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Stability of Davison functional equation with n- variables over non-Archimedean (n, β) normed spaces

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Abstract

In this paper, we discuss the Hyers-Ulam stability of the Davison functional equation

$$\sum_{i=1}^{n-1} m(x_i x_n) + \sum_{i=1}^{n-1} m(x_i + x_n) = \sum_{i=1}^{n-1} m(x_i x_n + x_i) + m(x_n)$$

with n-variables over non-Archimedean (n, β) normed spaces (NA$n\beta$NS). Also, discuss some results for the same with suitable counter-example.

Keywords: Hyers-Ulam stability, Davison functional equation, non-Archimedean (n, β) normed spaces (NA$n\beta$NS).

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1. Introduction

Ulam famously raised the topic of functional equation stability. Instead of a functional equation, imagine a functional inequality and pose the following question: When may we say that the inequality's solutions are close to those of the equation? The problem is: "Given a group $(K, *)$, a metric group $(K, .)$ with the metric d and a mapping $g : K \to K'$, does there exist $\delta > 0$ such that if

$$d(g(x * y), g(x).g(y)) \leqslant \delta$$

for all $x, y \in K$, then there is a homomorphism $i : K \to K'$ such that

$$d(g(x), i(x)) \leqslant \epsilon$$

for all $x \in K$?" ([35]).

Hyers [13, 15] presented the first mathematical solution on functional equation stability in 1941. He provided a great solution to Ulam's classic question about the stability of functional equations when G

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and H are considered to be Banach spaces. Further, Hyers[14] studied the stability of functional equations in several variables. Davison [9] put forward a functional equation,

$$m(x+y)+m(xy)=m(xy+x)+m(y).$$

Bourgin [6] studied classes of transformations and bordering transformations in 1951. Moreover, the theory of functional equations and its inequalities developed by many authors [3, 8, 22, 31, 32] and also some further generalizations of the Ulam-Hyers-Rassias stability of functional equations given by Wang [16]. Jung et al. [18, 17] proved Hyers-Ulam stability of the functional equations. Later, Jung et al. discussed a linear functional equation of the third order associated to the Fibonacci numbers [19] and trigonometric type functional equations obtained by Jung et al. [20].

Gajda [11] studied stability of additive mappings in 1951. Subsequently, Czerwik[7] developed the stability of the quadratic mapping in normed spaces whereas Alessa et al. [4] investigated Hyers-Ulam stability of functional equation deriving from quadratic mappings in NAnβNS. In 2002, Trif [34] showed the stability of a functional equation deriving from an inequality of Popoviciu for convex functions. Smajdor [33] analyzed notes on a Jensen-type functional equation in 2003.

Jung and Sahoo [21] investigated the Hyers-Ulam stability of an equation of Davison although its generalizations were studied by Girgensohn and Lajko [12]. In 2012, the stability of mixed additive and cubic functional equations in many variables in non-Archimedean spaces was examined by Ebadian and Zolfaghari [10]. Afterward, Lee et al. [25, 24] established the stability of the n-dimensional mixed type additive and quadratic functional equation in non-Archimedean normed space. In 2014, the stability of a functional equation associated with the Fibonacci numbers obtained by Mortici et al. [28].

Yang et al. [37] proved the stability of functional equation in (n, β) normed spaces in 2015. In 2017, Kim and Son [23] studied approximate Cauchy-Jensen mappings in non-Archimedean Quasi β normed space. Thereafter, Abdollahpour [2, 1] obtained the Hyers-Ulam stability of differential equations. In 2019, Park and Rassias [30] discussed additive functional equations and partial multipliers in C*-algebras as well as Liu et al. [27] gave stability of an AQCQ functional equation in NAnβNS. Recently, Najati and Sahoo [29] discussed two Pexiderized functional equation of Davison type in 2023.

This study examines the stability of the Davison functional equation

$$\sum_{i=1}^{n-1} m(x_i x_n)+\sum_{i=1}^{n-1} m(x_i+x_n)=\sum_{i=1}^{n-1} m(x_i x_n+x_i)+m(x_n)$$

with n-variables over NAnβNS. Also, discuss some results for the same.

2. Preliminaries

Definition 2.1 ([5, 36])**.** Let $|.| : \mathbb{K} \to \mathbb{R}$ be a function said to non-Archimedean (NA) valuation for any field $|.| : \mathbb{K} \to \mathbb{R}$, satisfying the following conditions. If $\mathbb{K}$ is any field, then a valuation (of rank 1) is a function $|.| : \mathbb{K} \to \mathbb{R}$ satisfying the following axioms:

(i) $|d| \geqslant 0$, when $d \neq 0$;
(ii) $|d| = 0$, when $d = 0$;
(iii) $|de| = |d||e|$;
(iv) $|d+e| \leqslant \max\{|d|, |e|\}$.

