1,t_2]}\left|\frac{1}{\Gamma(\delta)}\int_{t_1}^{t_2}(t-\wp)^{\delta-1}\mathbf{P}(\wp, \mathbb{F}(\wp))d\wp - \frac{1}{\Gamma(\delta)}\int_{t_1}^{t_2}(t-\wp)^{\delta-1}\mathbf{P}(\wp, \bar{\mathbb{F}}(\wp))d\wp\right| \\ &\leqslant \frac{T^{\delta}}{\Gamma(\delta+1)}L_{\mathbf{P}}\|\mathbb{F} - \bar{\mathbb{F}}\|.\end{aligned} \tag{3.6}$$

From (3.6), we have

$$\|\mathbb{V}(\mathbb{F}) - \mathbb{V}(\bar{\mathbb{F}})\| \leqslant \frac{T^{\delta}}{\Gamma(\delta+1)}L_{G}\|\mathbb{F} - \bar{\mathbb{F}}\|. \tag{3.7}$$

Let, $\max\left\{TL_{\mathbf{P}}, \frac{T^{\delta}}{\Gamma(\delta+1)}L_{\mathbf{P}}\right\} = \Delta$. So by (3.5) and (3.7) the model under consideration has only one solution on each sub-intervals,

$$\|\mathbb{V}(\mathbb{F}) - \mathbb{V}(\bar{\mathbb{F}})\| \leqslant \Delta\|\mathbb{F} - \bar{\mathbb{F}}\|.$$

Therefore, $\mathbb{V}$ is a contraction map. As a result, within the scope of the Banach contraction principle, the suggested model has a only one solution in the second sub-interval. □

4. Numerical method

Here, we present numerical scheme for the suggested piecewise model (1.3). The numerical scheme for the two sub-interval of $[0, T]$, in classical and Caputo sense, respectively, is presented in this section. Applying the piece-wise integral to equation (1.3) in classical and Caputo sense, we obtain

$$\begin{aligned}S(t) &= \begin{cases} S_0 + \int_0^{t_1} G_1(\wp, S)d\wp, & 0 < t \leqslant t_1, \\ S(t_1) + \frac{1}{\Gamma(\delta)}\int_{t_1}^{t_2}(t-\wp)^{\delta-1}G_1(\wp, S)d\wp, & t_1 < t \leqslant t_2, \end{cases} \\ S_L(t) &= \begin{cases} S_L(0) + \int_0^{t_1} G_2(\wp, S_L)d\wp, & 0 < t \leqslant t_1, \\ S_L(t_1) + \frac{1}{\Gamma(\delta)}\int_{t_1}^{t_2}(t-\wp)^{\delta-1}G_2(\wp, S_L)d\wp, & t_1 < t \leqslant t_2, \end{cases} \\ I(t) &= \begin{cases} I_0 + \int_0^{t_1} G_3(\wp, I)d\wp, & 0 < t \leqslant t_1, \\ I(t_1) + \frac{1}{\Gamma(\delta)}\int_{t_1}^{t_2}(t-\wp)^{\delta-1}G_3(\wp, I)d\wp, & t_1 < t \leqslant t_2, \end{cases}\end{aligned} \tag{4.1}$$

$$I_L(t)) = \begin{cases} I_L(0) + \int_0^{t_1} G_4(\wp, I_L)d\wp, & 0 < t \leqslant t_1, \\ I_L(t_1) + \frac{1}{\Gamma(\delta)} \int_{t_1}^{t_2} (t-\wp)^{\delta-1} G_4(\wp, I_L)d\wp, & t_1 < t \leqslant t_2, \end{cases}$$

$$L(t)) = \begin{cases} L_0 + \int_0^{t_1} G_5(\wp, L)d\wp, & 0 < t \leqslant t_1, \\ L(t_1) + \frac{1}{\Gamma(\delta)} \int_{t_1}^{t_2} (t-\wp)^{\delta-1} G_5(\wp, L)d\wp, & t_1 < t \leqslant t_2. \end{cases}$$

Further, we prove the technique for the first state variable of model (4.1) and the same can be obtained for the remaining state variables. At $t = t_{n+1}$,

$$\S(t_{n+1})) = \begin{cases} \S_0 + \int_0^{t_1} G_1(S, S_L, I, I_L, L, \wp)d\wp, & 0 < t \leqslant t_1, \\ \S(t_1) + \frac{1}{\Gamma(\delta)} \int_{t_1}^{t_{n+1}} (t-\wp)^{\delta-1} G_1(S, S_L, I, I_L, L, \wp)d\wp, & t_1 < t \leqslant t_2. \end{cases} \tag{4.2}$$

