

Properties and applications of Bell polynomials of two variables



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Abstract

In this article, we introduce Bell polynomials of two variables within the framework of generating functions and explore various properties associated with them. Specifically, we delve into explicit representations, summation formulae, recurrence relations, and addition formulas. Additionally, we present the matrix form and product formula for these polynomials. Finally, we introduce the two-variable Bell-based Stirling polynomials of the second kind and outline their corresponding results. This study contributes to a deeper understanding of the properties and applications of Bell polynomials in mathematical analysis.

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1. Introduction and preliminaries

Special polynomials refer to a class of polynomials that exhibit unique properties or have specific significance in various mathematical contexts. Examples of special polynomials include well-known families such as Legendre polynomials, Chebyshev polynomials, Hermite polynomials, Bell polynomials, Touchard polynomials, Hermite polynomials and others. These polynomials often arise in mathematical physics, engineering, computer science, and other scientific disciplines, see for instance [1, 3, 4, 6, 8, 18–20, 22].

The research on general cases of special polynomials has led to the discovery of new properties associated with these polynomials. This implies that mathematicians have identified novel characteristics, relationships, or applications of special polynomials that were not previously known. These discoveries

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contribute to the enrichment of mathematical knowledge and may have implications in various scientific and applied fields. The study of special polynomials is significant within the broader field of mathematics for several reasons.

- (i) Special polynomials often arise in the solution of differential equations, orthogonal polynomial theory, numerical analysis, and other mathematical and computational problems.
- (ii) Special polynomials often possess specific algebraic structures and recurrence relations that make them amenable to analysis. Understanding these structures contributes to the broader study of algebra and mathematical structures.
- (iii) Special polynomials often have deep connections to other areas of mathematics, including combinatorics, number theory, and analysis. These connections facilitate interdisciplinary research and the development of mathematical theories.

The use of exponential operators is highlighted as a powerful tool, particularly in the context of dealing with differential equations. Exponential operators often simplify the analysis of differential equations and offer a convenient way to express solutions. In [8], a comprehensive examination of the foundational formalism has been presented. Notably, it has been demonstrated that, through a suitable change of variable, the impact of the operator on a given function of x can be regarded as that of a conventional shift operator. Thus, for any parameter μ , the shift operator $\exp(\mu\partial_x)$ when operated on any function of x produces the following:

$$\exp(\mu\partial_x)\{f(x)\} = \sum_{r=0}^{\infty} \partial_x^r f(x) \frac{\mu^r}{r!} = \sum_{r=0}^{\infty} f^{(r)}(x) \frac{\mu^r}{r!} = f(x + \mu), \quad (1.1)$$

where $\partial_x^r = \frac{\partial^r}{\partial x^r}$. The following identities are exploited from (1.1):

$$\begin{aligned} \exp(\mu x^2 \partial_x)\{f(x)\} &= f\left(\frac{x}{1 - \mu x}\right), \\ \exp(\mu \partial_x)\{x_1^r\} &= (x + \mu)^r, \\ \exp(\mu \partial_x^r)\{e^x\} &= e^{x+\mu}, \\ \exp(\mu x \partial_x)f\{x\} &= f(e^x \mu). \end{aligned}$$

One of the important classes of special polynomials is Bell polynomials [6]. Bell polynomials, named after mathematician Eric Temple Bell, are a sequence of polynomials that arise in combinatorics. They are denoted as $\mathcal{B}_r(x)$ and are defined by the following exponential generating function:

$$\sum_{r=0}^{\infty} \mathcal{B}_r(x) \frac{\xi^r}{r!} = e^{x(e^\xi - 1)} \quad (1.2)$$

and for $x = 1$, the Bell polynomials reduce to the Bell numbers given by the relation:

$$\sum_{r=0}^{\infty} \mathcal{B}_r \frac{\xi^r}{r!} = e^{e^\xi - 1}.$$

The Bell polynomials are used to express the partial Bell polynomials, which in turn represent the partial derivatives of the exponential generating function. The Bell polynomials have applications in combinatorics, probability theory, and the analysis of algorithms. They are particularly useful in counting and enumerating combinatorial structures, such as partitions of sets and compositions of integers, see for instance [2, 10–12, 14, 15].

Another important class of polynomial sequences and numbers are Stirling numbers. These are a family of numbers that arise in combinatorics, particularly in problems related to permutations, combinations,

and partitions, for instance, see [2, 5, 9–12, 14, 15]. There are two main types of Stirling numbers: Stirling numbers of the first kind, denoted as $S_1(r, \epsilon)$, and Stirling numbers of the second kind, denoted as $S_2(r, \epsilon)$. The Stirling numbers of the first kind $S_1(r, \epsilon)$ represents the number of permutations of (r) elements with exactly (ϵ) cycles. In other words, it counts how many ways you can arrange (r) distinct elements into (ϵ) cycles. The Stirling numbers of the second kind $S_2(r, \epsilon)$ represents the number of ways to partition a set of (r) distinct elements into (ϵ) non-empty subsets. Each subset is treated as an indistinguishable set, and the order of subsets matters. Stirling numbers have numerous applications in combinatorics, including counting problems involving permutations, combinations, and partitions. They also appear in the analysis of algorithms and in the study of various combinatorial structures. It's important to note that there are different notations and conventions for Stirling numbers, and the ones described above are commonly used.

