J. Math. Computer Sci., 35 (2024), 270–290

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

The uniqueness and existence of solutions in a new complex function space for Kannan nonlinear dynamical systems

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Abstract

This paper explores the topological and geometric characteristics of a novel complex function space. We do this by constructing a weighted binomial matrix within the Nakano space of absolute type. The fixed points of Kannan contraction and non-expansive operators are associated with these structures. To solve Volterra-type summable equations, one can analyze practical instances and their applications.

Keywords: Pre-quasi norm, binomial matrix, Kannan non-expansive operators.

2020 MSC: 47B10, 46C05, 46E30, 47H09.

1. Introduction

Summable equations come up in many situations in critical point theory for non-smooth energy functionals, mathematical physics, control theory, bio-mathematics, difference variational inequalities, fuzzy set theory [34], probability theory [14], and traffic problems, to mention but a few. In particular, Volterra-type summable equations are known to be of great importance in investigating dynamical systems [1] and stochastic processes [19, 23]. Some instances are in the fields of granular systems, sweeping processes, oscillation problems, control problems, and decision-making problems [10]. The solution of summable equations is contained in a certain sequence space. So there is a great interest in mathematics to construct new sequence spaces, see [25]. Mursaleen and Noman [26] examined some new sequence spaces of non-absolute type related to the spaces ℓ_p and ℓ_∞, and Mursaleen and Başar [24] constructed and investigated the domain of C_1 matrix in some spaces of double sequences. The Banach contraction principle (BCP) is a noteworthy result from 1922 by the Polish mathematician Banach [8] that is fundamental to a theory of metric fixed points. Banach's work is well appreciated and has flexible repercussions in the theory of fixed points. Since BCP's initial work in this area in 1922, numerous other researchers have built upon their findings. The nonlinear analysis places significant weight on the BCP, which is a potent instrument for nonlinear analysis [9, 13, 29, 32, 33]. Kannan [21] showed one of the most important generalizations of

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doi: 10.22436/jmcs.035.03.02

Received: 2024-02-14 *Revised:* 2024-03-04 *Accepted:* 2024-04-15

the BCP. He showed a group of operators with the same fixed point actions as contractions, but this group is not continuous. Once, in Reference [18], an attempt was made to explain Kannan operators in modular vector spaces. That was the only attempt ever made. Bakery and Mohamed [5] researched the idea of a pre-quasi norm on the Nakano sequence space with a variable exponent that fell somewhere in the range $(0, 1]$. They talked about the settings that must be met to generate pre-quasi Banach and closed space when it is endowed with a specified pre-quasi norm, as well as the Fatou property of various pre-quasi norms. They also determined a fixed point for Kannan pre-quasi norm contraction operators, in addition to the ideal of pre-quasi Banach operators derived from s-numbers in this sequence space. Both of these ideals were established. In addition, several fixed point findings of Kannan non-expansive operators on generalized Cesàro backward difference sequence space of a non-absolute type were discovered in [7]. Among the Kannan fixed point results, we refer the reader to Gaba et al. [17] and Aydi et al. [3]. By $\mathcal{N}$, $\mathfrak{R}$, and $\mathbb{C}$, we will indicate the sets of non-negative integers, real and complex numbers, respectively. We will describe the spaces of real and positive real sequences by $\mathfrak{R}^{\mathcal{N}}$ and $\mathfrak{R}^{+\mathcal{N}}$, respectively. The binomial formula is defined by

$$(u+v)^a = \sum_{k=0}^{a} \begin{bmatrix} a \\ k \end{bmatrix} u^k v^{a-k},$$

where $u, v \geqslant 0$, and $\begin{bmatrix} a \\ k \end{bmatrix} = \frac{a!}{k!(a-k)!}$. Given that the proof of many fixed point theorems in a given space requires either growing the space itself or expanding the self-mapping that acts on it, both of these options are viable, we have constructed the space, $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$, which is the domain of absolute type weighted binomial matrix in Nakano complex functions space of formal power series $\left(\ell^{\mathcal{P}}_{((\tau_a))}\right)$, where the weighted binomial matrix, $E_{u,v} = ((\lambda_{u,v})_{ak}(q))$, is defined as:

$$(\lambda_{u,v})_{ak}(q) = \begin{cases} \frac{A(a,k)q_k}{(u+v)^a}, & 0 \leqslant k \leqslant a, \\ 0, & k > a, \end{cases}$$

where $q_k \in (0, \infty)$, for all $k \in \mathcal{N}$ and $A(a, k) = \begin{bmatrix} a \\ k \end{bmatrix} u^k v^{a-k}$. If $\tau = (\tau_a) \in \mathfrak{R}^{+\mathcal{N}}$, then the domain of absolute type weighted binomial matrix in Nakano complex functions space of formal power series is denoted by: $E^{\mathcal{P}}_{u,v}(q,\tau) = \left\{ f \in \mathbb{C}^{\mathbb{C}} : f(y) = \sum_{v=0}^{\infty} \widehat{f_v} y^v \text{ and } \varphi(\mu f) < \infty \text{ , for some } \mu > 0 \right\}$, when $\varphi(f) = \sum_{a=0}^{\infty} \left(\frac{\sum_{k=0}^{a} A(a,k) q_k |\widehat{f_k}|}{(u+v)^a} \right)^{\tau_a}$. Please refer to [15, 16, 20, 31] for further information on formal power series spaces and their associated behaviors.

Lemma 1.1 ([2]). *If* $\tau_p > 0$ *and* $y_p, z_p \in \mathfrak{R}$, *for every* $p \in \mathcal{N}$, *then* $|y_p + z_p|^{\tau_p} \leqslant 2^{K-1}(|y_p|^{\tau_p} + |z_p|^{\tau_p})$, *where* $K = \max\{1, \sup_p \tau_p\}$.

In this article, we have constructed vast spaces of solutions to a variety of nonlinear summable and difference equations. The following is an outline of the objectives of this research. In Section 2, we investigate sufficient setups of $E^{\mathcal{P}}_{u,v}(q,\tau)$ equipped with the definite function φ to be pre-quasi Banach (ssfps). We have examined some topological and geometric structures of $E^{\mathcal{P}}_{u,v}(q,\tau)$, which are connected with the fixed point property. In Section 3, we look at how to configure $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$ with different φ so that there is only one fixed point of Kannan contraction mapping. We give some illustrative examples to clarify our results. Specifically, in Section 4, we present the requirements of this function space with definite φ for the Kannan non-expansive mapping to have a fixed point on it, and we do this by introducing it. In Section 5, we discuss many applications of solutions to summable equations and illustrative examples of our findings.

2. Structure of $E^{\mathcal{P}}_{u,v}(q,\tau)$

We have presented in this section some topological and geometric structures of $E^{\mathcal{P}}_{u,v}(q,\tau)$, which are connected with the fixed points of Kannan contraction mapping.

Theorem 2.1. *Suppose that* $(\tau_b) \in \mathfrak{R}^{+\mathcal{N}} \cap \ell_\infty$, *then*

$$E^{\mathcal{P}}_{u,v}(q,\tau) = \Big\{g \in \mathbb{C}^{\mathbb{C}} : g(z) = \sum_{b=0}^{\infty} \widehat{g_b} z^b \text{ and } \varphi(\iota g) < \infty \text{ , for every } \iota > 0\Big\}.$$

Proof. Evidently, as (τ_b) is a bounded sequence. □

Suppose ϑ is the zero function of $\mathcal{P}$ and

$$\mathfrak{F} = \Big\{g \in \mathbb{C}^{\mathbb{C}} : \exists k \in \mathcal{N} \text{ with } g(z) = \sum_{b=0}^{k} \widehat{g_b} z^b\Big\}.$$

Definition 2.2 ([27]). A mapping $h : \mathcal{P} \to [0,\infty)$ is said to be modular, where $\mathcal{P}$ is a vector space, if the followings are satisfied:

(a) if $f \in \mathcal{P}$, then $\varphi(f) \geqslant 0$ and $\varphi(f) = 0 \Leftrightarrow f = \vartheta$;
(b) assume $f \in \mathcal{P}$ and $|\eta| = 1$, then $\varphi(\eta f) = \varphi(f)$;
(c) suppose $\eta \in [0,1]$ and $f, g \in \mathcal{P}$, then $\varphi(\eta f + (1-\eta)g) \leqslant \varphi(f) + \varphi(g)$.

Definition 2.3 ([4]). The space $\mathcal{P} = \Big\{f \in \mathbb{C}^{\mathbb{C}} : f(y) = \sum_{a=0}^{\infty} \widehat{f_a} y^a\Big\}$ is said to be a special space of formal power series (ssfps), if the followings are satisfied:

(1) if $q \in \mathcal{P}$ and $|\widehat{p_b}| \leqslant |\widehat{q_b}|$, so that $b \in \mathcal{N}$, then $p \in \mathcal{P}$;
(2) $e^{(l)} \in \mathcal{P}$, for every $l \in \mathcal{N}$, so that $e^{(l)}(z) = \sum_{b=0}^{\infty} \widehat{e_b^{(l)}} z^b = z^l$;
(3) suppose $f \in \mathcal{P}$, then $f_{[.]} \in \mathcal{P}$, where $f_{[.]}(y) = \sum_{b=0}^{\infty} \widehat{f_{[\frac{b}{2}]}} y^b$ and $[\frac{b}{2}]$ marks the integral part of $\frac{b}{2}$.

Definition 2.4 ([4]). The function $h : \mathcal{P} \to [0,\infty)$ is said to be pre-modular on $\mathcal{P}$, if the followings are satisfied:

(i) if $f \in \mathcal{P}$, then $\varphi(f) \geqslant 0$ and $\varphi(f) = 0 \Leftrightarrow f = \vartheta$;
(ii) assume $g \in \mathcal{P}$ and $\varepsilon \in \mathbb{C}$, then $Q \geqslant 1$ with $\varphi(\varepsilon g) \leqslant |\varepsilon| Q \varphi(g)$;
(iii) let $f, g \in \mathcal{P}$, one has $P \geqslant 1$ such that $\varphi(f+g) \leqslant P(\varphi(f) + \varphi(g))$;
(iv) suppose $|\widehat{f_b}| \leqslant |\widehat{g_b}|$, for every $b \in \mathcal{N}$, then $\varphi(f) \leqslant \varphi(g)$;
(v) there is $P_0 \geqslant 1$ with $\varphi(f) \leqslant \varphi(f_{[.]}) \leqslant P_0 \varphi(f)$;
(vi) the closure of $\mathfrak{F} = \mathcal{P}_\varphi$;
(vii) there is $\xi > 0$ so that $\varphi(\lambda e^{(0)}) \geqslant \xi |\lambda| \varphi(e^{(0)})$, for every $\lambda \in \mathbb{C}$.

The space $\mathcal{P}_\varphi$ is called a pre-modular ssfps. When $\mathcal{P}_\varphi$ is complete, then $\mathcal{P}_\varphi$ is said to be a pre-modular Banach ssfps. Pre-modular vector spaces are more general than modular vector spaces. Following is an example of a pre-modular vector space and a modular vector space.

Example 2.5. The space $E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)_{\varphi}$, where

$$\varphi(f) = \inf\Big\{\alpha > 0 : \sum_{a=0}^{\infty} \left(\frac{\sum_{k=0}^{a} |\frac{\widehat{f_k}}{\alpha}|}{(u+v)^a}\right)^{\frac{2a+3}{a+2}} \leqslant 1\Big\}.$$

Following is an example of pre-modular vector space, but not modular vector space.