Definition 2.2 ([5])**.** Let p be a positive prime number. Every non-zero rational number y can be expressed in the form of

$$y = p^{\beta}.\frac{g}{h}$$

where g, h, β are integers.

$$|y|_p = \frac{1}{p^\beta} \text{ when } y \neq 0, \quad |0|_p = 0 \text{ when } y = 0.$$

This is called a p-adic valuation.

Example 2.3. Let $y = \frac{676}{17}$. The 13-adic absolute value is

$$y = 169.\frac{4}{17} = 13^2.\frac{4}{7},$$

which means $|y|_{13} = \frac{1}{13^2}$. We also find the 17-adic absolute value for y as

$$y = 17^{-1} \cdot 676, \quad |y|_{17} = \frac{1}{17^{-1}} = 17.$$

Definition 2.4 ([5, 36])**.** Let a function $\|.\| : L \to \mathbb{R}$ be called a non-Archimedean norm if it satisfies the following conditions:

(i) $\|d\| = 0$ iff $d = 0$ for all $d \in L$;
(ii) $\|\alpha d\| = |\alpha|\|d\|$ for all $d \in L$ and $\alpha \in \mathbb{K}$;
(iii) $\|d + e\| \leqslant \max\{\|d\|, \|e\|\}$ for all $d, e \in L$,

where L is a vector space over a field $\mathbb{K}$.

Definition 2.5 ([26])**.** Let L be a real vector space with $\dim L \geqslant n$ over scalar field K with a non-Archimedean nontrivial valuation $|.|$, where n is a positive integer. β is a constant and $0 < \beta \leqslant 1$. A real-valued function $\|, \ldots, \|_\beta : L_n \to \mathbb{R}$ is called a (n, β)-norm on L if the following conditions hold:

(i) $\|d_1, \ldots, d_n\|_\beta = 0$ if and only if $d_1, \ldots, d_n$ are linearly dependent;
(ii) $\|d_1, \ldots, d_n\|_\beta$ is invariant under permutations of $d_1, \ldots, d_n$;
(iii) $\|\gamma d_1, d_2, \ldots, d_n\|_\beta = |\gamma|_\beta \|d_1, d_2, \ldots, d_n\|_\beta$;
(iv) $\|d_0 + d_1, d_2, \ldots, d_n\|_\beta \leqslant \max\{\|d_0, d_2, \ldots, d_n\|_\beta, \|d_1, d_2, \ldots, d_n\|_\beta\}$,

for all $\gamma \in K$ and $d_0, d_1, \ldots, d_n \in L$. Then $(L, \|, \ldots, \|_\beta)$ is called a NAnβNS.

Definition 2.6. A sequence $\{x_n\}$ in a NAnβNS L is a Cauchy sequence if and only if $\{x_{n+1} - x_n\}$ converges to zero.

Throughout this paper, let T and U be a additive group and complete NAnβNS, respectively. Let

$$E(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n-1} m(x_i x_n) + \sum_{i=1}^{n-1} m(x_i + x_n) - \sum_{i=1}^{n-1} m(x_i x_n + x_i) - m(x_n).$$

3. Main results

Theorem 3.1. *Let $\phi : T^n \to [0, \infty)$ be a function such that*

$$\lim_{s\to\infty} \frac{1}{|2|^{s\beta}} \tilde{\phi}(2^s x) = 0 \tag{3.1}$$

and let for each $x \in T$, $\psi : U^{n-1} \to [0, \infty)$ be a function, then

$$\lim_{s\to\infty} \max\left\{\frac{1}{|2|^{a\beta}} \tilde{\phi}(2^a x) : 0 \leqslant a < s\right\} \tag{3.2}$$

denoted by $\overline{\phi}(x)$ *exists. Suppose that* $m : T \to U$ *is a mapping satisfying* $m(0) = 0$ *such that*

$$\|E(x_1, x_2, \dots, x_n), y_1, y_2, \dots, y_{n-1}\|_\beta \leqslant \phi(x_1, x_2, \dots, x_n)\psi(y_1, y_2, \dots, y_{n-1}), \tag{3.3}$$

then there is an additive mapping $V : T \to U$ *so that*

$$\|m(x) - V(x), y_1, y_2, \dots, y_{n-1}\|_\beta \leqslant \max\left\{\frac{1}{|2|^\beta}\overline{\phi}(x), \phi(x, 0, 0, \dots, 0)\right\}\psi(y_1, y_2, \dots, y_{n-1}), \tag{3.4}$$

for each $y_1, y_2, \dots, y_{n-1} \in U$*. Moreover, if*

$$\lim_{r\to\infty}\lim_{s\to\infty}\max\left\{\frac{1}{|2|^{a\beta}}\tilde{\phi}(2^a x) : r \leqslant a < r+s\right\} = 0,$$

