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Discrete version of fundamental theorems of fractional order integration for nabla operator

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Abstract

The goal of this paper is to develop and present a precise theory for integer and fractional order ℓ-nabla integration and its fundamental theorems. In our research work, we take two forms of higher order difference equation such as closed form and summation form. But most of the authors are focused only on the summation part only. Instead of finding the solution for the summation part, finding the solution for the closed gives the exact solution. To find the closed form solution for the integer order using the ℓ-nabla operator, we used the factorial-coefficient method. For developing the theory of fractional order ℓ-nabla operator and its integration, we introduce a function called N_ν-type function. If the summation series is huge, this approach can help us to find the solution quickly. Suitable examples are provided for verification. Finally, we provide the application for detecting viral transmission using the nabla operator.

Keywords: Mathematical operators, problem solving, nonlinear systems, iterative methods, discrete-nabla fractional calculus, $\mathcal{N}_\nu$-type function.

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1. Introduction

Fractional calculus has a vital influence in applied physics like quantum mechanics, thermodynamics and control theory in the past few decades. Nowadays, the application of fractional calculus has been

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extended for identifying the spread of disease caused by virus, COVID19 etc, in medical field also see [10, 20, 23, 25]. The fractional calculus was formulated shortly in 1695 after the development of classical calculus. In the olden days, the theory of fractional calculus was used for pure mathematics, but recently the concept of fractional calculus has been changed. In 2011, Machado [11] introduced that the fractional calculus concept is suitable for applied mathematics especially for applications in population growth, transforms, and medical field sciences. Here, the derivatives of discrete fractional calculus are based on forward difference operator and we use the notation $\mathbb{N}(1) = \{1, 2, 3, \ldots\}$. For more details on the applications of discrete fractional calculus, one can refer to [5–9] and for more details on other related fields refer [1–19]. For Example, in [1] (see equation 2.1), the ν^{th} order fractional sum of given function f based at a is defined as

$$\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} \frac{\Gamma(t-s)}{\Gamma(t-s+1-\nu)} f(s), \tag{1.1}$$

where $\nu > 0$ and a real valued function f is defined for $s \in a + \mathbb{N}(1)$ and $\Delta^{-\nu} f$ is defined for $t = a + \nu + \mathbb{N}(1)$. The authors in [12–15] have developed several theorems based on equation (1.1) and also the generalized ℓ-delta operator denoted as Δ_ℓ.

The importance of this work arises from the partial development of various fractional order sums. In the field of discrete fractional calculus, the ν^{th} fractional sum of f is defined in equation (1.1). The solution of $\Delta_a^{-\nu}$ is possible for any given function f using the equation (1.1). Most of the authors are focusing only on the summation part of the equation (1.1). Instead of finding the solution for the summation part, finding the solution for the closed form gives exact solution for equation (1.1) in quick manner. This motivate us to develop some fundamental theorems in discrete fractional calculus in the case of backward difference operator, say ℓ-nabla operator, which is denoted as Δ_ℓ. For finding these closed form solutions, we have used a new approach, called factorial-coefficient method for integer order and N_ν-type function for fractional order. This method will be used to reduce the running time of the computer when evaluating the summation part of equation (1.1), if the distance between a and t is very large. Through this research, we have introduced the fundamental theorem of discrete ℓ-nabla fractional calculus related to ℓ-nabla operator. To derive these type of fundamental identities for any $\nu > 0$, we need to analyze the concept of $\overset{-\nu}{\nabla}$ and $\underset{-1}{\overset{-1}{\nabla}} f(t)$. Hence, we present the basic theory of $\underset{\ell}{\nabla}$ and its inverse operator $\underset{\ell}{\overset{-1}{\nabla}}$. The theory developed in this paper is based on the concept available in [5] and also some concepts used by Atici and Eloe [1–3] and Holm [16].

This paper is organized as follows. Section 1 is Introduction. In Section 2, we present the basic concepts of ℓ-nabla operator. In Section 3, we develop several fundamental theorems for integer order using the ℓ-nabla operator and in Section 4, we developed the fractional order theorems by utilizing the N_ν-type function. Section 5 deals with the Application of nabla operator. Finally, Section 6 devotes to conclusion.

2. Preliminaries of ℓ-nabla operator

Here, we give some basic definitions of ℓ-nabla and its inverse operators. Also, we have derived first order anti-difference principle to $\underset{\ell}{\nabla}$ and also provide some properties, lemmas, and theorems related to nabla operator that will be used in the subsequent theorems. For $\ell > 0$, let J_ℓ denotes a set of real numbers satisfying the condition that $t \in J_\ell$ if and only if $t \pm \ell \in J_\ell$. Throughout this paper, we use the notation $J_\ell = \{t, t \pm \ell, t \pm 2\ell, t \pm 3\ell, \ldots\} \subseteq \mathbb{R}$.

Definition 2.1 ([13]). Assume that $f, g : J_\ell \to \mathbb{R}$ are two functions. The ℓ-nabla and its anti-difference nabla operators on f and g are respectively defined as

$$\underset{\ell}{\nabla} f(t) = f(t) - f(t-\ell), \; t \in J_\ell, \tag{2.1}$$

and

$$\underset{-\ell}{\nabla} f(t) = f(t) - f(t+\ell),\ t \in J_\ell.$$

and if $\underset{\ell}{\nabla} g(t) = f(t)$ for $t \in J_\ell$, then we denote $g(t) = \overset{-1}{\underset{\ell}{\nabla}} f(t) + c$, c is constant, and

$$\overset{-1}{\underset{\ell}{\nabla}} f(s)\Big|_{s=t-n\ell}^{s=t} = \overset{-1}{\underset{\ell}{\nabla}} f(t-n\ell) - \overset{-1}{\underset{\ell}{\nabla}} f(t) = g(t-n\ell) - g(t),\ t \in J_\ell,\ m \in \mathbb{N}. \tag{2.2}$$

Note that $\underset{\ell}{\nabla}$ is a linear operator.

