J. Math. Computer Sci., 35 (2024), 1–15

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

An innovation diffusion model with two innovations under two simultaneous effects in a competitive market: coexistence through optimal control

Sonu Chugh[a,c,*], Joydip Dhar[b], Rangan Kumar Guha[c]

[a]Department of Mathematics, D. M. College, Moga, 142001, Punjab, India.

[b]Department of Applied Science, ABV-Indian Institute of Information Technology and Management, Gwalior, 474015, Madhya Pradesh, India.

[c]Department of Mathematics, Sant Longowal Institute of Engineering and Technology, Longowal, 148106, Punjab, India.

Abstract

Since mass media plays an important role in influencing the non-adopter population, the adoption rate changes with the media awareness rate. Motivated by this concept, in this paper, a model of three non-intersecting classes of non-adopters, adopters for product-I and adopters for product-II is proposed. Under the influence of media coverage and word-of-mouth, the dynamic behaviour of the system is investigated. The basic influence numbers $\mathcal{R}_{0_1}$ and $\mathcal{R}_{0_2}$ associated with the first and second innovations help in performing stability analysis. It is observed from stability analysis that adopter-free equilibrium is conditionally stable. Also, the system has no stable interior equilibrium point. The basic influence numbers determine the sustainability of a particular product in the market. The optimal control theory is used to reduce the frustration rate in both adopter classes. The Hamiltonian function is constructed using the extended optimum control model and is then solved according to Pontryagin's maximum principle to get the cost. Also, coexistence is possible with the implementation of optimal control. Sensitivity analysis has been performed for both the basic influence numbers $\mathcal{R}_{0_1}$ and $\mathcal{R}_{0_2}$. Lastly, numerical experimentations have been executed to assist analytical findings with distinct sets of parameters.

Keywords: Basic influence number, word-of-mouth effect, impact of media, sensitivity analysis, optimal control.

2020 MSC: 37C75, 93D20, 37C20, 35F21.

1. Introduction

In the present article, we have strived to present an innovation dissemination model in a competitive market. The innovation dissemination model is mainly helpful in predicting product life cycles and trends in product purchase. The Bass model is a significant device for predicting the adoption of an innovation for which no close substitute exists in the marketplace [1]. The model attempts to forecast how many customers will eventually adopt the new product and when they will do so. Researchers in this field are relaxing the assumptions of this model and presenting sophisticated models. Jain et al. [12] looked at

*Corresponding author

Email address: sonubarnala0001@gmail.com (Sonu Chugh)

doi: 10.22436/jmcs.035.01.01

Received: 2023-08-16 *Revised:* 2023-10-04 *Accepted:* 2024-02-19

the impact of pricing on the adoption of novel durable goods. Jones et al. [13] introduced a new model for the spread of products that had two concurrent acceptance processes: one for retailers and one for consumers. This model assumes that there is an essential interaction between these two processes and that the customer will buy the product only from retail outlets. The Lotka-Volterra equation [16] has been used in markets where two or more closely substituted products are competing for their sustainability. Modis [22] examined the behaviour of company stocks as if they were competing for investors' resources. Kekana et al. [14] analyzed the global stability of equilibria for two competing products in an innovation diffusion model. Yu et al. [30] presented the mathematical model for three competitive products with advertisement effect, interpersonal valid contact rate, and return rate from adopter class to non-adopter class as key parameters and discussed the global stability of equilibria. Kim et al. [15] presented the situation of the Korean mobile phone market and observed a commensal relationship. Horsky et al. [10] examined the effect of advertising on the sales growth of new products that were purchased infrequently. In recent years, the bifurcation analysis of different innovation diffusion models has been studied [18, 19]. Many models regarding innovation diffusion have been developed and analyzed in recent years [7, 11, 20, 26, 27, 31].

Motivation and novelty in the paper

Numerous disciplines, including engineering, environmental management, energy systems, economics, and medicine, can greatly benefit from optimal control theory. It helps in optimizing control strategies and resource allocation, leading to improved system performance, efficient resource utilisation, and decision-making. It is extensively used in engineering systems, including electrical, automotive, and aeronautical systems. It aids in designing control strategies that optimize system performance while minimizing costs or energy consumption. It is used in environmental management, including the prevention of pollution and the management of natural resources. It assists in choosing the best regulations and preventative measures to reduce pollution, save resources, and advance sustainable development. To maximize energy production and distribution, it is used in energy systems, including power grids and renewable energy systems. Using mathematical modelling and optimization approaches to create plans for preventing the spread of infectious illnesses is known as optimal control in epidemiology. To lessen the impact of the disease on a community, it entails determining the optimal distribution of control measures, such as vaccination campaigns, quarantine measures, or social isolation. Mathematical modelling helps in comprehending the dynamics of disease transmission and its control strategy. In epidemic models, the theory of optimal control is particularly fascinating and helpful [7, 17, 25, 28]. As far as innovation diffusion modelling is concerned, Chugh et al. [2] analyzed a four-compartmental system with the help of optimal control theory to examine the interaction and market dissemination of two product categories. Its usefulness motivates us to adopt this theory in innovation diffusion modelling. In our present article, we will discuss this theory.

Dhar et al. [4] studied a model considering only the word-of-mouth effect and ignoring the advertisement effect on innovation diffusion. Chugh et al. [3] modified the model [4] and presented a model with the concept of cooperativeness in a competitive market. But both research papers ignored the media effect [3, 4]. In this paper, we consider the effects of word-of-mouth as well as media alerts.

Structure of the paper

In the present paper, using the stability theory of differential equations and two competing products, we offer a non-linear mathematical model to investigate the impact of advertising and word-of-mouth on innovation dissemination. This article has been organized in the following way. In Section 2, a realistic mathematical model is framed with two competing products under internal and external influence. Section 3 deals with basic influence numbers. The study regarding the stability and Hopf-bifurcation analysis is presented in Sections 4 and 5, respectively. In Section 6, we use optimal control theory to reduce the frustration rate in both adopter classes. We run a numerical simulation to examine the impacts of optimal control and show a fall in the cost function. Sensitivity analysis and numerical validation are performed in Sections 7 and 8, respectively. Finally, a brief conclusion of the model is presented in Section 9.

2. Mathematical assumptions and proposed model

In this paper, a population is classified into three non-intersecting classes namely, $N(t); A_1(t); A_2(t)$, where $N(t)$ is the non-adopter population, $A_1(t)$ is the adopter population of innovation-I, $A_2(t)$ is the adopter population of innovation-II at time 't' and a non-linear dynamical mathematical model is proposed. Our proposed model hinges on the following assumptions.

1. Adopters will adopt only one innovation at a time. The situation in which an adopter uses more than one innovation simultaneously is ignored.
2. It is assumed that Λ is the constant enrollment rate in the non-user class and δ is the constant fatality rate for all classes of the population.
3. Adopters will only positively impact non-adopters.
4. The rate at which non-users contact users of the first (second) product before the media alert is denoted by $\beta_1(\beta_3)$ and $\beta_2(\beta_4)$ is the additional contact rate due to the media alert for the adopter $A_1(A_2)$. Hence, $\beta_1 + \frac{\beta_2 A_1}{m+A_1}(\beta_3 + \frac{\beta_4 A_2}{m+A_2})$ is the total contact rate after media alert. We chose these contact rates to model the media alert with the assumption that $\frac{\beta_2 A_1}{m+A_1}(\frac{\beta_4 A_2}{m+A_2})$ will reflect the transmission rate when adopter individuals appear and are reported. When $A_1(A_2) \to \infty$, the increased value of the transmission rate approaches its maximum $\beta_2(\beta_4)$, and the increased value of the transmission rate equals half of the maximum $\beta_2(\beta_4)$ when the reported adopter arrives at 'm' [5]. Here 'm' is half media saturation rate.
5. The rate at which users of the first (second) product frustrate and join the non-adopter class is $\gamma_1(\gamma_2)$.
6. Competition rate $\delta_1(\delta_2)$ causes a reduction in the adoption of the first (second) adopter population.