then V *is unique.*

Proof. Put $x_2, x_3, \dots, x_{n-1} = 0$ in (3.3), we get

$$\begin{aligned} &\|m(x_1x_n) + m(x_1 + x_n) - m(x_n) - m(x_1x_n + x_1), y_1, y_2, \dots, y_{n-1}\|_\beta \\ &\quad \leqslant \phi(x_1, 0, \dots 0, x_n)\psi(y_1, y_2, \dots, y_{n-1}), \end{aligned} \tag{3.5}$$

for each $x_1, x_n \in T$, $y_1, \dots, y_{n-1} \in U$. If we replace x_n by x_{n+1} in (3.5), we get

$$\begin{aligned} &\|m(x_1x_n + x_1) + m(x_1 + x_n + 1) - m(x_1x_n + 2x_1) - m(x_n + 1), y_1, y_2, \dots, y_{n-1}\|_\beta \\ &\quad \leqslant \phi(x_1, 0, \dots, 0, x_n + 1)\psi(y_1, y_2, \dots, y_{n-1}), \end{aligned} \tag{3.6}$$

for each $x_1, x_n \in T$, $y_1, \dots, y_{n-1} \in U$. Using (3.5) and (3.6),

$$\begin{aligned} &\|m(x_1x_n) + m(x_1 + x_n) + m(x_1 + x_n + 1) - m(x_n) - m(x_1x_n + 2x_1) - m(x_n + 1), y_1, y_2, \dots, y_{n-1}\|_\beta \\ &\quad \leqslant \max\left\{\phi(x_1, 0, \dots, 0, x_n), \phi(x_1, 0, \dots, 0, x_n + 1)\right\}\psi(y_1, y_2, \dots, y_{n-1}), \end{aligned} \tag{3.7}$$

for each $x_1, x_n \in T$, $y_1, \dots, y_{n-1} \in U$. Replacing x_1 by $\frac{x_1}{2}$ and x_n by $2x_n$ in (3.7),

$$\begin{aligned} &\left\|m(x_1x_n) + m\left(\frac{x_1}{2} + 2x_n\right) + m\left(\frac{x_1}{2} + 2x_n + 1\right) - m(x_1x_n + x_1) - m(2x_n + 1), y_1, y_2, \dots, y_{n-1}\right\|_\beta \\ &\quad \leqslant \max\left\{\phi\left(\frac{x_1}{2}, 0, \dots, 0, 2x_n\right), \phi\left(\frac{x_1}{2}, 0, \dots, 0, 2x_n + 1\right)\right\}\psi(y_1, y_2, \dots, y_{n-1}), \end{aligned} \tag{3.8}$$

for each $x_1, x_n \in T$, $y_1, \dots, y_{n-1} \in U$. From (3.5) and (3.8), we have

$$\begin{aligned} &\left\|m\left(\frac{x_1}{2} + 2x_n\right) + m\left(\frac{x_1}{2} + 2x_n + 1\right) - m(x_1 + x_n) - m(2x_n) - m(2x_n + 1) + m(x_n), y_1, y_2, \dots, y_{n-1}\right\|_\beta \\ &\quad \leqslant \max\left\{\phi\left(\frac{x_1}{2}, 0, \dots, 0, 2x_n\right), \phi\left(\frac{x_1}{2}, 0, \dots, 0, 2x_n + 1\right), \phi(x_1, x_2, \dots, x_n)\right\}\psi(y_1, y_2, \dots, y_{n-1}), \end{aligned} \tag{3.9}$$

for each $x_1, x_n \in T$, $y_1, \dots, y_{n-1} \in U$. If we replace x_1 by $x_1 - x_n$ in (3.9), we have

$$\begin{aligned} &\left\|m\left(\frac{x_1}{2} + \frac{3x_n}{2}\right) + m\left(\frac{x_1}{2} + \frac{3x_n}{2} + 1\right) - m(x_1) - m(2x_n) - m(2x_n + 1) + m(x_n), y_1, y_2, \dots, y_{n-1}\right\|_\beta \\ &\quad \leqslant \max\left\{\phi\left(\frac{x_1}{2} - \frac{x_n}{2}, 0, \dots, 0, 2x_n\right), \phi\left(\frac{x_1}{2} - \frac{x_n}{2}, 0, \dots, 0, 2x_n + 1\right), \phi(x_1 - x_n, x_2, \dots, x_n)\right\} \\ &\quad\quad \times \psi(y_1, y_2, \dots, y_{n-1}), \end{aligned} \tag{3.10}$$

for each $x_1, x_n \in T$, $y_1, \ldots, y_{n-1} \in U$. Put x_n as $\frac{x_n}{3}$ in (3.10), we have