Applying the Newton interpolation formula as presented in [5] to equation (4.2), we obtain the following

$$S(t_{n+1}) = \begin{Bmatrix} S_0 + \left\{ \sum_{k=2}^{i} \left[\frac{5}{12} G_1(S^{k-2}, t_{k-2})\Delta t - \frac{4}{3} G_1(S^{k-1}, t_{k-1})\Delta t + G_1(S^k, t_k) \right], \right. \\ S(t_1) + \begin{cases} \frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+1)} \sum_{k=i+3}^{n} \left[G_1(S^{k-2}, t_{k-2}) \right] \Pi, \\ + \frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+2)} \sum_{k=i+3}^{n} \left[G_1(S^{k-1}, t_{k-1}) - G_1(S^{k-2}, t_{k-2}) \right] \sum, \\ + \frac{\delta(\Delta t)^{\delta-1}}{2\Gamma(\delta+3)} \sum_{k=i+3}^{n} \left[G_1(S^k, t_k) - 2G_1(S^{k-1}, t_{k-1}) + G_1(S^{k-2}, t_{k-2}) \right] \Delta, \end{cases} \end{Bmatrix}.$$

For the rest of the classes, we have

$$S_L(t_{n+1}) = \begin{Bmatrix} S_L(0) + \left\{ \sum_{k=2}^{i} \left[\frac{5}{12} G_2(S_L^{k-2}, t_{k-2})\Delta t - \frac{4}{3} G_2(S_L^{k-1}, t_{k-1})\Delta t + G_2(S_L^k, t_k) \right], \right. \\ S_L(t_1) + \begin{cases} \frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+1)} \sum_{k=i+3}^{n} \left[G_2(S_L^{k-2}, t_{k-2}) \right] \Pi, \\ + \frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+2)} \sum_{k=i+3}^{n} \left[G_2(S_L^{k-1}, t_{k-1}) - G_2(S_L^{k-2}, t_{k-2}) \right] \sum, \\ + \frac{\delta(\Delta t)^{\delta-1}}{2\Gamma(\delta+3)} \sum_{k=i+3}^{n} \left[G_2(S_L^k, t_k) - 2G_2(S_L^{k-1}, t_{k-1}) + G_2(S_L^{k-2}, t_{k-2}) \right] \Delta, \end{cases} \end{Bmatrix},$$

$$I(t_{n+1}) = \begin{Bmatrix} I_0 + \left\{ \sum_{k=2}^{i} \left[\frac{5}{12} G_3(I^{k-2}, t_{k-2})\Delta t - \frac{4}{3} G_3(I^{k-1}, t_{k-1})\Delta t + G_3(I^k, t_k) \right], \right. \\ I(t_1) + \begin{cases} \frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+1)} \sum_{k=i+3}^{n} \left[G_3(I^{k-2}, t_{k-2}) \right] \Pi, \\ + \frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+2)} \sum_{k=i+3}^{n} \left[G_3(I^{k-1}, t_{k-1}) - G_3(I^{k-2}, t_{k-2}) \right] \sum, \\ + \frac{\delta(\Delta t)^{\delta-1}}{2\Gamma(\delta+3)} \sum_{k=i+3}^{n} \left[G_3(I^k, t_k) - 2G_3(I^{k-1}, t_{k-1}) + G_3(I^{k-2}, t_{k-2}) \right] \Delta, \end{cases} \end{Bmatrix},$$

$$I_L(t_{n+1}) = \begin{Bmatrix} I_L(0) + \left\{ \sum_{k=2}^{i} \left[\frac{5}{12} G_4(I_L^{k-2}, t_{k-2})\Delta t - \frac{4}{3} G_4(I_L^{k-1}, L^{k-1}, t_{k-1})\Delta t + G_4(I_L^k, t_k) \right], \right. \\ I_L(t_1) + \begin{cases} \frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+1)} \sum_{k=i+3}^{n} \left[G_4(I_L^{k-2}, t_{k-2}) \right] \Pi, \\ + \frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+2)} \sum_{k=i+3}^{n} \left[G_4(I_L^{k-1}, t_{k-1}) - G_4(I_L^{k-2}, t_{k-2}) \right] \sum, \\ + \frac{\delta(\Delta t)^{\delta-1}}{2\Gamma(\delta+3)} \sum_{k=i+3}^{n} \left[G_4(I_L^k, t_k) - 2G_4(I_L^{k-1}, t_{k-1}) + G_4(I_L^{k-2}, t_{k-2}) \right] \Delta, \end{cases} \end{Bmatrix},$$

