J. Math. Computer Sci., 34 (2024), 361–380

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Existence and controllability results for neutral fractional Volterra-Fredholm integro-differential equations

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Abstract

This paper delves into the investigation of a Volterra-Fredholm integro-differential equation enhanced with Caputo fractional derivatives subject to specific order conditions. The study rigorously establishes the existence of solutions through the application of the Schauder fixed-point theorem. Furthermore, it encompasses neutral Volterra-Fredholm integro-differential equations, thereby extending the applicability of the findings. In addition, the paper explores the concept of controllability for the obtained solutions, offering valuable insights into how these solutions behave over extended time periods.

Keywords: Volterra-Fredholm integro-differential equations, fractional derivatives, controllability, Schauder fixed point theorem.

2020 MSC: 26A33, 26D10, 34A12, 45G10, 45J05, 47G20.

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1. Introduction

The realm of fractional integro-differential equations represents a captivating and rapidly evolving branch of mathematics, finding profound applications across scientific and engineering disciplines. These equations, delving into complex processes with memory and long-range dependence, stand as a testament to the limitations of traditional differential equations [37]. Fractional calculus, championed by D'Alembert, Euler, and Liouville [6], now thrives with new generalized definitions incorporating non-singular kernels, presenting exciting avenues in studying differential equations with fractional derivatives [20]. Pioneering works by Kilbas et al. [24] and Zhou et al. [50] form the theoretical cornerstone, paving the way for further exploration in fractional calculus and integro-differential equations, navigating a complex web of mathematical relationships.

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doi: 10.22436/jmcs.034.04.04

Received: 2023-12-05 *Revised:* 2023-12-24 *Accepted:* 2024-02-21

In the quest for solutions, Ahmad and Sivasundaram [2] and Wu and Liu [49] laid groundwork exploring existence, uniqueness, and solutions for fractional integro-differential equations, unveiling their mathematical properties. Contributions by Hamoud and Ghadle [17], Ndiaye and Mansal [36], Dahmani [9], Hamoud [16], and Hamoud et al. [18] shed light on the diversity of solutions, especially for fractional Volterra-Fredholm equations, spanning applications across diverse fields from population dynamics to finance. Novel definitions for fractional derivatives by Hattaf [20] and Atangana and Baleanu [4] significantly expanded the mathematical toolbox, enriching the description of complex phenomena. The prominence of the Caputo fractional derivative, devoid of singular kernels, established by Caputo and Fabrizio [6], played a pivotal role in advancing the study of fractional integro-differential equations.

The theoretical underpinnings spurred exploration into practical implementations and controllability aspects of fractional integro-differential systems. Researchers like Feckan et al. [11], Hernández and O'Regan [21], Dineshkumar et al. [10], and Chalishajar [7] addressed controllability concerns within Banach spaces, emphasizing the pivotal role of control in engineering and scientific applications. Addressing practical challenges, Toma and Postavaru [46] developed numerical methods crucial for simulating real-world systems described by fractional integro-differential equations. Contributions by Russell [38] and Jurdjevic and Quinn [23] further enriched our understanding of controllability and stability in differential equations, emphasizing their connections to fractional integro-differential equations, ultimately aiming to bridge theoretical developments with practical applications. Columbu et al. [8] studied properties of unbounded solutions in a class of chemotaxis models. Their work focuses on understanding the behavior and properties of solutions within this specific class of models, shedding light on potential instability in chemotaxis systems. Li et al. [25] explored the combined effects ensuring boundedness in an attraction-repulsion chemotaxis model involving production and consumption. This investigation likely touches upon crucial stability aspects that govern the behavior of these systems under different conditions. Li et al. [26] investigated properties of solutions to porous medium problems with various sources and boundary conditions. This exploration likely contributes insights into stability aspects in systems described by porous medium problems, possibly shedding light on factors influencing system behavior. Li and Viglialoro [30] delved into boundedness considerations for a nonlocal reaction chemotaxis model, even in attraction-dominated scenarios. This study might offer valuable perspectives on stability aspects within such models, especially in regimes where attraction dynamics dominate. Agarwal et al. [1] provided remarks on oscillation of second-order neutral differential equations. While not directly related to PDEs, their insights into oscillatory behavior could inform discussions on system dynamics and stability in certain differential equation models. Bohner and Li [5] studied the oscillation of second-order p-Laplace dynamic equations with nonpositive neutral coefficients. Their findings on oscillatory behavior could potentially contribute to understanding the stability properties of certain dynamic equations. Li and Rogovchenko's works [27–29] provided oscillation criteria for various types of second and third-order neutral differential equations. Although not directly related to PDEs, these criteria may offer valuable insights into stability conditions governing differential equation models. Moaaz et al. [34] explored oscillation criteria for even-order neutral differential equations with distributed deviating arguments. While focused on differential equations, their findings might offer parallels or insights applicable to stability analysis in certain PDE systems.

Oscillation criteria in differential equations have garnered substantial attention in recent mathematical research. Santra's work from 2015 explores oscillation criteria for first-order nonlinear neutral differential equations with multiple delays [40]. Tripathy and Santra present necessary and sufficient conditions for oscillations in second-order neutral differential equations with impulses, contributing crucial insights into these complex systems [47]. Investigations by Alzabut et al. delve into higher-order Nabla difference equations with forcing terms, elucidating non-oscillatory solutions in these intricate mathematical models [3]. These studies shed light on various differential equation models, including partial differential equations, fractional order equations, impulsive systems, and delay difference equations, enriching our understanding of oscillatory behavior in diverse systems.