$$\begin{aligned}
&\left\| m\left(\frac{x_1}{2}+\frac{x_n}{2}\right)+m\left(\frac{x_1}{2}+\frac{x_n}{2}+1\right)-m(x_1)-m\left(\frac{2x_n}{3}\right)-m\left(\frac{2x_n}{3}+1\right)+m\left(\frac{x_n}{3}\right), y_1, y_2, \ldots, y_{n-1}\right\|_\beta \\
&\quad\leqslant \max\left\{\phi\left(\frac{x_1}{2}-\frac{x_n}{6}, 0, \ldots, 0, \frac{2x_n}{3}\right), \phi\left(\frac{x_1}{2}-\frac{x_n}{6}, 0, \ldots, 0, \frac{2x_n}{3}+1\right),\right. \\
&\qquad \left.\times\, \phi\left(x_1-x_n, x_2, \ldots, \frac{x_n}{3}\right)\right\}\psi(y_1, y_2, \ldots, y_{n-1}),
\end{aligned} \tag{3.11}$$

for each $x_1, x_n \in T$, $y_1, \ldots, y_{n-1} \in U$. Let us define functions $g, h : T \to U$ by

$$g(x)=m\left(\frac{2x}{3}\right)+m\left(\frac{2x}{3}+1\right)-m\left(\frac{x}{3}\right), \qquad h(x)=m\left(\frac{x}{2}\right)+m\left(\frac{x}{2}+1\right), \tag{3.12}$$

for each $x \in T$, $y_1, \ldots, y_{n-1} \in U$. From (3.11) and (3.12), we obtain the following inequality concerning the Pexider equation:

$$\begin{aligned}
&\|h(x_1+x_n)-m(x_n)-g(x_n), y_1, y_2, \ldots, y_{n-1}\|_\beta \\
&\quad\leqslant \max\left\{\phi\left(\frac{x_1}{2}-\frac{x_n}{6}, 0, \ldots, \frac{2x_n}{3}\right), \phi\left(\frac{x_1}{2}-\frac{x_n}{6}, 0, \ldots, \frac{2x_n}{3}+1\right), \phi\left(x_1-x_n, 0, \ldots, 0, \frac{x_n}{3}\right)\right\} \\
&\quad \times \psi(y_1, y_2, \ldots, y_{n-1}),
\end{aligned} \tag{3.13}$$

for each $x_1, x_n \in T$, $y_1, \ldots, y_{n-1} \in U$. Put $x_n = 0$ in (3.13), we obtain

$$\begin{aligned}
&\|h(x_1)-m(x_1)-g(0), y_1, y_2, \ldots, y_{n-1}\|_\beta \\
&\quad\leqslant \max\left\{\phi\left(\frac{x_1}{2}, 0, \ldots, 0\right), \phi\left(\frac{x_1}{2}, 0, \ldots, 0, 1\right) \phi(x_1, 0, \ldots, 0)\right\}\psi(y_1, y_2, \ldots, y_{n-1}),
\end{aligned} \tag{3.14}$$

for each $x_1, x_n \in T$, $y_1, \ldots, y_{n-1} \in U$. Put $x_1 = 0$ in (3.13), we obtain

$$\begin{aligned}
&\|h(x_n)-m(0)-g(x_n), y_1, y_2, \ldots, y_{n-1}\|_\beta \\
&\quad\leqslant \max\left\{\phi\left(\frac{-x_n}{6}, 0, \ldots, 0, \frac{2x_n}{3}\right), \phi\left(\frac{-x_n}{6}, 0, \ldots, 0, \frac{2x_n}{3}+1\right), \phi\left(-x_n, 0, \ldots, 0, \frac{x_n}{3}\right)\right\} \\
&\quad \times \psi(y_1, y_2, \ldots, y_{n-1}),
\end{aligned} \tag{3.15}$$

for each $x_n \in T$, $y_1, \ldots, y_{n-1} \in U$. Define $H : T \to U$ by

$$H(x) = h(x) - m(0) - g(0), \tag{3.16}$$

for each $x \in T$. From (3.13)-(3.16), we have

$$\begin{aligned}
&\|H(x_1+x_n)-H(x_1)-H(x_n), y_1, y_2, \ldots, y_{n-1}\|_\beta \\
&\quad\leqslant \max\left\{\phi\left(\frac{x_1}{2}-\frac{x_n}{6}, 0, \ldots, \frac{2x_n}{3}\right), \phi\left(\frac{x_1}{2}-\frac{x_n}{6}, 0, \ldots, \frac{2x_n}{3}+1\right), \phi\left(x_1-x_n, 0, \ldots, 0, \frac{x_n}{3}\right),\right. \\
&\qquad \phi\left(\frac{x_1}{2}, 0, \ldots, 0\right), \phi\left(\frac{x_1}{2}, 0, \ldots, 0, 1\right), \phi(x_1, 0, \ldots, 0), \phi\left(\frac{-x_n}{6}, 0, \ldots, 0, \frac{2x_n}{3}\right), \\
&\qquad \left.\phi\left(\frac{-x_n}{6}, 0, \ldots, 0, \frac{2x_n}{3}+1\right), \phi\left(-x_n, 0, \ldots, 0, \frac{x_n}{3}\right)\right\}\psi(y_1, y_2, \ldots, y_{n-1}),
\end{aligned} \tag{3.17}$$