Stirling polynomials of the second kind denoted as $S_2(r, \epsilon; x)$, are associated with exponential generating functions. The exponential generating function for Stirling polynomials of the second kind is given by:

$$\sum_{r=0}^{\infty} \frac{S_2(r, \epsilon; x) \xi^r}{r!} = \frac{(e^{\xi x} - 1)^\epsilon}{\epsilon!}$$

and for $x = 1$ the exponential generating function for Stirling numbers of the second kind reduces to:

$$\sum_{r=0}^{\infty} \frac{S_2(r, \epsilon) \xi^r}{r!} = \frac{(e^\xi - 1)^\epsilon}{\epsilon!}. \quad (1.3)$$

Further, the recurrence relation for Stirling numbers of the second kind $S_2(r, \epsilon)$ can be computed using the recurrence relation [7, 17]:

$$x^r = \sum_{\epsilon=0}^r S_2(r, \epsilon) (x)_\epsilon$$

or

$$(x)_r = \sum_{\epsilon=0}^r S_2(r, \epsilon) x^\epsilon,$$

where $(x)_\epsilon = x(x-1)(x-2) \cdots (x-(\epsilon-1))$ is the falling factorial. Also, for each integer $\epsilon \in \mathbb{N}_0$, the expression

$$S_\epsilon(r) = \sum_{l=0}^r l^\epsilon$$

is called the sum of integer powers and the exponential generating function for $S_\epsilon(r)$ is given by:

$$\sum_{\epsilon=0}^{\infty} S_\epsilon(r) \frac{\xi^\epsilon}{\epsilon!} = \frac{e^{(r+1)\xi} - 1}{e^\xi - 1}.$$

These concepts are fundamental in combinatorics and have various applications in counting problems related to partitions and subsets of distinguishable objects.

Bell polynomials offer a versatile and powerful framework for understanding combinatorial structures, particularly in the context of partitioning, generating functions, and algebraic combinatorics. Their importance is widespread in mathematics, which makes them an effective tool for counting issues and discrete structure analysis. Motivated by the above applications, here, we develop the generating relation of two variable Bell polynomials in the form:

$$\sum_{r=0}^{\infty} \mathcal{B}_r(x, y) \frac{\xi^r}{r!} = e^{x(e^\xi - 1) + y(e^\xi - 1)^2}. \quad (1.4)$$

The first five two variable Bell polynomials are as follows:

$$\mathcal{B}_0(x, y) = 1,$$

$$\mathcal{B}_1(x, y) = x,$$

$$\mathcal{B}_2(x, y) = x^2 + x + 2y,$$

$$\mathcal{B}_3(x, y) = x^3 + 3x^2 + x + 6y + 6xy,$$

$$\mathcal{B}_4(x, y) = x^4 + 6x^3 + 7x^2 + x + 12y^2 + 14y + 12x^2y + 6xy.$$

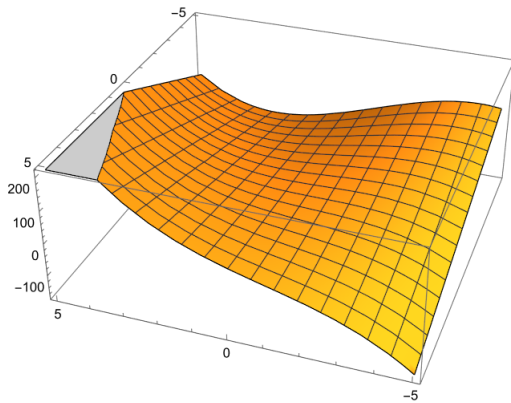


Figure 1: $x^3 + 3x^2 + x + 6y + 6xy$.

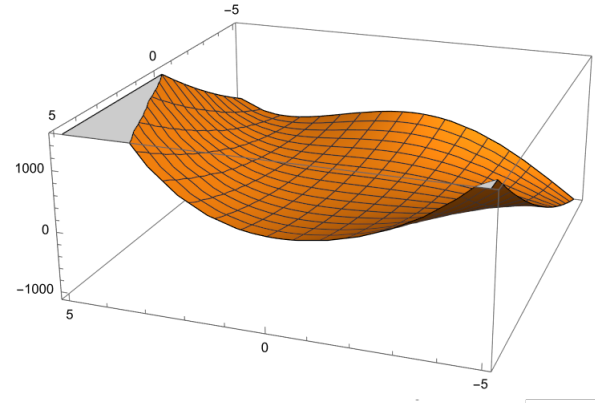


Figure 2: $x^4 + 6x^3 + 7x^2 + x + 12y^2 + 14y + 12x^2y + 6xy$.