Definition 2.2 ([12])**.** Assuming $n \in N$ and $t, \ell \in R$, the ℓ-falling and raising factorial polynomials $t_\ell^{(n)}$ and $t_{-\ell}^{(n)}$ are respectively defined by

$$t_\ell^{(n)} = \prod_{r=0}^{n-1}(t - r\ell) \quad \& \quad t_{-\ell}^{(n)} = \prod_{r=0}^{n-1}(t + r\ell). \tag{2.3}$$

Using the ℓ-falling factorial polynomial for any positive integer n, equation (2.3) can be expressed as

$$t_\ell^{(n)} = \prod_{r=0}^{n-1}\left(\frac{t}{\ell} - r\right)\ell^n = \left(\frac{t}{\ell}\right)_1^{(n)}\ell^n.$$

Applying the property of Gamma function in $\left(\frac{t}{\ell}+1\right)$ and $\left(\frac{t}{\ell}+n\right)$, we get

$$t_\ell^{(n)} = \frac{\Gamma\left(\frac{t}{\ell}+1\right)}{\Gamma\left(\frac{t}{\ell}-n+1\right)}\ell^n$$

and

$$t_{-\ell}^{(n)} = \frac{\Gamma\left(\frac{t}{\ell}+n\right)}{\Gamma\left(\frac{t}{\ell}\right)}\ell^n,$$

$\frac{t}{\ell}+1$ & $\frac{t}{\ell}-n+1 \notin -\mathbb{N}(0)$ and $\frac{t}{\ell}$ & $\frac{t}{\ell}+n \notin -\mathbb{N}(0)$.

Definition 2.3 (Generalized factorial polynomial)**.** Let $t \in (-\infty, \infty)$ and $\ell, \nu > 0$. Then the ℓ-falling and raising factorial polynomials $t_\ell^{(\nu)}$ and $t_{-\ell}^{(\nu)}$ are respectively defined as

$$t_\ell^{(\nu)} = \frac{\Gamma\left(\frac{t}{\ell}+1\right)}{\Gamma\left(\frac{t}{\ell}-\nu+1\right)}\ell^\nu, \quad \frac{t}{\ell}+1, \quad \text{and} \quad \frac{t}{\ell}-\nu+1 \notin \{0,-1,-2,-3,\ldots\}, \tag{2.4}$$

and

$$t_{-\ell}^{(\nu)} = \frac{\Gamma\left(\frac{t}{\ell}+\nu\right)}{\Gamma\left(\frac{t}{\ell}\right)}\ell^\nu, \quad \frac{t}{\ell}, \quad \text{and} \quad \frac{t}{\ell}+\nu \notin \{0,-1,-2,-3,\ldots\}.$$

Theorem 2.4. *For $t, r \in \mathbb{R}$ and $m \in \mathbb{N}$, then the ℓ-nabla power rule is obtained by*

$$\underset{\ell}{\nabla}(t+r)_\ell^{(m)} = m\ell(t+r-\ell)_\ell^{(m-1)} \quad \textit{(falling factorial)} \tag{2.5}$$

and

$$\underset{\ell}{\nabla}(t+r)_{-\ell}^{(m)} = m\ell(t+r)_{-\ell}^{(m-1)} \quad \textit{(raising factorial).} \tag{2.6}$$

Proof. Using the equations (2.1) and (2.3), we get

$$\underset{\ell}{\nabla}(t+r)_{\ell}^{(m)} = [(t+r)-(t+r-m\ell)]\Big(\prod_{j=1}^{m-1}(t+r-j\ell)\Big),$$

which yields (2.5). Similarly, by applying the equations (2.1) and (2.3) in $\underset{\ell}{\nabla}(t+r)_{-\ell}^{(m)}$, we get (2.6). □

The following theorem is a *first order anti-difference principle* related to $\underset{\ell}{\nabla}$.

Theorem 2.5. *Let* $n \in \mathbb{N}$ *and* f *and* g *be as defined in the Definition 2.1. Then,*

$$\underset{\ell}{\overset{-1}{\nabla}} f(\tau)\Big|_{t-n\ell}^{t} = g(t) - g(t-n\ell) = \sum_{r=0}^{n-1} f(t-r\ell). \tag{2.7}$$

Proof. Assuming $\underset{\ell}{\nabla} g(t) = f(t)$, and then applying $\underset{\ell}{\nabla}$ operator on $g(t)$ gives

$$g(t) = f(t) + g(t-\ell). \tag{2.8}$$

For $n = 1, 2, \ldots, m-1$ and substituting t by $t-n\ell$ in (2.8), we obtain $g(t-n\ell)$ and again applying $g(t-n\ell)$ in (2.8) repeatedly, we derive

$$g(t) - g(t-n\ell) = \sum_{r=0}^{n-1} f(t-r\ell).$$

Now the proof of (2.7) follows by equation (2.2). □

Corollary 2.6. *Assume that* $\underset{\ell}{\overset{-1}{\nabla}} f(s)\Big|_{k=t}$ *is denoted as* $\underset{\ell}{\overset{-1}{\nabla}} f(t)$. *If* $\lim_{t\to-\infty} \underset{\ell}{\overset{-1}{\nabla}} f(t) = 0$ *and* $\sum_{p=0}^{\infty} f(t-p\ell)$ *is convergent, then*

$$\underset{\ell}{\overset{-1}{\nabla}} f(t) = \sum_{p=0}^{\infty} f(t-p\ell). \tag{2.9}$$

Proof. By taking $n \to \infty$ in (2.7), we get (2.9). □

Corollary 2.7. *Consider the conditions given in Corollary 2.6. Then,*

$$\underset{-1}{\overset{-1}{\nabla}} f(t\ell)\Big|_{t=0} = \sum_{p=0}^{\infty} f(p\ell).$$

