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Characterization of almost filters and their fuzzifications of semigroups

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Abstract

In this paper, we define and explore the properties and relationships of almost filters and fuzzy almost filters within semigroups. We examine whether the connection of two filters remains an almost filter in semigroups and verify if similar results hold within the framework of class fuzzifications. Furthermore, we investigate the connections between almost filters and fuzzy almost filters. We also define and demonstrate the characteristics of prime, semiprime, and strongly prime almost filters, including their applications as bi-filters in semigroups.

Keywords: Almost filter, almost bi-filters, fuzzy almost filters, fuzzy almost bi-filters.

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1. Introduction

The fundamental concept of a fuzzy set, introduced by Zadeh in his seminal paper [26] of 1965, has emerged as a significant mathematical tool for modeling systems that are too complex or ill-defined for precise mathematical description using traditional methods. The application of fuzzy sets in semigroups was advocated by Kuroki, enhancing the understanding of logical algebras as demonstrated by the filter theory [17]. The notions of left, right, and two-sided almost ideals in semigroups were first defined in 1980 by Grosek and Satko [5], who further explored minimal and smallest almost ideals in subsequent works [4, 16]. In 1981, Bogdanovic [1] initiated the concept of almost bi-ideals in semigroups, expanding on the ideas of almost ideals and bi-ideals. Further advancements were made in 2018 when Wattanatripop et al. [25] examined the properties of fuzzy almost bi-ideals in semigroups. By 2020, Kaopusek et al. [8] delineated the concepts of almost interior ideals and weakly almost interior ideals, exploring their interrelationships within semigroups. The following year, Krailoet et al. [11] investigated fuzzy almost interior ideals. Recent studies in 2022 by Chinram and Nakkhasen [2, 3] delved into almost bi-quasi-interior ideals and their fuzzy counterparts, as well as almost bi-interior ideals.

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Research on various types of almost ideals and fuzzy almost ideals has also been conducted in specialized areas such as Γ-semigroups [7, 18, 19], ordered semigroups [20, 23], ternary semigroups [22], semihypergroups [6, 13, 15, 21, 24], and LA-semihypergroups [14], including studies on bipolar fuzzy sets [9, 10].

The structure of this paper is as follows. Section 2 reviews essential concepts and preliminaries concerning types of ideals, filters, fuzzy sets, fuzzy ideals, and fuzzy filters in semigroups. Section 3 defines and discusses almost left (right) filters and bi-filters in semigroups, including their basic properties. Sections 4-6 explore fuzzy almost left (right) filters and bi-filters, examining their characteristics and defining minimal, maximal, prime, semiprime, and strongly prime almost filter (bi-filters). The paper concludes by establishing the relationships between almost filters and fuzzy almost filters in semigroups.

2. Preliminaries

In this section, we introduce key concepts and findings related to semigroups and fuzzy semigroups, which will be instrumental for the discussions in subsequent sections.

Definition 2.1. Suppose $\mathfrak{F}$ be any non-empty subset of semigroup $\mathfrak{S}$, it is called

(1) a *subsemigroup* of $\mathfrak{S}$ if $\mathfrak{F}^2 \subseteq \mathfrak{F}$;
(2) a *left ideal* (LI) of $\mathfrak{S}$ if $\mathfrak{S}\mathfrak{F} \subseteq \mathfrak{F}$;
(3) a *right ideal* (RI) of $\mathfrak{S}$ if $\mathfrak{F}\mathfrak{S} \subseteq \mathfrak{F}$;
(4) an *ideal* (ID) $\mathfrak{F}$ of a semigroup $\mathfrak{S}$ which means that $\mathfrak{F}$ is both an LI and an RI of $\mathfrak{S}$;
(5) a *bi-ideal* (BI) of $\mathfrak{S}$ if $\mathfrak{F}$ is a subsemigroup of $\mathfrak{S}$ and $\mathfrak{F}\mathfrak{S}\mathfrak{F} \subseteq \mathfrak{F}$;
(6) a *left almost ideal* (LAI) of $\mathfrak{S}$ if $\mathfrak{r}\mathfrak{F} \cap \mathfrak{F} \neq \emptyset$ for all $\mathfrak{r} \in \mathfrak{S}$;
(7) a *right almost ideal* (RAI) of $\mathfrak{S}$ if $\mathfrak{F}\mathfrak{r} \cap \mathfrak{F} \neq \emptyset$ for all $\mathfrak{r} \in \mathfrak{S}$;
(8) an *almost ideal* (AI) $\mathfrak{F}$ of a semigroup $\mathfrak{S}$ mean that $\mathfrak{F}$ is both an LAI and an RAI of $\mathfrak{S}$;
(9) an *almost bi-ideal* (ABI) of $\mathfrak{S}$ if $\mathfrak{F}\mathfrak{r}\mathfrak{F} \cap \mathfrak{F} \neq \emptyset$ for all $\mathfrak{r} \in \mathfrak{S}$.

Definition 2.2 ([17])**.** A subsemigroup $\mathfrak{F}$ of a semigroup $\mathfrak{S}$ is called

(1) a *left filter* (LFR) of $\mathfrak{S}$ if $\mathfrak{r}\mathfrak{e} \in \mathfrak{F} \Rightarrow \mathfrak{e} \in \mathfrak{F}$ for all $\mathfrak{r}, \mathfrak{e} \in \mathfrak{S}$;
(2) a *right filter* (RFR) of $\mathfrak{S}$ if $\mathfrak{r}\mathfrak{e} \in \mathfrak{F} \Rightarrow \mathfrak{r} \in \mathfrak{F}$ for all $\mathfrak{r}, \mathfrak{e} \in \mathfrak{S}$;
(3) a *filter* (FR) of $\mathfrak{S}$ if $\mathfrak{r}\mathfrak{e} \in \mathfrak{F} \Rightarrow \mathfrak{r}, \mathfrak{e} \in \mathfrak{F}$ for all $\mathfrak{r}, \mathfrak{e} \in \mathfrak{S}$;
(4) a *bi-filter* (BFR) of $\mathfrak{S}$ if $\mathfrak{r}\mathfrak{e}\mathfrak{r} \in \mathfrak{F} \Rightarrow \mathfrak{r} \in \mathfrak{F}$ for all $\mathfrak{r}, \mathfrak{e} \in \mathfrak{S}$.

For any $\mathfrak{h}_i \in [0,1]$, $i \in \mathfrak{F}$, define $\bigvee\limits_{i\in\mathfrak{F}} \mathfrak{h}_i := \sup_{i\in\mathfrak{F}}\{\mathfrak{h}_i\}$ and $\bigwedge\limits_{i\in\mathfrak{F}} \mathfrak{h}_i := \inf_{i\in\mathfrak{F}}\{\mathfrak{h}_i\}$. We see that for any $\mathfrak{h}, \mathfrak{r} \in [0,1]$, we have $\mathfrak{h} \vee \mathfrak{r} = \max\{\mathfrak{h}, \mathfrak{r}\}$ and $\mathfrak{h} \wedge \mathfrak{r} = \min\{\mathfrak{h}, \mathfrak{r}\}$. Any function ζ from $\mathfrak{S}$ to closed interval $[0,1]$ is called *fuzzy set* (FS).

For any two FSs ζ and ξ of a non-empty set $\mathfrak{S}$, define the inclusion relation as follows:

(1) $\zeta \geqslant \xi \Leftrightarrow \zeta(\mathfrak{r}) \geqslant \xi(\mathfrak{r})$ for all $\mathfrak{r} \in \mathfrak{S}$;
(2) $\zeta = \xi \Leftrightarrow \zeta \geqslant \xi$ and $\xi \geqslant \zeta$;
(3) $(\zeta \wedge \xi)(\mathfrak{r}) = \zeta(\mathfrak{r}) \wedge \xi(\mathfrak{r}) = \min\{\zeta(\mathfrak{r}), \xi(\mathfrak{r})\}$;
(4) $(\zeta \vee \xi)(\mathfrak{r}) = \zeta(\mathfrak{r}) \vee \xi(\mathfrak{r}) = \max\{\zeta(\mathfrak{r}), \xi(\mathfrak{r})\}$ for all $\mathfrak{r} \in \mathfrak{S}$;
(5) the *support* of ζ instead of $\operatorname{supp}(\zeta) = \{\mathfrak{r} \in \mathfrak{S} \mid \zeta(\mathfrak{r}) \neq 0\}$.