For $y = 0$ in (1.4), the two-variable Bell polynomials $\mathcal{B}_r(x, y)$ reduce to the Bell polynomials given by (1.2). The rest of the article is as follows. In Section 2, we introduce Bell polynomials of two variables within the framework of generating functions and explore various properties associated with them. Specifically, we derive explicit representations, summation formulae, recurrence relations, and addition formulas, with a focus on their connection to Stirling polynomials of the second kind. Section 3 presents the matrix form and product formula for these polynomials, offering insights into their structural properties. In Section 4, we introduce the two-variable Bell-based Stirling polynomials of the second kind and outline their corresponding results, further enriching our understanding of their properties. Finally, the conclusion section underscores the significance of our study in advancing the comprehension of the properties and applications of Bell polynomials in mathematical analysis.

2. Two variable Bell polynomials

Here, in this section, we derive the explicit forms and certain other properties of two-variable Bell polynomials denoted by $\mathcal{B}_r(x, y)$. The explicit forms and summation formulae derived in this study are of paramount significance in enhancing our understanding and utilization of Bell polynomials in mathematical analysis. These explicit representations provide concise and structured expressions for Bell polynomials of two variables, facilitating their computational implementation and theoretical analysis. Moreover, the derived summation formulae offer systematic methods for efficiently computing the sums of these polynomials, enabling researchers to address complex mathematical problems with greater ease and accuracy. By elucidating the explicit forms and summation properties of Bell polynomials, this study not only contributes to the theoretical foundation of the field but also enhances the practical applicability of these polynomials in various mathematical and scientific disciplines. The results are derived as under.

Theorem 2.1. *The two-variable Bell polynomials denoted by $\mathcal{B}_r(x, y)$ satisfy the listed explicit form:*

$$\mathcal{B}_r(x, y) = \sum_{s=0}^r \binom{r}{s} \mathcal{B}_{r-s}(x) \frac{S_2(s, k) (e^x - 1)^k}{1 - y}. \quad (2.1)$$

Proof. The generating expression (1.4) can be written as

$$e^{x(e^\xi-1)+y(e^\xi-1)^2} = e^{x(e^\xi-1)} e^{y(e^\xi-1)^2}.$$

Inserting expressions (1.2) and (1.3) in the r.h.s. of the previous expression, we find

$$e^{x(e^\xi-1)+y(e^\xi-1)^2} = \sum_{r=0}^{\infty} \mathcal{B}_r(x) \frac{\xi^r}{r!} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} y^m S_2(s, k) \frac{\xi^s}{s!} (e^\xi - 1)^k.$$

Inserting r.h.s. of the expression (1.4) in the l.h.s. of the previous expression and simplifying the r.h.s., it follows that

$$\sum_{r=0}^{\infty} \mathcal{B}_r(x, y) \frac{\xi^r}{r!} = \sum_{r=0}^{\infty} \mathcal{B}_r(x) \sum_{s=0}^{\infty} \frac{S_2(s, k) (e^\xi - 1)^k}{1 - y} \frac{\xi^{r+s}}{r! s!}.$$

Taking $r - s$ in place of r in the r.h.s. of the previous expression by using series rearrangement, it follows that

$$\sum_{r=0}^{\infty} \mathcal{B}_r(x, y) \frac{\xi^r}{r!} = \sum_{r=0}^{\infty} \sum_{s=0}^{[r]} \binom{r}{s} \mathcal{B}_{r-s}(x) \frac{S_2(s, k) (e^\xi - 1)^k}{1 - y} \frac{\xi^r}{r!}.$$

Comparing the like powers of $\frac{\xi^r}{r!}$ on both sides of the previous expression, we get the assertion (2.1). $\square$

Theorem 2.2. The two-variable Bell polynomials denoted by $\mathcal{B}_r(x, y)$ satisfy the listed series representation form:

$$\mathcal{B}_r(x, y) = \sum_{s=0}^r \binom{r}{s} \frac{S_2(r-s, l)}{1-x} \frac{S_2(s, k) (e^\xi - 1)^k}{1-y}. \quad (2.2)$$

Proof. The generating expression (1.4) can be written as

$$e^{x(e^\xi-1)+y(e^\xi-1)^2} = e^{x(e^\xi-1)} e^{y(e^\xi-1)^2}.$$

Inserting expression (1.3) in the r.h.s. of the previous expression, we find

$$e^{x(e^\xi-1)+y(e^\xi-1)^2} = \sum_{l=0}^{\infty} \sum_{r=0}^{\infty} x^l S_2(r, l) \frac{\xi^r}{r!} (e^\xi - 1)^k r! \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} y^r S_2(s, k) \frac{\xi^s}{s!} (e^\xi - 1)^k.$$

Inserting r.h.s. of the expression (1.4) in the l.h.s. of the previous expression and simplifying the r.h.s., it follows that

$$\sum_{r=0}^{\infty} \mathcal{B}_r(x, y) \frac{\xi^r}{r!} = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{S_2(r, l)}{1-x} \frac{S_2(s, k) (e^\xi - 1)^k}{1-y} \frac{\xi^{r+s}}{r! s!}.$$

Taking $r - s$ in place of r in the r.h.s. of the previous expression by using series rearrangement, it follows that

$$\sum_{r=0}^{\infty} \mathcal{B}_r(x, y) \frac{\xi^r}{r!} = \sum_{r=0}^{\infty} \sum_{s=0}^{[r]} \binom{r}{s} \frac{S_2(r-s, l)}{1-x} \frac{S_2(s, k) (e^\xi - 1)^k}{1-y} \frac{\xi^r}{r!}.$$