Based on the assumptions, Figure 1 illustrates the system's schematic flow.

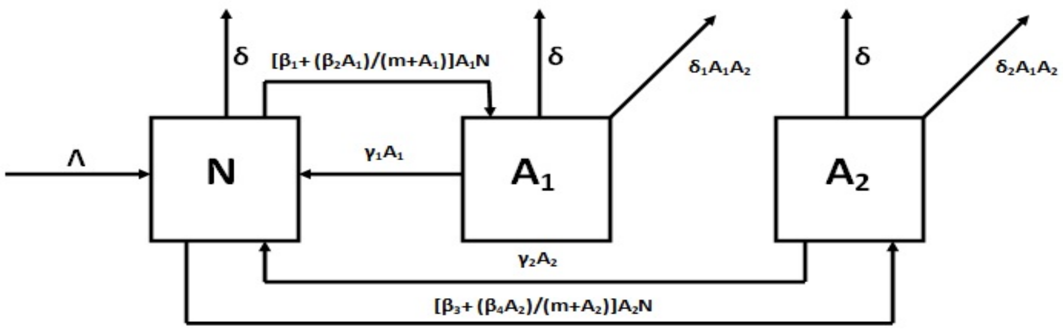

Figure 1: Schematic flow diagram of the suggested model (N, A_1, A_2).

In view of the schematic flow, the proposed mathematical model is hegemonized by a system of ordinary differential equations given below:

$$\frac{dN}{dt} = \Lambda - \left(\beta_1 + \frac{\beta_2 A_1}{m + A_1}\right) N A_1 - \left(\beta_3 + \frac{\beta_4 A_2}{m + A_2}\right) N A_2 + \gamma_1 A_1 + \gamma_2 A_2 - \delta N, \tag{2.1}$$

$$\frac{dA_1}{dt} = \left(\beta_1 + \frac{\beta_2 A_1}{m + A_1}\right) N A_1 - \gamma_1 A_1 - \delta_1 A_1 A_2 - \delta A_1, \tag{2.2}$$

$$\frac{dA_2}{dt} = \left(\beta_3 + \frac{\beta_4 A_2}{m + A_2}\right) N A_2 - \gamma_2 A_2 - \delta_2 A_1 A_2 - \delta A_2, \tag{2.3}$$

with initial conditions $N(0) \geqslant 0$, $A_1(0) \geqslant 0$, and $A_2(0) \geqslant 0$.

Positivity and boundedness of proposed model

The positivity and boundedness of the solutions to system (2.1)-(2.3) are discussed. Here, positivity means that the population will survive, whereas boundedness means that natural expansion will be constrained by the availability of resources. For boundedness and positivity, the following theorem has been stated and proved.

Theorem 2.1. *The solutions of the system* (2.1)-(2.3), *with the initial conditions, are non-negative and ultimately bounded.*

Proof. It is clear that

$$\left.\frac{dN}{dt}\right|_{N=0} = \Lambda + \gamma_1 A_1 + \gamma_2 A_2 > 0, \quad \left.\frac{dA_1}{dt}\right|_{A_1=0} = 0 \geqslant 0, \quad \text{and} \quad \left.\frac{dA_2}{dt}\right|_{A_2=0} = 0 \geqslant 0.$$

On the boundary of hyperplanes, the aforementioned rates are positive. Any point we choose that initiates in the interior of $\mathbf{R}^3_+$ will always stay in the closed plane. This suggests that all possible system solutions are non-negative. Also, it can be perceived easily from system (2.1)-(2.3) that

$$\frac{d}{dt}(N + A_1 + A_2) \leqslant \Lambda - \delta(N + A_1 + A_2),$$

and $\limsup_{t\to\infty}(N + A_1 + A_2) \leqslant \frac{\Lambda}{\delta}$. It shows the upper boundedness of the system, and the feasible region for the system is $\Omega = \{(N, A_1, A_2) : 0 \leqslant N + A_1 + A_2 \leqslant \frac{\Lambda}{\delta}, N \geqslant 0, A_1 \geqslant 0, A_2 \geqslant 0\}$. This set is known as the positively invariant region for all solutions of system (2.1)-(2.3). □

3. Basic influence number

The basic influence number is defined and calculated in this section. It is a concept of epidemiology, where it is known as the basic reproduction number. This number helps in knowing whether innovation will proliferate or not in the population. It is usually symbolized by $\mathcal{R}_0$. When $\mathcal{R}_0 < 1$, the innovation will run out of the market in the future as per the value of $\mathcal{R}_0$, and when $\mathcal{R}_0 > 1$, the innovation may expand in the market. The rate of expansion completely depends upon the value of $\mathcal{R}_0$. $E_1(\frac{\Lambda}{\delta}, 0, 0)$, the adopter-free equilibrium of the system (2.1)-(2.3) is discussed in section 4. Let $\mathcal{F}$ be the vector representing the new adopters from the direct (word-of-mouth) or indirect (media coverage) contact of adopter population with non-adopter population from system equations of adopter compartments (i.e., (2.2)-(2.3, [24]). The remaining transfer terms of adopter compartments (i.e., (2.2)-(2.3), [24]) are being represented by $\mathcal{V}$, and we have

$$\mathcal{F} = \begin{pmatrix} (\beta_1 + \frac{\beta_2 A_1}{m+A_1})NA_1 \\ (\beta_3 + \frac{\beta_4 A_2}{m+A_2})NA_2 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} \gamma_1 A_1 + \delta_1 A_1 A_2 + \delta A_1 \\ \gamma_2 A_2 + \delta_2 A_1 A_2 + \delta A_2 \end{pmatrix}.$$

And Jacobian matrices around the adopter-free equilibrium E_1 are given by

$$F = J(\mathcal{F})_{E_1} = \begin{pmatrix} \frac{\beta_1\Lambda}{\delta} & 0 \\ 0 & \frac{\beta_3\Lambda}{\delta} \end{pmatrix}, \quad V = J(\mathcal{V})_{E_1} = \begin{pmatrix} \gamma_1 + \delta & 0 \\ 0 & \gamma_2 + \delta \end{pmatrix}.$$

Here $K = FV^{-1}$ is the next generation matrix and given by

$$FV^{-1} = \begin{pmatrix} \frac{\beta_1\Lambda}{\delta(\gamma_1+\delta)} & 0 \\ 0 & \frac{\beta_3\Lambda}{\delta(\gamma_2+\delta)} \end{pmatrix}.$$

The spectral radius of matrix FV^{-1} is denoted by $\rho(FV^{-1})$. As calculated in [29], the basic influence number connected with our proposed mathematical model (2.1)-(2.3) is the spectral radius of matrix FV^{-1}, denoted by $\mathcal{R}_0$. Therefore, $\mathcal{R}_0 = \max(\mathcal{R}_{0_1}, \mathcal{R}_{0_2})$. The basic influence numbers associated with first and second innovation respectively in the absence of each other are denoted by $\mathcal{R}_{0_1}$ and $\mathcal{R}_{0_2}$ and given by $\mathcal{R}_{0_1} = \frac{\Lambda\beta_1}{\delta(\delta+\gamma_1)}$ and $\mathcal{R}_{0_2} = \frac{\Lambda\beta_3}{\delta(\delta+\gamma_2)}$.