Proof. By applying the operator $\underset{-1}{\overset{-1}{\nabla}}$ to the function $f(t)$, we get

$$\underset{-1}{\overset{-1}{\nabla}} f(t) = \sum_{p=0}^{\infty} f(t+p). \tag{2.10}$$

Replacing t by $t\ell$ in equation (2.10), we obtain

$$\underset{-1}{\overset{-1}{\nabla}} f(t\ell) = f(t\ell) + f((t+1)\ell) + f((t+2)\ell) + f((t+3)\ell) + f((t+4)\ell) + \cdots. \tag{2.11}$$

Taking $t = 0$ in (2.11), we get the expression as

$$\underset{-1}{\overset{-1}{\nabla}} f(0) = \sum_{p=0}^{\infty} f(p\ell). \tag{2.12}$$

Then the proof completes by taking $\underset{-1}{\overset{-1}{\nabla}} f(0)$ as $\underset{-1}{\overset{-1}{\nabla}} f(t\ell)\Big|_{t=0}$ in (2.12). □

Corollary 2.8. *Consider the conditions given in Corollary 2.6. Then,*

$$\overset{-1}{\underset{-1}{\nabla}} f(s-t\ell)\Big|_{t=0} = \sum_{p=0}^{\infty} f(s-p\ell). \tag{2.13}$$

Here t *is variable and* s *is constant.*

Proof. By replacing $t\ell$ by $s-t\ell$ in corollary 2.7, we get the proof of (2.13). □

Remark 2.9. Consider the conditions given in Corollary 2.8 and if the limit $t=0$ does not exists, then we find

$$\overset{-1}{\underset{-1}{\nabla}} f(s-k\ell)\Big|_{k=t} = \sum_{p=t}^{\infty} f(s-p\ell), \tag{2.14}$$

where t is variable. Here $\overset{-1}{\underset{-1}{\nabla}} f(s-k\ell)\Big|_{k=t}$ can be denoted as $\overset{-1}{\underset{-1}{\nabla}} f(s-t\ell)$.

Lemma 2.10 (Product rule)**.** *Assume that* $f, g : J_\ell \to \mathbb{R}$ *and* $\ell \neq 0$. *Then,*

$$\overset{-1}{\underset{\ell}{\nabla}}\{f(t)h(t)\} = f(t)\overset{-1}{\underset{\ell}{\nabla}} h(t) - \overset{-1}{\underset{\ell}{\nabla}}\{\overset{-1}{\underset{\ell}{\nabla}} h(t-\ell)\underset{\ell}{\nabla} f(t)\}. \tag{2.15}$$

Proof. By applying the operator $\underset{\ell}{\nabla}$ to the function $f(t)h(t)$ and then adding and subtracting $f(t)h(t-\ell)$ yield

$$\underset{\ell}{\nabla}\{f(t)h(t)\} = f(t)\underset{\ell}{\nabla} h(t) + h(t-\ell)\underset{\ell}{\nabla} f(t). \tag{2.16}$$

By taking $g(t) = \overset{-1}{\underset{\ell}{\nabla}} h(t)$ and $\underset{\ell}{\nabla} g(t) = h(t)$ in (2.16), we get

$$\underset{\ell}{\nabla}\{f(t)\overset{-1}{\underset{\ell}{\nabla}} h(t)\} = f(t)h(t) + \overset{-1}{\underset{\ell}{\nabla}} h(t-\ell)\underset{\ell}{\nabla} f(t).$$

Hence (2.15) follows by applying $\overset{-1}{\underset{\ell}{\nabla}}$ on both sides of above expression. □

The following Corollary 2.11 will be used in the subsequent theorems and examples.

Corollary 2.11. *For any two real valued functions* f *and* g, *we have*

$$\overset{-1}{\underset{-1}{\nabla}}\{f(t)h(t)\} = f(t)\overset{-1}{\underset{-1}{\nabla}} h(t) - \overset{-1}{\underset{-1}{\nabla}}\{\overset{-1}{\underset{-1}{\nabla}} h(t+1)\underset{-1}{\nabla} f(t)\}.$$

Proof. The proof follows by taking $\ell = -1$ in (2.15). □

Lemma 2.12. *If we denote the falling factorial polynomial* $t_1^{(n)}$ *as* $t^{(n)}$. *The sum of* r^{th} *factorial polynomials of first* t *natural number is*

$$1^{(r)} + 2^{(r)} + 3^{(r)} + \cdots + t^{(r)} = \frac{(t+1)^{(r+1)}}{(r+1)}. \tag{2.17}$$

Proof. Using the $\underset{\ell}{\Delta}$ and its inverse operator $\overset{-1}{\underset{\ell}{\Delta}}$, we get (2.17) (refer to [12]). □

3. Fundamental theorems in discrete nabla calculus

Here, we establish higher order $(m \in \mathbb{N})$ anti-difference principle related to $\underset{\ell}{\nabla}$ from which, we develop fundamental theorems in discrete calculus. Suitable examples are also provided in this section for verification.

Definition 3.1. Let $s, \ell, t \in \mathbb{R}$, $m \in \mathbb{N}$ such that $s - t\ell \in J_\ell$ and $f : J_\ell \to \mathbb{R}$ be a function. Then the factorial-coefficient of f at t on (m, s, ℓ) is defined as

$$f_{m,s,\ell}(t) = \frac{(t+m-1)^{(m-1)} f(s-t\ell)}{(m-1)!}. \tag{3.1}$$

The following theorem is the Higher order anti-difference principle related to $\underset{\ell}{\nabla}$.