Comparing the like powers of $\frac{\xi^r}{r!}$ on both sides of the previous expression, we get the assertion (2.2). $\square$

Theorem 2.3. Assume the two-variable Bell polynomials denoted by $\mathcal{B}_r(x, y)$. Then the following summation formulas hold:

$$\mathcal{B}_r(x + y, z) = \sum_{k=0}^r \binom{r}{k} \mathcal{B}_k(x, z) \mathcal{B}_{r-k}(y).$$

Proof. By (1.4) and (1.2), we have

$$\begin{aligned} \sum_{r=0}^{\infty} \mathcal{B}_r(x+y, z) \frac{\xi^r}{r!} &= e^{(x+y)(e^\xi-1)+z(e^\xi-1)^2} \\ &= \sum_{r=0}^{\infty} \mathcal{B}_r(x, z) \frac{\xi^r}{r!} \sum_{r=0}^{\infty} \mathcal{B}_r(y) \frac{\xi^r}{r!} = \sum_{r=0}^{\infty} \left[\sum_{k=0}^r \binom{r}{k} \mathcal{B}_k(x, z) \mathcal{B}_{r-k}(y) \right] \frac{\xi^r}{r!}. \end{aligned}$$

Finally equating the coefficients of $\frac{\xi^r}{r!}$ of both sides, we get the asserted Theorem 2.3. $\square$

Theorem 2.4. For any arbitrary $r \in \mathbb{N}$, the following relation holds true:

$$\mathcal{B}_r(x+1, y) - \mathcal{B}_r(x, y) = \sum_{k=0}^r \binom{r}{k} \mathcal{B}_{r-k}(x, y) \mathcal{B}_k - \mathcal{B}_r(x, y). \quad (2.3)$$

Proof. Using the relation (1.4), we get

$$\begin{aligned} \sum_{r=0}^{\infty} [\mathcal{B}_r(x+1, y) - \mathcal{B}_r(x, y)] \frac{\xi^r}{r!} &= e^{(x+1)(e^\xi-1)+y(e^\xi-1)^2} - e^{x(e^\xi-1)+y(e^\xi-1)^2} \\ &= e^{x(e^\xi-1)+y(e^\xi-1)^2} [e^{e^\xi-1} - 1] \\ &= \sum_{r=0}^{\infty} \left[\sum_{k=0}^r \binom{r}{k} k \mathcal{B}_{r-k}(x, y) \mathcal{B}_k - \mathcal{B}_r(x, y) \right] \frac{\xi^r}{r!}. \end{aligned}$$

By equating both sides, we obtained the result (2.3). $\square$

Theorem 2.5. For $r > 1$, let $\mathcal{B}_r(x, y)$ be the two-variable Bell polynomials. Then we have

$$\frac{\partial}{\partial x} \mathcal{B}_{r-1}(x, y) = \frac{1}{(r^2 + r)} \sum_{k=0}^r \binom{r+1}{k} \mathcal{B}_k(x, y) \quad (2.4)$$

and

$$\frac{\partial}{\partial y} \mathcal{B}_r(x, y) = \sum_{k=0}^r \binom{r}{k} 2! \mathcal{B}_k(x, y) \lambda_{r-k} S_2(r-k, 2), \quad (2.5)$$

where

$$\lambda_r = \begin{cases} 0, & \text{if } r = 0, 1, \\ 1, & \text{if } r \geq 2. \end{cases}$$

Proof. (see (2.4)). Deriving partially with respect to the variable x on both sides of equation (1.4), we have

$$\begin{aligned} \frac{\partial}{\partial x} \left[\sum_{r=0}^{\infty} \mathcal{B}_r(x, y) \frac{\xi^r}{r!} \right] &= \frac{\partial}{\partial x} [e^{x(e^\xi-1)+y(e^\xi-1)^2}] \\ &= e^{x(e^\xi-1)+y(e^\xi-1)^2} (e^\xi - 1) \\ &= \sum_{k=0}^{\infty} \mathcal{B}_k(x, y) \frac{\xi^k}{k!} \sum_{r=0}^{\infty} \frac{\xi^r}{r!} = \sum_{r=0}^{\infty} \sum_{k=0}^r \binom{r+1}{k} \mathcal{B}_k(x, y) \frac{\xi^{r+1}}{(r+1)!}. \end{aligned}$$

So,

$$\frac{\partial}{\partial x} \mathcal{B}_{r-1}(x, y) = \frac{1}{(r^2 + r)} \sum_{k=0}^r \binom{r+1}{k} \mathcal{B}_k(x, y).$$

Further, to demonstrate equation (2.5), we can simply take partial derivatives concerning the variable y on both sides of the equation (1.4). By leveraging the ratio of the second species Stirling numbers, as provided in [13], we attain the desired result. $\square$