for each $x_1, x_n \in T$, $y_1, \ldots, y_{n-1} \in U$. Substituting $(x_1, x_2, \ldots, x_n)$ as $(x, 0, \ldots, 0, x)$ in (3.17):

$$\|H(2x) - 2H(x), y_1, y_2, \ldots, y_{n-1}\|_\beta \leqslant \tilde{\phi}(x)\psi(y_1, y_2, \ldots, y_{n-1}), \tag{3.18}$$

for each $x \in T$, $y_1, \ldots, y_{n-1} \in U$, where

$$\tilde{\phi}(x) = \max\left\{\phi\left(\frac{x}{3}, 0, \ldots, 0, \frac{2x}{3}\right), \phi\left(\frac{x}{3}, 0, \ldots, 0, \frac{2x}{3} + 1\right), \phi(0, \ldots, 0, x), \phi\left(\frac{x}{2}, 0, \ldots, 0, 1\right), \phi\left(\frac{x}{2}, 0, \ldots, 0\right),\right.$$
$$\left.\phi(x, 0, \ldots, 0), \phi\left(\frac{-x}{6}, 0, \ldots, 0, \frac{2x}{3}\right), \phi\left(\frac{-x}{6}, 0, \ldots, 0, \frac{2x}{3} + 1\right), \phi\left(-x, 0, \ldots, 0, \frac{x}{3}\right)\right\},$$

for each $x \in T$, $y_1, \ldots, y_{n-1}$. Replacing x by $2^{s-1}x$ and dividing by $2^{s\beta}$ in (3.18),

$$\left\|\frac{H(2^s x)}{2^s} - \frac{H(2^{s-1}x}{2^{s-1}}), y_1, y_2, \ldots, y_{n-1}\right\|_\beta \leqslant \frac{1}{|2|^{s\beta}}\tilde{\phi}(2^{s-1}x)\psi(y_1, y_2, \ldots, y_{n-1}),$$

for each $x \in T$, $y_1, \ldots, y_{n-1} \in U$. Hence $\{\frac{H(2^s x)}{2^s}\}$ is a Cauchy sequence. Define

$$V(x) = \lim_{s\to\infty} \frac{H(2^s x)}{2^s}, \tag{3.19}$$

for each $x \in T$, $y_1, \ldots, y_{n-1} \in U$. By induction,

$$\left\|\frac{H(2^s x)}{2^s} - H(x), y_1, y_2, \ldots, y_{n-1}\right\|_\beta \leqslant \max\{\frac{1}{|2|^{a\beta}}\tilde{\phi}(2^a x) : 0 \leqslant a < s\}, \tag{3.20}$$

for each $x \in T$, $y_1, \ldots, y_{n-1} \in U$. By taking limit $s \to \infty$ in (3.20), using (3.2), we get (3.4). Replacing x_1 and x_n by $2^s x_1$ and $2^s x_n$ in (3.17) and by (3.1) and (3.3), we have

$$\begin{aligned}
&\|H(2^s(x_1 + x_n)) - H(2^s x_1) - H(2^s x_n), y_1, y_2, \ldots, y_{n-1}\|_\beta \\
&\leqslant \frac{1}{|2|^s\beta}\max\left\{\phi\left(\frac{2^s x_1}{2} - \frac{2^s x_n}{6}, 0, \ldots, \frac{2^{s+1}x_n}{3}\right), \phi\left(\frac{2^s x_1}{2} - \frac{2^s x_n}{6}, 0, \ldots, \frac{2^{s+1}x_n}{3} + 1\right),\right. \\
&\qquad \phi\left(2^s x_1 - 2^s x_n, 0, \ldots, 0, \frac{2^s x_n}{3}\right), \phi\left(\frac{2^s x_1}{2}, 0, \ldots, 0\right), \phi\left(\frac{2^s x_1}{2}, 0, \ldots, 0, 1\right), \\
&\qquad \phi(2^s x_1, 0, \ldots, 0), \phi\left(\frac{-2^s x_n}{6}, 0, \ldots, 0, \frac{2^{s+1}x_n}{3}\right), \phi\left(\frac{-2^s x_n}{6}, 0, \ldots, 0, \frac{2^{s+1}x_n}{3} + 1\right), \\
&\qquad \left.\phi\left(-2^s x_n, 0, \ldots, 0, \frac{2^s x_n}{3}\right)\right\}\psi(y_1, y_2, \ldots, y_{n-1}),
\end{aligned}$$

for each $x_1, x_n \in T$, $y_1, \ldots, y_{n-1} \in U$. This means H satisfies additive. Let V' be another additive function satisfying (3.4),

$$\|V(x) - V'(x), y_1, y_2, \ldots, y_{n-1}\|_\beta \leqslant \lim_{r\to\infty}\lim_{s\to\infty}\max\left\{\frac{1}{|2|^{a\beta}}\tilde{\phi}(2^a x) : r \leqslant a < r + s\right\}\psi(y_1, y_2, \ldots, y_{n-1}),$$