Example 2.6. The space $E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+4}\right)_{k=0}^{\infty}\right)_{\varphi}$, where $\varphi(f) = \sum_{b=0}^{\infty} \left(\frac{\sum_{k=0}^{b} |\widehat{f_k}|}{(u+v)^b}\right)^{\frac{2b+3}{b+4}}$. Since for all $f, g \in E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+4}\right)_{k=0}^{\infty}\right)_{\varphi}$, one has

$$\varphi\left(\frac{f+g}{2}\right) = \sum_{a=0}^{\infty} \left(\frac{\sum_{k=0}^{a} |\frac{\widehat{f_k}+\widehat{g_k}}{2}|}{(u+v)^a}\right)^{\frac{2a+3}{a+4}} \leqslant \frac{2}{\sqrt[4]{8}}(\varphi(f) + \varphi(g)).$$

Definition 2.7 ([4])**.** A subspace $\mathcal{P}_\varphi$ of the ssfps is said to be a pre-quasi normed ssfps whenever the function $h : \mathcal{P} \to [0, \infty)$ satisfies the parts (i), (ii), and (iii) of Definition 2.4.

Presume $\mathcal{P}_\varphi$ is complete, then $\mathcal{P}_\varphi$ is called a pre-quasi Banach ssfps.

Theorem 2.8 ([6])**.** *Every quasi-normed ssfps is a pre-quasi-normed ssfps.*

Example 2.9. The function $\varphi(f) = \sqrt[d]{\sum_{a\in\mathcal{N}} \left(\frac{\sum_{k=0}^{a}|\widehat{f_k}|}{(u+v)^a}\right)^d}$ is a quasi norm on $E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty}, (d)_{k=0}^{\infty}\right)$, for $0 < d < 1$. But the function $\varphi(f)$ is not a norm, since $\varphi(f+g) \leqslant 2^{\frac{1}{d}-1}(\varphi(f)+\varphi(g))$, for every $\varepsilon \in \mathbb{C}$ and $f, g \in E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty}, \left(\frac{3k+2}{k+1}\right)_{k=0}^{\infty}\right)$.

Example 2.10. The function $\varphi(f) = \sum_{a\in\mathcal{N}} \left(\frac{\sum_{k=0}^{a}|\widehat{f_k}|}{(u+v)^a}\right)^{\frac{3a+2}{a+1}}$ is a pre-quasi norm on $E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty}, \left(\frac{3k+2}{k+1}\right)_{k=0}^{\infty}\right)$. But the function $\varphi(f)$ is not a quasi norm, since $\varphi(\varepsilon f) \neq |\varepsilon|\varphi(f)$, for every $\varepsilon \in \mathbb{C}$ and $f \in E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty}, \left(\frac{3k+2}{k+1}\right)_{k=0}^{\infty}\right)$.

Example 2.11. If $(\tau_a) \in [1, \infty)^{\mathcal{N}}$, then the function $\varphi(f) = \inf\left\{\alpha > 0 : \sum_{a\in\mathcal{N}} \left(\frac{\sum_{k=0}^{a} A(a,k)q_k|\frac{\widehat{f_k}}{\alpha}|}{(u+v)^a}\right)^{\tau_a} \leqslant 1\right\}$ is a Luxemburg norm on $E^{\mathcal{P}}_{u,v}(q, \tau)$. Therefore, the function $\varphi(f)$ is a norm, quasi norm, and pre-quasi norm.

Theorem 2.12 ([6])**.** *The space $\mathcal{P}_\varphi$ is a pre-quasi normed ssfps whenever it is a pre-modular ssfps.*

Definition 2.13.

(i) If $\delta_\varphi(\Omega) := \sup\left\{\varphi(f-g) : f, g \in \Omega\right\} < \infty$, then $\Omega \subset \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$ is called φ-bounded.

(ii) A pre-quasi norm φ on $E^{\mathcal{P}}_{u,v}(q,\tau)$ satisfies the Fatou property, if for every $\{g_p\}_{p=1}^{\infty} \subseteq \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$ so that $\lim_{p\to\infty} \varphi(g_p - g) = 0$ and every $f \in \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$, one has $\varphi(f-g) \leqslant \sup_r \inf_{p\geqslant r} \varphi(f-g_p)$.

(iii) Consider $\mathbf{B}_\varphi(h,\varepsilon) := \left\{t \in \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi : \varphi(h-t) \leqslant \varepsilon\right\}$ for the φ-ball of radius $\varepsilon \geqslant 0$ and center h, for every $h \in \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$.

Let $\mathbf{I}$ and $\mathbf{D}$ be the spaces of all increasing and decreasing sequences of real numbers, respectively. Presume $\mathbf{I}^+$ is the space for all increasing positive reals sequences.

Theorem 2.14. *$\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$, where $\varphi(f) = \left[\sum_{a=0}^{\infty} \left(\frac{\sum_{k=0}^{a} A(a,k)q_k|\widehat{f_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}}$, for all $f \in E^{\mathcal{P}}_{u,v}(q,\tau)$, is a pre-modular (ssfps), if the followings are satisfied:*

(b1) $u + v > 1$;

(b2) $(\tau_p)_{p\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}^+$;

(b3) $(A(a,k)q_k)_{k=0}^{\infty} \in \mathbf{D}$ *or,* $(A(a,k)q_k)_{k=0}^{\infty} \in \mathbf{I} \cap \ell_\infty$ *and there exists* $C \geqslant 1$ *such that*

$$A(a, 2k+1)q_{2k+1} \leqslant CA(a,k)q_k.$$

Proof.

(i) Evidently, $\varphi(f) = 0 \Leftrightarrow f = \vartheta$ and $\varphi(f) \geqslant 0$.

(1-i) If $f, g \in E^{\mathcal{P}}_{u,v}(q,\tau)$, then $(f+g)(y) = \sum_{v=0}^{\infty}(\widehat{f_v} + \widehat{g_v})y^v \in \mathbb{C}$ so that

$$\begin{aligned}\varphi(f+g) &= \left[\sum_{a=0}^{\infty}\left(\frac{\sum_{k=0}^{a} A(a,k)q_k|\widehat{f_k}+\widehat{g_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}}\\ &\leqslant \left[\sum_{a=0}^{\infty}\left(\frac{\sum_{k=0}^{a} A(a,k)q_k|\widehat{f_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}} + \left[\sum_{a=0}^{\infty}\left(\frac{\sum_{k=0}^{a} A(a,k)q_k|\widehat{g_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}} = \varphi(f)+\varphi(g) < \infty,\end{aligned}$$

so $f+g \in E^{\mathcal{P}}_{u,v}(q,\tau)$.

(iii) There are $P \geqslant 1$ so that $\varphi(f+g) \leqslant P(\varphi(f)+\varphi(g))$, for every $f, g \in E^{\mathcal{P}}_{u,v}(q,\tau)$.

(1-ii) Let $\alpha \in \mathbb{C}$ and $f \in E^{\mathcal{P}}_{u,v}(q,\tau)$, then

$$\begin{aligned}\varphi(\alpha f) &= \left[\sum_{a=0}^{\infty}\left(\frac{\sum_{k=0}^{a} A(a,k)q_k|\alpha\widehat{f_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}}\\ &\leqslant \sup_a |\alpha|^{\frac{\tau_a}{K}}\left[\sum_{a=0}^{\infty}\left(\frac{\sum_{k=0}^{a} A(a,k)q_k|\widehat{f_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}} \leqslant Q|\alpha|\varphi(f) < \infty.\end{aligned}$$

Since $\alpha f \in E^{\mathcal{P}}_{u,v}(q,\tau)$, from parts (1-i) and (1-ii), then $E^{\mathcal{P}}_{u,v}(q,\tau)$ is linear. Obviously, $e^{(p)} \in E^{\mathcal{P}}_{u,v}(q,\tau)$, for every $p \in \mathcal{N}$, as $\varphi(e^{(p)}) = \left[\sum_{a=0}^{\infty}\left(\frac{\sum_{k=0}^{a} A(a,k)q_k\left|\widehat{e_k^{(p)}}\right|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}} = \left[\sum_{a=p}^{\infty}\left(\frac{A(a,p)q_p}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}} < \infty$.

(ii) One has $Q = \max\left\{1, \sup_a |\alpha|^{\frac{\tau_a}{K}-1}\right\} \geqslant 1$ so that $\varphi(\alpha f) \leqslant Q|\alpha|\varphi(f)$, for every $f \in E^{\mathcal{P}}_{u,v}(q,\tau)$ and $\alpha \in \mathbb{C}$.

(2) If $|f_a| \leqslant |g_a|$, for every $a \in \mathcal{N}$ and $g \in E^{\mathcal{P}}_{u,v}(q,\tau)$, then

$$\varphi(f) = \left[\sum_{a=0}^{\infty}\left(\frac{\sum_{k=0}^{a} A(a,k)q_k|\widehat{f_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}} \leqslant \left[\sum_{a=0}^{\infty}\left(\frac{\sum_{k=0}^{a} A(a,k)q_k|\widehat{g_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}} = \varphi(g) < \infty,$$

so $f \in E^{\mathcal{P}}_{u,v}(q,\tau)$.

(iv) Clearly, from (2).

(3) Assume $(f_a) \in E^{\mathcal{P}}_{u,v}(q,\tau)$ and $(A(a,k)q_k)_{k=0}^{\infty} \in \mathbf{D}$, then

$$\begin{aligned}\varphi\left(f_{[\frac{a}{2}]}\right) &= \left[\sum_{a=0}^{\infty}\left(\frac{\sum_{k=0}^{a} A(a,k)q_k|\widehat{f_{[\frac{k}{2}]}}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}}\\ &= \left[\sum_{a=0}^{\infty}\left(\frac{\sum_{k=0}^{2a} A(a,k)q_k|\widehat{f_{[\frac{k}{2}]}}|}{(u+v)^{2a}}\right)^{\tau_{2a}}\right]^{\frac{1}{K}} + \left[\sum_{a=0}^{\infty}\left(\frac{\sum_{k=0}^{2a+1} A(a,k)q_k|\widehat{f_{[\frac{k}{2}]}}|}{(u+v)^{2a+1}}\right)^{\tau_{2a+1}}\right]^{\frac{1}{K}}\\ &\leqslant \left[\sum_{a=0}^{\infty}\left(\frac{A(2a,k)q_{2a}|\widehat{f_a}| + 2\sum_{k=0}^{a} A(a,k)q_k|\widehat{f_k}|}{(u+v)^a}\right)^{\tau_a} + \sum_{a=0}^{\infty}\left(\frac{2\sum_{k=0}^{a}|\widehat{f_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}}\\ &\leqslant \left[\sum_{a=0}^{\infty}\left(\frac{3\sum_{k=0}^{a} A(a,k)q_k|\widehat{f_k}|}{(u+v)^a}\right)^{\tau_a} + \sum_{a=0}^{\infty}\left(\frac{2\sum_{k=0}^{a}|\widehat{f_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}} = \left(3^K+2^K\right)^{\frac{1}{K}}\varphi\left(f\right),\end{aligned}$$

then $(f_{[\frac{a}{2}]}) \in E^{\mathcal{P}}_{u,v}(q,\tau)$.

(v) From (3), there exists $P_0 = (3^K + 2^K)^{\frac{1}{K}} \geqslant 1$.

(vi) Obviously, the closure of $\mathfrak{F} = E^{\mathcal{P}}_{u,v}(q,\tau)$.

(vii) One gets $0 < \sigma \leqslant \sup_p |\alpha|^{\frac{\tau_p}{K}-1}$, for $\alpha \neq 0$ or $\sigma > 0$, for $\alpha = 0$ so that $\varphi(\alpha e^{(0)}) \geqslant \sigma|\alpha|\varphi(e^{(0)})$. □

Theorem 2.15. *Let the settings of Theorem 2.14 be satisfied, then* $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$ *is a pre-quasi Banach (ssfps), where* $\varphi(h) = \left[\sum_{a=0}^\infty \left(\frac{\sum_{k=0}^a A(a,k)q_k|\widehat{h_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}}$*, for every* $h \in E^{\mathcal{P}}_{u,v}(q,\tau)$.