Theorem 2.6. For $r \geq 0$, let $\{\mathcal{B}_r(x, y)\}_{r \geq 0}$ be the sequences of two-variable Bell polynomials in the variable x, y and z , they satisfy the following relation

$$\sum_{k=0}^r \binom{r}{k} [\mathcal{B}_k(x+z, y) \mathcal{B}_{r-k}(y) - \mathcal{B}_{r-k}(x, y) \mathcal{B}_r(z, y)] = 0.$$

Proof. Let's consider the following expressions

$$e^{x(e^\xi-1)+y(e^\xi-1)^2} = \sum_{r=0}^{\infty} \mathcal{B}_r(x, y) \frac{\xi^r}{r!} \quad (2.6)$$

and

$$e^{z(e^\xi-1)+y(e^\xi-1)^2} = \sum_{r=0}^{\infty} \mathcal{B}_r(z, y) \frac{\xi^r}{r!}. \quad (2.7)$$

From (2.6) and (2.7), we have

$$\begin{aligned} e^{(x+z)(e^\xi-1)+y(e^\xi-1)^2} e^{y(e^\xi-1)^2} &= \left(\sum_{r=0}^{\infty} \mathcal{B}_r(x, y) \frac{\xi^r}{r!} \right) \left(\sum_{r=0}^{\infty} \mathcal{B}_r(z, y) \frac{\xi^r}{r!} \right), \\ \left(\sum_{r=0}^{\infty} \mathcal{B}_r(x+z, y) \frac{\xi^r}{r!} \right) \left(\sum_{r=0}^{\infty} \mathcal{B}_r(0, y) \frac{\xi^r}{r!} \right) &= \left(\sum_{r=0}^{\infty} \mathcal{B}_r(x, y) \frac{\xi^r}{r!} \right) \left(\sum_{r=0}^{\infty} \mathcal{B}_r(z, y) \frac{\xi^r}{r!} \right), \\ \sum_{r=0}^{\infty} \sum_{k=0}^r \binom{r}{k} \mathcal{B}_k(x+z, y) \mathcal{B}_{r-k}(y) \frac{\xi^r}{r!} &= \sum_{r=0}^{\infty} \sum_{k=0}^r \binom{r}{k} \mathcal{B}_{r-k}(x, y) \mathcal{B}_r(z, y) \frac{\xi^r}{r!}, \\ \sum_{k=0}^r \binom{r}{k} \mathcal{B}_k(x+z, y) \mathcal{B}_{r-k}(y) &= \sum_{k=0}^r \binom{r}{k} \mathcal{B}_{r-k}(x, y) \mathcal{B}_r(z, y). \end{aligned}$$

Therefore,

$$\sum_{k=0}^r \binom{r}{k} [\mathcal{B}_k(x+z, y) \mathcal{B}_{r-k}(y) - \mathcal{B}_{r-k}(x, y) \mathcal{B}_r(z, y)] = 0.$$

$\square$

Theorem 2.7. For $r \geq 0$, let $\{\mathcal{B}_r(x, y)\}_{r \geq 0}$ be the sequences of two-variable Bell polynomials in the variables x, y, z , and w , they satisfy the following relation

$$\mathcal{B}_r(x+y, z+w) = \sum_{k=0}^r \binom{r}{k} \mathcal{B}_{r-k}(x, z) \mathcal{B}_k(y, w). \quad (2.8)$$

Proof. Utilizing the subsequent identity

$$\begin{aligned} e^{(x+y)(e^\xi-1)+(z+w)(e^\xi-1)^2} &= e^{x(e^\xi-1)+y(e^\xi-1)+z(e^\xi-1)^2+w(e^\xi-1)^2} \\ &= e^{x(e^\xi-1)+z(e^\xi-1)^2} e^{y(e^\xi-1)+w(e^\xi-1)^2}, \end{aligned}$$

and applying the aforementioned identity to equation (1.4), we obtain

$$\begin{aligned} \sum_{r=0}^{\infty} \mathcal{B}_r(x+y, z+w) \frac{\xi^r}{r!} &= e^{(x+y)(e^\xi-1) + (z+w)(e^\xi-1)^2} \\ &= e^{x(e^\xi-1) + z(e^\xi-1)^2} e^{y(e^\xi-1) + w(e^\xi-1)^2} = \sum_{r=0}^{\infty} \mathcal{B}_r(x, z) \frac{\xi^r}{r!} \sum_{r=0}^{\infty} \mathcal{B}_r(y, w) \frac{\xi^r}{r!}. \end{aligned}$$

Using the series rearrangement technique, we obtain

$$\sum_{r=0}^{\infty} \mathcal{B}_r(x+y, z+w) \frac{\xi^r}{r!} = \sum_{r=0}^{\infty} \sum_{k=0}^r \binom{r}{k} \mathcal{B}_{r-k}(x, z) \mathcal{B}_k(y, w) \frac{\xi^r}{r!}.$$

Now, equating both sides, we obtained the result (2.8). $\square$

3. The two-variable Bell polynomials matrix

The matrix form of special polynomials holds profound significance in mathematical analysis and practical applications. By expressing these polynomials in matrix notation, we can leverage powerful mathematical tools and techniques from linear algebra to study their properties and behaviors. This representation allows for a unified and compact framework to analyze various aspects, such as recurrence relations, orthogonality properties, and computational algorithms. Furthermore, the matrix form facilitates efficient manipulation and computation of special polynomials, enabling researchers to develop novel algorithms, solving intricate problems, and exploring new applications across diverse fields, including signal processing, image reconstruction, and numerical analysis. Thus, the matrix representation of special polynomials not only deepens our theoretical understanding but also enhances their utility in practical scenarios, contributing to advancements in both theory and application domains. Inspired by the articles [16, 23] in which the authors introduce the generalized Apostol-type polynomial matrix, in this section we focus our attention on the two-variable Bell polynomial matrix.