Theorem 3.2. *Let* $f, g : J_\ell \to \mathbb{R}$, $\sum\limits_{r=0}^{\infty} f(s-r\ell)$ *is convergent,* $f_{m,s,\ell}(t)$ *is as given in* (3.1) *and* $\underset{-1}{\nabla} f_{m,s,\ell}(t) = f_{m,s,\ell}(t) - f_{m,s,\ell}(t+1)$. *If* $\overset{m}{\underset{\ell}{\nabla}} g(t) = f(t)$, $\lim\limits_{t\to-\infty} g(t) = 0$ *for* $\ell > 0$ ($\lim\limits_{t\to\infty} g(t) = 0$ *for* $\ell < 0$), $\overset{-m}{\underset{\ell}{\nabla}} f(k)\Big|_{k=s} = \overset{-m}{\underset{\ell}{\nabla}} f(s)$ *and* $\overset{-1}{\underset{-1}{\nabla}} f_{m,s,\ell}(t)$ *exists, then we denote* $\overset{-m}{\underset{\ell}{\nabla}} f(t) = g(t)$ *and*

$$\overset{-m}{\underset{\ell}{\nabla}} f(s) - \overset{-1}{\underset{-1}{\nabla}} f_{m,s,\ell}(t) = \sum_{r=0}^{t-1} f_{m,s,\ell}(r). \tag{3.2}$$

Proof.
Case (i). Let $\ell > 0$ and $\lim\limits_{t\to-\infty} g(t) = 0$. The series (2.9) can be expressed as

$$\overset{-1}{\underset{\ell}{\nabla}} f(s) = \sum_{r=0}^{t-1} f(s-r\ell) + \sum_{r=t}^{\infty} f(s-r\ell). \tag{3.3}$$

Applying the equation (2.14) to the last term of (3.3), we obtain

$$\overset{-1}{\underset{\ell}{\nabla}} f(s) - \overset{-1}{\underset{-1}{\nabla}} f(s-t\ell) = \sum_{r=0}^{t-1} f(s-r\ell).$$

Taking $\overset{-1}{\underset{\ell}{\nabla}}$ on both sides of (3.3), we get

$$\overset{-2}{\underset{\ell}{\nabla}} f(s) = \sum_{r=0}^{\infty} \overset{-1}{\underset{\ell}{\nabla}} f(s-r\ell). \tag{3.4}$$

Since $\overset{-1}{\underset{\ell}{\nabla}}$ is linear, by our assumption $\lim\limits_{t\to-\infty} g(t) = 0$, (3.4) is valid. Now, replacing t by s and then replacing s by $s - j\ell$ in the equation (2.9), we get

$$\overset{-1}{\underset{\ell}{\nabla}} f(s-j\ell) = \sum_{r=0}^{\infty} f(s-j\ell-r\ell), \; j = 0, 1, 2, 3, \ldots. \tag{3.5}$$

Substituting (3.5) in (3.4), we obtain

$$\overset{-2}{\underset{\ell}{\nabla}} f(s) = \sum_{j=0}^{\infty} \Big(\sum_{r=0}^{\infty} f(s-j\ell-r\ell) \Big),$$

which is same as,

$$\overset{-2}{\underset{\ell}{\nabla}} f(s) = \sum_{r=0}^{t-1}(r+1)f(s-r\ell) + \sum_{r=t}^{\infty}(r+1)f(s-r\ell). \tag{3.6}$$

By applying the equation (2.14) for the function $(t+1)f(s-t\ell)$ to the infinite series term of (3.6), we get

$$\overset{-2}{\underset{\ell}{\nabla}} f(s) - \overset{-1}{\underset{-1}{\nabla}}\{(t+1)f(s-t\ell)\} = \sum_{r=0}^{t-1}(r+1)f(s-r\ell).$$

Again, taking $\overset{-1}{\underset{\ell}{\nabla}}$ on both sides of the equation (3.4), and then proceeding the similar steps upto (3.5), we get

$$\overset{-3}{\underset{\ell}{\nabla}} f(s) = f(s) + (1+2)f(s-\ell) + (1+2+3)f(s-2\ell) + (1+2+3+4)f(s-3\ell) + \cdots.$$

Applying (2.17) in the above and then grouping into two sums, we obtain

$$\begin{aligned}\overset{-3}{\underset{\ell}{\nabla}} f(s) = &\Big[\frac{2^{(2)}}{2}f(s) + \frac{3^{(2)}}{2}f(s-\ell) + \frac{4^{(2)}}{2}f(s-2\ell) + \cdots + \frac{(t+1)^{(2)}}{2}f(s-(t-1)\ell)\Big]\\ &+ \Big[\frac{(t+2)^{(2)}}{2}f(s-t\ell) + \frac{(t+3)^{(2)}}{2}f(s-(t+1)\ell) + \cdots\Big],\end{aligned}$$

which is same as

$$\overset{-3}{\underset{\ell}{\nabla}} f(s) = \sum_{r=0}^{t-1}\frac{(r+2)^{(2)}}{2}f(s-r\ell) + \sum_{r=t}^{\infty}\frac{(r+3)^{(2)}}{2}f(s-r\ell).$$

Applying the equation (2.14) for $\frac{(t+2)^{(2)}}{2}f(s-t\ell)$ to the above equation gives

$$\overset{-3}{\underset{\ell}{\nabla}} f(s) - \overset{-1}{\underset{-1}{\nabla}}\left\{\frac{(t+2)^{(2)}}{2}f(s-t\ell)\right\} = \sum_{r=0}^{t-1}\frac{(r+2)^{(2)}}{2}f(s-r\ell).$$

Proceeding like this upto n-times and by the use of (2.17), we get the general form as

$$\overset{-n}{\underset{\ell}{\nabla}} f(s) = \sum_{r=0}^{t-1}\frac{(r+n-1)^{(n-1)}}{(n-1)!}f(s-r\ell) + \sum_{r=t}^{\infty}\frac{(r+n-1)^{(n-1)}}{(n-1)!}f(s-r\ell). \tag{3.7}$$

By applying (2.14) to the last term of (3.7), we arrive

$$\overset{-n}{\underset{\ell}{\nabla}} f(s) - \overset{-1}{\underset{-1}{\nabla}}\left\{\frac{(t+n-1)^{(n-1)}}{(n-1)!}f(s-t\ell)\right\} = \sum_{r=0}^{t-1}\frac{(r+n-1)^{(n-1)}}{(n-1)!}f(s-r\ell). \tag{3.8}$$

Hence, (3.2) arrives by applying (3.1) for $f_{m,s,\ell}(t)$ in (3.8).