Proof. The space $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$, given by Theorems 2.14 and 2.12, is a pre-quasi normed (ssfps). Presume $h^l = (h^l_k)_{k=0}^\infty$ is a Cauchy sequence in $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$, then for all $\varepsilon \in (0,1)$, one has $l_0 \in \mathcal{N}$. One obtains for all $l, m \geqslant l_0$ that

$$\varphi(h^l - h^m) = \left[\sum_{a=0}^\infty \left(\frac{\sum_{k=0}^a A(a,k)q_k|\widehat{h^l_k} - \widehat{h^m_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}} < \varepsilon.$$

Therefore $|\widehat{h^l_k} - \widehat{h^m_k}| < \varepsilon$. Fix $k \in \mathcal{N}$, then $(\widehat{h^m_k})$ is a Cauchy sequence in $\mathbb{C}$. Then $\lim_{m\to\infty} |\widehat{h^m_k} - \widehat{h^0_k}| = 0$. We get $\varphi(h^l - h^0) < \varepsilon$, for every $l \geqslant l_0$. Since $\varphi(h^0) \leqslant \varphi(h^l - h^0) + \varphi(h^l) < \infty$. Hence $h^0 \in E^{\mathcal{P}}_{u,v}(q,\tau)$. □

Theorem 2.16. *The function* $\varphi(f) = \left[\sum_{a=0}^\infty \left(\frac{\sum_{k=0}^a A(a,k)q_k|\widehat{f_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}}$ *verifies the Fatou property, whenever the settings of Theorem 2.14 are confirmed, for every* $f \in E^{\mathcal{P}}_{u,v}(q,\tau)$.

Proof. If $\{g^p\} \subseteq \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$ so that $\lim_{p\to\infty} \varphi(g^p - g) = 0$. As $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$ is a pre-quasi closed space, we have $g \in \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$. For all $f \in \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$, then

$$\begin{aligned}
\varphi(f-g) &= \left[\sum_{a=0}^\infty \left(\frac{\sum_{k=0}^a A(a,k)q_k|\widehat{f_k} - \widehat{g_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}} \\
&\leqslant \left[\sum_{a=0}^\infty \left(\frac{\sum_{k=0}^a A(a,k)q_k|\widehat{f_k} - \widehat{g^p_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}} + \left[\sum_{a=0}^\infty \left(\frac{\sum_{k=0}^a A(a,k)q_k|\widehat{g^p_k} - \widehat{g_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}} \\
&\leqslant \sup_m \inf_{p\geqslant m} \varphi(f - g^p).
\end{aligned}$$

□

Theorem 2.17. *The function* $\varphi(f) = \sum_{a=0}^\infty \left(\frac{\sum_{k=0}^a A(a,k)q_k|\widehat{f_k}|}{(u+v)^a}\right)^{\tau_a}$ *does not verify the Fatou property, for all* $f \in E^{\mathcal{P}}_{u,v}(q,\tau)$*, when the settings of Theorem 2.14 are confirmed with* $\sup_a \tau_a > 1$.

Proof. Suppose $\{g^p\} \subseteq \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$ with $\lim_{p\to\infty} \varphi(g^p - g) = 0$. As $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$ is a pre-quasi closed space, we obtain $g \in \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$. For all $f \in \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$, then

$$\begin{aligned}
\varphi(f-g) &= \sum_{a=0}^\infty \left(\frac{\sum_{k=0}^a A(a,k)q_k|\widehat{f_k} - \widehat{g_k}|}{(u+v)^a}\right)^{\tau_a} \\
&\leqslant 2^{K-1}\left(\sum_{a=0}^\infty \left(\frac{\sum_{k=0}^a A(a,k)q_k|\widehat{f_k} - \widehat{g^p_k}|}{(u+v)^a}\right)^{\tau_a} + \sum_{a=0}^\infty \left(\frac{\sum_{k=0}^a A(a,k)q_k|\widehat{g^p_k} - \widehat{g_k}|}{(u+v)^a}\right)^{\tau_a}\right) \\
&\leqslant 2^{K-1} \sup_m \inf_{p\geqslant m} \varphi(f - g^p).
\end{aligned}$$

□

We'll use that $\varphi(g) = \left[\sum_{a=0}^{\infty}\left(\frac{\sum_{k=0}^{a} A(a,k)q_k|\widehat{g_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}}$, for all $g \in E_{u,v}^{\mathcal{P}}(q,\tau)$, in the next part of this section. By $S(\mathcal{P}_\varphi)$ and $B(\mathcal{P}_\varphi)$, we indicate the unit sphere and the unit ball of $\mathcal{P}_\varphi$, respectively.

Definition 2.18 ([12]). A function $\varphi \in \delta_2$, if for all $\varepsilon > 0$ we have $k \geqslant 2$ and $a > 0$ so that

$$\varphi(2g) \leqslant k\varphi(g) + \varepsilon \text{ for each } g \in \mathcal{P}_h, \text{ and } \varphi(g) \leqslant a.$$

Suppose $\varphi \in \delta_2$, for all $a > 0$ so that $k \geqslant 2$ depending on a, then $\varphi \in \delta_2^s$.

Theorem 2.19 ([12]). *Assume $\varphi \in \delta_2^s$, then for all $L > 0$ and $\varepsilon > 0$ one has $\delta > 0$ with $|\varphi(h+t) - \varphi(h)| < \varepsilon$, $h, t \in \mathcal{P}_h$, with $\varphi(h) \leqslant L$ and $\varphi(t) \leqslant \delta$.*

Theorem 2.20. *Presume the settings of Theorem 2.14 are verified, then for all $\varepsilon > 0$ and $L > 0$ one has $\delta > 0$ with $|\varphi(h+t) - \varphi(h)| < \varepsilon$, for all $h, t \in \left(E_{u,v}^{\mathcal{P}}(q,\tau)\right)_\varphi$, so that $\varphi(h) \leqslant L$ and $\varphi(t) \leqslant \delta$.*

Proof. In view of Theorems 2.14 and 2.19, we have $\varphi \in \delta_2^s$. □

Definition 2.21 ([28]). The function φ is strictly convex, (SC), when for all $h, t \in \mathcal{P}_\varphi$ with $\varphi(h) = \varphi(t)$ and $\varphi\left(\frac{h+t}{2}\right) = \frac{\varphi(h)+\varphi(t)}{2}$, then $h = t$.

Definition 2.22 ([22]). A sequence $\{g_p\} \subseteq \mathcal{P}_\varphi$, is said to be ε-separated for some $\varepsilon > 0$, if

$$\operatorname{sep}(g_p) = \inf\{\varphi(g_p - g_t) : p \neq t\} > \varepsilon.$$

Definition 2.23 ([22]). A Banach space $\mathcal{P}_\varphi$ is said to be k-nearly uniformly convex (k-NUC), for an integer $k \geqslant 2$, if for all $\varepsilon > 0$ one has $\delta \in (0,1)$ so that for all sequences $\{g_p\} \subseteq B(\mathcal{P}_\varphi)$, with $\operatorname{sep}(g_p) \geqslant \varepsilon$, there are $p_1, p_2, p_3, \ldots, p_k \in \mathbb{N}$. So that

$$\varphi\left(\frac{g_{p_1} + g_{p_2} + g_{p_3} + \cdots + g_{p_k}}{k}\right) < 1 - \delta.$$

Recall that k-NUC implies reflexivity.

Theorem 2.24. *If the settings of Theorem 2.14 are confirmed so that $\tau_0 > 1$, then $\left(E_{u,v}^{\mathcal{P}}(q,\tau)\right)_\varphi$ is k-NUC, for every integer $k \geqslant 2$.*

Proof. Suppose $\varepsilon \in (0,1)$ and $\{f_n\} \subseteq B\left(\left(E_{u,v}^{\mathcal{P}}(q,\tau)\right)_\varphi\right)$, where $f_n(y) = \sum_{i=0}^{\infty} \widehat{f_n(i)} y^i$ so that $\operatorname{sep}(f_n) \geqslant \varepsilon$. For every $m \in \mathbb{N}$, assume $f_n^m(y) = \sum_{i=0}^{\infty} \widehat{f_n^m(i)} y^i$, where $\left(\widehat{f_n^m(i)}\right)_{i=0}^{\infty} = \left(0, 0, 0, \ldots, \widehat{f_n(m)}, \widehat{f_n(m+1)}, \ldots\right)$. Since for every $i \in \mathbb{N}$, $\left(\widehat{f_n(i)}\right)_{n=0}^{\infty} \in \ell_\infty$. By the diagonal method, we have (f_{n_j}) of (f_n) with $\left(\widehat{f_{n_j}(i)}\right)$ converges for every $i \in \mathbb{N}$, $0 \leqslant i \leqslant m$. One has an increasing sequence of positive integers (t_m) so that $\operatorname{sep}\left((f_{n_j}^m)_{j > t_m}\right) \geqslant \varepsilon$. Therefore, one has a sequence of positive integers $(v_m)_{m=0}^{\infty}$ under $v_0 < v_1 < v_2 < \cdots$, with

$$\varphi^K(f_{v_m}^m) \geqslant \frac{\varepsilon}{2}, \tag{2.1}$$

for all $m \in \mathbb{N}$. Fix $k \geqslant 2$, assume $\varepsilon_1 = \left(\frac{k^{p_0-1}-1}{(k-1)k^{p_0}}\right)\left(\frac{\varepsilon}{4}\right)$ by Theorem 2.20 there exists $\delta > 0$ so that

$$|\varphi^K(f+g) - \varphi^K(f)| < \varepsilon_1. \tag{2.2}$$

Presume $\varphi^K(f) \leqslant 1$ and $\varphi^K(g) \leqslant \delta$. Since $\varphi^K(f_n) \leqslant 1$, for every $n \in \mathbb{N}$, one obtains positive integers $m_i (i = 0, 1, 2, \ldots, k-2)$ so that $m_0 < m_1 < m_2 < \cdots < m_{k-2}$ and $\varphi^K(f_i^{m_i}) \leqslant \delta$. Suppose $m_{k-1} =$