4. The stability analysis of the model

The system (2.1)-(2.3) has the following four feasible steady states, namely

1. $E_1(\frac{\Lambda}{\delta},0,0)$: corresponds to adopter-free situation;
2. $E_2(\widetilde{N},\widetilde{A_1},0)$: corresponds to a situation in which A_2 will dissipate, for $\frac{\Lambda\beta_1}{\delta(\delta+\gamma_1)}=\mathcal{R}_{0_1}>1$, $\widetilde{N}=\frac{\Lambda}{\delta}-\widetilde{A_1}$, $\widetilde{A_1}=\frac{-B_2+\sqrt{B_2^2-4B_1B_3}}{2B_1}$, $B_1=\delta(\beta_1+\beta_2)$, $B_2=\delta(\delta+\gamma_1)(1-\mathcal{R}_{0_1})+\delta m\beta_1-\Lambda\beta_2$ and $B_3=\delta m(\delta+\gamma_1)(1-\mathcal{R}_{0_1})$;
3. $E_3(\widehat{N},0,\widehat{A_2})$: corresponds to a situation in which A_1 will dissipate, for $\frac{\Lambda\beta_3}{\delta(\delta+\gamma_2)}=\mathcal{R}_{0_2}>1$, $\widehat{N}=\frac{\Lambda}{\delta}-\widehat{A_2}$, $\widehat{A_2}=\frac{-C_2+\sqrt{C_2^2-4C_1C_3}}{2C_1}$, $C_1=\delta(\beta_3+\beta_4)$, $C_2=\delta(\delta+\gamma_2)(1-\mathcal{R}_{0_2})+\delta m\beta_3-\Lambda\beta_4$ and $C_3=\delta m(\delta+\gamma_2)(1-\mathcal{R}_{0_2})$;
4. $E_4(\overbrace{N},\overbrace{A_1},\overbrace{A_2})$: corresponds to a situation in which both A_1 and A_2 will sustain in the market.

Theorem 4.1. *The adopter-free equilibrium $E_1(\frac{\Lambda}{\delta},0,0)$ is stable for $\mathcal{R}_{0_1}<1$ and $\mathcal{R}_{0_2}<1$.*

Proof. Firstly, the general variational matrix associated with the system is calculated and given below

$$J=\begin{pmatrix} K_1 & L_1 & M_1\\ K_2 & L_2 & M_2\\ K_3 & L_3 & M_3\end{pmatrix},$$

where $K_1=-\delta-(\beta_1+\frac{\beta_2A_1}{m+A_1})A_1-(\beta_3+\frac{\beta_4A_2}{m+A_2})A_2$, $L_1=-(\beta_1+\frac{\beta_2A_1}{m+A_1})N-A_1N(\frac{m\beta_2}{(m+A_1)^2})+\gamma_1$, $M_1=-(\beta_3+\frac{\beta_4A_2}{m+A_2})N-A_2N(\frac{m\beta_4}{(m+A_2)^2})+\gamma_2$, $K_2=(\beta_1+\frac{\beta_2A_1}{m+A_1})A_1$, $L_2=(\beta_1+\frac{\beta_2A_1}{m+A_1})N+A_1N(\frac{m\beta_2}{(m+A_1)^2})-\gamma_1-\delta_1A_2-\delta$, $M_2=-A_1\delta_1$, $K_3=(\beta_3+\frac{\beta_4A_2}{m+A_2})A_2$, $L_3=-\delta_2A_2$, $M_3=(\beta_3+\frac{\beta_4A_2}{m+A_2})N+A_2N(\frac{m\beta_4}{(m+A_2)^2})-\gamma_2-\delta_2A_1-\delta$. The characteristic equation about the adopter-free equilibrium E_1 is given below

$$(\delta+\lambda)[(\delta+\gamma_1)(\mathcal{R}_{0_1}-1)-\lambda][(\delta+\gamma_2)(\mathcal{R}_{0_2}-1)-\lambda]=0. \tag{4.1}$$

The roots of the above equation, commonly known as characteristic values of matrix are $\lambda_1=-\delta$, $\lambda_2=(\delta+\gamma_1)(\mathcal{R}_{0_1}-1)$, and $\lambda_3=(\delta+\gamma_2)(\mathcal{R}_{0_2}-1)$. For $\mathcal{R}_{0_1}<1$ and $\mathcal{R}_{0_2}<1$, all the roots of (4.1) contain negative real parts. Therefore the equilibrium point E_1 is always locally asymptotically stable under the conditions $\mathcal{R}_{0_1}<1$ and $\mathcal{R}_{0_2}<1$. This situation has been graphically shown in Figure 2. □

Theorem 4.2. *For $\mathcal{R}_{0_1}>1$, there exists unique adopter-I dominating equilibrium $E_2(\widetilde{N},\widetilde{A_1},0)$ and it is conditionally stable.*

Proof. First of all, we will prove uniqueness of this equilibrium. The values of $\widetilde{A_1}$ are given by a quadratic equation $B_1\widetilde{A_1}^2+B_2\widetilde{A_1}+B_3=0$, where $B_1=\delta(\beta_1+\beta_2)$, $B_2=\delta(\delta+\gamma_1)(1-\mathcal{R}_{0_1})+\delta m\beta_1-\Lambda\beta_2$, $B_3=\delta m(\delta+\gamma_1)(1-\mathcal{R}_{0_1})$. The value of $\widetilde{N}$ can be obtained by $\widetilde{N}+\widetilde{A_1}=\frac{\Lambda}{\delta}$. It is clear that B_1 is always positive. For $\mathcal{R}_{0_1}>1$, B_3 will be negative, hence by Descartes rule of signs, the above quadratic equation will have only one positive solution irrespective of the sign of B_2 and it ends first part of the Theorem 4.2.

Secondly, about the adopter-I dominating equilibrium $E_2(\widetilde{N},\widetilde{A_1},0)$, the characteristic equation of general variational matrix J gives roots $\lambda_1=-\delta$, $\lambda_2=\widetilde{A_1}[\widetilde{N}(\frac{m\beta_2}{(m+\widetilde{A_1})^2})-(\beta_1+\frac{\beta_2\widetilde{A_1}}{m+\widetilde{A_1}})]$, $\lambda_3=(\gamma_2+\delta)(\mathcal{R}_{0_2}-1)-(\beta_3+\delta_2)\widetilde{A_1}$ and all the eigenvalues must have negative real parts if $\mathcal{R}_{0_2}<1$ and $(\frac{\Lambda}{\delta}-\widetilde{A_1})(\frac{m\beta_2}{(m+\widetilde{A_1})^2})<\beta_1+\frac{\beta_2\widetilde{A_1}}{m+\widetilde{A_1}}$. So the system will be conditionally stable under the conditions $\mathcal{R}_{0_2}<1$ and $(\frac{\Lambda}{\delta}-\widetilde{A_1})(\frac{m\beta_2}{(m+\widetilde{A_1})^2})<\beta_1+\frac{\beta_2\widetilde{A_1}}{m+\widetilde{A_1}}$. This situation has been graphically represented in Figure 3 of numerical simulation section. □

Theorem 4.3. *For $\mathcal{R}_{0_2} > 1$, there exists unique adopter-II dominating equilibrium $E_3(\widehat{N}, 0, \widehat{A_2})$ and it is conditionally stable.*

Proof. Proof of this theorem is similar as that of Theorem 4.2. This situation has been graphically represented in Figure 4 of numerical simulation section. □

Theorem 4.4. *The positive interior equilibrium $E_4(\widehat{N}, \widehat{A_1}, \widehat{A_2})$ is always unstable.*