Let ζ and ξ be fuzzy sets of a semigroup $\mathfrak{S}$. The product of fuzzy subsets ζ and ξ of $\mathfrak{S}$, for all $\mathfrak{r} \in \mathfrak{S}$, is defined as

$$(\zeta \circ \xi)(\mathfrak{r}) = \begin{cases} \bigvee_{\mathfrak{r}=\mathfrak{k}\mathfrak{p}}\{\zeta(\mathfrak{k}) \wedge \xi(\mathfrak{p})\}, & \text{if } \mathfrak{r} = \mathfrak{k}\mathfrak{p}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathfrak{F}$ be a non-empty subset of $\mathfrak{S}$ is a fuzzy set of $\mathfrak{S}$, *the characteristic function* is denoted by

$$\lambda_{\mathfrak{F}}(\mathfrak{r}) = \begin{cases} 1, & \text{if } \mathfrak{r} \in \mathfrak{F}, \\ 0, & \text{if } \mathfrak{r} \notin \mathfrak{F}, \end{cases}$$

for all $\mathfrak{r} \in \mathfrak{S}$.

Lemma 2.3 ([25]). *Let $\mathfrak{K}$ and $\mathfrak{L}$ be non-empty subsets of a semigroup $\mathfrak{S}$. Then the following holds:*

(1) $\lambda_{\mathfrak{K}} \wedge \lambda_{\mathfrak{L}} = \lambda_{\mathfrak{K} \cap \mathfrak{L}}$;
(2) $\lambda_{\mathfrak{K}} \circ \lambda_{\mathfrak{L}} = \lambda_{\mathfrak{K}\mathfrak{L}}$.

Definition 2.4 ([12]). An FS ζ of a semigroup $\mathfrak{S}$ is said to be

(1) a *fuzzy subsemigroup* (FSG) of $\mathfrak{S}$ if $\zeta(\mathfrak{r}\mathfrak{e}) \geqslant \zeta(\mathfrak{r}) \wedge \zeta(\mathfrak{e})$, for all $\mathfrak{r}, \mathfrak{e} \in \mathfrak{S}$;
(2) a *fuzzy left ideal* (FLI) of $\mathfrak{S}$ if $\zeta(\mathfrak{r}\mathfrak{e}) \geqslant \zeta(\mathfrak{r})$, for all $\mathfrak{r}, \mathfrak{e} \in \mathfrak{S}$;
(3) a *fuzzy right ideal* (FRI) of $\mathfrak{S}$ if $\zeta(\mathfrak{r}\mathfrak{e}) \geqslant \zeta(\mathfrak{e})$, for all $\mathfrak{r}, \mathfrak{e} \in \mathfrak{S}$;
(4) a *fuzzy ideal* (FI) of $\mathfrak{S}$ if it is both an FLI and an FRI of $\mathfrak{S}$;
(5) a *fuzzy bi-ideal* (FBI) of $\mathfrak{S}$ if ζ is an FSG of $\mathfrak{S}$ and $\zeta(\mathfrak{r}\mathfrak{e}\mathfrak{k}) \geqslant \zeta(\mathfrak{r}) \wedge \zeta(\mathfrak{k})$ for all $\mathfrak{r}, \mathfrak{e}, \mathfrak{k} \in \mathfrak{S}$.

Definition 2.5 ([17]). An FSG ζ of a semigroup $\mathfrak{S}$ is called

(1) a *fuzzy left filter* (FLFR) of $\mathfrak{S}$ if $\zeta(\mathfrak{r}\mathfrak{e}) \leqslant \zeta(\mathfrak{e})$ for all $\mathfrak{r}, \mathfrak{e} \in \mathfrak{S}$;
(2) a *fuzzy right filter* (FRFR) of $\mathfrak{S}$ if $\zeta(\mathfrak{r}\mathfrak{e}) \leqslant \zeta(\mathfrak{r})$ for all $\mathfrak{r}, \mathfrak{e} \in \mathfrak{S}$;
(3) a *fuzzy filter* (FFR) of $\mathfrak{S}$ if $\zeta(\mathfrak{r}\mathfrak{e}) \leqslant \zeta(\mathfrak{r}) \wedge \zeta(\mathfrak{e})$ for all $\mathfrak{r}, \mathfrak{e} \in \mathfrak{S}$;
(4) a *fuzzy bi-filter* (FBFR) of $\mathfrak{S}$ if $\zeta(\mathfrak{r}\mathfrak{e}\mathfrak{r}) \leqslant \zeta(\mathfrak{r})$ for all $\mathfrak{r}, \mathfrak{e} \in \mathfrak{S}$.

For every element $\mathfrak{v}$ of a non-empty set $\mathfrak{S}$ and $\iota \in (0, 1]$, a *fuzzy point* (FP) $\mathfrak{v}_\iota$ of a set $\mathfrak{S}$ is an FS of $\mathfrak{S}$, for any $\mathfrak{r} \in \mathfrak{S}$, defined by

$$\mathfrak{v}_\iota(\mathfrak{r}) = \begin{cases} \iota, & \text{if } \mathfrak{r} = \mathfrak{v}, \\ 0, & \text{if } \mathfrak{r} \neq \mathfrak{v}. \end{cases}$$

Definition 2.6 ([25]). An FS ζ of a semigroup $\mathfrak{S}$ is said to be

(1) a *fuzzy left almost ideal* (FLAI) of $\mathfrak{S}$ if $(\mathfrak{v}_\iota \circ \zeta) \wedge \zeta \neq 0$ for all FP $\mathfrak{v}_\iota$;
(2) a *fuzzy right almost ideal* (FRAI) of $\mathfrak{S}$ if $(\zeta \circ \mathfrak{v}_\iota) \wedge \zeta \neq 0$ for all FP $\mathfrak{v}_\iota$;
(3) a *fuzzy almost ideal* (FAI) of $\mathfrak{S}$ if it is both an FLAI and an FRAI of $\mathfrak{S}$;
(4) a *fuzzy almost bi-ideal* (FABI) of $\mathfrak{S}$ if $(\mathfrak{v}_\iota \circ \zeta \circ \mathfrak{v}_\iota) \wedge \zeta \neq 0$ for all FP $\mathfrak{v}_\iota$.

3. Almost filters in semigroups

In this section, we define the concepts of almost left (right) filters and bi-filters in semigroups and prove the fundamental properties of various types of filters within semigroups.

Definition 3.1. Let $\mathfrak{F}$ be a non-empty subset of a semigroup $\mathfrak{S}$. Then

(1) $\mathfrak{F}$ is called an *almost left filter* (ALFR) if $\mathfrak{r}\mathfrak{S} \cap \mathfrak{F} \neq \emptyset \Rightarrow \mathfrak{r} \in \mathfrak{F}$ for all $\mathfrak{r} \in \mathfrak{S}$;
(2) $\mathfrak{F}$ is called an *almost right filter* (ARFR) if $\mathfrak{S}\mathfrak{r} \cap \mathfrak{F} \neq \emptyset \Rightarrow \mathfrak{r} \in \mathfrak{F}$ for all $\mathfrak{r} \in \mathfrak{S}$;
(3) $\mathfrak{F}$ is called an *almost filter* (AFR) if $\mathfrak{r}\mathfrak{S} \cap \mathfrak{F} \neq \emptyset$ and $\mathfrak{S}\mathfrak{r} \cap \mathfrak{F} \neq \emptyset \Rightarrow \mathfrak{r} \in \mathfrak{F}$ for all $\mathfrak{r} \in \mathfrak{S}$;
(4) $\mathfrak{F}$ is called an *almost bi-filter* (ABFR) if $\mathfrak{r}\mathfrak{S}\mathfrak{r} \cap \mathfrak{F} \neq \emptyset$ implies $\mathfrak{r} \in \mathfrak{F}$ for all $\mathfrak{r} \in \mathfrak{S}$.

Example 3.2. Let $\mathfrak{S} = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$. We define

$\cdot$	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{d}$
$\mathfrak{a}$	$\mathfrak{a}$	$\mathfrak{a}$	$\mathfrak{a}$	$\mathfrak{a}$
$\mathfrak{b}$	$\mathfrak{a}$	$\mathfrak{a}$	$\mathfrak{a}$	$\mathfrak{a}$
$\mathfrak{c}$	$\mathfrak{a}$	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{b}$
$\mathfrak{d}$	$\mathfrak{a}$	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{b}$

Then $\mathfrak{S}$ is a semigroup. Let $\mathfrak{F} = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$. Then $\mathfrak{F}$ is an ALF of $\mathfrak{S}$.