for each $x \in T$, $y_1, \ldots, y_{n-1} \in U$. Hence, from (3.14), (3.16), (3.18), (3.19), we have

$$\|m(x) - m(0) - H(x), y_1, y_2, \ldots, y_{n-1}\|_\beta \leqslant \max\left\{\frac{1}{|2|^\beta}\overline{\phi}(x), \phi(x, 0, 0, \ldots, 0)\right\}\psi(y_1, y_2, \ldots, y_{n-1}),$$

for each $x \in T$, $y_1, \ldots, y_{n-1} \in U$. Hence the proof. □

Corollary 3.2. *Let l, θ be positive real numbers and $l > 1$. If a mapping $m : T \to U$ satisfies*

$$\|E(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_{n-1})\|_\beta \leqslant \theta \sum_{i=1}^{n} \|x_i\|_\beta^l,$$

then there is a unique additive mapping $V : T \to U$ so that

$$\|m(x) - V(x), y_1, y_2, \dots, y_{n-1}\|_\beta \leqslant \theta \max\left\{\frac{1}{|2|^\beta}\overline{\phi}(x), \phi(x, 0, 0, \dots, 0)\right\}\psi(y_1, y_2, \dots, y_{n-1}),$$

where

$$\overline{\phi}(x) = \lim_{s\to\infty} \max\left\{\frac{1}{|2|^{a\beta}}\tilde{\phi}(2^a x) : 0 \leqslant a < s\right\},$$

$$\tilde{\phi}(x) = \theta \max\left\{\left\|\frac{x}{3}\right\|_\beta^l + \left\|\frac{2x}{3}\right\|_\beta^l, \left\|\frac{x}{3}\right\|_\beta^l + \left\|\frac{2x}{3} + 1\right\|_\beta^l, \left\|x\right\|_\beta^l, \left\|\frac{x}{2}\right\|_\beta^l, \right.$$
$$\left. \left\|\frac{-x}{6}\right\|_\beta^l + \left\|\frac{2x}{3}\right\|_\beta^l, \left\|\frac{-x}{6}\right\|_\beta^l + \left\|\frac{2x}{3} + 1\right\|_\beta^l, \left\|-x\right\|_\beta^l + \left\|\frac{x}{3}\right\|_\beta^l\right\},$$

for each $x \in T$, $y_1, \dots, y_{n-1} \in U$.

Example 3.3. Let $T = U = \mathbb{Q}_p$ and define $m(x) = x + k$ for any constant k. Let $|2|_p^t = 1$, $\theta > k, t \in \mathbb{Z}$, $l > 1$, $p > 2$, where p is prime. If $m : T \to U$ is an odd mapping satisfying $m(0) = 0$ and

$$\|E(x_1, 0, \dots, 0, x_n, y_1, y_2, \dots, y_{n-1})\|_\beta = 0 \leqslant \theta \sum_{i=1}^{n} \|x_i\|_\beta^l \psi(y_1, y_2, \dots, y_{n-1}),$$

then there is an additive mapping $V : T \to U$ so that

$$\|m(x) - V(x), y_1, y_2, \dots, y_{n-1}\|_\beta = k \leqslant \max\left\{\frac{1}{|2|^\beta}\overline{\phi}(x), \phi(x, 0, 0, \dots, 0)\right\}\psi(y_1, y_2, \dots, y_{n-1}),$$

which is unique for each $x \in T$, $y_1, \dots, y_{n-1} \in U$.

In the case of $l = 1$, we have following counter-example.

Example 3.4. Let $T = U = \mathbb{Q}_p$ and define $m(x) = x + \frac{1}{x}$. Let $|2|_p^t = 1$, $t \in \mathbb{Z}$, $p > 2$, where p is prime. If $m : T \to U$ is an odd mapping satisfying $m(0) = 0$ and

$$\|E(x_1, 0, \dots, 0, x_n, y_1, y_2, \dots, y_{n-1})\|_\beta = 0 \leqslant \theta \sum_{i=1}^{n} \|x_i\|_\beta^l \psi(y_1, y_2, \dots, y_{n-1}),$$

$$\left\|\frac{H(2^s x)}{2^s} - \frac{H(2^{s-1}x)}{2^{s-1}}, y_1, y_2, \dots, y_{n-1}\right\|_\beta = \frac{|3|^{l\beta}}{|2|^{2s\beta}\|x\|_\beta^l} \neq 0,$$

for each $x \in T$, $y_1, \dots, y_{n-1} \in U$, hence $\left\{\frac{H(2^s x)}{2^s}\right\}$ is not a Cauchy.