Case (ii). Let $\ell < 0$ and $\lim_{t\to\infty} g(t) = 0$. By replacing ℓ by $-\ell$ in Case (i) and by (2.3), we get Case (ii). □

Corollary 3.3. *By denoting* $f_{m,s,1}(t)$ *as* $f_{m,s}(t)$ *in* (3.2), *we get*

$$\overset{-m}{\nabla} f(s) - \overset{-1}{\underset{-1}{\nabla}} f_{m,s}(t) = \sum_{r=0}^{t-1} f_{m,s}(r).$$

Proof. The proof completes by taking $\ell = 1$ in (3.2). □

The following example illustrates the Theorem 3.2 for $\ell > 0$ and $\ell < 0$.

Example 3.4.

Case (i). Taking $m = 3$, $\ell = 2$ ($\ell > 0$), and $f(t) = 2^t$ in (3.2), then we have

$$\underset{2}{\overset{-3}{\nabla}} 2^s - \underset{-1}{\overset{-1}{\nabla}} f_{3,s,2}(t) = \sum_{r=0}^{t-1} f_{3,s,2}(r). \tag{3.9}$$

Applying the $\underset{\ell}{\nabla}$ operator on the function 2^s and taking $\ell = 2$, we obtain the value as $\underset{2}{\nabla} 2^s = 2^s(1-2^{-2})$. From this, we obtain $\underset{2}{\overset{-1}{\nabla}} 2^s = \dfrac{2^s}{(1-2^{-2})}$. When proceeding like this and taking $s = 5$, the first term of (3.9) becomes

$$\underset{2}{\overset{-3}{\nabla}} 2^s\Big|_{s=5} = \frac{2^s}{(1-2^{-2})^3}\Big|_{s=5} = \frac{2^5 \cdot 4^3}{3^3}. \tag{3.10}$$

Using the Definition 3.1, the second term of (3.9) becomes

$$\underset{-1}{\overset{-1}{\nabla}} f_{3,5,2}(t) = \underset{-1}{\overset{-1}{\nabla}} \left\{ \frac{(t+1)^{(2)}}{2!} 2^{5-2t} \right\} = \frac{2^5}{2!} \underset{-1}{\overset{-1}{\nabla}} \left\{ (t+1)^{(2)} 2^{-2t} \right\} = 2^4 \underset{-1}{\overset{-1}{\nabla}} \left\{ (t+1)^{(2)} 2^{-2t} \right\}.$$

By taking $f(t) = (t+1)^{(2)}$ and $g(t) = 2^{-2t}$ in (2.15), we get

$$2^4 \underset{-1}{\overset{-1}{\nabla}} \left\{ (t+1)^{(2)} 2^{-2t} \right\} = 2^4 \left\{ (t+2)^{(2)} \underset{-1}{\overset{-1}{\nabla}} 2^{-2t} - \underset{-1}{\overset{-1}{\nabla}} \left(\underset{-1}{\overset{-1}{\nabla}} 2^{-2(t+1)} \underset{-1}{\nabla} (t+2)^{(2)} \right) \right\}.$$

Solving this, we derive the above equation as

$$2^4 \underset{-1}{\overset{-1}{\nabla}} \left\{ (t+1)^{(2)} 2^{-2t} \right\} = 2^4 \left\{ \frac{(t+2)^{(2)} \cdot 2^{-2t}}{(1-2^{-2})} + \frac{2(t+2) \cdot 2^{-2(t+1)}}{(1-2^{-2})^2} + \frac{2 \cdot 2^{-2(t+2)}}{(1-2^{-2})^3} \right\}. \tag{3.11}$$

Applying $t = 3$ in (3.11), then we obtain

$$\underset{-1}{\overset{-1}{\nabla}} f_{3,5,2}(t)\Big|_{t=3} = 2^4 \underset{-1}{\overset{-1}{\nabla}} \left\{ (t+1)^{(2)} 2^{-2t} \right\}\Big|_{t=3} = 2^4 \left\{ \frac{4(20)(2^{-6})}{3} + \frac{4^2(10)(2^{-8})}{3^2} + \frac{4^3(2^{-10})}{3^3} \right\}. \tag{3.12}$$

From the relation (3.9), and expanding it, we find

$$\sum_{r=0}^{t-1} f_{3,5,2}(r) = \sum_{r=0}^{2} \frac{(r+1)^{(2)}}{2!} 2^{5-2r} = \frac{2^{(2)}}{2!} 2^5 + \frac{3^{(2)}}{2!} 2^3 + \frac{4^{(2)}}{2!} 2 = \frac{136}{2}. \tag{3.13}$$

Now, (3.9) is verified by (3.10), (3.12), and (3.13).

Case (ii). In (3.2), taking $f(t) = 2^{-t}$, $m = 3$, and $\ell = -2$ ($\ell < 0$), we derive $\underset{-2}{\nabla} \dfrac{2^{-t}}{(1-2^{-2})} = 2^{-t}$, $\underset{-2}{\overset{2}{\nabla}} \dfrac{2^{-t}}{(1-2^{-2})^2} = 2^{-t}$, $\underset{-2}{\overset{3}{\nabla}} \dfrac{2^{-t}}{(1-2^{-2})^3} = 2^{-t}$, and

$$\underset{-2}{\overset{-3}{\nabla}} 2^{-s} - \underset{-1}{\overset{-1}{\nabla}} f_{3,s,-2}(t) = \sum_{r=0}^{t-1} f_{3,s,-2}(r). \tag{3.14}$$