Theorem 3.3. *Let $\mathfrak{S}$ be semigroup. Then the following statements hold.*

(1) *Every LFR of $\mathfrak{S}$ is an ALFR of $\mathfrak{S}$.*
(2) *Every RFR of $\mathfrak{S}$ is an ARFR of $\mathfrak{S}$.*
(3) *Every FR of $\mathfrak{S}$ is an AFR of $\mathfrak{S}$.*
(4) *Every BFR of $\mathfrak{S}$ is an ABFR of $\mathfrak{S}$.*

Proof. Suppose that $\mathfrak{F}$ is a LFR of $\mathfrak{S}$ and $\mathfrak{r}, \mathfrak{e} \in \mathfrak{S}$. Let $\mathfrak{r}\mathfrak{e} \in \mathfrak{F}$. Then $\emptyset \neq \mathfrak{r}\mathfrak{e} \cap \mathfrak{F} \subseteq \mathfrak{r}\mathfrak{S} \cap \mathfrak{F}$. Thus, $\mathfrak{e} \in \mathfrak{F}$. Hence, $\mathfrak{F}$ is an ALFR of $\mathfrak{S}$. The proofs of (2), (3), and (4) are similar to the proof of (1). □

The converse of Theorem 3.3 does not hold in general.

Example 3.4. Assume that a semigroup $\mathfrak{S} = \mathbb{Z}_6$ under the addition on $\mathbb{Z}_6$ and $\mathfrak{F} = \{\bar{2}, \bar{3}, \bar{5}\}$. We can prove similarly as in [23], so $\mathfrak{F}$ is an LAFR of $\mathbb{Z}_6$. But $\mathfrak{L}$ is not LFR of $\mathbb{Z}_6$.

Theorem 3.5. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be non-empty subsets of a semigroup of $\mathfrak{S}$ with $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$. Then, the following statements hold.*

(1) *If $\mathfrak{F}_1$ is an ALFR of $\mathfrak{S}$, then $\mathfrak{F}_2$ is an ALFR of $\mathfrak{S}$.*
(2) *If $\mathfrak{F}_1$ is an ARFR of $\mathfrak{S}$, then $\mathfrak{F}_2$ is an ARFR of $\mathfrak{S}$.*
(3) *If $\mathfrak{F}_1$ is an AFR of $\mathfrak{S}$, then $\mathfrak{F}_2$ is an AFR of $\mathfrak{S}$.*
(4) *If $\mathfrak{F}_1$ is an ABFR of $\mathfrak{S}$, then $\mathfrak{F}_2$ is an ABFR of $\mathfrak{S}$.*

Proof. Suppose that $\mathfrak{F}_1$ is an ALFR of $\mathfrak{S}$ and $\mathfrak{r} \in \mathfrak{S}$. Let $\mathfrak{r}\mathfrak{S} \cap \mathfrak{F}_1 \neq \emptyset$. Since $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ we have $\emptyset \neq \mathfrak{r}\mathfrak{S} \cap \mathfrak{F}_1 \subseteq \mathfrak{r}\mathfrak{S} \cap \mathfrak{F}_2$. Thus, $\mathfrak{r} \in \mathfrak{F}_1 \subseteq \mathfrak{F}_2$ so $\mathfrak{r} \in \mathfrak{F}_2$. Hence $\mathfrak{F}_2$ is an ALFR of $\mathfrak{S}$.

The proofs of (2), (3), and (4) are similar to the proof of (1). □

Theorem 3.6. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be non-empty subsets of a semigroup of $\mathfrak{S}$. Then the following statements hold.*

(1) *If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are ALFR of $\mathfrak{S}$, then $\mathfrak{F}_1 \cup \mathfrak{F}_2$ is an ALFR of $\mathfrak{S}$.*
(2) *If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are ARFR of $\mathfrak{S}$, then $\mathfrak{F}_1 \cup \mathfrak{F}_2$ is an ARF of $\mathfrak{S}$.*
(3) *If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are AFR of $\mathfrak{S}$, then $\mathfrak{F}_1 \cup \mathfrak{F}_2$ is an AFR of $\mathfrak{S}$.*
(4) *If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are ABFR of $\mathfrak{S}$, then $\mathfrak{F}_1 \cup \mathfrak{F}_2$ is an ABFR of $\mathfrak{S}$.*

Proof. Since $\mathfrak{F}_1 \subseteq \mathfrak{F}_1 \cup \mathfrak{F}_2$ we have $\mathfrak{F}_1 \cup \mathfrak{F}_2$ is an ALFR of $\mathfrak{S}$, by Theorem 3.5.

The proofs of the other assertions are similar to the proof of (1). □

The following result is an immediate consequence of Theorem 3.6.

Corollary 3.7. *Let $\mathfrak{S}$ be a semigroup. Then the following statements hold.*

(1) *The finite union ALFRs of $\mathfrak{S}$ is an ALFR of $\mathfrak{S}$.*
(2) *The finite union ARFRs of $\mathfrak{S}$ is an ARFR of $\mathfrak{S}$.*
(3) *The finite union ABFRs of $\mathfrak{S}$ is an ABFR of $\mathfrak{S}$.*

The following example shows that the above property is not true in the case of the intersection.

Example 3.8. Assume a semigroup $\mathfrak{S} = \mathbb{Z}_6$ under the addition on $\mathbb{Z}_6$ and $\mathfrak{F}_1 = \{\bar{1}, \bar{3}, \bar{4}\}$, $\mathfrak{F}_2 = \{\bar{1}, \bar{3}, \bar{4}\}$. Then $\mathfrak{F}_1$ and $\mathfrak{F}_2$ is a left AF of $\mathbb{Z}_6$ but $\mathfrak{F}_1 \cap \mathfrak{F}_2 = \{3\}$ is not a ALFR of $\mathbb{Z}_6$.

Theorem 3.9. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be non-empty subsets of a semigroup of $\mathfrak{S}$. Then the following statements hold.*

(1) *If $\mathfrak{F}$ is an ALFR, then $\mathfrak{F} \cup \mathfrak{H}$ is an ALFR of $\mathfrak{S}$.*
(2) *If $\mathfrak{F}$ is an ARFR, then $\mathfrak{F} \cup \mathfrak{H}$ is an ARFR of $\mathfrak{S}$.*
(3) *If $\mathfrak{F}$ is an AFR, then $\mathfrak{F} \cup \mathfrak{H}$ is an AFR of $\mathfrak{S}$.*

(4) *If $\mathfrak{F}$ is an ABFR, then $\mathfrak{F} \cup \mathfrak{H}$ is an ABFR of $\mathfrak{S}$.*

Proof. By Theorem 3.6 and $\mathfrak{F} \subseteq \mathfrak{F} \cup \mathfrak{H}$, $\mathfrak{F} \cup \mathfrak{H}$ is an ALFR of $\mathfrak{S}$.

The demonstrations of (2), (3), and (4) follow the approach used in proving (1). □

Corollary 3.10. *Let $\{\mathfrak{F}_i \mid i \in \mathfrak{I}\}$ be non-empty subset of a semigroup $\mathfrak{S}$. Then*

(1) *the union $\bigcup_{i\in\mathfrak{I}} \mathfrak{F}_i$ is an ALFR of $\mathfrak{S}$ if there exists an ALFR $\mathfrak{F}_i$ for some $i \in \mathfrak{I}$;*
(2) *the union $\bigcup_{i\in\mathfrak{I}} \mathfrak{F}_i$ is an ARFR of $\mathfrak{S}$ if there exists an ARFR $\mathfrak{F}_i$ for some $i \in \mathfrak{I}$;*
(3) *the union $\bigcup_{i\in\mathfrak{I}} \mathfrak{F}_i$ is an AFR of $\mathfrak{S}$ if there exists an AFR $\mathfrak{F}_i$ for some $i \in \mathfrak{I}$;*
(4) *the union $\bigcup_{i\in\mathfrak{I}} \mathfrak{F}_i$ is an ABFR of $\mathfrak{S}$ if there exists an ABFR $\mathfrak{F}_i$ for some $i \in \mathfrak{I}$.*

Proof. Assume that there exists an ALFR $\mathfrak{F}_i$ for some $i \in \mathfrak{I}$. Then $\mathfrak{F}_i \subseteq \bigcup_{i\in\mathfrak{I}} \mathfrak{F}_i$. By Theorem 3.9, $\bigcup_{i\in\mathfrak{I}} \mathfrak{F}_i$ is an ALFR of $\mathfrak{S}$.