Proof. The variational matrix around the positive interior equilibrium $E_4(\widehat{N}, \widehat{A_1}, \widehat{A_2})$ is given by

$$J=\begin{pmatrix} -\delta-\widehat{A_1}S-\widehat{A_2}T & \gamma_1-\widehat{N}S-\widehat{A_1}\widehat{N}U & \gamma_2-\widehat{N}T-\widehat{A_2}\widehat{N}V \\ \widehat{A_1}S & W & -\delta_1\widehat{A_1} \\ \widehat{A_2}T & -\delta_2\widehat{A_2} & X \end{pmatrix},$$

where $W = -\delta-\gamma_1+\widehat{N}S+\widehat{A_1}\widehat{N}U-\delta_1\widehat{A_2}$, $X = -\delta-\gamma_2+\widehat{N}T+\widehat{A_2}\widehat{N}V-\delta_2\widehat{A_1}$, $S = \beta_1+\frac{\beta_2\widehat{A_1}}{m+\widehat{A_1}}$, $T = \beta_3+\frac{\beta_4\widehat{A_2}}{m+\widehat{A_2}}$, $U = \frac{m\beta_2}{(m+\widehat{A_1})^2}$, $V = \frac{m\beta_4}{(m+\widehat{A_2})^2}$.

Let us assume that the characteristic equation about the interior equilibrium $E_4(\widehat{N}, \widehat{A_1}, \widehat{A_2})$ is

$$\lambda^3+a_1\lambda^2+a_2\lambda+a_3=0,$$

where $a_1 = \delta+\widehat{A_1}S+\widehat{A_2}T$, $a_2 = -\delta_1\delta_2\widehat{A_1}\widehat{A_2}+\widehat{A_1}S(\delta+\delta_1\widehat{A_2})+\widehat{A_2}T(\delta+\delta_2\widehat{A_1})$, $a_3 = -\widehat{A_1}\widehat{A_2}[\delta_1\delta_2(-\delta-\widehat{A_1}S-\widehat{A_2}T)+\delta_1T(-\delta-\delta_1\widehat{A_2})+\delta_2S(-\delta-\delta_2\widehat{A_1})]$. Using the conditions $\widehat{N}S+\widehat{A_1}\widehat{N}U-\delta_1\widehat{A_2}-\delta=\gamma_1$, $\widehat{N}T+\widehat{A_2}\widehat{N}V-\delta_2\widehat{A_1}-\delta=\gamma_2$. It is clear that $a_1, a_3 > 0$, but after simplification, we get $a_1a_2-a_3 < 0$ is always. The system is therefore unstable around E_4 according to the Routh-Hurwitz criterion. □

Remark 4.5. The system (2.1)-(2.3) has no stable interior equilibrium E_4. In long run, either system will be adopter-free E_1 or one of the innovations will survive, i.e., E_2 or E_3 will be stable.

5. Hopf-bifurcation analysis of E_2 and E_3

Now, we investigate the system's potential for Hopf-bifurcation, by taking "δ" (i.e., the constant fatality rate of all classes of the population) as the bifurcation parameter. The characteristic equation about the $E_2(\widetilde{N}, \widetilde{A_1}, 0)$ is

$$\lambda^3+b_1\lambda^2+b_2\lambda+b_3=0, \tag{5.1}$$

where values of b_1, b_2, and b_3 are given in Appendix A. The necessary and sufficient conditions for the existence of the Hopf-bifurcation are, if there exist $\delta = \delta^*$ such that (i) $b_j(\delta^*) > 0$, $j = 1,2,3$; (ii) $b_1(\delta^*)b_2(\delta^*)$-$b_3(\delta^*)$=0; and (iii) if we consider the eigen values of the characteristic equation (5.1) are of the form λ_j =u_j + ιv_j, then $\frac{d}{d\lambda}(u_j) \neq 0$, $j = 1,2,3$. Putting $\lambda = u+\iota v$ in (5.1) we get

$$(u+\iota v)^3+b_1(u+\iota v)^2+b_2(u+\iota v)+b_3=0. \tag{5.2}$$

On separating the real and imaginary parts of equation (5.2) and eliminating v between real and imaginary parts, we get

$$8u^3+8b_1u^2+2(b_1^2+b_2)u+b_1b_2-b_3=0. \tag{5.3}$$

It is clear from above that $u(\delta^*) = 0$ if and only if $b_1(\delta^*)b_2(\delta^*)$-$b_3(\delta^*)$=0. The existence of threshold value $\delta = \delta^*$ is ensured by positive root of (ii). Hence the discriminant of $8u^2+8b_1u+2(b_1^2+b_2) = 0$ is $64b_1^2-64(b_1^2+b_2) < 0$, which ensures that $\frac{d}{d\lambda}(b_1b_2-b_3) \neq 0$ at $\delta = \delta^*$. Again differentiating (5.3) with respect to δ, we have $(24u^2+16b_1u+2(b_1^2+b_2))\frac{du}{d\lambda}+(8u^2+4b_1u)\frac{db_1}{d\lambda}+2u\frac{db_2}{d\lambda}+\frac{d}{d\lambda}(b_1b_2-b_3) = 0$. Now since at $\delta = \delta^*$, $u(\delta^*) = 0$, we get $[\frac{du}{d\lambda}]_{\delta=\delta^*} = \frac{-\frac{d}{d\lambda}(b_1b_2-b_3)}{2(b_1^2+b_2)} \neq 0$, which satisfies transversality condition of hopf-bifurcation. The analytical value of δ is difficult to determine from (ii). Numerically, it is observed that for realistic value of δ, there is no Hopf-bifurcation for set-1, 2, 3 in Table 2. A very small change in the value of δ, switches equilibrium point E_2 into E_3 and vice versa. Hopf bifurcation analysis at equilibrium point E_3 is similar as we discussed the same for E_2.

6. Optimal control of innovation diffusion model

As we know, the presence of a high frustration rate in the market is not good for the health of businesses. Here, we try to control this frustration rate. Frustration with a product jeopardizes its existence. So in order to remain in the market, we need to reduce its frustration rate. To achieve this goal, we have to make some efforts and bear some costs. We will form a strategy in such a way that we get the most out of our goal with the least effort and cost. By using Pontryagin's maximal principle [23], we aim to identify the necessary conditions for the optimal control of the innovation diffusion model in this part. Our main objective is to reduce the frustration rate in both adopter classes. We suggest two time-dependent control variables in the range $[0, t_f]$ called $u(t)$ and $v(t)$, where t_f is the final time. Keeping this strategy in mind, we have the following innovation diffusion control model

$$\begin{cases} \dot{N} = \Lambda - \left(\beta_1 + \frac{\beta_2 A_1}{m+A_1}\right) NA_1 - \left(\beta_3 + \frac{\beta_4 A_2}{m+A_2}\right) NA_2 + (\gamma_1 - u(t))A_1 + (\gamma_2 - v(t))A_2 - \delta N, \\ \dot{A_1} = \left(\beta_1 + \frac{\beta_2 A_1}{m+A_1}\right) NA_1 - (\gamma_1 - u(t))A_1 - \delta_1 A_1 A_2 - \delta A_1, \\ \dot{A_2} = \left(\beta_3 + \frac{\beta_4 A_2}{m+A_2}\right) NA_2 - (\gamma_2 - v(t))A_2 - \delta_2 A_1 A_2 - \delta A_2. \end{cases} \tag{6.1}$$

The objective functional that this study takes into consideration is

$$J(u, v, A_1, A_2) = \int_0^{t_f} \left(p_1 A_1 + p_2 A_2 + \frac{1}{2} p_3 u^2 + \frac{1}{2} p_4 v^2\right) dt. \tag{6.2}$$

The control function $u(t)$ and $v(t)$ are bounded, Lebesgue-integrable function. Here, our aim is to minimize the frustration rate in both adopter classes. We want to get the maximum result with the minimum cost and effort.