$m_{k-2}+1$. By inequality (2.1), one gets $\varphi(f^{m_k}_{v_{m_k}}) \geqslant \frac{\varepsilon}{2}$. Assume $p_i = i$ for $0 \leqslant i \leqslant k-2$ and $p_{k-1} = v_{m_{k-1}}$. By inequalities (2.1), (2.2), and $J_n(u) = |u|^{\tau_n}$ is convex for every $n \in \mathcal{N}$, then

$$
\begin{aligned}
&\varphi^K\left(\frac{f_{p_0}+f_{p_1}+f_{p_2}+\cdots+f_{p_{k-1}}}{k}\right)\\
&=\sum_{n=0}^{\infty}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\frac{\widehat{f_{p_0}(i)}+\widehat{f_{p_1}(i)}+\cdots+\widehat{f_{p_{k-1}}(i)}}{k}\right|}{(u+v)^n}\right)^{\tau_n}\\
&=\sum_{n=0}^{m_1-1}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\frac{\widehat{f_{p_0}(i)}+\widehat{f_{p_1}(i)}+\cdots+\widehat{f_{p_{k-1}}(i)}}{k}\right|}{(u+v)^n}\right)^{\tau_n}\\
&\quad+\sum_{n=m_1}^{\infty}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\frac{\widehat{f_{p_0}(i)}+\widehat{f_{p_1}(i)}+\cdots+\widehat{f_{p_{k-1}}(i)}}{k}\right|}{(u+v)^n}\right)^{\tau_n}\\
&\leqslant\sum_{n=0}^{m_1-1}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\frac{\widehat{f_{p_0}(i)}+\widehat{f_{p_1}(i)}+\cdots+\widehat{f_{p_{k-1}}(i)}}{k}\right|}{(u+v)^n}\right)^{\tau_n}\\
&\quad+\sum_{n=m_1}^{\infty}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\frac{\widehat{f_{p_1}(i)}+\widehat{f_{p_2}(i)}+\cdots+\widehat{f_{p_{k-1}}(i)}}{k}\right|}{(u+v)^n}\right)^{\tau_n}+\varepsilon_1\\
&\leqslant\sum_{n=0}^{m_1-1}\frac{1}{k}\sum_{j=0}^{k-1}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\widehat{f_{p_j}(i)}\right|}{(u+v)^n}\right)^{\tau_n}+\sum_{n=m_1}^{m_2-1}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\frac{\widehat{f_{p_1}(i)}+\widehat{f_{p_2}(i)}+\cdots+\widehat{f_{p_{k-1}}(i)}}{k}\right|}{(u+v)^n}\right)^{\tau_n}\\
&\quad+\sum_{n=m_2}^{\infty}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\frac{\widehat{f_{p_1}(i)}+\widehat{f_{p_2}(i)}+\cdots+\widehat{f_{p_{k-1}}(i)}}{k}\right|}{(u+v)^n}\right)^{\tau_n}+\varepsilon_1\\
&\leqslant\sum_{n=0}^{m_1-1}\frac{1}{k}\sum_{j=0}^{k-1}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\widehat{f_{p_j}(i)}\right|}{(u+v)^n}\right)^{\tau_n}+\sum_{n=m_1}^{m_2-1}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\frac{\widehat{f_{p_1}(i)}+\widehat{f_{p_2}(i)}+\cdots+\widehat{f_{p_{k-1}}(i)}}{k}\right|}{(u+v)^n}\right)^{\tau_n}\\
&\quad+\sum_{n=m_2}^{\infty}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\frac{\widehat{f_{p_2}(i)}+f_{p_3}(i)+\cdots+\widehat{f_{p_{k-1}}(i)}}{k}\right|}{(u+v)^n}\right)^{\tau_n}+2\varepsilon_1\\
&\leqslant\sum_{n=0}^{m_1-1}\frac{1}{k}\sum_{j=0}^{k-1}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\widehat{f_{p_j}(i)}\right|}{(u+v)^n}\right)^{\tau_n}+\sum_{n=m_1}^{m_2-1}\frac{1}{k}\sum_{j=1}^{k-1}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\widehat{f_{p_j}(i)}\right|}{(u+v)^n}\right)^{\tau_n}\\
&\quad+\sum_{n=m_2}^{m_3-1}\frac{1}{k}\sum_{j=2}^{k-1}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\widehat{f_{p_j}(i)}\right|}{(u+v)^n}\right)^{\tau_n}+\cdots+\sum_{n=m_{k-1}}^{m_k-1}\frac{1}{k}\sum_{j=k-2}^{k-1}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\widehat{f_{p_j}(i)}\right|}{(u+v)^n}\right)^{\tau_n}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\widehat{\frac{f_{p_k}(i)}{k}}\right|}{(u+v)^n}\right)^{\tau_n}+(k-1)\varepsilon_1\\
&\leqslant \frac{h^K(f_{p_0}+f_{p_1}+f_{p_2}+\cdots+f_{p_{k-2}})}{k}+\frac{1}{k}\sum_{n=0}^{m_k-1}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\widehat{f_{p_k}(i)}\right|}{(u+v)^n}\right)^{\tau_n}\\
&+\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\widehat{\frac{f_{p_k}(i)}{k}}\right|}{(u+v)^n}\right)^{\tau_n}+(k-1)\varepsilon_1\\
&\leqslant \frac{k-1}{k}+\frac{1}{k}\sum_{n=0}^{m_k-1}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\widehat{f_{p_k}(i)}\right|}{(u+v)^n}\right)^{\tau_n}+\frac{1}{k^{\tau_0}}\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\widehat{f_{p_k}(i)}\right|}{(u+v)^n}\right)^{\tau_n}+(k-1)\varepsilon_1\\
&\leqslant 1-\frac{1}{k}+\frac{1}{k}\left(1-\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\widehat{f_{p_k}(i)}\right|}{(u+v)^n}\right)^{\tau_n}\right)\\
&+\frac{1}{k^{\tau_0}}\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\widehat{f_{p_k}(i)}\right|}{(u+v)^n}\right)^{\tau_n}+(k-1)\varepsilon_1\\
&=1+(k-1)\varepsilon_1-\left(\frac{k^{\tau_0-1}-1}{k^{\tau_0}}\right)\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}A(n,i)q_i\left|\widehat{f_{p_k}(i)}\right|}{(u+v)^n}\right)^{\tau_n}\\
&\leqslant 1+(k-1)\varepsilon_1-\left(\frac{k^{\tau_0-1}-1}{k^{\tau_0}}\right)\frac{\varepsilon}{2}=1-\left(\frac{k^{\tau_0-1}-1}{k^{\tau_0}}\right)\frac{\varepsilon}{4}.
\end{aligned}
$$

Hence $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$ is k-NUC. □

By fixing Ω a nonempty φ-closed, φ-bounded, and φ-convex subset of $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$ we have the following.

Definition 2.25. The space $\mathcal{P}_{\varphi}$ satisfies the property (R), if and only if, for each decreasing sequence $\{\Omega_j\}_{j\in\mathcal{N}}$ of φ-closed and φ-convex nonempty subsets of $\mathcal{P}_{\varphi}$ so that $\sup_{j\in\mathcal{N}}\mathfrak{K}_{\varphi}(f,\Omega_j)<\infty$, where $\mathfrak{K}_{\varphi}(f,\Omega)=\inf\left\{\varphi(f-g):g\in\Omega\right\}$, for some $f\in\mathcal{P}_{\varphi}$, we have $\bigcap_{j\in\mathcal{N}}\Omega_j\neq\emptyset$.

Theorem 2.26. *Presume the settings of Theorem 2.14 are confirmed so that $\tau_0>1$, then*

(i) *if $f\in\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$ so that $\mathfrak{K}_{\varphi}(f,\Omega)<\infty$, one gets a unique $\lambda\in\Omega$ with $\mathfrak{K}_{\varphi}(f,\Omega)=\varphi(f-\lambda)$;*

(ii) *$\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$ verifies the property* (R).

Proof. For (i), if $f\notin\Omega$ as Ω is φ-closed, then $C:=\mathfrak{K}_{\varphi}(f,\Omega)>0$. Therefore, for every $i\in\mathcal{N}$, we have $g_i\in\Omega$ so that $\varphi(f-g_i)<C(1+\frac{1}{i})$. Assume $\{\frac{g_i}{2}\}$ is not φ-Cauchy. We have a $\{\frac{g_{m(i)}}{2}\}$ and $l_0>0$ so that $\varphi\left(\frac{g_{m(i)}-g_{m(j)}}{2}\right)\geqslant l_0$, for all $i>j\geqslant 0$. As

$$
\max\left(\varphi(f-g_{m(i)}),\varphi(f-g_{m(j)})\right)\leqslant C\left(1+\frac{1}{m(j)}\right)
$$

and

$$\varphi\left(\frac{g_{m(i)}-g_{m(j)}}{2}\right) \geqslant l_0 \geqslant \frac{l_0}{2}\left(1+\frac{1}{m(j)}\right),$$

for all $i > j \geqslant 0$. As $(\tau_a)_{a\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}^+$ so that $\tau_0 > 1$, hence the function $V_n(u) = |u|^{\tau_n}$ is strictly convex, for every $n \in \mathcal{N}$. So the space $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$ is strictly convex, then

$$\varphi\left(f-\frac{g_{m(i)}+g_{m(j)}}{2}\right) < C\left(1+\frac{1}{m(j)}\right).$$

So

$$C = \mathfrak{K}_\varphi(f,\Omega) < C\left(1+\frac{1}{m(j)}\right),$$

for every $j \in \mathcal{N}$. Let $j \to \infty$, then there is a contradiction. Hence $\{\frac{g_i}{2}\}$ is φ-Cauchy. Since $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$ is φ-complete, so $\{\frac{g_i}{2}\}$ φ-converges to some g. For every $j \in \mathcal{N}$, we obtain $\{\frac{g_i+g_j}{2}\}$ φ-converges to $g+\frac{g_j}{2}$. As Ω is φ-closed and φ-convex, so $g+\frac{g_j}{2} \in \Omega$. As $g+\frac{g_j}{2}$ φ-converges to $2g$, then $2g \in \Omega$. By Theorem 2.16 and if $\lambda = 2g$, as φ satisfies the Fatou property, then

$$\begin{aligned}\mathfrak{K}_\varphi(f,\Omega) \leqslant \varphi(f-\lambda) &\leqslant \sup_i \inf_{j\geqslant i} \varphi\left(f-\left(g+\frac{g_j}{2}\right)\right)\\ &\leqslant \sup_i \inf_{j\geqslant i} \sup_i \inf_{p\geqslant i} \varphi\left(f-\frac{g_p+g_j}{2}\right)\\ &\leqslant \frac{1}{2}\sup_i \inf_{p\geqslant i} \sup_i \inf_{p\geqslant i}\left[\varphi(f-g_p)+\varphi(f-g_j)\right] = \mathfrak{K}_\varphi(f,\Omega).\end{aligned}$$

Hence $\varphi(f-\lambda) = \mathfrak{K}_\varphi(f,\Omega)$. As φ is (SC), we have the uniqueness of λ. To prove (ii), if $f \notin \Omega_{i_0}$, for some $i_0 \in \mathcal{N}$, as $(\mathfrak{K}_\varphi(f,\Omega_i))_{i\in\mathcal{N}} \in \ell_\infty$ is increasing, let $\lim_{i\to\infty}\mathfrak{K}_\varphi(f,\Omega_i) = C$. If $C > 0$. Otherwise $f \in \Omega_i$, for every $i \in \mathcal{N}$. By part (i), one has one point $g_i \in \Omega_i$ so that $\mathfrak{K}_\varphi(f,\Omega_i) = \varphi(f-g_i)$, for all $i \in \mathcal{N}$. Obviously, $\{\frac{g_i}{2}\}$ φ-converges to some $g \in \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$. Since $\{\Omega_i\}$ are φ-convex, decreasing and φ-closed, then $2g \in \cap_{i\in\mathcal{N}}\Omega_i$. □

Definition 2.27 ([35])**.** Presume $\mathcal{P}_\varphi$ is a Banach space, then

$$WCS(\mathcal{P}_\varphi) := \inf\left\{A(\{f_n\}) : \{f_n\}_{n=0}^\infty \subset S(\mathcal{P}_\varphi), A(\{f_n\}) = A_1(\{f_n\}), f_n \xrightarrow{w} 0\right\},$$

where

$$A(\{f_n\}) = \limsup_{n\to\infty}\{\varphi(f_a - f_b):\ a,b \geqslant n, a \neq b\}$$

and

$$A_1(\{f_n\}) = \liminf_{n\to\infty}\{\varphi(f_a - f_b):\ a,b \geqslant n, a \neq b\}.$$

Definition 2.28. $\mathcal{P}_\varphi$ verifies the φ-normal structure-property, if and only if, for every $\Omega \subseteq \mathcal{P}_\varphi$ is not decreased to one point, then $h \in \Omega$ such that $\sup_{t\in\Omega}\varphi(h-t) < \delta_\varphi(\Omega) := \sup\left\{\varphi(h-t) : h,t \in \Omega\right\} < \infty$.