$$L(t_{n+1}) = \begin{cases} L_0 + \left\{ \sum_{k=2}^{i} \left[\frac{5}{12} G_4(L^{k-2}, t_{k-2})\Delta t - \frac{4}{3} G_5(L^{k-1}, t_{k-1})\Delta t + G_5(L^k, t_k) \right], \right. \\ L(t_1) + \left\{ \begin{array}{l} \frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+1)} \sum_{k=i+3}^{n} \left[G_5(L^{k-2}, t_{k-2}) \right] \Pi, \\ + \frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+2)} \sum_{k=i+3}^{n} \left[G_4(L^{k-1}, t_{k-1}) - G_5(L^{k-2}, t_{k-2}) \right] \sum, \\ + \frac{\delta(\Delta t)^{\delta-1}}{2\Gamma(\delta+3)} \sum_{k=i+3}^{n} \left[G_4(L^k, t_k) - 2G_5(L^{k-1}, t_{k-1}) + G_5(L^{k-2}, t_{k-2}) \right] \Delta, \end{array} \right\} \end{cases}.$$

5. Results and discussion

This part of the manuscript contains simulations of mathematical approximations performed utilizing the piecewise notion, classical and the Caputo operator, and the Newton's polynomial approach. For this, we take two sub-intervals $[0, t_1] = [0, 20]$ and $[t_1, T] = [20, 200]$, which makes the whole interval $[0, T]$. In the interval $[0, 200]$, we have used the classical operator, while in the interval $[20, 200]$ Caputo derivative is used. For the simulations, the parameters used are given in Table 2 and initial conditions are $S = 200,\ S_L = 300,\ I = 100,\ I_L = 500,\ L = 200$.

Table 2: Parameters' values of the model (1.2).

Parameter	Value	Source	Parameter	Value	Source
Λ	400	[33]	β	0.000017	[31]
λ_1	0.0002	[33]	λ_2	0.002	Estimated
d	0.0096	[7]	γ_1	0.16979	[2]
γ_2	0.16979	[2]	θ_1	0.2	[33]
α_1	0.03275	[2]	θ_2	0.02	Estimated
α_2	0.03275	[2]	μ	0.0005	[33]
ϕ	0.06	[33]			

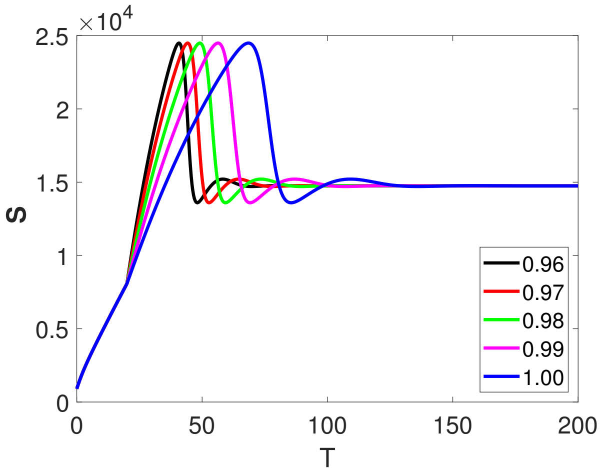

Figure 1: Dynamics of S for the model (1.2) on different fractional-order δ for both sub intervals with time 20-200.

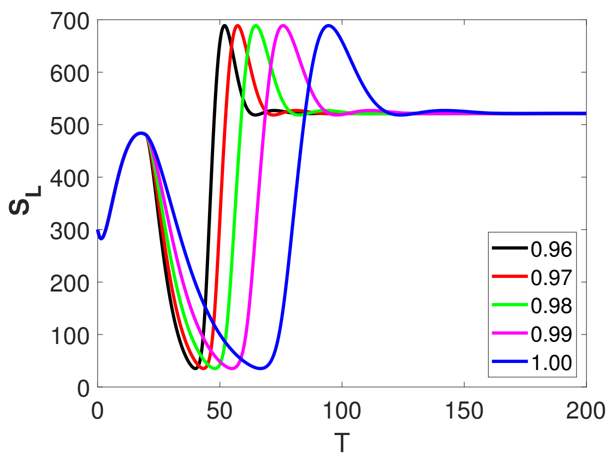

Figure 2: Dynamics of S_L for the model (1.2) on different fractional-order δ for both sub intervals with time 20-200.