In the above-mentioned objective functional, the quantity $p_1(p_2)$ represents the cost associated with reducing the frustration rate of compartment $A_1(A_2)$, p_3, and p_4 are positive weight parameter, and t_f is the extent of the intervention period. The main objective is to find an optimal control pair (u^*, v^*) such that

$$J(u^*, v^*) = \min_{\mathfrak{U}} J(u, v)$$

where the control set $\mathfrak{U}$ defined as follows

$$\mathfrak{U} = \{(u, v) | 0 \leqslant u(t), v(t) \leqslant 1, t \in [0, t_f]\},$$

is Lebesgue measurable. The optimal control problem is solved using Pontryagin's maximal principle, which specifies the necessary conditions that an optimal solution must meet. In its application, the initial stage is to demonstrate that the system (6.1) has an optimal control, and then we derive the optimality system. Furthermore, the Lagrangian and Hamiltonian are defined for the optimal control problem (6.1)-(6.2). The Lagrangian expression is

$$\mathbf{L} = p_1 A_1 + p_2 A_2 + \frac{1}{2} p_3 u^2 + \frac{1}{2} p_4 v^2.$$

And, we express the Hamiltonian in the following way in order to minimise this Lagrangian:

$$\begin{aligned} \mathbf{H} = p_1 A_1 + p_2 A_2 + \frac{1}{2} p_3 u^2 + \frac{1}{2} p_4 v^2 + \lambda_1 \Bigg[\Lambda - \left(\beta_1 + \frac{\beta_2 A_1}{m + A_1}\right) NA_1 - \left(\beta_3 + \frac{\beta_4 A_2}{m + A_2}\right) NA_2 \\ + (\gamma_1 - u(t))A_1 + (\gamma_2 - v(t))A_2 - \delta N\Bigg] + \lambda_2 \Bigg[\left(\beta_1 + \frac{\beta_2 A_1}{m + A_1}\right) NA_1 - (\gamma_1 - u(t))A_1 \\ - \delta_1 A_1 A_2 - \delta A_1\Bigg] + \lambda_3 \Bigg[\left(\beta_3 + \frac{\beta_4 A_2}{m + A_2}\right) NA_2 - (\gamma_2 - v(t))A_2 - \delta_2 A_1 A_2 - \delta A_2\Bigg]. \end{aligned}$$

Here λ_1, λ_2, and λ_3 are the adjoint or co-state variables. If we assume $x_1 = N$, $x_2 = A_1$, and $x_3 = A_2$, the system of (6.1) can be determined again using the following formula:

$$\dot{x_i} = \frac{\partial \mathbf{H}}{\partial \lambda_i}, \quad i = 1, 2, 3.$$

Pontryagin's maximum principle [23] transforms (6.1)-(6.2) into a Hamiltonian's point-wise minimization problem [9] with regard to u and v.

Theorem 6.1. *Given the optimal control* u^*, v^* *and state variables* N, A_1, *and* A_2 *of the corresponding state system* (6.1)-(6.2), *which minimizes* $J(u, v)$ *over* $\mathfrak{U}$, *there exist adjoint variables* λ_1, λ_2, *and* λ_3 *satisfying*

$$\frac{d\lambda_i}{dt} = -\frac{\partial \mathbf{H}}{\partial \lambda_i}, \quad i = 1, 2, 3,$$

with transversality conditions

$$\lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = 0,$$

and

$$u^* = \max\{\min\{A_1 \frac{(\lambda_1 - \lambda_2)}{p_3}, u^{\max}\}, 0\}, \quad v^* = \max\{\min\{A_2 \frac{(\lambda_1 - \lambda_3)}{p_4}, v^{\max}\}, 0\}.$$

Proof. We will show the existence of optimal control by using Corollary (4.1) of Fleming and Rishel [6]. We can infer that the model (6.1) is bounded because our state variables are non-negative and a super-solution of the system of equations (6.1) is taken into account. In addition, the control set $\mathfrak{U}$ is by definition convex and closed. As a result, the optimal system (6.1)-(6.2) is bounded, ensuring the compactness required for optimal control to exist. Thus, all conditions for the existence of controls have been met. One can follow [21] for more information in-depth.

We have the following in accordance with Pontryagin's maximum principle:

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= (\lambda_1 - \lambda_2)\left(\beta_1 + \frac{\beta_2 A_1}{m + A_1}\right)A_1 + (\lambda_1 - \lambda_3)\left(\beta_3 + \frac{\beta_4 A_2}{m + A_2}\right)A_2 + \lambda_1 \delta, \\
\frac{d\lambda_2}{dt} &= (\lambda_1 - \lambda_2)\left[\left(\beta_1 + \frac{\beta_2 A_1}{m + A_1}\right)N + \frac{m\beta_2 N A_1}{(m + A_1)^2} - (\gamma_1 - u(t))\right] + \delta_1 \lambda_2 A_2 + \delta \lambda_2 + \lambda_3 \delta_2 A_2 - p_1, \\
\frac{d\lambda_3}{dt} &= (\lambda_1 - \lambda_3)\left[\left(\beta_3 + \frac{\beta_4 A_2}{m + A_2}\right)N + \frac{m\beta_4 N A_2}{(m + A_2)^2} - (\gamma_2 - v(t))\right] + \delta_1 \lambda_2 A_1 + \delta \lambda_3 + \lambda_3 \delta_2 A_1 - p_2.
\end{aligned}$$

We take into account three situations to describe our control. First, consider the set $\{t | 0 < u^*(t) < u^{\max}\}$, in which we get

$$0 = \left.\frac{\partial \mathbf{H}}{\partial u}\right|_{u^*} = p_3 u^* - (\lambda_1 - \lambda_2) A_1.$$

From the above equation, we get

$$u^* = \frac{\lambda_1 - \lambda_2}{p_3} A_1.$$

Secondly, consider the set $\{t | u^*(t) = 0\}$, in which we have

$$0 \leqslant \left.\frac{\partial \mathbf{H}}{\partial u}\right|_{u^*} = p_3 u^* - (\lambda_1 - \lambda_2) A_1,$$

thus

$$0 \geqslant \frac{\lambda_1 - \lambda_2}{p_3} A_1.$$

And then

$$u^* = \max\left\{\frac{\lambda_1 - \lambda_2}{p_3} A_1, 0\right\}$$

holds on this set. Finally, consider the set $\{t|u^*(t) = u^{max}\}$, in which we get

$$0 \geqslant \frac{\partial \mathbf{H}}{\partial u}\bigg|_{u^*} = p_3 u^{max} - (\lambda_1 - \lambda_2)A_1,$$

which gives

$$u^{max} \leqslant \frac{\lambda_1 - \lambda_2}{p_3} A_1.$$

So

$$u^* = \min\left\{A_1 \frac{(\lambda_1 - \lambda_2)}{p_3}, u^{max}\right\}.$$

Based on these three situations, we characterize optimal control as

$$u^* = \max\left\{\min\left\{A_1 \frac{(\lambda_1 - \lambda_2)}{p_3}, u^{max}\right\}, 0\right\}.$$