Theorem 2.29 ([11])**.** *A reflexive Banach space $\mathcal{P}_\varphi$ so that $WCS(\mathcal{P}_\varphi) > 1$ satisfies the normal structure-property.*

Theorem 2.30. *Presumr the settings of Theorem 2.14 are confirmed so that $\tau_0 > 1$, then $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$ verifies the φ-normal structure-property.*

Proof. Assume $\{f_n\} \subset S\left(\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}\right)$ is an asymptotic equidistant sequence so that $f_n \overset{w}{\to} 0$ and $\varepsilon > 0$. If $v_0 = f_0$, we have $i_0 \in \mathcal{N}$ so that $\varphi\left(\sum_{i=i_0}^{\infty} \widehat{v_0(i)} e^{(i)}\right) < \varepsilon$. Since $f_n \to 0$ coordinate-wise, there exists $n_1 \in \mathcal{N}$ with $\varphi\left(\sum_{i=0}^{i_0-1} \widehat{f_n(i)} e^{(i)}\right) < \varepsilon$. For $n \geqslant n_1$, let $v_1 = f_{n_1}$, one has $i_1 > i_0$ with $\varphi\left(\sum_{i=i_1}^{\infty} \widehat{v_0(i)} e^{(i)}\right) < \varepsilon$. Since $f_n(i) \to 0$ coordinate-wise, one has $n_2 \in \mathcal{N}$ with $\varphi\left(\sum_{i=0}^{i_1-1} \widehat{f_n(i)} e^{(i)}\right) < \varepsilon$. For $n \geqslant n_2$, by induction, we have a subsequence $\{v_n\}$ of $\{f_n\}$ so that

$$\varphi\left(\sum_{i=i_n}^{\infty} \widehat{v_n(i)} e^{(i)}\right) < \varepsilon \quad \text{and} \quad \varphi\left(\sum_{i=0}^{i_n-1} \widehat{v_{n+1}(i)} e^{(i)}\right) < \varepsilon.$$

Put $z_n = \sum_{i=i_{n-1}}^{i_n-1} \widehat{v_n(i)} e^{(i)}$ for $n = 1, 2, 3, \ldots$. Hence

$$\begin{aligned} 1 \geqslant \varphi(z_n) &= \varphi\left(\sum_{i=0}^{\infty} \widehat{v_n(i)} e^{(i)} - \sum_{i=0}^{i_{n-1}-1} \widehat{v_n(i)} e^{(i)} - \sum_{i=i_n}^{\infty} \widehat{v_n(i)} e^{(i)}\right) \\ &\geqslant \varphi\left(\sum_{i=0}^{\infty} \widehat{v_n(i)} e^{(i)}\right) - \varphi\left(\sum_{i=1}^{i_{n-1}-1} \widehat{v_n(i)} e^{(i)}\right) - \varphi\left(\sum_{i=i_n}^{\infty} \widehat{v_n(i)} e^{(i)}\right) > 1 - 2\varepsilon. \end{aligned}$$

For every $n, m \in \mathcal{N}$ with $n \neq m$, one has

$$\begin{aligned} \varphi(v_n - v_m) &= \varphi\left(\sum_{i=0}^{\infty} \widehat{v_n(i)} e^{(i)} - \sum_{i=0}^{\infty} \widehat{v_m(i)} e^{(i)}\right) \\ &\geqslant \varphi\left(\sum_{i=i_{n-1}}^{i_n-1} \widehat{v_n(i)} e^{(i)} - \sum_{i=i_{m-1}}^{i_m-1} \widehat{v_m(i)} e^{(i)}\right) - \varphi\left(\sum_{i=0}^{i_{n-1}-1} \widehat{v_n(i)} e^{(i)}\right) \\ &\quad - \varphi\left(\sum_{i=i_n}^{\infty} \widehat{v_n(i)} e^{(i)}\right) - \varphi\left(\sum_{i=0}^{i_{m-1}-1} \widehat{v_m(i)} e^{(i)}\right) - \varphi\left(\sum_{i=i_m}^{\infty} \widehat{v_m(i)} e^{(i)}\right) \\ &\geqslant \varphi(z_n - z_m) - 4\varepsilon. \end{aligned}$$

Therefore, $A(\{f_n\}) = A(\{v_n\}) \geqslant A(\{z_n\}) - 4\varepsilon$. Put $\lambda_n = \frac{z_n}{\|z_n\|}$, for $n = 1, 2, 3, \ldots$ Hence

$$\lambda_n \in S\left(\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}\right), \qquad A(\{f_n\}) \geqslant 1 - \varepsilon A(\{\lambda_n\}) - 4\varepsilon. \tag{2.3}$$

On the other hand $\varphi(v_n - v_m) \leqslant \varphi(z_n - z_m) + 4\varepsilon \leqslant \varphi(\lambda_n - \lambda_m) + 4\varepsilon$, for all $n, m \in \mathcal{N}$ with $n \neq m$. Then

$$A(\{\lambda_n\}) \geqslant A(\{f_n\}) - 4\varepsilon. \tag{2.4}$$

As $\varepsilon > 0$, from (2.3) and (2.4), one has $WCS\left(\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}\right) = \inf\{A(\{\lambda_n\})\}$, with

$$\lambda_n = \sum_{i=i_{n-1}}^{i_n-1} \widehat{\lambda_n(i)} e^{(i)} \in S\left(\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}\right), 0 = i_{-1} < i_0 < \cdots, \lambda_n \overset{w}{\to} 0 \text{ and } \{\lambda_n\} \text{ is asymptotic equidistant.}$$

Fix $m \in \mathcal{N}$ large enough so that $\sum_{k=i_{m-1}}^{\infty} \left(\frac{b}{(u+v)^k}\right)^{\tau_k} < \varepsilon$, where $b := \sum_{i=i_{n-1}}^{i_n-1} A(k,i) q_i |\widehat{\lambda_n(i)}|$. We obtain for $n < m$,

$$\begin{aligned}
\varphi^K(\lambda_n - \lambda_m) &= \sum_{k=i_{n-1}}^{i_{m-1}-1} \left(\frac{1}{(u+v)^k} \sum_{i=0}^{k} A(k,i) q_i \left|\widehat{\lambda_n(i)}\right| \right)^{\tau_k} \\
&\quad + \sum_{k=i_{m-1}}^{\infty} \left(\frac{1}{(u+v)^k} \left(b + \sum_{i=0}^{k} A(k,i) q_i \left|\widehat{\lambda_m(i)}\right| \right) \right)^{\tau_k} \\
&\geqslant \sum_{k=i_{n-1}}^{i_{m-1}-1} \left(\frac{1}{(u+v)^k} \sum_{i=0}^{k} A(k,i) q_i \left|\widehat{\lambda_n(i)}\right| \right)^{\tau_k} + \sum_{k=i_{m-1}}^{\infty} \left(\frac{1}{(u+v)^k} \sum_{i=0}^{k} A(k,i) q_i \left|\widehat{\lambda_m(i)}\right| \right)^{\tau_k} \\
&= \sum_{k=i_{n-1}}^{\infty} \left(\frac{1}{(u+v)^k} \sum_{i=0}^{k} A(k,i) q_i \left|\widehat{\lambda_n(i)}\right| \right)^{\tau_k} - \sum_{k=i_{m-1}}^{\infty} \left(\frac{b}{(u+v)^k} \right)^{\tau_k} \\
&\quad + \sum_{k=i_{m-1}}^{\infty} \left(\frac{1}{(u+v)^k} \sum_{i=0}^{k} A(k,i) q_i \left|\widehat{\lambda_m(i)}\right| \right)^{\tau_k} > 1 - \varepsilon + 1 = 2 - \varepsilon,
\end{aligned}$$

that is $A_n(\{\lambda_n\}) \geqslant (2-\varepsilon)^{\frac{1}{K}}$. Recall that

$$\begin{aligned}
&\left[\sum_{k=i_{m-1}}^{\infty} \left(\frac{1}{(u+v)^k} \left(b + \sum_{i=0}^{k} A(k,i) q_i \left|\widehat{\lambda_m(i)}\right| \right) \right)^{\tau_k} \right]^{\frac{1}{K}} \\
&\quad \leqslant \left[\sum_{k=i_{m-1}}^{\infty} \left(\frac{b}{(u+v)^k} \right)^{\tau_k} \right]^{\frac{1}{K}} + \left[\sum_{k=i_{m-1}}^{\infty} \left(\frac{1}{(u+v)^k} \sum_{i=0}^{k} A(k,i) q_i \left|\widehat{\lambda_m(i)}\right| \right)^{\tau_k} \right]^{\frac{1}{K}} < \varepsilon^{\frac{1}{K}} + 1.
\end{aligned}$$

Hence

$$\begin{aligned}
&\varphi^K(\lambda_n - \lambda_m) \\
&= \sum_{k=i_{n-1}}^{i_{m-1}-1} \left(\frac{1}{(u+v)^k} \sum_{i=0}^{k} A(k,i) q_i \left|\widehat{\lambda_m(i)}\right| \right)^{\tau_k} + \sum_{k=i_{m-1}}^{\infty} \left(\frac{1}{(u+v)^k} \left(b + \sum_{i=0}^{k} A(k,i) q_i \left|\widehat{\lambda_m(i)}\right| \right) \right)^{\tau_k} \\
&\leqslant \sum_{k=i_{n-1}}^{\infty} \left(\frac{1}{(u+v)^k} \sum_{i=0}^{k} A(k,i) q_i \left|\widehat{\lambda_m(i)}\right| \right)^{\tau_k} + \sum_{k=i_{m-1}}^{\infty} \left(\frac{1}{(u+v)^k} \left(b + \sum_{i=0}^{k} A(k,i) q_i \left|\widehat{\lambda_m(i)}\right| \right) \right)^{\tau_k} \\
&\leqslant 1 + (1 + \varepsilon^{\frac{1}{K}})^K,
\end{aligned}$$

with $n, m \in \mathcal{N}$ and $n \neq m$. So $A_n(\{\lambda_n\}) \leqslant \left(1 + (1+\varepsilon^{\frac{1}{K}})^K\right)^{\frac{1}{K}}$ and, as this is true for every $\varepsilon > 0$, one gets $WCS\left(\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}\right) = 2^{\frac{1}{K}}$. According to Theorems 2.24 and 2.29, hence the function space $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$ has the φ-normal structure-property. □

3. Kannan contraction operator

We have presented in this section the space $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$ with different φ so that one has a unique fixed point of Kannan contraction operator. We also explain some illustrative examples to clarify our results.

Definition 3.1. A mapping $V : \mathcal{P}_\varphi \to \mathcal{P}_\varphi$ is said to be a Kannan φ-contraction, when one has $\alpha \in [0, \frac{1}{2})$ with $\varphi(Vf - Vg) \leqslant \alpha(\varphi(Vf - f) + \varphi(Vg - g))$, for all $f, g \in \mathcal{P}_\varphi$. The mapping V is said to be Kannan φ-non-expansive, if $\alpha = \frac{1}{2}$. An element $f \in \mathcal{P}_\varphi$ is said to be a fixed point of V, when $V(f) = f$.