Theorem 3.5. *Let $\phi : T^n \to [0, \infty)$ be a function so that*

$$\lim_{s\to\infty} |2|^{s\beta}\tilde{\phi}\left(\frac{x}{2^s}\right) = 0 \tag{3.21}$$

and let for each $x \in T$, $\psi : U^{n-1} \to [0,\infty)$ *be a function so that the limit*

$$\lim_{s\to\infty} \max\left\{ |2|^{a\beta} \tilde{\phi}\left(\frac{x}{2^{a+1}}\right) : 0 \leqslant a < s \right\} \tag{3.22}$$

denoted by $\overline{\phi}(x)$ *exists. Suppose that* $m : T \to U$ *is a mapping satisfying* $m(0) = 0$ *and*

$$\|E(x_1, x_2, \dots, .x_n), y_1, y_2, \dots, y_{n-1}\|_\beta \leqslant \phi(x_1, x_2, \dots, x_n)\psi(y_1, y_2, \dots, y_{n-1}), \tag{3.23}$$

then there is an additive mapping $V : T \to U$ *so that*

$$\|m(x) - V(x), y_1, y_2, \dots, y_{n-1}\|_\beta \leqslant \max\left\{\overline{\phi}(x), \phi(x, 0, 0, \dots, 0)\right\}\psi(y_1, y_2, \dots, y_{n-1}), \tag{3.24}$$

for each $y_1, y_2, \dots, y_{n-1} \in U$. *Moreover, if*

$$\lim_{r\to\infty}\lim_{s\to\infty} \max\left\{ |2|^{a\beta} \tilde{\phi}\left(\frac{x}{2^{a+1}}\right) : r \leqslant a < r+s \right\} = 0,$$

then V *is unique.*

Proof. Using Theorem 3.1, replacing x by $\frac{x}{2^s}$ and multiplying by $|2|^{(s-1)\beta}$ in (3.18),

$$\left\|2^{s-1}H\left(\frac{x}{2^{s-1}}\right) - 2^sH\left(\frac{x}{2^s}\right), y_1, y_2, \dots, y_{n-1}\right\|_\beta \leqslant |2|^{(s-1)\beta}\tilde{\phi}\left(\frac{x}{2^s}\right)\psi(y_1, y_2, \dots, y_{n-1}),$$

for each $x \in T$, $y_1, \dots, y_{n-1} \in U$. Hence $\left\{2^sH\left(\frac{x}{2^s}\right)\right\}$ is a Cauchy sequence. Define

$$V(x) = \lim_{s\to\infty} 2^sH\left(\frac{x}{2^s}\right),$$

for each $x \in T$, $y_1, \dots, y_{n-1} \in U$. By induction,

$$\left\|2^sH(\frac{x}{2^s}) - H(x), y_1, y_2, \dots, y_{n-1}\right\|_\beta \leqslant \max\left\{ |2|^{a\beta} \tilde{\phi}\left(\frac{x}{2^{a+1}}\right) : 0 \leqslant a < s \right\}, \tag{3.25}$$

for each $x \in T$, $y_1, \dots, y_{n-1} \in U$. By taking limit $s \to \infty$ in (3.25), using (3.22), we get (3.24). Replacing x_1 and x_n by $\frac{x_1}{2^s}$ and $\frac{x_n}{2^s}$ in (3.17) and by (3.21) and (3.23), we have

$$\begin{aligned}
&\left\|H\left(\frac{x_1+x_n}{2^s}\right) - H\left(\frac{x_1}{2^s}\right) - H\left(\frac{x_n}{2^s}\right), y_1, y_2, \dots, y_{n-1}\right\|_\beta \\
&\leqslant |2|^{s\beta} \max\Bigg\{ \phi\left(\frac{x_1}{2^{s+1}} - \frac{x_n}{2^s.6}, 0, \dots, \frac{2x_n}{2^s.3}\right), \phi\left(\frac{x_1}{2^{s+1}} - \frac{x_n}{2^s.6}, 0, \dots, \frac{2x_n}{2^s.3} + 1\right), \\
&\qquad \phi\left(\frac{x_1}{2^s} - \frac{x_n}{2^s}, 0, \dots, 0, \frac{x_n}{2^s.3}\right), \phi\left(\frac{x_1}{2^{s+1}}, 0, \dots, 0\right), \\
&\qquad \phi\left(\frac{x_1}{2^{s+1}}, 0, \dots, 0, 1\right), \phi(\frac{x_1}{2^s}, 0, \dots, 0), \phi\left(\frac{-x_n}{2^s.6}, 0, \dots, 0, \frac{2x_n}{2^s.3}\right), \\
&\qquad \phi\left(\frac{-x_n}{2^s.6}, 0, \dots, 0, \frac{2x_n}{2^s.3} + 1\right), \phi\left(\frac{-x_n}{2^s}, 0, \dots, 0, \frac{x_n}{2^s.3}\right) \Bigg\}\psi(y_1, y_2, \dots, y_{n-1}),
\end{aligned}$$

for each $x_1, x_n \in T$, $y_1, \dots, y_{n-1} \in U$. This means H satisfies additive. Let V' be another additive function satisfying (3.24),