Thus u^* verifies these standard control arguments involving the limits of the controls

$$u^* = \left\{\begin{array}{ll} 0, & \text{if } \xi^* \leqslant 0, \\ \xi^*, & \text{if } 0 < \xi^* < u^{max}, \\ u^{max}, & \text{if } \xi^* \geqslant u^{max}, \end{array}\right\}$$

where $\xi^* = \frac{\lambda_1 - \lambda_2}{p_3} A_1$. Additionally, $\frac{\partial^2 \mathbf{H}}{\partial u^2} > 0$ shows that the optimal control minimizes the Hamiltonian. Continuing like this, we have

$$v^* = \max\left\{\min\left\{A_1 \frac{(\lambda_1 - \lambda_3)}{p_4}, v^{max}\right\}, 0\right\}.$$

And, v^* verifies these standard control arguments involving the limits of the controls

$$v^* = \left\{\begin{array}{ll} 0, & \text{if } \eta^* \leqslant 0, \\ \eta^*, & \text{if } 0 < \eta^* < v^{max}, \\ v^{max}, & \text{if } \eta^* \geqslant v^{max}, \end{array}\right\}$$

where $\eta^* = \frac{\lambda_1 - \lambda_3}{p_4} A_2$. Moreover, $\frac{\partial^2 \mathbf{H}}{\partial v^2} > 0$ shows that the optimal control minimizes the Hamiltonian. □

We present and discuss the outcomes of the numerical simulation of the control measure (strategy) for the extended model (6.1) and objective functional (6.2) as a way of concluding this section. The state system is solved with some initial guesses using the forward fourth-order Runge-Kutta method, while the adjoint system is solved using the backward fourth-order Runge-Kutta methodology. We take a set of parametric values defined in Set-8 with the initial conditions $N(0) = 0$, $A_1(0) = 0.4$, and $A_2(0) = 0.85$. We draw two figures that correspond to Set-8. One is without control, and the other is with a control strategy. From Figures 9 and 10, it is clear that the level of the non-adopter population increases after the implementation of the control strategy. It is good because this non-adopter population is a potential buyer of innovations. Also, after a decrease in frustration rate, we observe that the face of the curve that corresponds to users of the second innovation, which was approaching zero, visible in Figure 9, transforms, and it starts increasing slowly, visible in Figure 10. We are achieving our target with the minimum cost, which is also shown in Figure 10. Thus, the coexistence of products in a competitive market is possible by adopting the above-mentioned strategy, which reduces the frustration rate.

7. Sensitivity analysis

In this section, sensitivity analysis has been performed for basic reproduction numbers $\mathcal{R}_{0_1}$ and $\mathcal{R}_{0_2}$ associated with first and second innovations, respectively. Sensitivity analysis describes the role of each parameter in the innovation diffusion dynamics. It is used to detect degree of impactness of parameters, i.e., high, moderate, or low. The basic influence number $\mathcal{R}_{0_1}(=\frac{\Lambda\beta_1}{\delta(\delta+\gamma_1)})$ is a function of four parameters.

The normalized sensitivity indices of basic influence numbers $\mathcal{R}_{0_1}(\mathcal{R}_{0_2})$ w.r.t. parameters are presented in Table 1. From Table 1, it is observed that Λ and β_1 have positive impact on $\mathcal{R}_{0_1}$, while γ_1 and δ have negative impact. It is also shown that Λ and β_3 have positive impact on $\mathcal{R}_{0_2}$, while γ_2 and δ have negative impact. Hence, significant changes in $\mathcal{R}_{0_1}(\mathcal{R}_{0_2})$ can be observed by small changes in these parameters.

Table 1: The normalized sensitivity indices for the basic influence numbers $\mathcal{R}_{0_1}(\mathcal{R}_{0_2})$.

Parameter(y_j)	Parametric Value	Sensitivity index of $\mathcal{R}_{0_1}$ ($\mathcal{R}_{0_2}$) w.r.t. y_j
Λ	0.30	1
$\beta_1(\beta_3)$	0.20	1
$\gamma_1(\gamma_2)$	0.10	-0.33
δ	0.20	-1.66

8. Numerical simulations

For ensuring the analytic conclusions, numerical simulations have been carried out with the help of MATLAB's built-in ode45, and bvp4c. In Table 2, sets of different parametric values are taken. In Table 3, the values of basic influence numbers and equilibrium points are calculated for different sets of parametric values. The Figures (2)-(9) show population distribution w.r.t. time for various cases. The simulation result of the model with optimal control is shown in Figure 10.

Table 2: Assumed different sets of parametric values.

Set No.	Λ	β_1	β_2	β_3	β_4	γ_1	γ_2	m	δ	δ_1	δ_2
Set-1	0.30	0.15	0.15	0.20	0.20	0.20	0.25	0.90	0.50	0.10	0.20
Set-2	0.30	0.30	0.30	0.25	0.25	0.10	0.20	0.50	0.20	0.10	0.20
Set-3	0.30	0.25	0.25	0.30	0.30	0.20	0.10	0.50	0.20	0.10	0.20
Set-4	0.25	0.25	0.30	0.22	0.25	0.30	0.35	0.50	0.10	0.10	0.20
Set-5	0.25	0.24	0.30	0.28	0.25	0.35	0.29	0.50	0.10	0.10	0.20
Set-6	0.30	0.28	0.30	0.28	0.25	0.30	0.35	0.50	0.152	0.10	0.20
Set-7	0.30	0.28	0.30	0.28	0.25	0.30	0.35	0.50	0.151	0.10	0.20
Set-8	0.38	0.28	0.30	0.28	0.25	0.30	0.35	0.50	0.125	0.10	0.20

Table 3: Equilibrium points $\mathcal{R}_{0_1}$ and $\mathcal{R}_{0_2}$ w.r.t. different sets of parametric values.

Parameters	$\mathcal{R}_{0_1}$	$\mathcal{R}_{0_2}$	Stable equilibrium point	Figure
Set-1	0.1286	0.16	E_1=(0.60,0,0)	2
Set-2	1.5	0.9375	E_2=(0.61,0.89,0)	3
Set-3	0.9375	1.5	E_3=(0.61,0,0.89)	4
Set-4	1.5625	1.2223	E_2=(0.8322,1.6678,0)	5
Set-5	1.3334	1.79487	E_3=(0.8256,0,1.6744)	6
Set-6	1.22264	1.10086	E_2=(0.937,1.0366,0)	7
Set-7	1.23346	1.11036	E_3=(1.1478,0,0.8390)	8
Set-8	2.00282	1.792	E_2=(0.8142,2.2002,0)	9

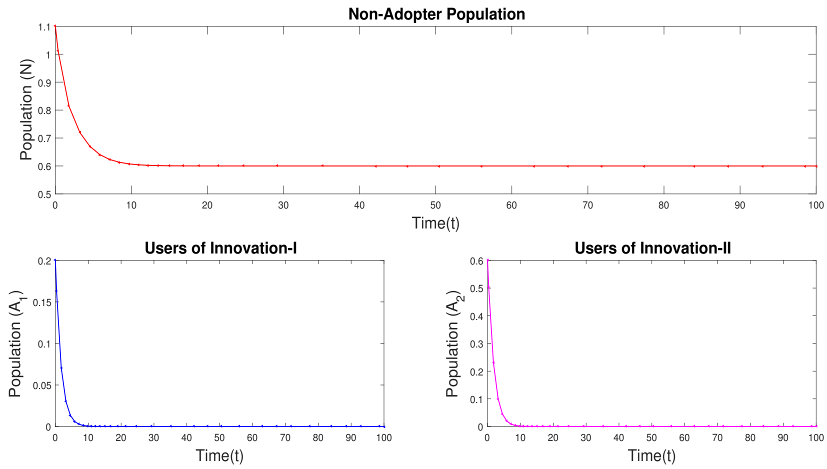

Figure 2: Corresponds to adopter-free equilibrium with $\mathcal{R}_{0_1} = 0.1286$ and $\mathcal{R}_{0_2} = 0.16$.