Theorem 3.2. *Suppose the settings of Theorem 2.14 are verified, and* $V : \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi} \to \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$ *is Kannan* φ*-contraction mapping, where* $\varphi(f) = \left[\sum_{a=0}^{\infty}\left(\frac{\sum_{k=0}^{a} A(a,k)q_k|\widehat{f_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}}$*, for all* $f \in E^{\mathcal{P}}_{u,v}(q,\tau)$*, then* V *has a unique fixed point.*

Proof. Assume $f \in E^{\mathcal{P}}_{u,v}(q,\tau)$, we have $V^p f \in E^{\mathcal{P}}_{u,v}(q,\tau)$. Since V is a Kannan φ-contraction mapping, one obtains

$$\varphi(V^{l+1}f - V^l f) \leqslant \alpha\left(\varphi(V^{l+1}f - V^l f) + \varphi(V^l f - V^{l-1}f)\right) \Rightarrow$$
$$\varphi(V^{l+1}f - V^l f) \leqslant \frac{\alpha}{1-\alpha}\varphi(V^l f - V^{l-1}f) \leqslant \left(\frac{\alpha}{1-\alpha}\right)^2 \varphi(V^{l-1}f - V^{l-2}f) \leqslant \cdots \leqslant \left(\frac{\alpha}{1-\alpha}\right)^l \varphi(Vf - f).$$

Hence for every $l, m \in \mathcal{N}$ with $m > l$, we have

$$\varphi(V^l f - V^m f) \leqslant \alpha\left(\varphi(V^l f - V^{l-1}f) + \varphi(V^m f - V^{m-1}f)\right) \leqslant \alpha\left(\left(\frac{\alpha}{1-\alpha}\right)^{l-1} + \left(\frac{\alpha}{1-\alpha}\right)^{m-1}\right)\varphi(Vf - f).$$

Hence $\{V^l f\}$ is a Cauchy sequence in $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$. Since the space $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$ is pre-quasi Banach space, then $g \in \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$ so that $\lim_{l\to\infty} V^l f = g$. To show that $Vg = g$, as φ verifies the Fatou property, we get

$$\varphi(Vg - g) \leqslant \sup_i \inf_{l\geqslant i} \varphi(V^{l+1}f - V^l f) \leqslant \sup_i \inf_{l\geqslant i} \left(\frac{\alpha}{1-\alpha}\right)^l \varphi(Vf - f) = 0,$$

hence $Vg = g$. Therefore, g is a fixed point of V. To prove the uniqueness, suppose $f, g \in \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$ are fixed points of V with $f \neq g$, then

$$\varphi(f - g) \leqslant \varphi(Vf - Vg) \leqslant \alpha\left(\varphi(Vf - f) + \varphi(Vg - g)\right) = 0.$$

Hence $f = g$. □

Corollary 3.3. *Presume the settings of Theorem 2.14 are established and* $V : \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi} \to \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$ *is Kannan* φ*-contraction mapping, where* $\varphi(f) = \left[\sum_{a=0}^{\infty}\left(\frac{\sum_{k=0}^{a} A(a,k)q_k|\widehat{f_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}}$*, for every* $f \in E^{\mathcal{P}}_{u,v}(q,\tau)$*, then* V *has unique fixed point* g *with* $\varphi(V^l f - g) \leqslant \alpha\left(\frac{\alpha}{1-\alpha}\right)^{l-1}\varphi(Vf - f)$.

Proof. By Theorem 3.2, we have a unique fixed point g of V. Hence

$$\varphi(V^l f - g) = \varphi(V^l f - Vg) \leqslant \alpha\left(\varphi(V^l f - V^{l-1}f) + \varphi(Vg - g)\right) = \alpha\left(\frac{\alpha}{1-\alpha}\right)^{l-1}\varphi(Vf - f).$$

□

Example 3.4. Consider $V : E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)_{\varphi} \to E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)_{\varphi}$, where $\varphi(g) = \sqrt{\sum_{a=0}^{\infty}\left(\frac{\sum_{k=0}^{a}|\widehat{g_k}|}{(u+v)^a}\right)^{\frac{2a+3}{a+2}}}$, for all $g \in E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)$ and

$$V(g) = \begin{cases} \frac{g}{4}, & \varphi(g) \in [0,1), \\ \frac{g}{5}, & \varphi(g) \in [1,\infty). \end{cases}$$

For all $g_1, g_2 \in E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)_{\varphi}$ so that $\varphi(g_1), \varphi(g_2) \in [0,1)$, we obtain

$$\varphi(Vg_1 - Vg_2) = \varphi(\frac{g_1}{4} - \frac{g_2}{4}) \leqslant \frac{1}{\sqrt[4]{27}}\Big(\varphi(\frac{3g_1}{4}) + \varphi(\frac{3g_2}{4})\Big) = \frac{1}{\sqrt[4]{27}}\Big(\varphi(Vg_1 - g_1) + \varphi(Vg_2 - g_2)\Big).$$

For every $g_1, g_2 \in E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)_{\varphi}$ so that $\varphi(g_1), \varphi(g_2) \in [1,\infty)$, we get

$$\varphi(Vg_1 - Vg_2) = \varphi(\frac{g_1}{5} - \frac{g_2}{5}) \leqslant \frac{1}{\sqrt[4]{64}}\Big(\varphi(\frac{4g_1}{5}) + \varphi(\frac{4g_2}{5})\Big) = \frac{1}{\sqrt[4]{64}}\Big(\varphi(Vg_1 - g_1) + \varphi(Vg_2 - g_2)\Big).$$

For every $g_1, g_2 \in E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)_{\varphi}$ so that $\varphi(g_1) \in [0,1)$ and $\varphi(g_2) \in [1,\infty)$, one has

$$\begin{aligned}\varphi(Vg_1 - Vg_2) = \varphi(\frac{g_1}{4} - \frac{g_2}{5}) &\leqslant \frac{1}{\sqrt[4]{27}}\varphi(\frac{3g_1}{4}) + \frac{1}{\sqrt[4]{64}}\varphi(\frac{4g_2}{5}) \\ &\leqslant \frac{1}{\sqrt[4]{27}}\Big(\varphi(\frac{3g_1}{4}) + \varphi(\frac{4g_2}{5})\Big) = \frac{1}{\sqrt[4]{27}}\Big(\varphi(Vg_1 - g_1) + \varphi(Vg_2 - g_2)\Big).\end{aligned}$$

So V is Kannan φ-contraction. Since φ verifies the Fatou property, by Theorem 3.2, we have V satisfies a unique fixed point $\vartheta \in E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)_{\varphi}$.

Definition 3.5. Assume $\mathcal{P}_{\varphi}$ is a pre-quasi normed (ssfps), $V : \mathcal{P}_{\varphi} \to \mathcal{P}_{\varphi}$, and $g \in \mathcal{P}_{\varphi}$. The mapping V is called φ-sequentially continuous at g, if and only if, when $\lim_{p\to\infty} \varphi(f_p - g) = 0$, then $\lim_{p\to\infty} \varphi(Vf_p - Vg) = 0$.

Example 3.6. Consider $V : E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty}, \left(\frac{k+1}{2k+3}\right)_{k=0}^{\infty}\right)_{\varphi}$, where $\varphi(g) = \sum_{a=0}^{\infty}\left(\frac{\sum_{k=0}^{a}|\widehat{g_k}|}{(u+v)^a}\right)^{\frac{a+1}{2a+3}}$, for all $g \in E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty}, \left(\frac{k+1}{2k+3}\right)_{k=0}^{\infty}\right)$ and

$$V(g) = \begin{cases} \frac{1}{18}(1+g), & \widehat{g_0} \in [0, \frac{1}{17}), \\ \frac{1}{17}, & \widehat{g_0} = \frac{1}{17}, \\ \frac{1}{18}, & \widehat{g_0} \in (\frac{1}{17}, 1]. \end{cases}$$

Evidently, V is φ-sequentially continuous and discontinuous at $\frac{1}{17} \in \left(E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty}, \left(\frac{k+1}{2k+3}\right)_{k=0}^{\infty}\right)\right)_{\varphi}$.

Example 3.7. If V is defined as in Example 3.4, presume $\{g^{(n)}\} \subseteq E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)_{\varphi}$ is so that $\lim_{n\to\infty} \varphi(g^{(n)} - g^{(0)}) = 0$, where $g^{(0)} \in E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)_{\varphi}$ with $\varphi(g^{(0)}) = 1$. Since the pre-quasi norm φ is continuous, one gets

$$\lim_{n\to\infty} \varphi(Vg^{(n)} - Vg^{(0)}) = \lim_{n\to\infty} h\Big(\frac{g^{(n)}}{4} - \frac{g^{(0)}}{5}\Big) = h\Big(\frac{g^{(0)}}{20}\Big) > 0.$$

Hence V is not φ-sequentially continuous at $g^{(0)}$.

Theorem 3.8. *Presume the settings of Theorem 2.14 are established under $\tau_0 > 1$, and $V : \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi} \to \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$, where $\varphi(f) = \sum_{a=0}^{\infty}\left(\frac{\sum_{k=0}^{a} A(a,k)q_k|\widehat{f_k}|}{(u+v)^a}\right)^{\tau_a}$, for all $f \in E^{\mathcal{P}}_{u,v}(q,\tau)$. If*

(1) *V is Kannan φ-contraction mapping;*

(2) *V is φ-sequentially continuous at $g \in \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$;*

(3) *one has* $f \in \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$ *with* $\{V^{l}f\}$ *has* $\{V^{l_j}f\}$ *converging to* g,

then $g \in \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$ *is the only fixed point of* V

Proof. Let g be not a fixed point of V, we have $Vg \neq g$. By settings (2) and (3), then

$$\lim_{l_j\to\infty} \varphi(V^{l_j}f-g)=0 \quad \text{and} \quad \lim_{l_j\to\infty} \varphi(V^{l_j+1}f-Vg)=0.$$

Since V is Kannan φ-contraction, we have

$$\begin{aligned}0<\varphi(Vg-g)&=\varphi\left((Vg-V^{l_j+1}f)+(V^{l_j}f-g)+(V^{l_j+1}f-V^{l_j}f)\right)\\ &\leqslant 2^{2\sup_i\tau_i-2}\varphi\left(V^{l_j+1}f-Vg\right)+2^{2\sup_i\tau_i-2}\varphi\left(V^{l_j}f-g\right)+2^{\sup_i\tau_i-1}\alpha\left(\frac{\alpha}{1-\alpha}\right)^{l_j-1}\varphi(Vf-f).\end{aligned}$$

By letting $l_j\to\infty$, one has a contradiction. So g is a fixed point of V. To prove the uniqueness, assume $g,f \in \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$ are fixed points of V with $g\neq f$, then

$$\varphi(g-f)\leqslant\varphi(Vg-Vf)\leqslant\alpha\left(\varphi(Vg-g)+\varphi(Vf-f)\right)=0.$$

Therefore, $g=f$. □

Example 3.9. Consider V is defined as in Example 3.4. Suppose $\varphi(f)=\sum_{a=0}^{\infty}\left(\frac{\sum_{k=0}^{a}|\widehat{f_k}|}{(u+v)^a}\right)^{\frac{2a+3}{a+2}}$, for every $f\in E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty},\left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)$. As for every $f_1,f_2\in E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty},\left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)_{\varphi}$ so that $\varphi(f_1),\varphi(f_2)\in[0,1)$, we obtain

$$\varphi(Vf_1-Vf_2)=\varphi(\frac{f_1}{4}-\frac{f_2}{4})\leqslant\frac{2}{\sqrt{27}}\Big(\varphi(\frac{3f_1}{4})+\varphi(\frac{3f_2}{4})\Big)=\frac{2}{\sqrt{27}}\Big(\varphi(Vf_1-f_1)+\varphi(Vf_2-f_2)\Big).$$