The demonstrations of (2), (3), and (4) follow the approach used in proving (1). □

4. Fuzzy almost filters in semigroups

In this section, we introduce the concepts of fuzzy left almost filters, fuzzy right almost filters, and fuzzy almost bi-filters in semigroups. We demonstrate the fundamental properties of these filters and explore the relationships between almost all types of filters and their fuzzy counterparts within semigroups.

Definition 4.1. Let ζ be an FS of a semigroup $\mathfrak{S}$. Then

(1) ζ is called a *fuzzy almost left filter* (FALFR) if $(\mathfrak{v}_\iota \circ \zeta) \wedge \zeta \neq 0 \Rightarrow \mathfrak{v}_\iota \in \zeta$ for all FP $\mathfrak{v}_\iota$ of $\mathfrak{S}$;
(2) ζ is called a *fuzzy almost right filter* (FARFR) if $(\zeta \circ \mathfrak{v}_\iota) \wedge \zeta \neq 0 \Rightarrow \mathfrak{v}_\iota \in \zeta$ for all FP $\mathfrak{v}_\iota$ of $\mathfrak{S}$;
(3) ζ is called a *fuzzy almost filter* (FAFR) if $(\mathfrak{v}_\iota \circ \zeta) \wedge \zeta \neq 0$ and $(\zeta \circ \mathfrak{v}_\iota) \wedge \zeta \neq 0 \Rightarrow \mathfrak{v}_\iota \in \zeta$ for all FP $\mathfrak{v}_\iota$ of $\mathfrak{S}$;
(4) ζ is called a *fuzzy almost bi-filter* (FABF) if $(\nu_\iota \circ \zeta \circ \mathfrak{v}_\iota) \wedge \zeta \neq 0 \Rightarrow \mathfrak{v}_\iota \in \zeta$ for all for all FP $\mathfrak{v}_\iota$ of $\mathfrak{S}$.

Theorem 4.2. *Let ζ and ξ be a nonzero FS of an ordered semigroup $\mathfrak{S}$ with $\zeta \leqslant \xi$. Then the following statements hold.*

(1) *If ζ is an FALFR of $\mathfrak{S}$, then ξ is an FALFR of $\mathfrak{S}$.*
(2) *If ζ is an FARFR of $\mathfrak{S}$, then ξ is an FARFR of $\mathfrak{S}$.*
(3) *If ζ is an FAFR of $\mathfrak{S}$, then ξ is an FAFR of $\mathfrak{S}$.*
(4) *If ζ is an FABFR of $\mathfrak{S}$, then ξ is an FABFR of $\mathfrak{S}$.*

Proof. Suppose that ζ is an FALFR of $\mathfrak{S}$ and $\mathfrak{v}_\iota$ is an FP of $\mathfrak{S}$. Let $(\nu_\iota \circ \zeta) \wedge \zeta \neq 0$. Then there exists $\mathfrak{r} \in \mathfrak{S}$ such that $0 \neq (\mathfrak{v}_\iota \circ \zeta)(\mathfrak{r}) \wedge \zeta(\mathfrak{r}) \leqslant (\mathfrak{v}_\iota \circ \xi)(\mathfrak{r}) \wedge \xi(\mathfrak{r})$. Thus $0 \neq (\mathfrak{v}_\iota \circ \zeta) \wedge \zeta \leqslant (\mathfrak{v}_\iota \circ \xi) \wedge \xi$ implies $\mathfrak{v}_\iota \in \zeta \leqslant \xi$ so $\mathfrak{v}_\iota \in \xi$. Hence, ξ is an FALFR of $\mathfrak{S}$.

The demonstrations of (2), (3), and (4) follow the approach used in proving (1). □

Theorem 4.3. *Let ζ and ξ be an FS a semigroup of $\mathfrak{S}$. Then the following statements hold.*

(1) *If ζ and ξ are FALFRs, then $\zeta \vee \xi$ is an FALFR of $\mathfrak{S}$.*
(2) *If ζ and ξ are FARFRs, then$\zeta \vee \xi$ is an FARFR of $\mathfrak{S}$.*
(3) *If ζ and ξ are FAFRs, then$\zeta \vee \xi$ is an FAFR of $\mathfrak{S}$.*
(4) *If ζ and ξ are FABFRs, then $\zeta \vee \xi$ is an FABFR of $\mathfrak{S}$.*

Proof. Since $\zeta \leqslant \zeta \vee \xi$ we have $\zeta \vee \xi$ is an FALFR of $\mathfrak{S}$, by Theorem 4.2.

The demonstrations of (2), (3), and (4) follow the approach used in proving (1). □

The following result is an immediate consequence of Theorem 4.3.

Corollary 4.4. *Let $\mathfrak{S}$ be a semigroup. Then the following statements hold.*

(1) *The finite union FALFRs of* $\mathfrak{S}$ *is an FALFR of* $\mathfrak{S}$.
(2) *The finite union FARFRs of* $\mathfrak{S}$ *is an FARFR of* $\mathfrak{S}$.
(3) *The finite union FAFRs of* $\mathfrak{S}$ *is an FAFR of* $\mathfrak{S}$.
(4) *The finite union FABFRs of* $\mathfrak{S}$ *is an FABFR of* $\mathfrak{S}$.

Theorem 4.5. *Let* ζ *and* ξ *be an FS of a semigroup* $\mathfrak{S}$. *Then the following statements hold.*

(1) *If* ζ *is an FALFR, then* $\zeta \vee \xi$ *is an FALFR of* $\mathfrak{S}$.
(2) *If* ζ *is an FARFR, then*$\zeta \vee \xi$ *is an FARFR of* $\mathfrak{S}$.
(3) *If* ζ *is an FAFR, then*$\zeta \vee \xi$ *is an FAFR of* $\mathfrak{S}$.
(4) *If* ζ *is an FABFR, then* $\zeta \vee \xi$ *is an FABFR of* $\mathfrak{S}$.

Proof. By Theorem 4.3 and $\rho \leqslant \rho \vee \nu$. Thus, $\rho \vee \nu$ is an FALFR of $\mathfrak{S}$. □

Corollary 4.6. *Let* ρ_i *be fuzzy set of semigroup* $\mathfrak{S}$. *Then*

(1) *the maximum* $\bigvee_{i\in\mathfrak{I}} \rho_i$ *is an FALFR of* $\mathfrak{S}$ *if there exists an FALFR* ρ_i *for some* $i \in \mathfrak{I}$;
(2) *the maximum* $\bigvee_{i\in\mathfrak{I}} \rho_i$ *is an FARFR of* $\mathfrak{S}$ *if there exists an FARFR* ρ_i *for some* $i \in \mathfrak{I}$;
(3) *the maximum* $\bigvee_{i\in\mathfrak{I}} \rho_i$ *is an FAFR of* $\mathfrak{S}$ *if there exists an FAFR* ρ_i *for some* $i \in \mathfrak{I}$;
(4) *the maximum* $\bigvee_{i\in\mathfrak{I}} \rho_i$ *is an FABFR of* $\mathfrak{S}$ *if there exists an FARFR* ρ_i *for some* $i \in \mathfrak{I}$.

Proof. Assume that there exists an FALFR ρ_i for some $i \in \mathfrak{I}$. Consequently, $\rho_i \leqslant \bigvee_{i\in\mathfrak{I}} \rho_i$. According to Theorem 4.5, $\bigvee_{i\in\mathfrak{I}} \rho_i$ is an FALFR of $\mathfrak{S}$. □

In the following theorem, we prove the relationship between AFRs and FAFRs in semigroups.

Theorem 4.7. *Let* $\mathfrak{F}$ *be a non-empty subset of a semigroup* $\mathfrak{S}$. *Then, the following statement holds.*

(1) $\mathfrak{F}$ *is an ALFR of* $\mathfrak{S}$ *if and only if* $\lambda_{\mathfrak{F}}$ *is an FALFR of* $\mathfrak{S}$.
(2) $\mathfrak{F}$ *is an ARFR of* $\mathfrak{S}$ *if and only if* $\lambda_{\mathfrak{F}}$ *is an FARFR of* $\mathfrak{S}$.
(3) $\mathfrak{F}$ *is an AFR of* $\mathfrak{S}$ *if and only if* $\lambda_{\mathfrak{F}}$ *is an FAFR of* $\mathfrak{S}$.
(4) $\mathfrak{F}$ *is an ARFR of* $\mathfrak{S}$ *if and only if* $\lambda_{\mathfrak{F}}$ *is an FARFR of* $\mathfrak{S}$.