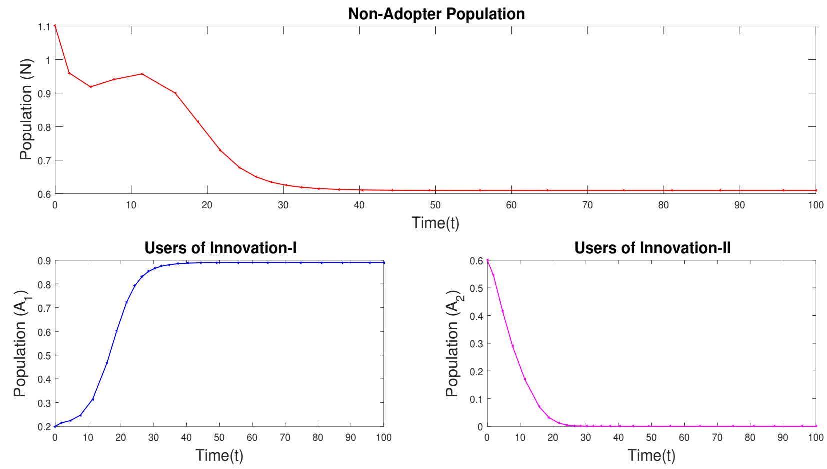

Figure 3: Corresponds to adopter-I dominating equilibrium with $\mathcal{R}_{0_1} = 1.5$ and $\mathcal{R}_{0_2} = 0.9375$.

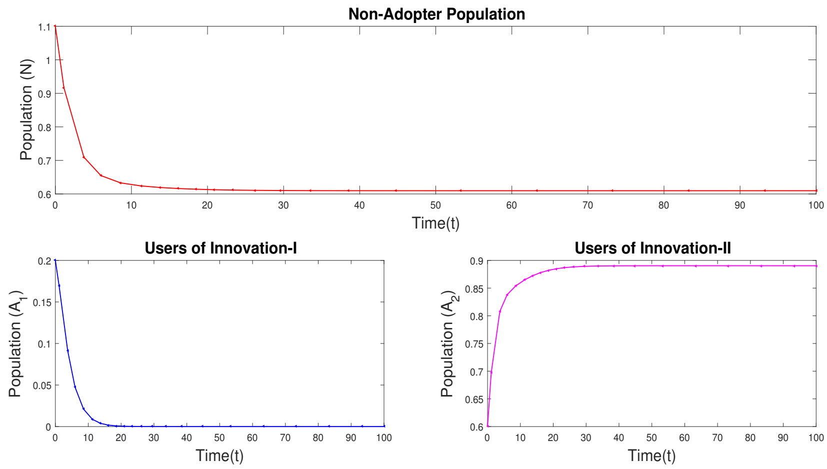

Figure 4: Corresponds to adopter-II dominating equilibrium with $\mathcal{R}_{0_1} = 0.9375$ and $\mathcal{R}_{0_2} = 1.5$.

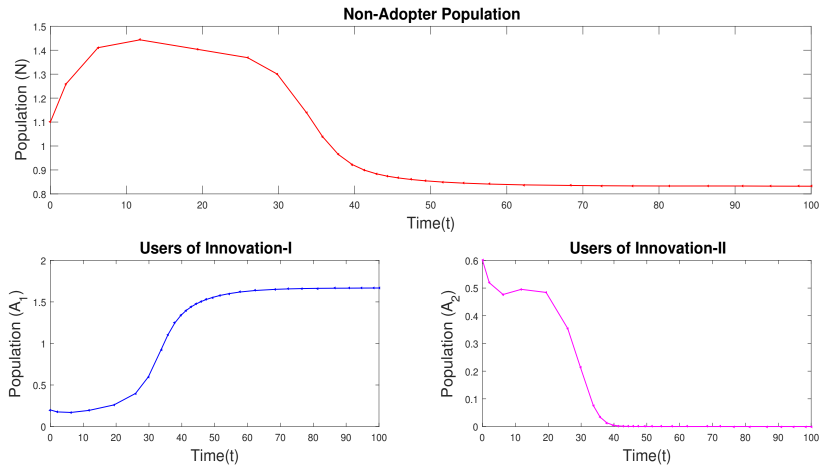

Figure 5: Corresponds to adopter-I dominating equilibrium in spite of $\mathcal{R}_{0_1} > \mathcal{R}_{0_2} > 1$.

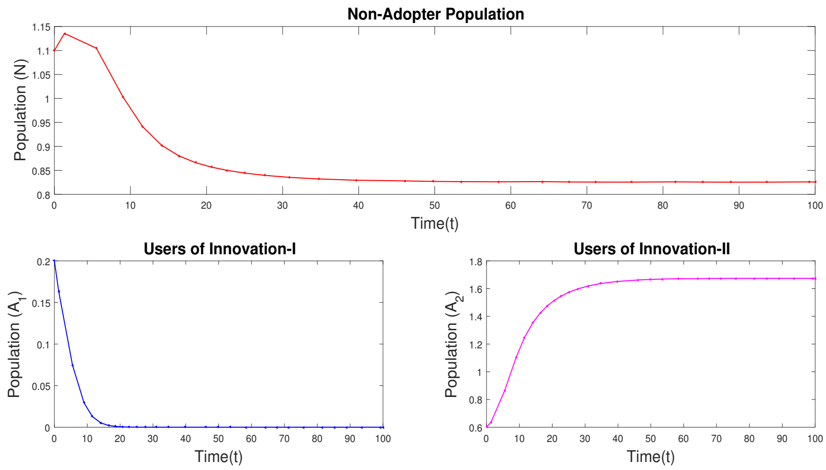

Figure 6: Corresponds to adopter-II dominating equilibrium in spite of $\mathcal{R}_{0_2} > \mathcal{R}_{0_1} > 1$.

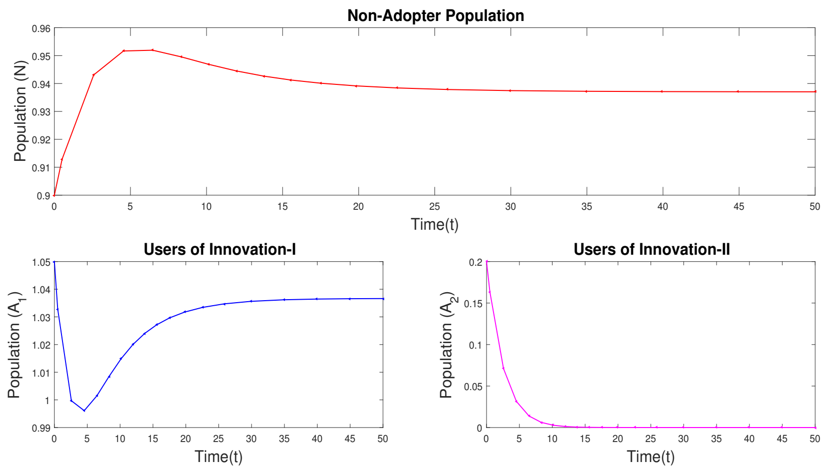

Figure 7: The system is stable with equilibrium point E_2=(0.937,1.0366,0) for the parametric values defined in Set-6 with δ=0.152.