Here, $g(t) = \dfrac{2^{-t}}{(1-2^{-2})^3}$ and $\lim_{t\to\infty} g(t) = 0$. When $s = 5$ and $t = 3$, as in the procedure of Case (i), we can easily find

$$\underset{-2}{\overset{-3}{\nabla}} 2^{-s}\Big|_{s=5} = \frac{2^{-5}}{(1-2^{-2})^3} = \frac{1}{2^5(1-2^{-2})^3} = \frac{64}{2^5 \cdot 3^3}, \tag{3.15}$$

$$\overset{-1}{\underset{-1}{\nabla}} f_{3,5,-2}(t)\Big|_{t=3} = \overset{-1}{\underset{-1}{\nabla}}\left\{\frac{(t+2)^{(2)}}{2!}2^{-(5+2t)}\right\} = \left(\frac{1}{2^6}\right)\overset{-1}{\underset{-1}{\nabla}}\{(t+2)^{(2)}2^{-2t}\}\Big|_{t=3},$$
$$\overset{-1}{\underset{-1}{\nabla}} f_{3,5,-2}(t)\Big|_{t=3} = \frac{1}{2^6}\left\{\frac{20\cdot 4}{3\cdot 2^6}+\frac{10\cdot 4^2}{3^2\cdot 2^8}+\frac{4^3}{3^3\cdot 2^{10}}\right\} = \frac{1}{2^6}\left\{\frac{80}{3\cdot 2^6}+\frac{160}{3^2\cdot 2^8}+\frac{64}{3^3\cdot 2^{10}}\right\}, \tag{3.16}$$

and

$$\sum_{r=0}^{t-1} f_{3,5,-2}(r) = \sum_{r=0}^{2}\frac{(r+2)^{(2)}}{2!}2^{-5-2r} = \frac{1}{2^5}+\frac{3}{2^7}+\frac{6}{2^9} = \frac{34}{2^9}. \tag{3.17}$$

Now, (3.14) is verified by (3.15), (3.16), and (3.17).

Theorem 3.5 (Finite sum for ℓ-nabla)**.** *Consider the conditions given in Theorem 3.2. Then, for every pair of real numbers* (x, y) *and positive integers* t *and* m*,*

$$\overset{-m}{\underset{\ell}{\nabla}} f(\tau)\Big|_y^x - \overset{-1}{\underset{-1}{\nabla}} f_{m,\tau,\ell}(t)\Big|_{\tau=y}^{\tau=x} = \sum_{r=0}^{t-1}\Big[f_{m,x,\ell}(r) - f_{m,y,\ell}(r)\Big]. \tag{3.18}$$

Proof.

Case (i). Let $\ell > 0$ and $\lim_{t\to-\infty} g(t) = 0$. Replacing s by x in (3.2), we get

$$\overset{-m}{\underset{\ell}{\nabla}} f(x) - \overset{-1}{\underset{-1}{\nabla}} f_{m,x,\ell}(t) = \sum_{r=0}^{t-1} f_{m,x,\ell}(r). \tag{3.19}$$

Similarly, replacing s by y in (3.2), we get

$$\overset{-m}{\underset{\ell}{\nabla}} f(y) - \overset{-1}{\underset{-1}{\nabla}} f_{m,y,\ell}(t) = \sum_{r=0}^{t-1} f_{m,y,\ell}(r). \tag{3.20}$$

Then the proof follows by subtracting (3.19) from (3.20).

Case (ii). Let $\ell < 0$ and $\lim_{t\to\infty} g(t) = 0$. By applying ℓ by $-\ell$ in Case (i) and by equation (2.3), we get the required result. □

The following corollary is the *fundamental theorem of discrete calculus related to nabla operator*.

Corollary 3.6. *By denoting* $f_{m,s,1}(t)$ *as* $f_{m,s}(t)$ *and taking* $\ell = 1$ *in* (3.18)*, we get*

$$\overset{-m}{\nabla} f(\tau)\Big|_y^x - \overset{-1}{\underset{-1}{\nabla}} f_{m,\tau}(t)\Big|_{\tau=y}^{\tau=x} = \sum_{r=0}^{t-1}\Big[f_{m,x}(r) - f_{m,y}(r)\Big].$$

4. Fundamental theorems in discrete nabla fractional calculus

In this section, we develop the generalized fractional order anti-difference principle from its integer order given in (3.2), by which we derive fundamental theorem of discrete fractional calculus. Here, for any $\nu > 0$, the generalized factorial-coefficient of f at t on (ν, s, ℓ) is defined as

$$f_{\nu,s,\ell}(t) = \frac{\Gamma(t+\nu)}{\Gamma(t+1)\Gamma(\nu)} f(s - t\ell). \tag{4.1}$$

Theorem 4.1 (Infinite sum)**.** *Let* $f, g : J_\ell \to \mathbb{R}$, $\overset{\nu}{\underset{\ell}{\nabla}} g(s) = f(s)$, $\nu > 0$, *and* $f_{\nu,s,\ell}(t)$ *be given in* (4.1)*. If* $\lim_{t\to-\infty} g(t) = 0$ *for* $\ell > 0$ ($\lim_{t\to\infty} g(t) = 0$ *for* $\ell < 0$)*, then*

$$\overset{-\nu}{\underset{\ell}{\nabla}} f(s) := g(s) = \sum_{t=0}^{\infty} f_{\nu,s,\ell}(t). \tag{4.2}$$

Proof. The equation (3.7) in the proof of Theorem 3.2 is

$$\overset{-m}{\underset{\ell}{\nabla}} f(s) = \sum_{t=0}^{\infty} \frac{(t+m-1)^{(m-1)}}{(m-1)!} f(s-t\ell).$$

For any real $\nu > 0$, using the equation (2.4) for $\ell = 1$, the generalized infinite summation takes the form

$$\overset{-\nu}{\underset{\ell}{\nabla}} f(s) = \sum_{t=0}^{\infty} \frac{\Gamma(t+\nu)}{\Gamma(t+1)\Gamma(\nu)} f(s-t\ell). \tag{4.3}$$