Moreover, recent research explores oscillation in second-order impulsive systems [42], numerical analysis of fractional order discrete Bloch equations [33], and oscillation results for half-linear delay difference equations of second order [22]. Contributions by Santra and Scapellato offer insights into necessary and sufficient conditions for the oscillation of second-order differential equations with mixed several delays [44], while Moaaz et al. investigate the asymptotic behavior of even-order noncanonical neutral differential equations [35]. Collectively, these studies contribute significantly to understanding oscillation phenomena in differential equations, paving the way for further exploration and application in various scientific domains.

As we delve into the investigation of the existence and controllability of solutions for Volterra-Fredholm integro-differential equations (IDEs), we employ the Schauder fixed-point theorem, a powerful mathematical tool for addressing existence and uniqueness issues. Furthermore, in an effort to extend the relevance of our findings to diverse systems and phenomena, we broaden our inquiry to include Volterra-Fredholm neutral integro-differential equations. This expansion is substantiated by relevant examples within the field, which serve to illustrate the underlying concepts.

2. Preliminaries

In this section, we concentrate on the prevalent definitions used in fractional calculus, including the Riemann-Liouville fractional derivative and the Caputo derivative, as previously discussed in academic literature [14, 15, 17, 24, 37, 50]. Let us consider the Banach space $C(\mathfrak{J}, \mathbb{R})$ equipped with the infinity norm defined as $\|\aleph\|_\infty = \sup\{|\aleph(x)| : x \in \mathfrak{J} = [t_0, b]\}$, where $\aleph$ belongs to $C(\mathfrak{J}, \mathbb{R})$.

Definition 2.1 ([24]). The fractional integral of a function ϕ with a Riemann-Liouville definition of order $\nu > 0$ is given by

$$J^\nu \phi(\rho) = \frac{1}{\Gamma(\nu)} \int_0^\rho (\rho - \zeta)^{\nu - 1} \phi(\zeta) d\zeta, \quad \text{for } \rho > 0, \nu \in \mathbb{R}^+, \tag{2.1}$$

where $\mathbb{R}^+$ denotes the set of positive real numbers, and $J^0 \phi(\rho) = \phi(\rho)$.

Definition 2.2 ([50]). The Riemann-Liouville derivative of order ν, where ν is confined to the interval $(0, 1)$ and the lower limit is set to zero, is defined for a function $\phi : [0, 1) \to \mathbb{R}$ as follows:

$$^L D^\nu \phi(\rho) = \frac{1}{\Gamma(1 - \nu)} \frac{d}{d\rho} \int_0^\rho \frac{\phi(\zeta)}{(\rho - \zeta)^\nu} d\zeta, \quad \rho > 0.$$

Definition 2.3 ([50]). The Caputo derivative of order ν, where ν falls within the range of 0 to 1, is applicable to a function $\phi : [0, 1) \to \mathbb{R}$. It can be represented as:

$$D^\nu \phi(\rho) = \frac{1}{\Gamma(1 - \nu)} \int_0^\rho \frac{\phi^{(0)}(\zeta)}{(\rho - \zeta)^\nu} d\zeta, \quad \rho > 0.$$

Definition 2.4 ([24]). The Caputo fractional derivative of the function $\phi(\rho)$ is defined as follows: for ν values between $n - 1$ and n (exclusive), it is given by:

$$^c D^\nu \phi(\rho) = \frac{1}{\Gamma(n - \nu)} \int_0^\rho (\rho - \zeta)^{n - \nu - 1} \frac{d^n \phi(\zeta)}{d\zeta^n} d\zeta.$$

For ν equal to n, it is simply the n-th derivative of $\phi(\rho)$:

$$^c D^\nu \phi(\rho) = \frac{d^n \phi(\rho)}{dx^n}.$$

The parameter ν in this definition can be a real or even complex number, representing the order of the derivative.

Definition 2.5 ([24]). The Riemann-Liouville fractional derivative of order $\nu > 0$ is typically expressed as:

$$D^{\nu}\phi(\rho) = D^{i}J^{i-\nu}\phi(\rho), \quad \text{where} \quad i-1 < \nu \leqslant i.$$

Theorem 2.6 (Arzela-Ascoli theorem, see [50]). *A sequence of functions that is both bounded and equicontinuous within the closed and bounded interval* $[a, b]$ *possesses a subsequence that converges uniformly.*

Theorem 2.7 (Schauder fixed point theorem, see [13, 19, 45]). *In a Banach space* E*, consider a nonempty subset* B *that is both closed and convex. Let* N *be a continuous mapping from* B *to itself, such that the image of* B *under* N *is relatively compact in* E*. Then,* N *possesses at least one fixed point within* B*.*

3. Volterra-Fredholm integro-differential equation

In this section, we will investigate the existence and controllability results for Volterra-Fredholm integro-differential equation, offering valuable insights for theoretical foundations.

3.1. Existence results

In this subsection, we explore into the Caputo fractional Volterra-Fredholm integro-differential equation, given by:

$$^{c}D^{\vartheta}\aleph(t) = \Upsilon(t)\aleph(t) + \xi(t, \aleph(t)) + \int_{t_0}^{t} \zeta_1(t, \sigma, \aleph(\sigma))ds + \int_{t_0}^{b} \zeta_2(t, \sigma, \aleph(\sigma))ds. \tag{3.1}$$

This equation is accompanied by the initial condition:

$$\aleph(t_0) = \aleph_0. \tag{3.2}$$

In the above expressions, $^{c}D^{\vartheta}$ denotes Caputo's fractional derivative with $0 < \vartheta \leqslant 1$, and $\aleph : \mathfrak{J} \to \mathbb{R}$, where $\mathfrak{J} = [t_0, b]$, represents the continuous function under consideration. Additionally, $\xi : \mathfrak{J} \times \mathbb{R} \to \mathbb{R}$ and $\zeta_i : \mathfrak{J} \times \mathfrak{J} \times \mathbb{R} \to \mathbb{R}$, where $i = 1, 2$, are continuous functions. Before commencing our main results and their proofs, we present the following lemma along with some essential hypotheses.