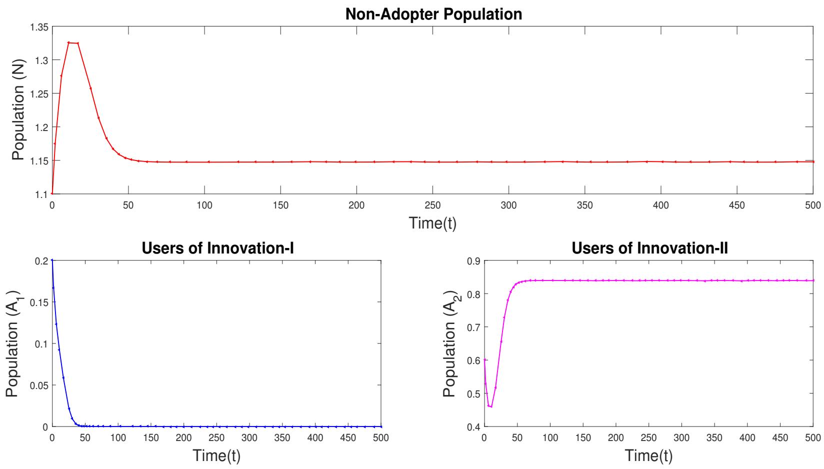

Figure 8: The system is stable with equilibrium point E_3=(1.1478,0,0.8390) for the parametric values defined in Set-7 with δ=0.151.

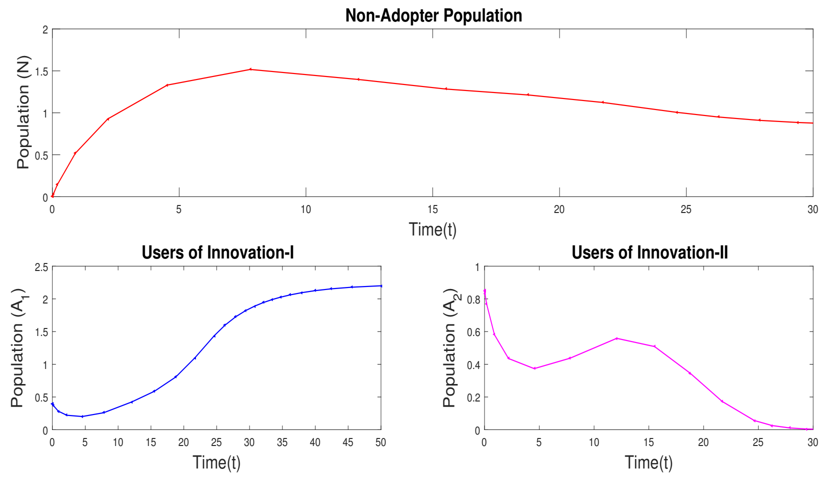

Figure 9: Distribution of non-adopter, adopter of Innovation-I, adopter of innovation-II populaion without control.

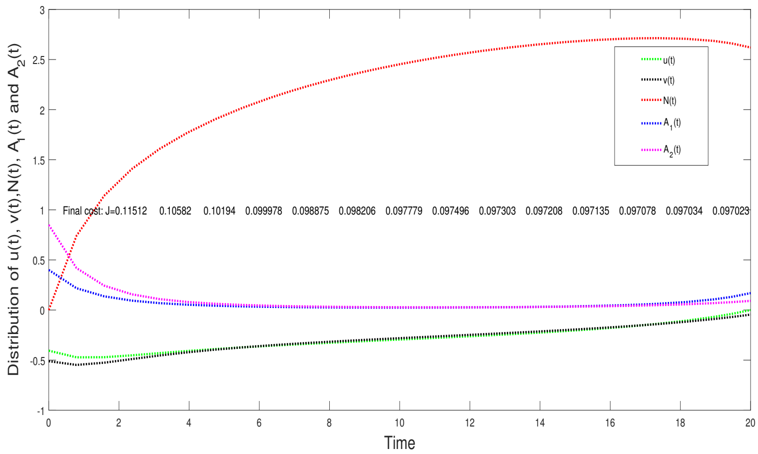

Figure 10: Simulation result of model with optimal control and optimal control curves.

9. Conclusions and future scope

The present model concludes that the sustainability of a product in the market depends mainly upon positive advertisement, media coverage, personal interaction, immigration, and emigration. The present model discusses the impact of competitiveness under the simultaneous effects of media coverage and personal interaction. From the qualitative study of the model, we obtained the following results and observations.

[i] It is observed that the system has four steady states, and the local stability of each equilibrium has been studied. The adopter-free equilibrium E_1 is stable, in which both products vanished from the market under the conditions mentioned in Theorem 4.1. The conditions for the stability of the other two boundary equilibria E_2 and E_3 are discussed in Theorems 4.2 and 4.3. Since the interior equilibrium E_4 is always unstable, as discussed in Theorem 4.4, the co-existence of the users of both products in the market is not always possible.

[ii] The use of an optimal control strategy makes coexistence possible. Following the widespread use of the control strategy, the non-adopter population level rises. It is advantageous since this non-adopting group has the capacity to purchase innovations. Additionally, we see that the face of the curve corresponding to users of the second invention, which was nearing zero, seen in Figure 9, alters and starts expanding gradually, as apparent in Figure 10. Thus, the coexistence of products in a competitive market is possible by adopting the optimal control strategy.

[iii] Sensitivity analysis shows the impact of parameters on basic influence numbers. Some parameters impact the basic influence numbers positively, while others do so negatively.

[iv] Numerical simulations executed with different sets of parameters assist in analytical findings. It is observed that if (a) $\mathcal{R}_{0_1} < 1$, $\mathcal{R}_{0_2} < 1$, then the adopter-free equilibrium E_1 is stable (see Figure 2); (b) $\mathcal{R}_{0_1} > 1$, $\mathcal{R}_{0_2} < 1$, then only the first adopter will survive, and E_2 is the stable equilibrium point (see Figure 3); (c) $\mathcal{R}_{0_1} < 1$, $\mathcal{R}_{0_2} > 1$, then only the second adopter will survive, and E_3 is the stable equilibrium point (see Figure 4); (d) $\mathcal{R}_{0_1} > \mathcal{R}_{0_2} > 1$, then E_2 may be stable and $\mathcal{R}_{0_2} > \mathcal{R}_{0_1} > 1$, then E_3 may be stable. It is shown in Figures 5, 6. A small change in Hopf bifurcation parameter "δ" switches the equilibrium point (see Figures 7, 8).

In the future, one can develop and analyze the said model with delays in adoption. Also, we can classify the non-adopter population on the basis of their income.

Acknowledgment

We express our heartfelt thanks to ABV-IIITM, SLIET, and the Principal of D.M. College Moga for providing us with the facilities required for research as well as a favorable environment.

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Appendix A

Here, we will give the values of b_1, b_2, and b_3 associated with equation (5.1), $b_1{=}-(\lambda_1+\lambda_2+\lambda_3)$, $b_2{=}\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1$, and $b_3{=}-\lambda_1\lambda_2\lambda_3$. And $\lambda_1{=}-\delta$, $\lambda_2{=}\widetilde{N}\beta_3-\delta-\gamma_2-\widetilde{A_1}\delta_2$,

$$\lambda_3 = -\frac{\widetilde{A_1}^3(\beta_1+\beta_2)+\widetilde{A_1}^2\left(\gamma_1+\delta+\beta_1(2m-\widetilde{N})+\beta_2(m-\widetilde{N})\right)+\widetilde{A_1}m\left(\beta_1(m-2\widetilde{N})+2\left(\gamma_1+\delta-\beta_2\widetilde{N}\right)\right)+m^2\left(\gamma_1+\delta-\beta_1\widetilde{N}\right)}{m^2+\widetilde{A_1}^2+2m\widetilde{A_1}}.$$