Now (4.2) follows form (4.3) and (4.1). □

Corollary 4.2. *Consider the conditions given in Theorem 4.1 for* $\ell = 1$*. Then,*

$$\overset{-\nu}{\nabla} f(s) = \sum_{t=0}^{\infty} f_{\nu,s}(t).$$

Theorem 4.3. *Let* $f, g : J_\ell \to \mathbb{R}$, $\sum_{r=0}^{\infty} f(s-r\ell)$ *converges and* $f_{m,s,\ell}(t)$ *be as given in* (3.1)*. If* $\overset{-m}{\underset{\ell}{\nabla}} g(t) = f(t)$ *and* $\lim_{t\to-\infty} g(t) = 0$ *for* $\ell > 0$ ($\lim_{t\to\infty} g(t) = 0$ *for* $\ell < 0$)*, then we denote* $\overset{-m}{\underset{\ell}{\nabla}} f(t) = g(t)$,

$$\overset{-m}{\underset{\ell}{\nabla}} f(s) - \sum_{r=0}^{\infty} \frac{(t+r+m-1)^{(m-1)}}{(m-1)!} f(s-(t+r)\ell) = \sum_{r=0}^{t-1} f_{m,s,\ell}(r). \tag{4.4}$$

Proof. In (3.7), the infinite series term will be expressed in the form

$$\sum_{r=t}^{\infty} \frac{(r+m-1)^{(m-1)}}{(m-1)!} f(s-r\ell) = \sum_{r=0}^{\infty} \frac{(t+r+m-1)^{(m-1)}}{(m-1)!} f(s-(t+r)\ell). \tag{4.5}$$

The proof of (4.4) completes by applying the equation (4.5) in (3.7). □

It is possible to express the infinite series $\sum_{r=0}^{\infty} \frac{(t+\nu+r)^{(\nu-1)}}{\Gamma(\nu)} f(s-(t+\nu+r+1)\ell)$ in terms of $f_{\nu,s,\ell}(t)$ for certain functions (like geometric, exponential, etc). Hence, we give the following definition for such type, say $\mathcal{N}_\nu$-type of functions.

Theorem 4.4 (Generalized version of finite summation for ℓ-nabla operator)**.** *Let* $f, g : J_\ell \to \mathbb{R}$ *be an* $\mathcal{N}_\nu$*-type function for any real* $\nu > 0$ *and* $\sum_{r=0}^{\infty} f(s-r\ell)$ *converges. If* $\overset{\nu}{\underset{\ell}{\nabla}} g(t) = f(t)$ *and* $\lim_{t\to-\infty} g(t) = 0$ *for* $\ell > 0$ ($\lim_{t\to\infty} g(t) = 0$ *for* $\ell < 0$)*, then*

$$\overset{-\nu}{\underset{\ell}{\nabla}} f(s) - \frac{(f_{\nu,s,\ell}(t))^2}{f_{\nu,s,\ell}(t) - f_{\nu,s,\ell}(t+1)} = \sum_{r=0}^{t-1} f_{\nu,s,\ell}(r). \tag{4.6}$$

Proof: Since $\overset{-\nu}{\underset{\ell}{\nabla}} f(-\infty) = 0$, *it is obvious that* f *is non-constant function and* $f_{\nu,s,\ell}(t) - f_{\nu,s,\ell}(t+1) \neq 0$ *(when* f *is zero, (4.6) is trivial). Now, expanding the second term of equation (4.4), we get*

$$\sum_{r=0}^{\infty} \frac{(t+r+m-1)^{(m-1)}}{(m-1)!} f(s-(t+r)\ell) = \frac{(t+m-1)^{(m-1)}}{(m-1)!} f(s-t\ell) + \frac{(t+m)^{(m-1)}}{(m-1)!} f(s-(t+1)\ell) + \frac{(t+m+1)^{(m-1)}}{(m-1)!} f(s-(t+2)\ell) + \cdots .$$

Using the property of geometric series, the above infinite series can be written in the form

$$\sum_{r=0}^{\infty} \frac{(t+r+m-1)^{(m-1)}}{(m-1)!} f(s-(t+r)\ell) = \frac{1}{(m-1)!} \cdot \frac{(t+m-1)^{(m-1)} f(s-t\ell)}{1 - \left(\frac{(t+m)^{(m-1)} f(s-(t+1)\ell)}{(t+m+1)^{(m-1)} f(s-t\ell)}\right)},$$

which is same as

$$\sum_{r=0}^{\infty} \frac{(t+r+m-1)^{(m-1)}}{(m-1)!} f(s-(t+r)\ell) = \frac{1}{(m-1)!} \cdot \frac{[(t+m-1)^{(m-1)} f(s-t\ell)]^2}{(t+m-1)^{(m-1)} f(s-t\ell) - (t+m)^{(m-1)} f(s-(t+1)\ell)}.$$

Since f *is* $\mathcal{N}_\nu$*-type function, then for any real* $\nu > 0$ *and by (2.4), the generalized version of the infinite series can be written as*

$$\sum_{r=0}^{\infty} \frac{\Gamma(t+r+\nu)}{\Gamma(t+r+1)\Gamma(\nu)} f(s-(t+r)\ell) = \frac{1}{\Gamma(\nu)} \cdot \frac{\left[\frac{\Gamma(t+\nu)}{\Gamma(t+1)} f(s-t\ell)\right]^2}{\frac{\Gamma(t+\nu)}{\Gamma(t+1)} f(s-t\ell) - \frac{\Gamma(t+\nu+1)}{\Gamma(t+2)} f(s-(t+1)\ell)}.$$

The proof completes by applying the equation (4.1) in above relation.

5. Applications

In this section, we develop an application which is related to COVID19. Before entering into the section, we revisit some of the description and parameters for solving the virus problem.

- $\mathbb{N}(t) \to$ Number of patients affected by particular virus at time t.
- $\mathbb{N}(t-1) \to$ Number of patients affected in previous time $(t-1)$.
- $\alpha \to$ Production rate of virus.
- $\beta \to$ Destruction rate of virus.