$$\|V(x) - V'(x), y_1, y_2, \dots, y_{n-1}\|_\beta \leqslant \lim_{r\to\infty}\lim_{s\to\infty} \max\left\{ |2|^{a\beta} \tilde{\phi}\left(\frac{x}{2^{a+1}}\right) : r \leqslant a < r+s \right\},$$

for each $x \in T$, $y_1, \ldots, y_{n-1} \in U$. Hence from (3.14), (3.16), (3.23), (3.24), we have

$$\|m(x) - m(0) - H(x), y_1, y_2, \ldots, y_{n-1}\|_\beta \leqslant \max\{\overline{\phi}(x), \phi(x, 0, 0, \ldots, 0)\}\psi(y_1, y_2, \ldots, y_{n-1}),$$

for each $x \in T$, $y_1, \ldots, y_{n-1} \in U$. Hence the proof. □

Corollary 3.6. *Let l, θ be positive real numbers and $l < 1$. If a mapping $m : T \to U$ satisfies*

$$\|E(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_{n-1})\|_\beta \leqslant \theta \sum_{i=1}^{n} \|x_i\|_\beta^l,$$

then there is a unique additive mapping $V : T \to U$ so that

$$\|m(x) - V(x), y_1, y_2, \ldots, y_{n-1}\|_\beta \leqslant \max\{\overline{\phi}(x), \phi(x, 0, 0, \ldots, 0)\}\psi(y_1, y_2, \ldots, y_{n-1}),$$

where

$$\begin{aligned}
\overline{\phi}(x) &= \lim_{s\to\infty} \max\left\{ |2|^{a\beta}\tilde{\phi}(\frac{x}{2^{a+1}}) : 0 \leqslant a < s \right\},\\
\tilde{\phi}(x) &= \theta \max\left\{ \left\|\frac{x}{3}\right\|_\beta^l + \left\|\frac{2x}{3}\right\|_\beta^l, \left\|\frac{x}{3}\right\|_\beta^l + \left\|\frac{2x}{3}+1\right\|_\beta^l, \left\|x\right\|_\beta^l, \left\|\frac{x}{2}\right\|_\beta^l, \left\|\frac{-x}{6}\right\|_\beta^l + \left\|\frac{2x}{3}\right\|_\beta^l, \right.\\
&\qquad \left. \left\|\frac{-x}{6}\right\|_\beta^l + \left\|\frac{2x}{3}+1\right\|_\beta^l, \left\|-x\right\|_\beta^l + \left\|\frac{x}{3}\right\|_\beta^l \right\},
\end{aligned}$$

for each $x \in T$, $y_1, \ldots, y_{n-1} \in U$.

Example 3.7. Let $T = U = \mathbb{Q}_p$ and define $m(x) = x + k$ for any constant k. Let $|2|_p^t = 1$, $\theta > k, t \in \mathbb{Z}$, $l > 1$, $p > 2$, where p is prime. If $m : T \to U$ is an odd mapping satisfying $m(0) = 0$ and

$$\|E(x_1, 0, \ldots 0, x_n, y_1, y_2, \ldots, y_{n-1})\|_\beta = 0 \leqslant \theta \sum_{i=1}^{n} \|x_i\|_\beta^l \psi(y_1, y_2, \ldots, y_{n-1}),$$

then there is a additive mapping $V : T \to U$ so that

$$\|m(x) - V(x), y_1, y_2, \ldots, y_{n-1}\|_\beta = k\theta \leqslant \max\{\overline{\phi}(x), \phi(x, 0, 0, \ldots, 0)\}\psi(y_1, y_2, \ldots, y_{n-1}),$$

which is unique, for each $x \in T$, $y_1, \ldots, y_{n-1} \in U$.

In the case of $l = 1$, we have the following counter-example.

Example 3.8. Let $T = U = \mathbb{Q}_p$ and define $m(x) = x + \frac{1}{x}$. Let $|2|_p^t = 1$, $t \in \mathbb{Z}$, $p > 2$, where p is prime. If $m : T \to U$ is an odd mapping satisfying $m(0) = 0$ and

$$\|E(x_1, 0, \ldots, 0, x_n, y_1, y_2, \ldots, y_{n-1})\|_\beta = 0 \leqslant \theta \sum_{i=1}^{n} \|x_i\|_\beta^l \psi(y_1, y_2, \ldots, y_{n-1}),$$

$$\left\|2^{s-1}H(\frac{x}{2^{s-1}}) - 2^s H(\frac{x}{2^s}), y_1, y_2, \ldots, y_{n-1}\right\|_\beta = \frac{|2|^{2s\beta}|3|^{l\beta}}{|4|^{l\beta}\|x\|_\beta^l} \neq 0,$$

for each $x \in T$, $y_1, \ldots, y_{n-1} \in U$, hence $\left\{2^s H\left(\frac{x}{2^s}\right)\right\}$ is not a Cauchy.

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