(A1) Consider continuous functions ζ_1 and $\zeta_2 : \mathfrak{J} \times \mathfrak{J} \times \mathbb{R} \to \mathbb{R}$ defined on the set $D = \{(t, \sigma) : 0 \leqslant t_0 \leqslant \sigma \leqslant t \leqslant b\}$. They satisfy the following conditions:

$$|\zeta_1(\rho, \sigma, \aleph_1(\sigma)) - \zeta_1(\rho, \sigma, \aleph_2(\sigma))| \leqslant L_{\zeta_1}\|\aleph_1(\sigma) - \aleph_2(\sigma)\|,$$
$$|\zeta_2(\rho, \sigma, \aleph_1(\sigma)) - \zeta_2(\rho, \sigma, \aleph_2(\sigma))| \leqslant L_{\zeta_2}\|\aleph_1(\sigma) - \aleph_2(\sigma)\|.$$

(A2) The function $\xi : \mathfrak{J} \times \mathbb{R} \to \mathbb{R}$ is continuous, and it satisfies the condition $|\xi(t, \aleph_1) - \xi(t, \aleph_2)| \leqslant L_{\xi}\|\aleph_1 - \aleph_2\|$.

(A3) The function $\Upsilon : \mathfrak{J} \to \mathbb{R}$ is continuous, and the constants L_{ζ_1}, L_{ζ_2}, and L_{ξ} are all positive.

(A4) The constants $L^*_{\zeta_1}$, $L^*_{\zeta_2}$ and L^*_{ξ} such that $L^*_{\zeta_1} = \max_{0\leqslant t_0\leqslant\sigma\leqslant t\leqslant b}\|\zeta_1(t, \sigma, 0)\|$, $L^*_{\zeta_2} = \max_{0\leqslant t_0\leqslant\sigma\leqslant t\leqslant b}\|\zeta_2(t, \sigma, 0)\|$, $L^*_{\xi} = \max_{t\in\mathfrak{J}}\|\xi(t, 0)\|$.

Lemma 3.1. *If the function* $\aleph_0(t) \in C(\mathfrak{J}, \mathbb{R})$*, then a function* $\aleph(t) \in C(\mathfrak{J}, \mathbb{R}^+)$ *is a solution to problem* (3.1)-(3.2) *if and only if it meets the following condition:*

$$\begin{aligned}\aleph(t) = \aleph_0 &+ \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\Upsilon(\sigma)\aleph(\sigma)ds + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\xi(\sigma, \aleph(\sigma))ds \\ &+ \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\int_{\sigma}^{t}\zeta_1(\rho, \sigma, \aleph(\sigma))d\rho ds + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho, \sigma, \aleph(\sigma))d\rho ds\end{aligned} \tag{3.3}$$

for $t \in \mathfrak{J}$*.*

Proof. This can be readily illustrated by employing the integral operator referenced as (2.1) to both sides of the equation denoted by (3.1). As a consequence, it yields the integral equation represented by (3.3). □

To establish the foundation of our investigation, we now present the following theorem that addresses the existence of solutions in the context of Volterra-Fredholm integro-differential equation, as described in equations (3.1)-(3.2).

Theorem 3.2. *Given that conditions* (A1)-(A4) *hold, it can be asserted that equation* (3.1) *possesses at least one solution.*

Proof. Let us consider the operator $\rho : C(\mathfrak{J}, \mathbb{R}) \to C(\mathfrak{J}, \mathbb{R})$. The corresponding integral equation (3.3) can be expressed in terms of this operator as follows:

$$(\rho\aleph)(t) = \aleph_0 + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\Upsilon(\sigma)\aleph(\sigma)ds + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\xi(\sigma,\aleph(\sigma))ds \\ + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\left(\int_{\sigma}^{t}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho + \int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho\right)ds.$$

Initially, we note that the operator ρ maps into itself. To illustrate, we select arbitrary values, denoting t from the set $\mathfrak{J}$ and $\aleph$ from the set Φ_γ. It follows that:

$$\begin{aligned}
|(\rho\aleph)(t)| &\leqslant |\aleph_0| + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}|\Upsilon(\sigma)||\aleph(\sigma)|ds \quad + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}|\xi(\sigma,\ \aleph(\sigma))|ds \\
&\quad + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\left(\int_{\sigma}^{t}|\zeta_1(\rho,\ \sigma,\ \aleph(\sigma))|d\rho + \int_{t_0}^{b}|\zeta_2(\rho,\ \sigma,\ \aleph(\sigma))|d\rho\right)ds \\
&\leqslant |\aleph_0| + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\|\Upsilon\|_\infty\|\aleph\|_\infty ds + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\big(|\xi(\sigma,\ \aleph(\sigma)) - \xi(\sigma,\ 0)| \\
&\quad + |\xi(\sigma,\ 0)|\big)ds + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\left(\int_{\sigma}^{t}\big(|\zeta_1(\rho,\ \sigma,\ \aleph(\sigma)) - \zeta_1(\rho,\ \sigma,\ 0)| + |\zeta_1(\rho,\ \sigma,\ O)|\big)d\rho\right. \\
&\quad \left. + \int_{t_0}^{b}\big(|\zeta_2(\rho,\ \sigma,\ \aleph(\sigma)) - \zeta_2(\rho,\ \sigma,\ 0)| + |\zeta_2(\rho,\ \sigma,\ 0)|\big)d\rho\right)ds \\
&\leqslant |\aleph_0| + \frac{\|\Upsilon\|_\infty b^{\vartheta}\gamma}{\Gamma(\vartheta+1)} + \frac{b^{\vartheta}}{\Gamma(\vartheta+1)}(L_\xi\gamma + L^*_\xi) + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_1}\gamma + L^*_{\zeta_1}) \\
&\quad + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_2}\gamma + L^*_{\zeta_2}) \\
&\leqslant |\aleph_0| + b^{\vartheta}\left(\frac{L^*_\xi}{\Gamma(\vartheta+1)} + \frac{(L^*_{\zeta_1} + L^*_{\zeta_2})b}{(\vartheta+1)\Gamma(\vartheta)}\right) + b^{\vartheta}\gamma\left(\frac{\|\Upsilon\|_\infty + L_\xi}{\Gamma(\vartheta+1)} + \frac{(L_{\zeta_1} + L_{\zeta_2})b}{(\vartheta+1)\Gamma(\vartheta)}\right), \\
|(\rho\aleph)(t) - \aleph_0| &\leqslant b^{\vartheta}\left(\frac{L^*_\xi}{\Gamma(\vartheta+1)} + \frac{(L^*_{\zeta_1} + L^*_{\zeta_2})b}{(\vartheta+1)\Gamma(\vartheta)}\right) + b^{\vartheta}\gamma\left(\frac{\|\Upsilon\|_\infty + L_\xi}{\Gamma(\vartheta+1)} + \frac{(L_{\zeta_1} + L_{\zeta_2})b}{(\vartheta+1)\Gamma(\vartheta)}\right) = \mathfrak{K}.
\end{aligned}$$

This establishes that ρ maps the set $\Phi_\gamma = \{\aleph \in C(\mathfrak{J},\mathbb{R}) : \|\aleph\|_\infty \leqslant \gamma\}$ onto itself. We will now demonstrate that the operator $\rho : \Phi_\gamma \to \Phi_\gamma$ fulfills all the criteria outlined in Theorem 2.7. The proof will be presented in multiple steps.

Step 1: ρ is continuous. Let $\aleph_n$ be a sequence such that $\aleph_n \to \aleph$ in Φ_γ.

$$\begin{aligned}
&|(\rho\aleph_n)(t) - (\rho\aleph)(t)| \\
&\quad\leqslant \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}|\Upsilon(\sigma)||\aleph_n(\sigma) - \aleph(\sigma)|ds + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}|\xi(\sigma,\aleph_n(\sigma)) - \xi(\sigma,\aleph(\sigma))|ds
\end{aligned}$$

$$+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\left(\int_{\sigma}^{t}|\zeta_1(\rho,\sigma,\aleph_n(\sigma))-\zeta_1(\rho,\sigma,\aleph(\sigma))|d\rho\right.$$
$$\left.+\int_{\sigma}^{b}|\zeta_2(\rho,\sigma,\aleph_n(\sigma))-\zeta_2(\rho,\sigma,\aleph(\sigma))|d\rho\right)ds$$
$$\leqslant\frac{\|\Upsilon\|_\infty b^\vartheta}{\Gamma(\vartheta+1)}\|\aleph_n(\sigma)-\aleph(\sigma)\|+\frac{L_\xi b^\vartheta}{\Gamma(\vartheta+1)}\|\aleph_n(\sigma)-\aleph(\sigma)\|+\frac{L_{\zeta_1}b^{\vartheta+1}+L_{\zeta_2}b^{\vartheta+1}}{(\vartheta+1)\Gamma(\vartheta)}\|\aleph_n(\sigma)-\aleph(\sigma)\|$$
$$=\left(\frac{\|\Upsilon\|_\infty+L_\xi}{\Gamma(\vartheta+1)}+\frac{(L_{\zeta_1}+L_{\zeta_2})b}{(\vartheta+1)\Gamma(\vartheta)}\right)b^\vartheta\|\aleph_n(\sigma)-\aleph(\sigma)\|.$$

Due to the continuity of ξ, ζ_1, and ζ_2, we can deduce that:

$$\|(\rho\aleph_n)(t)-(\rho\aleph)(t)\|\to 0 \text{ as } n\to\infty.$$

As a result, ρ exhibits continuity within the set Φ_γ.

Step 2: The set $\rho(\Phi_\gamma)$ possesses a uniform bound. This is evident as $\rho(\Phi_\gamma)\subset\Phi_\gamma$, which implies boundedness.

Step 3: We demonstrate that $\rho(\Phi_\gamma)$ exhibits equicontinuity.

Consider t_1 and t_2 belonging to the bounded set $[t_0,b]$ in $C(\mathfrak{J},\mathbb{R})$ as in Step 2, along with $\aleph$ from Φ_γ and $t_1<t_2$. In this context, we have:

$$\|(\rho\aleph)(t_2)-(\rho\aleph)(t_1)\|$$
$$=\Big\|\aleph_0+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_2}(t_2-\sigma)^{\vartheta-1}\Upsilon(\sigma)\aleph(\sigma)ds+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_2}(t_2-\sigma)^{\vartheta-1}\xi(\sigma,\aleph(\sigma))ds$$
$$+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_2}(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{t_2}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho ds$$
$$+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_2}(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho ds-\aleph_0-\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\Upsilon(\sigma)\aleph(\sigma)ds$$
$$-\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\xi(\sigma,\aleph(\sigma))ds-\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\int_{\sigma}^{t_1}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho ds$$
$$-\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho ds\Big\|$$
$$\leqslant\frac{1}{\Gamma(\vartheta)}\Big\|\int_{t_1}^{t_2}(t_2-\sigma)^{\vartheta-1}\Upsilon(\sigma)\aleph(\sigma)ds+\int_{t_1}^{t_2}(t_2-\sigma)^{\vartheta-1}\xi(\sigma,\aleph(\sigma))ds$$
$$+\int_{t_1}^{t_2}(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{t_2}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho ds+\int_{t_1}^{t_2}(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho ds$$
$$-\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\Upsilon(\sigma)\aleph(\sigma)ds-\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\xi(\sigma,\aleph(\sigma))ds$$
$$-\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\int_{\sigma}^{t_1}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho ds-\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho ds$$
$$+\int_{t_0}^{t_1}(t_2-\sigma)^{\vartheta-1}\Upsilon(\sigma)\aleph(\sigma)ds+\int_{t_0}^{t_1}(t_2-\sigma)^{\vartheta-1}\xi(\sigma,\aleph(\sigma))ds$$
$$+\int_{t_0}^{t_1}(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{t_2}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho ds+\int_{t_0}^{t_1}(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho ds\Big\|$$
$$\leqslant\frac{1}{\Gamma(\vartheta)}\Big\|\int_{t_1}^{t_2}(t_2-\sigma)^{\vartheta-1}\big[\Upsilon(\sigma)\aleph(\sigma)+\xi(\sigma,\aleph(\sigma))+\int_{\sigma}^{t_2}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho+\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho\big]ds$$

$$
\begin{aligned}
&-\int_{t_0}^{t_1}\left[(t_1-\sigma)^{\vartheta-1}-(t_2-\sigma)^{\vartheta-1}\right]\Upsilon(\sigma)\aleph(\sigma)ds-\int_{t_0}^{t_1}\left[(t_1-\sigma)^{\vartheta-1}-(t_2-\sigma)^{\vartheta-1}\right]\xi(\sigma,\aleph(\sigma))ds\\
&-\int_{t_0}^{t_1}\left[(t_1-\sigma)^{\vartheta-1}\int_{\sigma}^{t_1}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho-(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{t_2}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho\right]ds\\
&-\int_{t_0}^{t_1}\left[(t_1-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho-(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho\right]ds\Bigg\|.
\end{aligned}
$$

As t_2 approaches t_1, we observe that the right-hand side of the preceding equation tends to zero, denoted as $\|(\rho\aleph)(t_2)-(\rho\aleph)(t_1)\|\to 0$. Through the combined implications of steps 1-3 and the Arzela-Ascoli theorem, we establish the continuity and compactness of ρ. Applying Schauder's theorem, we consequently ascertain the existence of a fixed point $\aleph$ that acts as a solution to the problem (3.1)-(3.2). This completes the proof of the theorem. □

3.2. Controllability results

Controllability, a cornerstone of control theory in engineering and mathematics, deals with the manipulation of dynamic systems. It revolves around the ability to direct a system from an initial state to a desired state using external inputs, commonly known as control signals. In the realm of Volterra-Fredholm integro-differential equations, particularly crucial in physics, biology, and economics, this concept gains paramount importance. These equations encompass both differential and integral terms, capturing the influence of past states on the present behavior of the system [7, 10, 23, 38]. Now, we will establish the controllability of the Caputo fractional Volterra-Fredholm integro-differential equation with a control parameter of a specific form:

$$
{}^{c}D^{\vartheta}\aleph(t)=\Upsilon(t)\aleph(t)+\xi(t,\aleph(t))+\int_{t_0}^{t}\zeta_1(t,\sigma,\aleph(\sigma))ds+\int_{t_0}^{b}\zeta_2(t,\sigma,\aleph(\sigma))ds+(Bu)(t). \tag{3.4}
$$

This equation is accompanied by the initial condition:

$$
\aleph(t_0)=\aleph_0. \tag{3.5}
$$

In the given context, the state variable $x(\cdot)$ resides in the Banach space denoted as X, while the control function $\mu(\cdot)$ is defined in the space $L^2(\mathfrak{J},U)$, representing a Banach space of permissible control functions with U as its underlying Banach space. The operator B is bounded and linear, mapping from U to X. Consequently, for the equations (3.4)-(3.5), there exists a solution in a mild form, expressed as:

$$
\begin{aligned}
\aleph(t)=\aleph_0&+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\Upsilon(\sigma)\aleph(\sigma)ds+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\xi(\sigma,\aleph(\sigma))ds\\
&+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\int_{\sigma}^{t}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho ds+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho ds\\
&+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}Bu(\sigma)ds
\end{aligned} \tag{3.6}
$$

for $t\in\mathfrak{J}$.

Definition 3.3. The fractional system described by equations (3.4)-(3.5) is considered controllable over the interval $\mathfrak{J}$ if, for any given initial states $\aleph_0$ and $\aleph_1$ in the Banach space X, there exists a control function $\mu(t)$ in the space $L^2(\mathfrak{J},U)$. This control function ensures that the mild solution $\aleph(t)$ of equations (3.4)-(3.5) satisfies the conditions $\aleph(t_0)=\aleph_0$ and $\aleph(b)=\aleph_1$.

To establish the foundation of our investigation, we now present the following theorem that addresses the controllability results in the context of the Volterra-Fredholm integro-differential equation (3.4)-(3.5).

Theorem 3.4. *Assuming that hypotheses* (A1)-(A4) *are satisfied, we further postulate the following:*
(A5) *The bounded linear operator* $W : L^2(\mathfrak{J}, U) \to X$ *defined as*

$$Wu = \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{b} (b-\sigma)^{\vartheta-1} Bu(\sigma) ds$$

possesses an induced inverse operator W^{-1} *that operates within* $\frac{L^2(\mathfrak{J},U)}{kerW}$.