Then, the number of new cases at time t will be $\mathbb{N}(t) - \mathbb{N}(t-1) = (\alpha-\beta)\mathbb{N}(t)$, where $\mathbb{N}(t) - \mathbb{N}(t-1)$ is number of cases at time t, which is depending on the rate of $(\alpha-\beta)\mathbb{N}(t)$ with respect to the population. Hence, the corresponding difference equation for $\mathbb{N}(t) - \mathbb{N}(t-1) = (\alpha-\beta)\mathbb{N}(t)$ is given by

$$\nabla \mathbb{N}(t) = c\mathbb{N}(t), \tag{5.1}$$

with initial condition $\mathbb{N}(0) = \mathbb{N}_0$. Now, consider the exponential function given by

$$e^{ct} = 1 + \frac{ct}{1!} + \frac{c^2t^2}{2!} + \frac{c^3t^3}{3!} + \frac{c^4t^4}{4!} + \frac{c^5t^5}{5!} + \frac{c^6t^6}{6!} + \cdots . \tag{5.2}$$

Applying the ∇ operator on both sides of (5.2), we get

$$\nabla e^{ct} = \nabla\left[1+\frac{ct}{1!}+\frac{c^2t^2}{2!}+\frac{c^3t^3}{3!}+\frac{c^4t^4}{4!}+\frac{c^5t^5}{5!}+\frac{c^6t^6}{6!}+\cdots\right].$$

From the above equation, we can easily find

$$\nabla e^{ct} = c\{1+\frac{c(t-1)}{1!}+\frac{c^2(t-1)^2}{2!}+\frac{c^3(t-1)^3}{3!}+\frac{c^4(t-1)^4}{4!}+\frac{c^5(t-1)^5}{5!}+\cdots\}.$$

Hence,

$$\mathbb{N}(t) = \mathbb{N}_0 e^{c(t-1)}$$

is the solution for (5.1). When applying $\overset{-1}{\nabla}$ on both sides of (5.1), we obtain

$$\overset{-1}{\nabla}(\nabla\mathbb{N}(t))\Big|_0^t = c(\overset{-1}{\nabla}\mathbb{N}(t))\Big|_0^t,$$

which gives $\mathbb{N}(t)-\mathbb{N}(0) = c(\overset{-1}{\nabla}\mathbb{N}(t))\Big|_0^t$. But, the corresponding difference equation is valid only for consecutive days at the particular time t. So, we develop the ℓ-nabla operator for finding the new cases for long span of time. Here, we take $\mathbb{N}(t-\ell)$ be the number of patients affected in previous time $(t-\ell)$, where $\ell = \{1,2,3,4,\ldots\}$. Therefore, the number of new cases affected at the particular time t will be

$$\underset{\ell}{\nabla}\mathbb{N}(t) = \mathbb{N}(t)-\mathbb{N}(t-\ell) = \ell(\alpha-\beta)\mathbb{N}(t), \tag{5.3}$$

with initial condition $\mathbb{N}(0) = \mathbb{N}_0$. Consider the extorial function

$$e(ct_\ell^{(1)}) = 1+\frac{ct_\ell^{(1)}}{1!}+\frac{ct_\ell^{(2)}}{2!}+\frac{ct_\ell^{(3)}}{3!}+\frac{ct_\ell^{(4)}}{4!}+\frac{ct_\ell^{(5)}}{5!}+\cdots,$$

$$\underset{\ell}{\nabla}e(ct_\ell^{(1)}) = \underset{\ell}{\nabla}(1+\frac{ct_\ell^{(1)}}{1!}+\frac{ct_\ell^{(2)}}{2!}+\frac{ct_\ell^{(3)}}{3!}+\frac{ct_\ell^{(4)}}{4!}+\frac{ct_\ell^{(5)}}{5!}+\cdots).$$

Hence,

$$\underset{\ell}{\nabla}e(ct_\ell) = c\ell\{1+\frac{c(t-\ell)_\ell^{(1)}}{1!}+\frac{c^2(t-\ell)_\ell^{(2)}}{2!}+\frac{c^3(t-\ell)_\ell^{(3)}}{3!}+\frac{c^4(t-\ell)_\ell^{(4)}}{4!}+\cdots\}.$$

Thus, we obtain

$$\mathbb{N}(t) = \ell\mathbb{N}_0 e(c(t-\ell)_\ell),$$

which is the solution for (5.3). Now, applying the $\underset{\ell}{\overset{-1}{\nabla}}$ operator on (5.3), we get

$$\mathbb{N}(t)\Big|_{t-n\ell}^t = \ell\mathbb{N}_0\underset{\ell}{\overset{-1}{\nabla}}[e(c(t-\ell)_\ell)]\Big|_{t-n\ell}^t, \quad n\in\{0,1,2,3,\ldots\}. \tag{5.4}$$

If we take $t = n\ell$, then the limit varies from 0 to t in (5.4). By taking $f(t) = e(c(t-\ell)_\ell)$ in (2.7), we obtain the solution for $\mathbb{N}(t)$ from the particular time $t-n\ell$ to t. Similarly, if the virus gets mutation, then the newly arrived virus affected peoples $\underset{\ell}{\overset{-\nu}{\nabla}}$ can be find by (4.6).

6. Conclusion

In this paper, we have introduced factorial coefficient and gamma factorial functions, and also developed fundamental theorem of discrete fractional calculus using the anti-difference principle of fractional order of nabla operator. Also, we have arrived at several fundamental theorems on fractional order delta integration using the N_ν-type functions in the field of discrete fractional calculus. Along with this, we have approach the applications in context of virus spread and its mutation using the anti-difference principle of nabla operator. As a result, this paper's future work will focus on developing the fractional order theorems for $\ell(\alpha)$-nabla operator and then nabla operator with several parameters.

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Author contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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