In Figs. 1-5, the behavior of the system under consideration, is depicted using several colors and fractional orders are as (blue, 1.00), (magenta, 0.99), (green, 0.98), (red, 0.97), (black, 0.96). Fig. 1 shows the dynamics of the susceptible population, where it can be seen that the state variable S becomes stable more fastly with small fractional order.

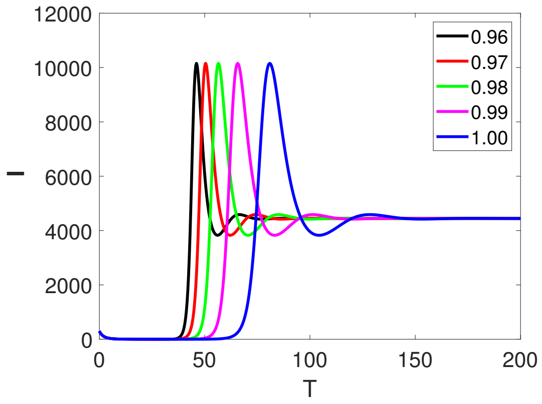

Figure 3: Dynamics of I for the model (1.2) on different fractional-order δ for both sub intervals with time 20-200.

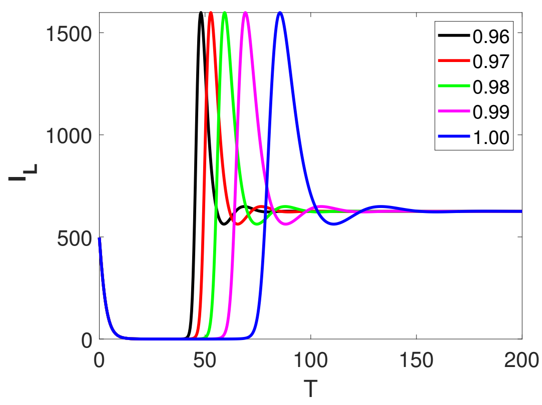

Figure 4: Dynamics of I_L for the model (1.2) on different fractional-order δ for both sub intervals with time 20-200.

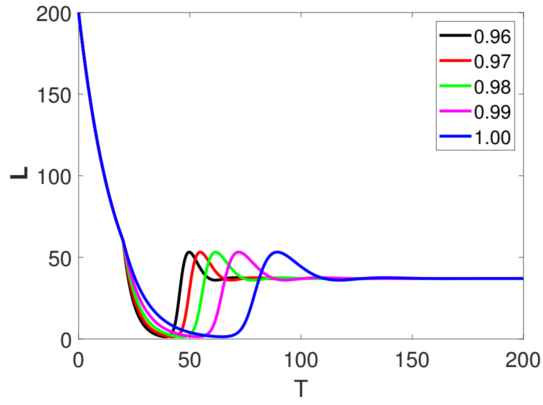

Figure 5: Dynamics of L for the model (1.2) on different fractional-order δ for both sub intervals with time 20-200.

Similarly, Fig. 2 demonstrates the population behavior of those susceptible individuals on which the lock down is imposed. Here it is observed that the individuals in S_L increases up-to 690 until $t = 100$, then gradually decreases at the point 520 and becomes stable, now when we see at lower fractional orders we observe that S_L becomes stable at $t = 70$. Further Figs. 3 and 4 project the behavior of the infected individuals I and those infected on which lock down is imposed I_L. From Fig. 3, we see that the there is very large amount of infections in the class I, where the lock down is not imposed. On the other hand Fig. 4 shows that there is less amount of infections after the iposition of increasing and decreasing during lock down. Finally Fig. 5 shows the cumulative density of the dynamics of lock down program.

6. Conclusion

In the considered study, we have analyzed the behavior of the COVID-19 model in which the lock down is imposed in piece-wise derivative sense. The novel piece-wise operator used for the analysis of the model is considered with classical-Caputo operator. By using the fixed point approach, existence as well as uniqueness results of the solution are presented. Numerically the proposed model is approximated using the Newton polynomial interpolation scheme. The numerical solution of the system is presented graphically with different fractional orders. We observed that the individuals in S_L increases up-to 690 till $t = 100$, then slowly decreases and becomes stable, with fractional orders it is observed that S_L becomes stable at $t = 70$. Moreover, it is observed that there is very large amount of infections in the class I, where the lock down is not imposed.

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