For every $f_1,f_2\in E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty},\left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)_{\varphi}$ so that $\varphi(f_1),\varphi(f_2)\in[1,\infty)$, we have

$$\varphi(Vf_1-Vf_2)=\varphi(\frac{f_1}{5}-\frac{f_2}{5})\leqslant\frac{1}{4}\Big(\varphi(\frac{4f_1}{5})+\varphi(\frac{4f_2}{5})\Big)=\frac{1}{4}\Big(\varphi(Vf_1-f_1)+\varphi(Vf_2-f_2)\Big).$$

For each $f_1,f_2\in E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty},\left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)_{\varphi}$ so that $\varphi(f_1)\in[0,1)$ and $\varphi(f_2)\in[1,\infty)$, we obtain

$$\begin{aligned}\varphi(Vf_1-Vf_2)&=\varphi(\frac{f_1}{4}-\frac{f_2}{5})\leqslant\frac{2}{\sqrt{27}}\varphi(\frac{3f_1}{4})+\frac{1}{4}\varphi(\frac{4f_2}{5})\\ &\leqslant\frac{2}{\sqrt{27}}\Big(\varphi(\frac{3f_1}{4})+\varphi(\frac{4f_2}{5})\Big)=\frac{2}{\sqrt{27}}\Big(\varphi(Vf_1-f_1)+\varphi(Vf_2-f_2)\Big).\end{aligned}$$

Hence V is Kannan φ-contraction and $V^p(f)=\begin{cases}\frac{f}{4^p}, & \varphi(f)\in[0,1),\\ \frac{f}{5^p}, & \varphi(f)\in[1,\infty).\end{cases}$ Evidently, V is φ-sequentially continuous at $\vartheta\in E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty},\left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)_{\varphi}$ and $\{V^pf\}$ verifies $\{V^{l_j}f\}$ converges to ϑ. From Theorem 3.8, the point $\vartheta\in E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)_{k=0}^{\infty},\left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)_{\varphi}$ is the unique fixed point of V.

4. Kannan non-expansive mapping on $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$

We give the sufficient settings of $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$, where $\varphi(f) = \left[\sum_{a=0}^{\infty}\left(\frac{\sum_{k=0}^{a} A(a,k)q_k|\widehat{f_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}}$, for every $f \in E^{\mathcal{P}}_{u,v}(q,\tau)$, such that the Kannan non-expansive mapping on it has a fixed point.

Lemma 4.1. *Assume $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$ satisfies the* (R) *property and the φ-quasi-normal property. If $V : \Omega \to \Omega$ is a Kannan φ-non-expansive mapping, suppose $G_t = \left\{f \in \Omega : \varphi(f - V(f)) \leqslant t\right\} \neq \emptyset$, so that $t > 0$. Take*

$$\Omega_t = \bigcap\left\{\mathbf{B}_\varphi(p,j) : V(G_t) \subset \mathbf{B}_\varphi(p,j)\right\} \cap \Omega.$$

Hence $\Omega_t \neq \emptyset$, φ-convex, φ-closed subset of Ω, $\delta_\varphi(\Omega_t) \leqslant t$ and $V(\Omega_t) \subset \Omega_t \subset G_t$

Proof. As $V(G_t) \subset \Omega_t$, hence $\Omega_t \neq \emptyset$. Since the φ-balls are φ-convex and φ-closed, then Ω_t is a φ-closed and φ-convex subset of Ω. To prove that $\Omega_t \subset G_t$, if $f \in \Omega_t$, suppose $\varphi(f - V(f)) = 0$, we get $f \in G_t$. Else, let $\varphi(f - V(f)) > 0$. Take

$$p = \sup\left\{\varphi\Big(V(g) - V(f)\Big) : g \in G_t\right\}.$$

By the definition of p, we have $V(G_t) \subset \mathbf{B}_\varphi\Big(V(f), p\Big)$. Hence $\Omega_t \subset \mathbf{B}_\varphi\Big(V(f), p\Big)$, then $\varphi\Big(f - V(f)\Big) \leqslant p$. Assume $l > 0$. We have $g \in G_t$ so that $p - l \leqslant \varphi\Big(V(g) - V(f)\Big)$. Hence

$$\varphi\Big(f - V(f)\Big) - l \leqslant p - l \leqslant \varphi\Big(V(g) - V(f)\Big) \leqslant \frac{1}{2}\Big(\varphi\Big(f - V(f)\Big) + \varphi\Big(g - V(g)\Big)\Big) \leqslant \frac{1}{2}\Big(\varphi\Big(f - V(f)\Big) + t\Big).$$

Since l is an arbitrary positive, then $\varphi\Big(f - V(f)\Big) \leqslant t$, so $f \in G_t$. As $V(G_t) \subset \Omega_t$, we have $V(\Omega_t) \subset V(G_t) \subset \Omega_t$, then Ω_t is V-invariant. To prove that $\delta_\varphi(\Omega_t) \leqslant t$, since

$$\varphi\Big(V(f) - V(g)\Big) \leqslant \frac{1}{2}\Big(\varphi\Big(f - V(f)\Big)\Big) + \varphi\Big(g - V(g)\Big)\Big),$$

for every $f, g \in G_t$, let $f \in G_t$. Hence $V(G_t) \subset \mathbf{B}_\varphi\Big(V(f), t\Big)$. The definition of Ω_t implies that $\Omega_t \subset \mathbf{B}_\varphi\Big(V(f), t\Big)$. Hence $V(f) \in \bigcap_{t \in \Omega_t} \mathbf{B}_\varphi(g, t)$. We have $\varphi\Big(g - f\Big) \leqslant t$, for every $g, f \in \Omega_t$, so $\delta_\varphi(\Omega_t) \leqslant t$. □

Theorem 4.2. *Presume $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$ verifies the φ-quasi-normal property and the (R) property. Assume $V : \Omega \to \Omega$ is a Kannan φ-non-expansive mapping. Then, V has a fixed point.*

Proof. Take $t_0 = \inf\left\{\varphi\Big(f - V(f)\Big) : f \in \Omega\right\}$ and $t_p = t_0 + \frac{1}{p}$, for every $p \geqslant 1$. From the definition of t_0, we have $G_{t_p} = \left\{f \in \Omega : \varphi\Big(f - V(f)\Big) \leqslant t_p\right\} \neq \emptyset$, for all $p \geqslant 1$. If Ω_{t_p} is defined as in Lemma 4.1. Obviously, $\{\Omega_{t_p}\}$ is a decreasing sequence of nonempty φ-bounded, φ-closed, and φ-convex subsets of Ω. The property (R) explains that $\Omega_\infty = \bigcap_{p \geqslant 1} \Omega_{t_p} \neq \emptyset$. Presume $f \in \Omega_\infty$, we have $\varphi\Big(f - V(f)\Big) \leqslant t_p$, for every $p \geqslant 1$. Assume $p \to \infty$, then $\varphi\Big(f - V(f)\Big) \leqslant t_0$, hence $\varphi\Big(f - V(f)\Big) = t_0$. So $G_{t_0} \neq \emptyset$. Then $t_0 = 0$. Else, $t_0 > 0$, then V fails to have a fixed point. Suppose Ω_{t_0} is defined in Lemma 4.1. Since V fails to have a fixed point and Ω_{t_0} is V-invariant, then Ω_{t_0} has more than one point, hence $\delta_\varphi(\Omega_{t_0}) > 0$. According to the φ-quasi-normal property, we get $f \in \Omega_{t_0}$ so that

$$\varphi\Big(f - g\Big) < \delta_\varphi(\Omega_{t_0}) \leqslant t_0,$$

for every $g \in \Omega_{t_0}$. By Lemma 4.1, one has $\Omega_{t_0} \subset G_{t_0}$. By definition of Ω_{t_0}, hence $V(f) \in G_{t_0} \subset \Omega_{t_0}$. So

$$\varphi\Big(f - V(f)\Big) < \delta_\varphi(\Omega_{t_0}) \leqslant t_0,$$

this contradicts the definition of t_0. Hence $t_0 = 0$, which implies that any point in G_{t_0} is a fixed point of V. □

By Theorems 2.26, 2.30, and 4.2, we have the following.

Corollary 4.3. *Presume the settings of Theorem 2.14 are established under $\tau_0 > 1$, and $V : \Omega \to \Omega$ is a Kannan φ-non-expansive, then V has a fixed point in $\Omega \subseteq \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$.*

Example 4.4. Consider $V : \Omega \to \Omega$ and $V(f) = \begin{cases} \frac{f}{4}, & \varphi(f) \in [0,1), \\ \frac{f}{5}, & \varphi(f) \in [1,\infty), \end{cases}$ where

$$\Omega = \left\{ f \in E^{\mathcal{P}}_{u,v}\left(\left(\left(\frac{1}{A(a,k)}\right)^\infty_{k=0}, \left(\frac{2k+3}{k+2}\right)^\infty_{k=0}\right)_\varphi : \widehat{f_0} = \widehat{f_1} = 0\right\}$$

and $\varphi(f) = \sqrt{\sum_{a=0}^\infty \left(\frac{\sum_{k=0}^a |\widehat{f_k}|}{(u+v)^a}\right)^{\frac{2a+3}{a+2}}}$, for all $f \in E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{A(a,k)}\right)^\infty_{k=0}, \left(\frac{2k+3}{k+2}\right)^\infty_{k=0}\right)_\varphi$. By using Example 3.4, V is Kannan φ-contraction. Therefore, it is Kannan φ-non-expansive. From Corollary 4.3, V has a fixed point ϑ in Ω.

5. Existence of solutions of Volterra-type summable equations

We have investigated a solution in $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$ to Volterra-type summable equations as (5.1), defined in [30], where the settings of Theorem 2.14 are established and $\varphi(f) = \left[\sum_{a=0}^\infty \left(\frac{\sum_{k=0}^a A(a,k) q_k |\widehat{f_k}|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}}$, for every $f \in \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$. Take the summable equation

$$\widehat{f_a} = \widehat{p_a} + \sum_{m=0}^\infty B(a,m) g(m, \widehat{f_m}). \tag{5.1}$$

Presume $W : \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi \to \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$ is defined as:

$$(W(f))(z) = \sum_{a=0}^\infty \left(\widehat{p_a} + \sum_{m=0}^\infty B(a,m) g(m, \widehat{f_m})\right) z^a. \tag{5.2}$$

Theorem 5.1. *The summable equation (5.1) has a unique solution in $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_\varphi$, when $B : \mathcal{N}^2 \to \mathfrak{R}$, $g : \mathcal{N} \times \mathbb{C} \to \mathbb{C}$, $\widehat{p} : \mathcal{N} \to \mathbb{C}$, $\widehat{\eta} : \mathcal{N} \to \mathbb{C}$, if one has $\lambda \in \mathfrak{R}$ so that $\sup_a |\lambda|^{\frac{\tau_a}{K}} \in [0, \frac{1}{2})$ and for every $a \in \mathcal{N}$, one has*

$$\begin{aligned} &\left|\sum_{k=0}^a \left(\sum_{m \in \mathcal{N}} B(k,m)[g(m,\widehat{f_m}) - g(m,\widehat{\eta_m})]\right) A(a,k) q_k\right| \\ &\leqslant |\lambda| \left|\sum_{k=0}^a \left(\widehat{p_k} - \widehat{f_k} + \sum_{m=0}^\infty B(k,m) g(m,\widehat{f_m})\right) A(a,k) q_k\right| \\ &\quad + |\lambda| \left|\sum_{k=0}^a \left(\widehat{p_k} - \widehat{\eta_k} + \sum_{m=0}^\infty B(k,m) g(m,\widehat{\eta_m})\right) A(a,k) q_k\right|. \end{aligned}$$

Proof. Suppose the mapping $W : \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi} \to \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$ is defined by equation (5.2). So

$$\begin{aligned}\varphi(Wf - W\eta) &= \left[\sum_{a=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{a}\left(\sum_{m\in\mathcal{N}} B(k,m)[g(m,\widehat{f_m}) - g(m,\widehat{\eta_m})]\right)A(a,k)q_k\right|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}}\\
&\leqslant \sup_a |\lambda|^{\frac{\tau_a}{K}}\left[\sum_{a=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{a}\left(\widehat{p_k} - \widehat{f_k} + \sum_{m=0}^{\infty} B(k,m)g(m,\widehat{f_m})\right)A(a,k)q_k\right|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}}\\
&\quad + \sup_a |\lambda|^{\frac{\tau_a}{K}}\left[\sum_{a=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{a}\left(\widehat{p_k} - \widehat{\eta_k} + \sum_{m=0}^{\infty} B(k,m)g(m,\widehat{\eta_m})\right)A(a,k)q_k\right|}{(u+v)^a}\right)^{\tau_a}\right]^{\frac{1}{K}}\\
&= \sup_a |\lambda|^{\frac{\tau_a}{K}}\left(\varphi(Wf - f) + \varphi(W\eta - \eta)\right).\end{aligned}$$

□

From Theorem 3.2, one has a unique solution of equation (5.1) in $\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$.