Definition 3.1. The $(n+1) \times (n+1)$ two-variable Bell polynomial matrix $\mathcal{B}(x, y)$ is defined by

$$\mathcal{B}_{i,j}(x, y) = \begin{cases} \binom{i}{j} \mathcal{B}_{i-j}(x, y), & i \geq j, \\ 0, & \text{otherwise,} \end{cases}$$

Let us consider $n = 3$. It follows from the Definition 3.1 that

$$\mathcal{B}(x, y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 + x + 2y & 2x & 1 & 0 \\ x^3 + 3x^2 + x + 6y + 6xy & 3(x^2 + x + 2y) & 3x & 1 \end{bmatrix}.$$

Theorem 3.2. The two-variable Bell polynomials matrix $\mathcal{B}(x, y)$ satisfies the following product formula

$$\mathcal{B}(x+y, z+w) = \mathcal{B}(x, z) \mathcal{B}(y, w) = \mathcal{B}(z, w) \mathcal{B}(x, y).$$

Proof. Let $W_{i,j}(x, y, z, w)$ be the (i, j) -th entry of the matrix product $\mathcal{B}(x, y) \mathcal{B}(z, w)$. Then, by the addition

formula (2.8), we have

$$\begin{aligned}
 W_{i,j}(x, y, z, w) &= \sum_{k=0}^n \binom{i}{k} \mathcal{B}_{i-k}(x, z) \binom{k}{j} \mathcal{B}_{k-j}(y, w) \\
 &= \sum_{k=j}^i \binom{i}{k} \mathcal{B}_{i-k}(x, z) \binom{k}{j} \mathcal{B}_{k-j}(z, w) \\
 &= \sum_{k=j}^i \binom{i}{j} \binom{i-j}{i-k} \mathcal{B}_{i-k}(x, z) \mathcal{B}_{k-j}(y, w) \\
 &= \binom{i}{j} \sum_{k=0}^{i-j} \binom{i-j}{k} \mathcal{B}_{i-j-k}(x, z) \mathcal{B}_k(y, w) = \binom{i}{j} \mathcal{B}_{i-j}(x+y, z+w).
 \end{aligned}$$

This establishes the first equality of the theorem. The second equality can be derived similarly. $\square$

Definition 3.3. The $(n+1) \times (n+1)$ two-variable Bell polynomial matrix $\Omega(x, y)$ is defined by

$$\Omega_{i,j}(x, y) = \begin{cases} \binom{i}{j} \mathcal{B}_{i-j}(-x, -y), & i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

Now, we show that the matrix $\Omega_{i,j}(x, y)$ given above is the inverse matrix of $\mathcal{B}(x, y)$ by proving the following theorem.

Theorem 3.4. The inverse matrix for the two-variable Bell polynomials matrix $\mathcal{B}(x, y)$ is given by

$$\mathcal{B}^{-1}(x, y) = \Omega_{i,j}(x, y).$$

Example 3.5. For a $(4) \times (4)$ matrix, we have

$$\begin{aligned}
 \mathcal{B}(x, y) \mathcal{B}^{-1}(x, y) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 + x + 2y & 2x & 1 & 0 \\ x^3 + 3x^2 + x + 6y + 6xy & 3(x^2 + x + 2y) & 3x & 1 \end{bmatrix} \\
 &\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ -x & 1 & 0 & 0 \\ x^2 - x - 2y & -2x & 1 & 0 \\ -x^3 + 3x^2 - x - 6y + 6xy & 3(x^2 - x - 2y) & -3x & 1 \end{bmatrix} = I_4.
 \end{aligned}$$

4. The two-variable Bell-based Stirling polynomials of the second kind

In this section, we introduce a new class of mathematical class of polynomials called two-variable Bell-based Stirling polynomials of the second kind and examine their basic characteristics and complex interactions in detail. These polynomials belong to a distinct class of functions in mathematics, and because of their unusual properties, they are interesting for both theoretical study and real-world use. The main purpose of our talk is to clarify their basic characteristics and provide insight into the fundamental ideas that drive their actions and operations. To aid comprehension, we offer a carefully constructed description of the second sort of Bell-based Stirling polynomials, which serves as a crucial foundation for deciphering the complexities included in this mathematical structure. The second sort of Bell-based Stirling polynomials are defined as follows.

Definition 4.1.

$$\sum_{r=0}^{\infty} \mathcal{BS}_2(r, \epsilon; x, y) \frac{\xi^r}{r!} = \frac{(e^\xi - 1)^\epsilon}{\epsilon!} e^{x(e^\xi - 1) + y(e^\xi - 1)^2}. \quad (4.1)$$

This definition acts as a conceptual cornerstone, laying the groundwork for further investigation and understanding of the significant ramifications and potential uses of these polynomials in the larger field of mathematics.