Additionally, there exist positive constants M_1 *and* M_2 *such that* $\|B\| \leqslant M_1$ *and* $\|W^{-1}\| \leqslant M_2$. *Given these conditions, the system described by* (3.4)-(3.5) *is controllable over the interval* $[t_0, b]$.

Proof. Consider the set of functions Φ_{γ_1}, which consists of all continuous functions denoted by $\aleph$ defined on an interval $\mathfrak{J}$ and taking real values, with the condition that $\|\aleph\|_\infty \leqslant \gamma_1$. With the assumptions provided in hypothesis (B5), we can establish a control based on the properties of an arbitrary function $\aleph(\cdot)$,

$$\mu(t) = W^{-1}\Big[\aleph_1 - \aleph_0 - \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{b} (b-\sigma)^{\vartheta-1} \Big[\Upsilon(\sigma)\aleph(\sigma) + \xi(\sigma, \aleph(\sigma)) + \int_{\sigma}^{t} \zeta_1(\rho, \sigma, \aleph(\sigma)) d\rho + \int_{\sigma}^{b} \zeta_2(\rho, \sigma, \aleph(\sigma)) d\rho\Big] ds\Big](t).$$

With the defined control, we will demonstrate that the operator Ψ mapping from the set Φ_{γ_1} to itself, which is defined as

$$\Psi\aleph(t) = \aleph_0 + \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{t} (t-\sigma)^{\vartheta-1} \Big[\Upsilon(\sigma)\aleph(\sigma) + \xi(\sigma, \aleph(\sigma)) + \int_{\sigma}^{t} \zeta_1(\rho, \sigma, \aleph(\sigma)) d\rho + \int_{\sigma}^{b} \zeta_2(\rho, \sigma, \aleph(\sigma)) d\rho\Big] ds + (B\mu)(\sigma). \tag{3.7}$$

We can conclude that there exists a fixed point for the operator Ψ, where $\mu(t)$ is defined as per equation (3.7). This fixed point serves as the mild solution to the control problem described by equations (3.4)-(3.5). Specifically, it's evident that $\Psi\aleph(b) = \aleph_1$, and this observation implies that the system represented by equations (3.4)-(3.5) is controllable over the interval $[t_0, b]$.

By applying the definition of the control function μ, we obtain the following expression,

$$\begin{aligned}\Psi\aleph(t) = \aleph_0 &+ \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{t} (t-\sigma)^{\vartheta-1} \Big[\Upsilon(\sigma)\aleph(\sigma) + \xi(\sigma, \aleph(\sigma)) + \int_{\sigma}^{t} \zeta_1(\rho, \sigma, \aleph(\sigma)) d\rho \\ &+ \int_{\sigma}^{b} \zeta_2(\rho, \sigma, \aleph(\sigma)) d\rho\Big] ds + BW^{-1}\Big[\aleph_1 - \aleph_0 - \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{b} (b-\sigma)^{\vartheta-1} \Big[\Upsilon(\sigma)\aleph(\sigma) + \xi(\sigma, \aleph(\sigma)) \\ &+ \int_{\sigma}^{t} \zeta_1(\rho, \sigma, \aleph(\sigma)) d\rho + \int_{\sigma}^{b} \zeta_2(\rho, \sigma, \aleph(\sigma)) d\rho\Big] ds\Big].\end{aligned} \tag{3.8}$$

Given that all the functions included in the operator's definition are continuous, we can conclude that the operator Ψ exhibits continuity.

Expanding upon equation (3.6), for any function $\aleph$ belonging to the set Φ_{γ_1} and for all values of t within the interval $[t_0, b]$, we can establish the following relationship:

$$\begin{aligned}\|\mu(t)\| &\leqslant \|W^{-1}\|\Big[\|\aleph_1\| + \|\aleph_0\| + \frac{\|\Upsilon\|_\infty b^\vartheta \gamma_1}{\Gamma(\vartheta+1)} + \frac{b^\vartheta}{\Gamma(\vartheta+1)}(L_\xi \gamma_1 + L_\xi^*) + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_1}\gamma_1 + L_{\zeta_1}^*) \\ &\quad + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_2}\gamma_1 + L_{\zeta_2}^*)\Big], \\ \|\mu(t)\| &\leqslant M_2\Big[\|\aleph_1\| + \|\aleph_0\| + \frac{\|\Upsilon\|_\infty b^\vartheta \gamma_1}{\Gamma(\vartheta+1)} + \frac{b^\vartheta}{\Gamma(\vartheta+1)}(L_\xi \gamma_1 + L_\xi^*) + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_1}\gamma_1 + L_{\zeta_1}^*) \\ &\quad + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_2}\gamma_1 + L_{\zeta_2}^*)\Big].\end{aligned} \tag{3.9}$$

Applying equations (3.8) and (3.9), we can derive the subsequent result:

$$\begin{aligned}
\|\Psi\aleph(t)\| &\leqslant \|\aleph_0\| + \frac{\|\Upsilon\|_\infty b^\vartheta \gamma_1}{\Gamma(\vartheta+1)} + \frac{b^\vartheta}{\Gamma(\vartheta+1)}(L_\xi\gamma_1 + L_\xi^*) + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_1}\gamma_1 + L_{\zeta_1}^*) \\
&\quad + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_2}\gamma_1 + L_{\zeta_2}^*) + \frac{b^\vartheta}{\Gamma(\vartheta+1)}M_1\|\mu(t)\| \\
&\leqslant \|\aleph_0\| + \frac{\|\Upsilon\|_\infty b^\vartheta \gamma_1}{\Gamma(\vartheta+1)} + \frac{b^\vartheta}{\Gamma(\vartheta+1)}(L_\xi\gamma_1 + L_\xi^*) + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_1}\gamma_1 + L_{\zeta_1}^*) \\
&\quad + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_2}\gamma_1 + L_{\zeta_2}^*) + \frac{b^\vartheta}{\Gamma(\vartheta+1)}M_1\Bigg[M_2\|\aleph_1\| + M_2\|\aleph_0\| + M_2\frac{\|\Upsilon\|_\infty b^\vartheta\gamma_1}{\Gamma(\vartheta+1)} \\
&\quad + M_2\frac{b^\vartheta}{\Gamma(\vartheta+1)}(L_\xi\gamma_1 + L_\xi^*) + M_2\frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_1}\gamma_1 + L_{\zeta_1}^*) + M_2\frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_2}\gamma_1 + L_{\zeta_2}^*)\Bigg] \\
&\leqslant \frac{b^\vartheta}{\Gamma(\vartheta+1)}M_1M_2\|\aleph_1\| + \left(1 + \frac{b^\vartheta}{\Gamma(\vartheta+1)}M_1M_2\right)\|\aleph_0\| + \frac{\|\Upsilon\|_\infty b^\vartheta\gamma_1}{\Gamma(\vartheta+1)}\left(1 + \frac{b^\vartheta}{\Gamma(\vartheta+1)}M_1M_2\right) \\
&\quad + \frac{b^\vartheta}{\Gamma(\vartheta+1)}(L_\xi\gamma_1 + L_\xi^*)\left(1 + \frac{b^\vartheta}{\Gamma(\vartheta+1)}M_1M_2\right) + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_1}\gamma_1 + L_{\zeta_1}^*)\left(1 + \frac{b^\vartheta}{\Gamma(\vartheta+1)}M_1M_2\right) \\
&\quad + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_2}\gamma_1 + L_{\zeta_2}^*)\left(1 + \frac{b^\vartheta}{\Gamma(\vartheta+1)}M_1M_2\right) \\
&\leqslant \frac{b^\vartheta}{\Gamma(\vartheta+1)}M_1M_2\|\aleph_1\| + \left(1 + \frac{b^\vartheta}{\Gamma(\vartheta+1)}M_1M_2\right)\Bigg[\|\aleph_0\| + \frac{\|\Upsilon\|_\infty b^\vartheta\gamma_1}{\Gamma(\vartheta+1)} + \frac{b^\vartheta}{\Gamma(\vartheta+1)}(L_\xi\gamma_1 + L_\xi^*) \\
&\quad + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_1}\gamma_1 + L_{\zeta_1}^*) + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_2}\gamma_1 + L_{\zeta_2}^*)\Bigg] \leqslant \mathfrak{K}_1.
\end{aligned}$$

This demonstrates that the operator Ψ maps the set $\Phi_{\gamma_1} = \{\aleph \in C(\mathfrak{J}, \mathbb{R}) : \|\aleph\|_\infty \leqslant \gamma_1\}$ onto itself. We will now proceed to demonstrate that the operator $\Psi : \Phi\gamma_1 \to \Phi_{\gamma_1}$ fulfills all the requirements of the Schauder theorem (Theorem 2.7). The proof will be presented in multiple steps.

Step 1: Ψ is continuous. Let $\aleph_n$ be a sequence such that $\aleph_n \to \aleph$ in Φ_{γ_1},

$$\begin{aligned}
|(\Psi\aleph_n)(t) - (\Psi\aleph)(t)| &\leqslant \frac{1}{\Gamma(\vartheta)}\int_{t_0}^t (t-\sigma)^{\vartheta-1}|\Upsilon(\sigma)||\aleph_n(\sigma) - \aleph(\sigma)|ds \\
&\quad + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^t (t-\sigma)^{\vartheta-1}|\xi(\sigma, \aleph_n(\sigma)) - \xi(\sigma, \aleph(\sigma))|ds \\
&\quad + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^t (t-\sigma)^{\vartheta-1}\left(\int_\sigma^t |\zeta_1(\rho, \sigma, \aleph_n(\sigma)) - \zeta_1(\rho, \sigma, \aleph(\sigma))|d\rho\right. \\
&\quad \left. + \int_\sigma^b |\zeta_2(\rho, \sigma, \aleph_n(\sigma)) - \zeta_2(\rho, \sigma, \aleph(\sigma))|d\rho\right) ds \\
&\quad + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^t (t-\sigma)^{\vartheta-1}BW^{-1}\Bigg[-\frac{1}{\Gamma(\vartheta)}\int_{t_0}^b (b-\sigma)^{\vartheta-1}\Big[\Upsilon(\sigma)|\aleph_n(\sigma) - \aleph(\sigma)| \\
&\quad + |\xi(\sigma, \aleph_n(\sigma)) - \xi(\sigma, \aleph(\sigma))| + \int_\sigma^t |\zeta_1(\rho, \sigma, \aleph_n(\sigma)) - \zeta_1(\rho, \sigma, \aleph(\sigma))|d\rho \\
&\quad + \int_\sigma^b |\zeta_2(\rho, \sigma, \aleph_n(\sigma)) - \zeta_2(\rho, \sigma, \aleph(\sigma))|d\rho\Big] ds\Bigg].
\end{aligned}$$

Due to the continuity of ξ, ζ_1, and ζ_2, we can deduce that: $\|(\Psi\aleph_n)(t) - (\Psi\aleph)(t)\| \to 0$ as $n \to \infty$. As a result, ρ exhibits continuity within the set Φ_{γ_1}.

Step 2: The set $\Psi(\Phi_{\gamma_1})$ possesses a uniform bound. This is evident as $\Psi(\Phi_{\gamma_1}) \subset \Phi_{\gamma_1}$, which implies boundedness.