Proof. Suppose that $\mathfrak{F}$ is an ALFR of $\mathfrak{S}$ with $\mathfrak{v}\mathfrak{F} \cap \mathfrak{F} \neq \emptyset$ for all $\mathfrak{v} \in \mathfrak{S}$ and $\iota \in (0, 1]$. To show that $\mathfrak{v}_\iota \in \lambda_{\mathfrak{F}}$, let $(\mathfrak{v}_\iota \circ \lambda_{\mathfrak{F}}) \wedge \lambda_{\mathfrak{F}} \neq 0$ for all FP $\mathfrak{v}_\iota$ of $\mathfrak{S}$. Then there exists $\mathfrak{r} \in \mathfrak{S}$ such that $\mathfrak{r} \in \mathfrak{v}\mathfrak{F} \cap \mathfrak{F}$. Thus, $((\mathfrak{v}_\iota \circ \lambda_{\mathfrak{F}}) \wedge \lambda_{\mathfrak{F}})(\mathfrak{r}) = 1$ where $\mathfrak{v} = \mathfrak{k}\mathfrak{e}$ for some $\mathfrak{k}, \mathfrak{e} \in \mathfrak{S}$. Assume that $\mathfrak{v}_\iota \notin \lambda_{\mathfrak{F}}$. Then $\mathfrak{v} \notin \mathfrak{F}$. Since $\mathfrak{F}$ is an ALF of $\mathfrak{S}$ we have $\mathfrak{v} \in \mathfrak{F}$. It is a contradiction, so $\mathfrak{v}_\iota \in \lambda_{\mathfrak{F}}$. Hence $\lambda_{\mathfrak{F}}$ is an FALFR of $\mathfrak{S}$.

For the converse, suppose that $\lambda_{\mathfrak{F}}$ is an FALFR of $\mathfrak{S}$ with $(\mathfrak{v}_\iota \circ \lambda_{\mathfrak{F}}) \wedge \lambda_{\mathfrak{F}} \neq 0$ for all FP $\mathfrak{v}_\iota$ of $\mathfrak{S}$. To show that $\mathfrak{x} \in \mathfrak{F}$, let $\mathfrak{x}\mathfrak{F} \cap \mathfrak{F} \neq \emptyset$ for all $\mathfrak{x} \in \mathfrak{S}$. Assume that $\mathfrak{x} \notin \mathfrak{F}$. Then $\lambda_{\mathfrak{F}}(\mathfrak{x}) = 0$. Thus, $((\mathfrak{v}_\iota \circ \lambda_{\mathfrak{F}}) \wedge \lambda_{\mathfrak{F}})(\mathfrak{x}) = 0$. It is a contradiction, so $\mathfrak{x} \in F$. Hence, $\mathfrak{F}$ is an ALFR of $\mathfrak{S}$.

The proofs for (2), (3), and (4) are conducted in a manner similar to the proof of (1). □

Theorem 4.8. *Let* ζ *be a nonzero FS of a semigroup* $\mathfrak{S}$. *Then, the following statement holds.*

(1) ζ *is an FALFR of* $\mathfrak{S}$ *if and only if* $\text{supp}(\zeta)$ *is an ALFR of* $\mathfrak{S}$.
(2) ζ *is an FARFR of* $\mathfrak{S}$ *if and only if* $\text{supp}(\zeta)$ *is an ARFR of* $\mathfrak{S}$.
(3) ζ *is an FAFR of* $\mathfrak{S}$ *if and only if* $\text{supp}(\zeta)$ *is an AFR of* $\mathfrak{S}$.
(4) ζ *is an FABFR of* $\mathfrak{S}$ *if and only if* $\text{supp}(\zeta)$ *is an ABFR of* $\mathfrak{S}$.

Proof. Suppose that ζ is an FALFR of $\mathfrak{S}$ with $(\mathfrak{v}_\iota \circ \zeta) \wedge \zeta \neq 0$ for all FP $\mathfrak{v}_\iota$ of $\mathfrak{S}$. To show that $\mathfrak{x} \in \text{supp}(\zeta)$, let $\mathfrak{x}\,\text{supp}(\zeta) \cap \text{supp}(\zeta) \neq \emptyset$. Assume that $\mathfrak{x} \notin \text{supp}(\zeta)$. Then $\zeta(\mathfrak{x}) = 0$. Thus, there exists $\mathfrak{x} \in \mathfrak{S}$ such that $((\mathfrak{v}_\iota \circ \zeta) \wedge \zeta)(\mathfrak{x}) = 0$ so $(\mathfrak{v}_\iota \circ \zeta) \wedge \zeta = 0$. It is a contradiction, so $\mathfrak{x} \in \text{supp}(\zeta)$. Hence $\text{supp}(\zeta)$ is an ALFR of $\mathfrak{S}$.

For the converse, assume that $\text{supp}(\zeta)$ is an ALFR of $\mathfrak{S}$ with $\mathfrak{x}\,\text{supp}(\zeta) \cap \text{supp}(\zeta) \neq \emptyset$. To show that $\mathfrak{v}_\iota \in \zeta$, let $(\mathfrak{v}_\iota \circ \zeta) \wedge \zeta \neq 0$ for all FP $\mathfrak{v}_\iota$ of $\mathfrak{S}$. Then there exists $\mathfrak{r} \in \mathfrak{S}$ such that $\mathfrak{r} \in \mathfrak{x}\mathfrak{F} \cap \mathfrak{F}$. Thus, $((\mathfrak{v}_\iota \circ \zeta) \wedge \zeta)(\mathfrak{r}) = 1$, where $\mathfrak{r} = \mathfrak{k}\mathfrak{e}$ for some $\mathfrak{k}, \mathfrak{e} \in \mathfrak{S}$. Assume that $\mathfrak{v}_\iota \notin \zeta$. Then $\zeta(\mathfrak{r}) = 0$ for all $\mathfrak{r} \in \mathfrak{S}$. Thus, $\mathfrak{r} \notin \text{supp}(\zeta)$ so $\mathfrak{x}\,\text{supp}(\zeta) \cap \text{supp}(\zeta) = \emptyset$. It is a contradiction, so $\mathfrak{v}_\iota \in \zeta$. Hence ζ is an FALFR of $\mathfrak{S}$.

The demonstrations of (2), (3), and (4) follow the approach used in proving (1). □

5. Minimality and maximality of fuzzy almost filters in semigorups

We define the minimalists of FAFR and FABFR in semigroups.

Definition 5.1. Let $\mathfrak{F}$ be a non-empty subset of a semigroup $\mathfrak{S}$. Then

(1) an AFR $\mathfrak{F}$ of $\mathfrak{S}$ is said to be *minimal almost filter* (MAFR) if for any AFR $\mathfrak{F}$ of $\mathfrak{S}$, we have $\mathfrak{M} = \mathfrak{F}$ whenever $\mathfrak{M} \subseteq \mathfrak{F}$;
(2) an ABFR $\mathfrak{F}$ of $\mathfrak{S}$ is said to be *minimal almost bi-filter* (MABFR) if for any ABFR $\mathfrak{F}$ of $\mathfrak{S}$, we have $\mathfrak{M} = \mathfrak{F}$ whenever $\mathfrak{M} \subseteq \mathfrak{F}$.

Definition 5.2. Let ζ be an FS of a semigroup $\mathfrak{S}$. Then

(1) an FAFR ζ of $\mathfrak{S}$ is said to be *minimal fuzzy almost filter* (MFAFR) if for any FAFR ξ of $\mathfrak{S}$, we have $\text{supp}(\zeta) = \text{supp}(\xi)$ whenever $\xi \leqslant \zeta$;
(2) an FABFR ζ of $\mathfrak{S}$ is said to be *minimal fuzzy almost bi-filter* (MFABFR) if for any FABF ξ of $\mathfrak{S}$, we have $\text{supp}(\zeta) = \text{supp}(\xi)$ whenever $\xi \leqslant \zeta$.

The relationship between MAFRs, MABFRs and MFAFRs, MFABFRs is investigated.