For $\epsilon = 2$, the first four two-variable Bell-based Stirling polynomials are as follows:

$$\begin{aligned} \mathcal{BS}_2(0, 2; x, y) &= \frac{1}{2}, \\ \mathcal{BS}_2(1, 2; x, y) &= \frac{1}{2}x + \frac{1}{2}, \\ \mathcal{BS}_2(2, 2; x, y) &= \frac{1}{2}x^2 + \frac{3}{2}x + y + \frac{7}{12}, \\ \mathcal{BS}_2(3, 2; x, y) &= \frac{1}{2}x^3 + 3x^2 + 3xy + 6y + \frac{3}{4}. \end{aligned}$$

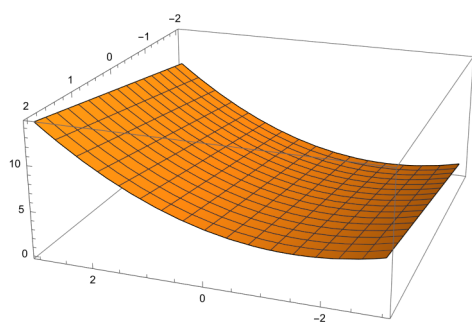


Figure 3: $\frac{1}{2}x^2 + \frac{3}{2}x + y + \frac{7}{12}$.

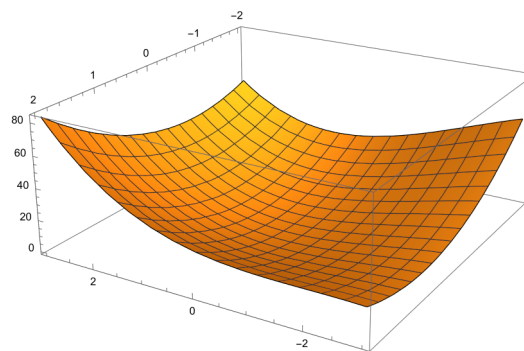


Figure 4: $\frac{1}{2}x^3 + 3x^2 + 3xy + 6y + \frac{3}{4}$.

Remark 4.2. When substituting $y = 0$ into the expression given by (4.1), we obtain a set of polynomials known as the Bell-Stirling polynomials of the second kind given by the expression:

$$\sum_{r=0}^{\infty} \mathcal{BS}_2(r, \epsilon; x) \frac{\xi^r}{r!} = \frac{(e^\xi - 1)^\epsilon}{\epsilon!} e^{x(e^\xi - 1)}.$$

Remark 4.3. Further substituting $x = y = 0$ into the expression given by (4.1), we obtain a set of polynomials known as the Stirling numbers of the second kind given in (1.3).

Theorem 4.4. For a non-negative integer r , the two variable Bell-based Stirling polynomials of the second kind satisfy the correlation:

$$\sum_{l=0}^r \binom{r}{l} \mathcal{BS}_2(l, \epsilon) \mathcal{BS}_{r-l}(x, y) = \mathcal{BS}_2(r, \epsilon; x, y).$$

Proof. The generating expression (4.1) can be written as:

$$\sum_{r=0}^{\infty} \mathcal{BS}_2(r, \epsilon; x, y) \frac{\xi^r}{r!} = \frac{(e^\xi - 1)^\epsilon}{\epsilon!} e^{x(e^\xi - 1) + y(e^\xi - 1)^2} = \sum_{r=\epsilon}^{\infty} \mathcal{BS}_2(r, \epsilon) \frac{\xi^r}{r!} \sum_{r=0}^{\infty} \mathcal{BS}_r(x, y) \frac{\xi^r}{r!},$$

which further can be expressed as

$$\sum_{r=0}^{\infty} \mathcal{B}S_2(r, \epsilon; x, y) \frac{\xi^r}{r!} = \sum_{r=0}^{\infty} \sum_{l=0}^r \binom{r}{l} S_2(l, \epsilon) \mathcal{B}_{r-l}(x, y) \frac{\xi^r}{r!}.$$

Comparing the exponents of the same powers of ξ , we get the desired result. $\square$

Remark 4.5. When substituting $y = 0$ into the expression given by (4.1), the Bell-based Stirling polynomials of the second kind satisfy the correlation:

$$\sum_{l=0}^r \binom{r}{l} S_2(l, \epsilon) \mathcal{B}_{r-l}(x) = \mathcal{B}S_2(r, \epsilon; x),$$

for a non-negative integer n .