Example 5.2. Take $\left(E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{(k+1)A(l,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)\right)_{\varphi}$, where $\varphi(f) = \sqrt{\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l}\frac{|\widehat{f_z}|}{z+1}}{(u+v)^l}\right)^{\frac{2l+3}{l+2}}}$, for every $f \in \left(E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{(k+1)A(l,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)\right)_{\varphi}$. Presume the Volterra-type summable equation

$$\widehat{f_a} = e^{-(3a+6i)} + \sum_{m=0}^{\infty} e^{a+m}\frac{\widehat{f^b_{a-2}}}{\widehat{f^d_{a-1}} + m^2 + 1}, \tag{5.3}$$

so that $i^2 = -1$, $b, d, \widehat{f_{-2}}, \widehat{f_{-1}} > 0$ and if

$$W : \left(E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{(k+1)A(l,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)\right)_{\varphi} \to \left(E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{(k+1)A(l,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)\right)_{\varphi},$$

is defined by

$$(W(f))(z) = \sum_{a=0}^{\infty}\left(e^{-(3a+6i)} + \sum_{m=0}^{\infty} e^{a+m}\frac{\widehat{f^b_{a-2}}}{\widehat{f^d_{a-1}} + m^2 + 1}\right)z^a. \tag{5.4}$$

One has $\lambda \in \mathfrak{R}$ such that $\sup_a |\lambda|^{\frac{2a+3}{2a+4}} \in [0, \frac{1}{2})$ and for every $a \in \mathcal{N}$, we obtain

$$\begin{aligned}&\left|\sum_{k=0}^{a}\left(\sum_{m=0}^{\infty} e^k\frac{\widehat{f^b_{k-2}}}{\widehat{f^d_{k-1}} + m^2 + 1}\left(e^m - e^m\right)\right)A(a,k)q_k\right|\\
&\leqslant |\lambda|\left|\sum_{k=0}^{a}\left(e^{-(3k+6i)} - f_k + \sum_{m=0}^{\infty} e^{k+m}\frac{\widehat{f^b_{k-2}}}{\widehat{f^d_{k-1}} + m^2 + 1}\right)A(a,k)q_k\right|\\
&\quad + |\lambda|\left|\sum_{k=0}^{a}\left(e^{-(3k+6i)} - \widehat{\eta_k} + \sum_{m=0}^{\infty} e^{k+m}\frac{\widehat{\eta^b_{k-2}}}{\widehat{\eta^d_{k-1}} + m^2 + 1}\right)A(a,k)q_k\right|.\end{aligned}$$

From Theorem 5.1, the Volterra-type summable equations (5.3) have a unique solution in $\left(E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{(k+1)A(l,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)\right)_{\varphi}$.

Example 5.3. Consider the Volterra-type summable equation (5.4) and $W:\Omega\to\Omega$ is defined as equation (5.4), where $\Omega = \left\{f \in \left(E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{(k+1)A(l,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)\right)_{\varphi} : \widehat{f_0} = \widehat{f_1} = 0\right\}$ and $\varphi(f) = \sqrt{\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l}\frac{|\widehat{f_z}|}{z+1}}{(u+v)^{l}}\right)^{\frac{2l+3}{l+2}}}$, for every $f \in \left(E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{(k+1)A(l,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)\right)_{\varphi}$. Obviously, Ω is a nonempty φ-bounded, φ-convex, and φ-closed subset of $\left(E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{(k+1)A(l,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)\right)_{\varphi}$. From Example 5.2, W is Kannan φ-contraction, therefore, it is Kannan φ-non-expansive. In view of Corollary 4.3, W has a fixed point in Ω.

Theorem 5.4. *Presume the summable equation (5.1), and if $W : \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi} \to \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$ is defined by (5.2), where the settings of Theorem 2.14 are confirmed so that $\tau_0 > 1$ and $\varphi(f) = \sum_{a=0}^{\infty}\left(\frac{\sum_{k=0}^{a}A(a,k)q_k|\widehat{f_k}|}{(u+v)^{a}}\right)^{\tau_a}$, for all $f \in E^{\mathcal{P}}_{u,v}(q,t)$. The summable equation (5.1) has a unique solution $\overline{Z} \in \left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$, if the next settings are established.*

(1) *Suppose $B:\mathcal{N}^2\to\mathfrak{R}$, $g:\mathcal{N}\times\mathbb{C}\to\mathbb{C}$, $\widehat{p}:\mathcal{N}\to\mathbb{C}$, $\widehat{\eta}:\mathcal{N}\to\mathbb{C}$, one has $\lambda\in\mathfrak{R}$ with $2^{K-1}\sup_a|\lambda|^{\tau_a}\in[0,\frac{1}{2})$ and for all $a\in\mathcal{N}$, then*

$$\begin{aligned}&\left|\sum_{k=0}^{a}\left(\sum_{m\in\mathcal{N}}B(k,m)[g(m,\widehat{f_m})-g(m,\widehat{\eta_m})]\right)A(a,k)q_k\right|\\ &\leqslant|\lambda|\left|\sum_{k=0}^{a}\left(\widehat{p_k}-\widehat{f_k}+\sum_{m=0}^{\infty}B(k,m)g(m,\widehat{f_m})\right)A(a,k)q_k\right|\\ &\quad+|\lambda|\left|\sum_{k=0}^{a}\left(\widehat{p_k}-\widehat{\eta_k}+\sum_{m=0}^{\infty}B(k,m)g(m,\widehat{\eta_m})\right)A(a,k)q_k\right|.\end{aligned}$$

(2) *W is φ-sequentially continuous at $t\in\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$.*

(3) *There is $f\in\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$ with $\{W^m f\}$ having $\{W^{m_j}f\}$ converging to t.*

Proof. We have

$$\begin{aligned}\varphi(Wf-W\eta)&=\sum_{a=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{a}\left(\sum_{m\in\mathcal{N}}B(k,m)[g(m,\widehat{f_m})-g(m,\widehat{\eta_m})]\right)A(a,k)q_k\right|}{(u+v)^{a}}\right)^{\tau_a}\\ &\leqslant 2^{K-1}\sup_a|\lambda|^{\tau_a}\sum_{a=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{a}\left(\widehat{p_k}-\widehat{f_k}+\sum_{m=0}^{\infty}B(k,m)g(m,\widehat{f_m})\right)A(a,k)q_k\right|}{(u+v)^{a}}\right)^{\tau_a}\\ &\quad+2^{K-1}\sup_a|\lambda|^{\tau_a}\sum_{a=0}^{\infty}\left(\frac{|\sum_{k=0}^{a}(\widehat{p_k}-\widehat{\eta_k}+\sum_{m=0}^{\infty}B(k,m)g(m,\widehat{\eta_m}))A(a,k)q_k|}{(u+v)^{a}}\right)^{\tau_a}\\ &=2^{K-1}\sup_a|\lambda|^{\tau_a}\left(\varphi(Wf-f)+\varphi(W\eta-\eta)\right).\end{aligned}$$

From Theorem 3.8, we obtain a unique solution $t\in\left(E^{\mathcal{P}}_{u,v}(q,\tau)\right)_{\varphi}$ of equation (5.1). □

Example 5.5. Presume $\left(E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{(k+1)A(l,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)\right)_{\varphi}$, where $\varphi(f) = \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l}\frac{|\widehat{f_z}|}{z+1}}{(u+v)^{l}}\right)^{\frac{2l+3}{l+2}}$, for every $f \in \left(E^{\mathcal{P}}_{u,v}\left(\left(\frac{1}{(k+1)A(l,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)\right)_{\varphi}$. Consider the summable equations (5.3),

$W : \left(E_{u,v}^{\mathcal{P}}\left(\left(\frac{1}{(k+1)A(l,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)\right)_{\varphi} \to \left(E_{u,v}^{\mathcal{P}}\left(\left(\frac{1}{(k+1)A(l,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)\right)_{\varphi}$ defined by (5.4). Suppose W is φ-sequentially continuous at $t \in \left(E_{u,v}^{\mathcal{P}}\left(\left(\frac{1}{(k+1)A(l,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)\right)_{\varphi}$, and there is $f \in \left(E_{u,v}^{\mathcal{P}}\left(\left(\frac{1}{(k+1)A(l,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)\right)_{\varphi}$ with $\{W^m f\}$ having $\{W^{m_j} f\}$ converging to t. Clearly, we have $\lambda \in \mathfrak{R}$ so that $2^{K-1}\sup_a |\lambda|^{\frac{2a+3}{a+2}} \in [0, \frac{1}{2})$ and for every $a \in \mathcal{N}$, we have

$$\begin{aligned}
&\left|\sum_{k=0}^{a}\left(\sum_{m=0}^{\infty} e^{k}\frac{\widehat{f_{k-2}^{b}}}{\widehat{f_{k-1}^{d}}+m^2+1}\left(e^{m}-e^{m}\right)\right)A(a,k)q_k\right| \\
&\leqslant |\lambda|\left|\sum_{k=0}^{a}\left(e^{-(3k+6i)}-f_k+\sum_{m=0}^{\infty} e^{k+m}\frac{\widehat{f_{k-2}^{b}}}{\widehat{f_{k-1}^{d}}+m^2+1}\right)A(a,k)q_k\right| \\
&\quad+|\lambda|\left|\sum_{k=0}^{a}\left(e^{-(3k+6i)}-\widehat{\eta_k}+\sum_{m=0}^{\infty} e^{k+m}\frac{\widehat{\eta_{k-2}^{b}}}{\widehat{\eta_{k-1}^{d}}+m^2+1}\right)A(a,k)q_k\right|.
\end{aligned}$$

From Theorem 5.4, the summable equations (5.3) have a unique solution $t \in \left(E_{u,v}^{\mathcal{P}}\left(\left(\frac{1}{(k+1)A(l,k)}\right)_{k=0}^{\infty}, \left(\frac{2k+3}{k+2}\right)_{k=0}^{\infty}\right)\right)_{\varphi}$.

6. Conclusion

In this article, several topological and geometric structures in the new complex function space $E_{u,v}^{\mathcal{P}}(q,\tau)$ are discussed. The nonlinear dynamical systems can be solved in a variety of ways. The variable exponent we discovered in the space we just discussed has bolstered several well-known ideas.

Acknowledgments

This work was funded by the University of Jeddah, Saudi Arabia, under grant No. (UJ-22-DR-5). The authors, therefore, acknowledge with thanks the University of Jeddah for its technical and financial support.

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