Theorem 5.3. *Let $\mathfrak{F}$ be a non-empty subset of a semigroup $\mathfrak{S}$. Then the following statements are equivalent:*

(1) *$\mathfrak{F}$ is a MAFR of $\mathfrak{S}$ if and only if $\lambda_{\mathfrak{F}}$ is a MFAFR of $\mathfrak{S}$;*
(2) *$\mathfrak{F}$ is a MABFR of $\mathfrak{S}$ if and only if $\lambda_{\mathfrak{F}}$ is a MFABFR of $\mathfrak{S}$.*

Proof. Suppose $\mathfrak{F}$ is an MAFR of $\mathfrak{S}$. Then $\mathfrak{F}$ is an AFR of $\mathfrak{S}$. Consequently, under Theorem 4.7, $\lambda_{\mathfrak{F}}$ is an FAFR of $\mathfrak{S}$. Let ξ be an FAFR of $\mathfrak{S}$ such that $\xi \leqslant \lambda_{\mathfrak{F}}$. Now, we know, by Theorem 4.8, that $\text{supp}(\xi)$ is an AFR of $\mathfrak{S}$. Since $\text{supp}(\xi) \subseteq \text{supp}(\lambda_F) = \mathfrak{F}$ and considering the minimality of $\mathfrak{F}$, we have $\text{supp}(\xi) = \text{supp}(\lambda_F)$. This shows that $\lambda_{\mathfrak{F}}$ is an MFAFR of $\mathfrak{S}$.

Conversely, assume that $\lambda_{\mathfrak{F}}$ is an MFAFR of $\mathfrak{S}$. Then $\lambda_{\mathfrak{F}}$ is an FAFR of $\mathfrak{S}$. Thus, by Theorem 4.7, $\mathfrak{F}$ is an AFR of $\mathfrak{S}$. Let $\mathfrak{F}$ be an AFR of $\mathfrak{S}$ such that $\mathfrak{M} \subseteq \mathfrak{F}$. Then, by Theorem 4.7, $\lambda_{\mathfrak{M}}$ is an FAFR of $\mathfrak{S}$ such that $\lambda_{\mathfrak{M}} \leqslant \lambda_{\mathfrak{F}}$. This implies that $\text{supp}(\lambda_{\mathfrak{M}}) \subseteq \text{supp}(\lambda_{\mathfrak{F}})$. By the minimality of $\lambda_{\mathfrak{F}}$, it follows that $\text{supp}(\lambda_{\mathfrak{M}}) = \text{supp}(\lambda_{\mathfrak{F}})$. That is, $\mathfrak{M} = \mathfrak{F}$. Therefore, $\mathfrak{F}$ is MAFR.

The proof of (2) is similar to the proof (1). □

Corollary 5.4. *Let $\mathfrak{S}$ be a semigroup. Then*

(1) *$\mathfrak{S}$ has no proper AFR if and only if* $\text{supp}(\zeta) = \mathfrak{S}$ *for every FAFR ζ of $\mathfrak{S}$;*
(2) *$\mathfrak{S}$ has no proper ABFR if and only if* $\text{supp}(\zeta) = S$ *for every FABFR ζ of $\mathfrak{S}$.*

Proof. Assume that $\mathfrak{S}$ has no proper AFR and let ζ be an FAFR of $\mathfrak{S}$. According to Theorem 4.8, $\text{supp}(\zeta)$ is an AFR of $\mathfrak{S}$. Given this assumption, it follows that $\text{supp}(\zeta) = \mathfrak{S}$.

Conversely, suppose that $\text{supp}(\zeta) = \mathfrak{S}$ and $\mathfrak{F}$ is a proper AFR of $\mathfrak{S}$. Then by Theorem 4.7, λ_F is an FAFR of $\mathfrak{S}$. Thus, $\text{supp}(\lambda_{\mathfrak{F}}) = \mathfrak{F} \neq \mathfrak{S}$. It is a contradiction. Hence, $\mathfrak{S}$ has no proper AFR.

The proof for (2) follows a similar approach to that of (1). □

Definition 5.5. Let $\mathfrak{F}$ be a non-empty subset of a semigroup. Then

(1) an AFR $\mathfrak{F}$ of $\mathfrak{S}$ is said to be *maximal almost filter* (MMAFR) if for all AFR $\mathfrak{F}$ of $\mathfrak{S}$ such that $\mathfrak{M} \subseteq \mathfrak{F}$ implies $\mathfrak{M} = \mathfrak{F}$;
(2) an ABFR $\mathfrak{F}$ of $\mathfrak{S}$ is said to be *maximal almost bi-filter* (MMABFR) if for all ABFR $\mathfrak{F}$ of $\mathfrak{S}$ such that $\mathfrak{M} \subseteq \mathfrak{F}$ implies $\mathfrak{M} = \mathfrak{F}$.

Definition 5.6. Let ζ be an FS of a semigroup $\mathfrak{S}$. Then

(1) an FAFR ζ of $\mathfrak{S}$ is said to be *maximal fuzzy almost filter* (MMFAFR) if for all FAFRs ξ of $\mathfrak{S}$ such that $\zeta \leqslant \xi$ implies $\text{supp}(\zeta) = \text{supp}(\xi)$;

(2) an FABFR ζ of $\mathfrak{S}$ is said to be *maximal fuzzy almost bi-filter* (MMFABFR) if for all FABFs ξ of $\mathfrak{S}$ such that $\zeta \leqslant \xi$ implies $\text{supp}(\zeta) = \text{supp}(\xi)$.

The relationship between MMAFRs, MMABFRs and MMFAFRs, MMFABFRs is investigated.

Theorem 5.7. *Let $\mathfrak{F}$ be a non-empty subset of $\mathfrak{S}$. Then the following statements are equivalent:*

(1) *$\mathfrak{F}$ is an MMAFR of $\mathfrak{S}$ if and only if $\lambda_{\mathfrak{M}}$ is an MMFAFR of $\mathfrak{S}$;*
(2) *$\mathfrak{F}$ is an MMABFR of $\mathfrak{S}$ if and only if $\lambda_{\mathfrak{M}}$ is an MMFABFR of $\mathfrak{S}$.*

Proof. Assume that $\mathfrak{F}$ is an MMAFR of $\mathfrak{S}$. Then $\mathfrak{F}$ is an AFR of $\mathfrak{S}$. Thus, by Theorem 4.7, $\lambda_{\mathfrak{M}}$ is an FAFR of $\mathfrak{S}$. Let ξ be an FAFR of $\mathfrak{S}$ such that $\lambda_{\mathfrak{M}} \leqslant \xi$. Now, we know, by Theorem 4.8, that $\text{supp}(\nu)$ is an AFR of $\mathfrak{S}$. Since $\text{supp}(\lambda_{\mathfrak{M}}) \subseteq \text{supp}(\xi) = \mathfrak{M}$, by the maximality of $\mathfrak{F}$, we have $\text{supp}(\xi) = \text{supp}(\lambda_{\mathfrak{M}})$. This shows that $\lambda_{\mathfrak{M}}$ is an MMFAFR of $\mathfrak{S}$.

For the converse, assume that $\lambda_{\mathfrak{M}}$ is an MMFAFR of $\mathfrak{S}$. Then $\lambda_{\mathfrak{M}}$ is an FAFR of $\mathfrak{S}$. Thus, by Theorem 4.7, $\mathfrak{M}$ is an AFR of $\mathfrak{S}$. Let L be an AFR of $\mathfrak{S}$ such that $\mathfrak{M} \subseteq \mathfrak{L}$. Then, by Theorem 4.7, $\lambda_{\mathfrak{L}}$ is an FAFR of $\mathfrak{S}$ such that $\lambda_{\mathfrak{M}} \leqslant \lambda_{\mathfrak{L}}$. This implies that $\text{supp}(\lambda_{\mathfrak{M}}) \subseteq \text{supp}(\lambda_{\mathfrak{L}})$. By the maximality of $\lambda_{\mathfrak{M}}$, we have $\text{supp}(\lambda_{\mathfrak{M}}) = \text{supp}(\lambda_{\mathfrak{L}})$. That is, $\mathfrak{M} = \mathfrak{L}$. Therefore, $\mathfrak{F}$ is MMAFR.

The proof of (2) is similar to the proof (1). □

6. Prime (Semiprime, strongly prime) filters in semigroups

We introduce various notions of prime almost filters and prime fuzzy almost filters in semigroups. Their fundamental related property is provided.