Theorem 4.6. The two-variable Bell-based Stirling polynomials are denoted by $\mathcal{B}S_2(r, \epsilon; x, y)$. Then the following summation formulas hold:

$$\mathcal{B}S_2(r, \epsilon; x + z, y) = \sum_{k=0}^r \binom{r}{k} \mathcal{B}S_2(r - k, \epsilon; x, y) \mathcal{B}_k(z).$$

Proof. By (4.1) and (1.2), we have

$$\begin{aligned} \sum_{r=0}^{\infty} \mathcal{B}S_2(r, \epsilon; x + z, y) \frac{\xi^r}{r!} &= \frac{(e^{\xi} - 1)^{\epsilon}}{\epsilon!} e^{(x+z)(e^{\xi}-1)+y(e^{\xi}-1)^2} \\ &= \sum_{r=0}^{\infty} \mathcal{B}S_2(r, \epsilon; x, y) \frac{\xi^r}{r!} \sum_{r=0}^{\infty} \mathcal{B}_r(z) \frac{\xi^r}{r!} = \sum_{r=0}^{\infty} \left[\sum_{k=0}^r \binom{r}{k} \mathcal{B}S_2(r - k, \epsilon; x, y) \mathcal{B}_k(z) \right] \frac{\xi^r}{r!}. \end{aligned}$$

Finally equating the coefficients of $\frac{\xi^r}{r!}$ of both sides, we get the asserted Theorem 4.6. $\square$

Theorem 4.7. For any arbitrary $r \in \mathbb{N}$, the following relation holds true:

$$\mathcal{B}S_2(r, \epsilon; x + 1, y) - \mathcal{B}S_2(r, \epsilon; x, y) = \sum_{k=0}^r \binom{r}{k} \mathcal{B}S_2(r - k, \epsilon; x, y) \mathcal{B}_k - \mathcal{B}S_2(r, \epsilon; x, y). \quad (4.2)$$

Proof. Using the relation (1.4), we get

$$\begin{aligned} \sum_{r=0}^{\infty} [\mathcal{B}S_2(r, \epsilon; x + 1, y) - \mathcal{B}S_2(r, \epsilon; x, y)] \frac{\xi^r}{r!} &= \frac{(e^{\xi} - 1)^{\epsilon}}{\epsilon!} e^{(x+1)(e^{\xi}-1)+y(e^{\xi}-1)^2} - \frac{(e^{\xi} - 1)^{\epsilon}}{\epsilon!} e^{x(e^{\xi}-1)+y(e^{\xi}-1)^2} \\ &= \frac{(e^{\xi} - 1)^{\epsilon}}{\epsilon!} e^{x(e^{\xi}-1)+y(e^{\xi}-1)^2} [e^{e^{\xi}-1} - 1] \\ &= \sum_{r=0}^{\infty} \left[\sum_{k=0}^r \binom{r}{k} \mathcal{B}S_2(r - k, \epsilon; x, y) \mathcal{B}_k - \mathcal{B}S_2(r, \epsilon; x, y) \right] \frac{\xi^r}{r!}. \end{aligned}$$

By equating both sides, we obtained the result (4.2). $\square$

Theorem 4.8. For $r > 1$, let $\mathcal{B}S_2(r, \epsilon; x, y)$ be the two-variable Bell-based Stirling polynomials. Then we have

$$\frac{\partial}{\partial x} \mathcal{B}S_2(r - 1, \epsilon; x, y) = \frac{1}{(r^2 + r)} \sum_{k=0}^r \binom{r+1}{k} \mathcal{B}S_2(k, \epsilon; x, y) \quad (4.3)$$

and

$$\frac{\partial}{\partial y} \mathcal{B}S_2(r, \epsilon; x, y) = \sum_{k=0}^r \binom{r}{k} 2! \mathcal{B}S_2(k, \epsilon; x, y) \lambda_{r-k} S_2(r-k, 2), \quad (4.4)$$

where

$$\lambda_r = \begin{cases} 0, & \text{if } r = 0, 1, \\ 1, & \text{if } r \geq 2. \end{cases}$$

Proof. (see (4.3)). Deriving partially with respect to the variable x on both sides of equation (4.1), we have

$$\begin{aligned} \frac{\partial}{\partial x} \left[\sum_{r=0}^{\infty} \mathcal{B}S_2(r, \epsilon; x, y) \right] &= \frac{\partial}{\partial x} \left[\frac{(e^{\xi} - 1)^{\epsilon}}{\epsilon!} e^{x(e^{\xi}-1)+y(e^{\xi}-1)^2} \right] \\ &= \frac{(e^{\xi} - 1)^{\epsilon}}{\epsilon!} e^{x(e^{\xi}-1)+y(e^{\xi}-1)^2} (e^{\xi} - 1) \\ &= \sum_{r=0}^{\infty} \mathcal{B}S_2(r, \epsilon; x, y) \frac{\xi^r}{r!} \sum_{r=0}^{\infty} \frac{\xi^r}{r!} = \sum_{r=0}^{\infty} \sum_{k=0}^r \binom{r+1}{k} \mathcal{B}S_2(k, \epsilon; x, y) \frac{\xi^{r+1}}{(r+1)!}. \end{aligned}$$

So

$$\frac{\partial}{\partial x} \mathcal{B}S_2(r-1, \epsilon; x, y) = \frac{1}{(r^2 + r)} \sum_{k=0}^r \binom{r+1}{k} \mathcal{B}S_2(k, \epsilon; x, y).$$

Further, to demonstrate equation (4.4), we can simply take partial derivatives concerning the variable y on both sides of the equation (4.1). By leveraging the ratio of the second species Stirling numbers, as provided in [13], we attain the desired result. $\square$

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