Definition 6.1. Let $\mathfrak{P}$ be an AFR of $\mathfrak{S}$. Then $\mathfrak{P}$ is said to be:

(1) *prime almost filter* (PAFR) if for any AFRs $\mathfrak{M}$ and $\mathfrak{L}$ of $\mathfrak{S}$, we have $\mathfrak{M} \subseteq \mathfrak{P}$ or $\mathfrak{L} \subseteq \mathfrak{P}$ whenever $\mathfrak{M}\mathfrak{L} \subseteq \mathfrak{P}$;
(2) *semiprime almost filter* (SAFR) if for any AFR $\mathfrak{M}$ of $\mathfrak{S}$, we have $\mathfrak{M} \subseteq \mathfrak{P}$ whenever $\mathfrak{M}^2 \subseteq \mathfrak{P}$;
(3) *strongly prime almost filter* (SPAFR) if for any AFRs $\mathfrak{M}$ and $\mathfrak{L}$ of $\mathfrak{S}$, we have $\mathfrak{M} \subseteq \mathfrak{P}$ or $\mathfrak{L} \subseteq \mathfrak{P}$ whenever $\mathfrak{M}\mathfrak{L} \cap \mathfrak{L}\mathfrak{M} \subseteq \mathfrak{P}$.

Definition 6.2. Let ζ be an FAFR of a semigroup $\mathfrak{S}$. Then ζ is said to be:

(1) *prime fuzzy almost filter* (PFAFR) if for any two FAFRs ξ and ϑ of $\mathfrak{S}$, we have $\xi \leqslant \zeta$ or $\vartheta \leqslant \zeta$ whenever $\xi \circ \vartheta \leqslant \zeta$.
(2) *semiprime fuzzy almost filter* (SFAFR) if for any FAFR ξ of $\mathfrak{S}$, we have $\xi \leqslant \eta$ whenever $\xi \circ \xi \leqslant \zeta$.
(3) *strongly prime fuzzy almost filter* (SPFAFR) if for any two FAFRs ξ and ϑ of $\mathfrak{S}$, we have $\xi \leqslant \zeta$ or $\vartheta \leqslant \zeta$ whenever $(\xi \circ \vartheta) \wedge (\vartheta \circ \xi) \leqslant \zeta$.

It is clear that every SPFAFR is a PFAFR, and every PFAFR is an SFAFR.
The relationship between PAFR and PFAFR is investigated.

Theorem 6.3. *Let $\mathfrak{P}$ be a non-empty subset of a semigroup $\mathfrak{S}$. Then*

(1) *$\mathfrak{P}$ is a PAFR AF of $\mathfrak{S}$ if and only if $\lambda_{\mathfrak{P}}$ is a PFAFR of $\mathfrak{S}$;*
(2) *$\mathfrak{P}$ is an SAFR of $\mathfrak{S}$ if and only if $\lambda_{\mathfrak{P}}$ is an SFAFR of $\mathfrak{S}$.*

Proof. Suppose that $\mathfrak{P}$ is a PAFR of a semigroup $\mathfrak{S}$. Then $\mathfrak{P}$ is an AFR of $\mathfrak{S}$. Thus by Theorem 4.7, $\lambda_{\mathfrak{P}}$ is an FAFR of $\mathfrak{S}$. Let ϑ and ξ be an FAFR of $\mathfrak{S}$ such that $\vartheta \circ \xi \leqslant \lambda_{\mathfrak{P}}$. Assume that $\vartheta \not\leqslant \lambda_{\mathfrak{P}}$ or $\xi \not\leqslant \lambda_{\mathfrak{P}}$. Then there exist $\mathfrak{h}, \mathfrak{r} \in \mathfrak{S}$ such that $\vartheta(\mathfrak{h}) \neq 0$ and $\xi(\mathfrak{r}) \neq 0$. While $\lambda_{\mathfrak{P}}(\mathfrak{h}) = 0$ and $\lambda_{\mathfrak{P}}(\mathfrak{r}) = 0$. Thus, $\mathfrak{h} \in \text{supp}(\vartheta)$ and $\mathfrak{r} \in \text{supp}(\xi)$, but $\mathfrak{h}, \mathfrak{r} \notin \mathfrak{P}$. So $\text{supp}(\vartheta) \nsubseteq \mathfrak{P}$ and $\text{supp}(\xi) \nsubseteq \mathfrak{P}$. Since $\text{supp}(\vartheta)$ and $\text{supp}(\xi)$ are AFRs of $\mathfrak{S}$ we have $\text{supp}(\vartheta)\,\text{supp}(\xi) \nsubseteq \mathfrak{P}$. Thus, there exists $\mathfrak{m} = \mathfrak{d}\mathfrak{e}$ for some $\mathfrak{d} \in \text{supp}(\vartheta)$ and $\mathfrak{e} \in \text{supp}(\xi)$ such that $\mathfrak{m} \in \mathfrak{P}$. Hence $\lambda_{\mathfrak{P}}(\mathfrak{m}) = 0$ implies that $(\vartheta \circ \xi)(\mathfrak{m}) = 0$. Since $\vartheta \circ \xi \leqslant \lambda_{\mathfrak{P}}$. Since $\mathfrak{d} \in \text{supp}(\vartheta)$ and

$\mathfrak{e} \in \text{supp}(\xi)$ we have $\vartheta(\mathfrak{d}) \neq 0$ and $\xi(\mathfrak{e}) \neq 0$. Thus $(\vartheta \circ \xi)(\mathfrak{m}) = \bigvee_{(\mathfrak{d}\mathfrak{e}) \in F_\mathfrak{m}} \{\vartheta(\mathfrak{d}) \wedge \xi(\mathfrak{e})\} \neq 0$. It is a contradiction so $\vartheta \leqslant \lambda_\mathfrak{P}$ or $\xi \leqslant \lambda_\mathfrak{P}$. Therefore $\lambda_\mathfrak{P}$ is a PFAFR of $\mathfrak{S}$.

For the converse, suppose that $\lambda_\mathfrak{P}$ is a PFAFR of $\mathfrak{S}$. Then $\lambda_\mathfrak{P}$ is an FAFR of $\mathfrak{S}$. Thus by Theorem 4.7, $\mathfrak{P}$ is an AFR of $\mathfrak{S}$. Let $\mathfrak{M}$ and $\mathfrak{L}$ be AFRs of $\mathfrak{S}$ such that $\mathfrak{M}\mathfrak{L} \subseteq \mathfrak{P}$. Then $\lambda_\mathfrak{M}$ and $\lambda_\mathfrak{L}$ are FAFRs of $\mathfrak{S}$. By Lemma 2.3, $\lambda_\mathfrak{M} \circ \lambda_\mathfrak{L} = \lambda_{\mathfrak{M}\mathfrak{L}} \subseteq \lambda_\mathfrak{P}$. By assumption, $\lambda_\mathfrak{M} \leqslant \lambda_\mathfrak{P}$ or $\lambda_\mathfrak{L} \leqslant \lambda_\mathfrak{P}$. Thus, $\mathfrak{M} \subseteq \mathfrak{P}$ or $\mathfrak{L} \subseteq \mathfrak{P}$. Thus, we determine that $\mathfrak{P}$ is a PAFR of $\mathfrak{S}$.

The proof for (2) is conducted similarly to the proof for (1). □

Theorem 6.4. *Let $\mathfrak{P}$ be a non-empty subset of $\mathfrak{S}$. Then $\mathfrak{P}$ is an SPAFR of $\mathfrak{S}$ if and only if $\lambda_\mathfrak{P}$ is an SFPAFR of $\mathfrak{S}$.*

Proof. Suppose that $\mathfrak{P}$ is an SPAFR of a semigroup $\mathfrak{S}$. Then $\mathfrak{P}$ is an AFR of $\mathfrak{S}$. Thus, by Theorem 4.7, $\lambda_\mathfrak{P}$ is an FAFR of $\mathfrak{S}$. Let ϑ and ξ be FAFRs of $\mathfrak{S}$ such that $(\vartheta \circ \xi) \wedge (\xi \circ \vartheta) \leqslant \lambda_\mathfrak{P}$. Assume that $\vartheta \nleqslant \lambda_\mathfrak{P}$ or $\xi \nleqslant \lambda_\mathfrak{P}$. Then there exist $\mathfrak{h}, \mathfrak{r} \in \mathfrak{S}$ such that $\vartheta(\mathfrak{h}) \neq 0$ and $\xi(\mathfrak{r}) \neq 0$. While $\lambda_\mathfrak{P}(\mathfrak{h}) = 0$ and $\lambda_\mathfrak{P}(\mathfrak{r}) = 0$. Thus $\mathfrak{h} \in \text{supp}(\vartheta)$ and $\mathfrak{r} \in \text{supp}(\xi)$, but $\mathfrak{h}, \mathfrak{r} \notin \mathfrak{P}$. So $\text{supp}(\vartheta) \nsubseteq \mathfrak{P}$ and $\text{supp}(\xi) \nsubseteq \mathfrak{P}$. Hence, there exists $\mathfrak{m} \in (\text{supp}(\vartheta)\,\text{supp}(\xi)) \cap (\text{supp}(\vartheta)\,\text{supp}(\xi))$ such that $\mathfrak{m} \notin \mathfrak{P}$. Thus, $\lambda_\mathfrak{P}(\mathfrak{m}) = 0$. Since $\mathfrak{m} \in \text{supp}(\vartheta)\,\text{supp}(\xi)$ and $\mathfrak{m} \in \text{supp}(\xi)\,\text{supp}(\vartheta)$, we have $\mathfrak{m} = \mathfrak{d}_1\mathfrak{e}_1$ and $\mathfrak{m} = \mathfrak{e}_2\mathfrak{d}_2$ for some $\mathfrak{d}_1, \mathfrak{d}_2 \in \text{supp}(\vartheta)$ and for some $\mathfrak{e}_1, \mathfrak{e}_2 \in \text{supp}(\xi)$. We have

$$(\vartheta \circ \xi)(\mathfrak{m}) = \bigvee_{(\mathfrak{d}_1\mathfrak{e}_1) \in F_\mathfrak{m}} \{\vartheta(\mathfrak{d}_1) \wedge \xi(\mathfrak{e}_1)\} \neq 0.$$

Similarly,

$$(\xi \circ \vartheta)(\mathfrak{m}) = \bigvee_{(\mathfrak{e}_2\mathfrak{d}_2) \in F_\mathfrak{m}} \{\xi(\mathfrak{e}_2) \wedge \vartheta(\mathfrak{d}_2)\} \neq 0.$$

It is a contradiction so $(\vartheta \circ \xi)(\mathfrak{m}) \wedge (\xi \circ \vartheta)(\mathfrak{m}) = 0$. Hence, $\vartheta \leqslant \lambda_\mathfrak{P}$ or $\xi \leqslant \lambda_\mathfrak{P}$. Therefore $\lambda_\mathfrak{P}$ is an SPFAFR of $\mathfrak{S}$.

For the converse, suppose that $\lambda_\mathfrak{P}$ is an SPFAFR of $\mathfrak{S}$. Then $\lambda_\mathfrak{P}$ is an FAFR of $\mathfrak{S}$. Thus, by Theorem 4.7, $\mathfrak{P}$ is an AFR of $\mathfrak{S}$. Let $\mathfrak{M}$ and $\mathfrak{L}$ be AFRs of $\mathfrak{S}$ such that $\mathfrak{M}\mathfrak{L} \cap \mathfrak{L}\mathfrak{M} \subseteq \mathfrak{P}$. Then $\lambda_\mathfrak{M}$ and $\lambda_\mathfrak{L}$ are FAFRs of $\mathfrak{S}$. By Lemma 2.3, $\lambda_{\mathfrak{M}\mathfrak{L}} = \lambda_\mathfrak{M} \circ \lambda_\mathfrak{L}$ and $\lambda_{\mathfrak{L}\mathfrak{M}} = \lambda_\mathfrak{L} \circ \lambda_\mathfrak{M}$. Thus, $(\lambda_\mathfrak{M} \circ \lambda_\mathfrak{L}) \wedge (\lambda_\mathfrak{L} \circ \lambda_\mathfrak{M}) = \lambda_{\mathfrak{M}\mathfrak{L}} \wedge \lambda_{\mathfrak{L}\mathfrak{M}} = \lambda_{\mathfrak{M}\mathfrak{L} \cap \mathfrak{L}\mathfrak{M}} \leqslant \lambda_\mathfrak{P}$. By assumption, $\lambda_\mathfrak{M} \leqslant \lambda_\mathfrak{P}$ or $\lambda_\mathfrak{L} \leqslant \lambda_\mathfrak{P}$. Thus, $\mathfrak{M} \subseteq \mathfrak{P}$ or $\mathfrak{L} \subseteq \mathfrak{P}$. We conclude that $\mathfrak{P}$ is an SPAFR of $\mathfrak{S}$. □

Next, we define prime almost bi-filter, semiprime almost bi-filter and strongly prime almost bi-filter analogous prime almost filter.

Definition 6.5. Let $\mathfrak{P}$ be an ABFR of $\mathfrak{S}$. Then $\mathfrak{P}$ is said to be:

(1) *prime almost bi-filter* (PABFR) if for any ABFRs $\mathfrak{M}$ and $\mathfrak{L}$ of $\mathfrak{S}$, we have $\mathfrak{M} \subseteq \mathfrak{P}$ or $\mathfrak{L} \subseteq \mathfrak{P}$ whenever $\mathfrak{M}\mathfrak{L} \subseteq \mathfrak{P}$;
(2) *semiprime almost bi-filter* (SABFR) if for any ABFR $\mathfrak{M}$ of $\mathfrak{S}$, we have $\mathfrak{M} \subseteq \mathfrak{P}$ whenever $\mathfrak{M}^2 \subseteq \mathfrak{P}$;
(3) *strongly prime almost bi-filter* (SPABFR) if for any ABFRs $\mathfrak{M}$ and $\mathfrak{L}$ of $\mathfrak{S}$, we have $\mathfrak{M} \subseteq \mathfrak{P}$ or $\mathfrak{L} \subseteq \mathfrak{P}$ whenever $\mathfrak{M}\mathfrak{L} \cap \mathfrak{L}\mathfrak{M} \subseteq \mathfrak{P}$.

Definition 6.6. Let ζ be an FABFR of a semigroup $\mathfrak{S}$. Then ζ is said to be:

(1) *prime fuzzy almost bi-filter* (PFABFR) if for any two FABFRs ξ and ϑ of $\mathfrak{S}$, we have $\xi \leqslant \zeta$ or $\vartheta \leqslant \zeta$ whenever $\xi \circ \vartheta \leqslant \zeta$;
(2) *semiprime fuzzy almost bi-filter* (SFABFR) if for any FABFR ξ of $\mathfrak{S}$, we have $\xi \leqslant \eta$ whenever $\xi \circ \xi \leqslant \zeta$;
(3) *strongly prime fuzzy almost bi-filter* (SPFABFR) if for any two FABFRs ξ and ϑ of $\mathfrak{S}$, we have $\xi \leqslant \zeta$ or $\vartheta \leqslant \zeta$ whenever $(\xi \circ \vartheta) \wedge (\vartheta \circ \xi) \leqslant \zeta$.

It is clear that every SPFABFR is a PFABFR and every PFABFR is an SFABFR.

Theorem 6.7. *Let $\mathfrak{P}$ be a non-empty subset of a semigroup $\mathfrak{S}$. Then*

(1) *$\mathfrak{P}$ is a PABFR of $\mathfrak{S}$ if and only if $\lambda_{\mathfrak{P}}$ is a PFABFR of $\mathfrak{S}$;*
(2) *$\mathfrak{P}$ is an SABFR of $\mathfrak{S}$ if and only if $\lambda_{\mathfrak{P}}$ is an SFABFR of $\mathfrak{S}$;*
(3) *$\mathfrak{P}$ is an SPABFR of $\mathfrak{S}$ if and only if $\lambda_{\mathfrak{P}}$ is an SFPABFR of $\mathfrak{S}$.*

Proof. The proof is similar to Theorems 6.3 and 6.4. □

7. Conclusion

In this paper, we explored the concept of almost all types of filters within semigroups. We demonstrated that the union of two almost filters, including bi-filters, also qualifies as an almost filter or bi-filter in semigroups, with consistent results in class fuzzifications. We also established several connections between almost types of filters and fuzzy almost types of filters in semigroups, as detailed in Theorems 4.7, 4.8, 5.3, 5.7, 6.3, and 6.4. We plan to investigate various almost filters and their fuzzifications across different algebraic structures in our future research.

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