Step 3: We demonstrate that $\Psi(\Phi_{\gamma_1})$ exhibits equicontinuity.

Consider t_1 and t_2 belonging to the bounded set $[t_0, b]$ in $C(\mathfrak{J}, \mathbb{R})$ as in Step 2, along with $\aleph$ from Φ_{γ_1} and $t_1 < t_2$. In this context, we have:

$$
\begin{aligned}
&\|(\Psi\aleph)(t_2) - (\Psi\aleph)(t_1)\| \\
&= \Bigg\| \aleph_0 + \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{t_2} (t_2-\sigma)^{\vartheta-1} \Upsilon(\sigma)\aleph(\sigma) ds + \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{t_2} (t_2-\sigma)^{\vartheta-1} \xi(\sigma, \aleph(\sigma)) ds \\
&\quad + \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{t_2} (t_2-\sigma)^{\vartheta-1} \int_{\sigma}^{t_2} \zeta_1(\rho, \sigma, \aleph(\sigma)) d\rho ds + \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{t_2} (t_2-\sigma)^{\vartheta-1} \int_{\sigma}^{b} \zeta_2(\rho, \sigma, \aleph(\sigma)) d\rho ds \\
&\quad + BW^{-1} \Bigg[\aleph_1 - \aleph_0 - \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{t_2} (t_2-\sigma)^{\vartheta-1} \Big[\Upsilon(\sigma)\aleph(\sigma) + \xi(\sigma, \aleph(\sigma)) + \int_{\sigma}^{t_2} \zeta_1(\rho, \sigma, \aleph(\sigma)) d\rho \\
&\quad + \int_{\sigma}^{b} \zeta_2(\rho, \sigma, \aleph(\sigma)) d\rho \Big] ds \Bigg] - \aleph_0 - \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{t_1} (t_1-\sigma)^{\vartheta-1} \Upsilon(\sigma)\aleph(\sigma) ds \\
&\quad - \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{t_1} (t_1-\sigma)^{\vartheta-1} \xi(\sigma, \aleph(\sigma)) ds - \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{t_1} (t_1-\sigma)^{\vartheta-1} \int_{\sigma}^{t_1} \zeta_1(\rho, \sigma, \aleph(\sigma)) d\rho ds \\
&\quad - \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{t_1} (t_1-\sigma)^{\vartheta-1} \int_{\sigma}^{b} \zeta_2(\rho, \sigma, \aleph(\sigma)) d\rho ds - BW^{-1} \Bigg[\aleph_1 - \aleph_0 \\
&\quad - \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{t_1} (t_1-\sigma)^{\vartheta-1} \Big[\Upsilon(\sigma)\aleph(\sigma) + \xi(\sigma, \aleph(\sigma)) + \int_{\sigma}^{t_1} \zeta_1(\rho, \sigma, \aleph(\sigma)) d\rho + \int_{\sigma}^{b} \zeta_2(\rho, \sigma, \aleph(\sigma)) d\rho \Big] ds \Bigg] \Bigg\| \\
&\leqslant \frac{1}{\Gamma(\vartheta)} \Bigg\| \int_{t_1}^{t_2} (t_2-\sigma)^{\vartheta-1} \big[\Upsilon(\sigma)\aleph(\sigma) + \xi(\sigma, \aleph(\sigma)) + \int_{\sigma}^{t_2} \zeta_1(\rho, \sigma, \aleph(\sigma)) d\rho + \int_{\sigma}^{b} \zeta_2(\rho, \sigma, \aleph(\sigma)) d\rho \big] ds \\
&\quad - \int_{t_0}^{t_1} \big[(t_1-\sigma)^{\vartheta-1} - (t_2-\sigma)^{\vartheta-1} \big] \Upsilon(\sigma)\aleph(\sigma) ds - \int_{t_0}^{t_1} \big[(t_1-\sigma)^{\vartheta-1} - (t_2-\sigma)^{\vartheta-1} \big] \xi(\sigma, \aleph(\sigma)) ds \\
&\quad - \int_{t_0}^{t_1} \big[(t_1-\sigma)^{\vartheta-1} \int_{\sigma}^{t_1} \zeta_1(\rho, \sigma, \aleph(\sigma)) d\rho - (t_2-\sigma)^{\vartheta-1} \int_{\sigma}^{t_2} \zeta_1(\rho, \sigma, \aleph(\sigma)) d\rho \big] ds \\
&\quad - \int_{t_0}^{t_1} \big[(t_1-\sigma)^{\vartheta-1} \int_{\sigma}^{b} \zeta_2(\rho, \sigma, \aleph(\sigma)) d\rho - (t_2-\sigma)^{\vartheta-1} \int_{\sigma}^{b} \zeta_2(\rho, \sigma, \aleph(\sigma)) d\rho \big] ds \\
&\quad + BW^{-1} \Bigg[\frac{1}{\Gamma(\vartheta)} \int_{t_1}^{t_2} (t_2-\sigma)^{\vartheta-1} \big[\Upsilon(\sigma)\aleph(\sigma) + \xi(\sigma, \aleph(\sigma)) + \int_{\sigma}^{t_2} \zeta_1(\rho, \sigma, \aleph(\sigma)) d\rho \\
&\quad + \int_{\sigma}^{b} \zeta_2(\rho, \sigma, \aleph(\sigma)) d\rho \big] ds - \int_{t_0}^{t_1} \big[(t_1-\sigma)^{\vartheta-1} - (t_2-\sigma)^{\vartheta-1} \big] \Upsilon(\sigma)\aleph(\sigma) ds \\
&\quad - \int_{t_0}^{t_1} \big[(t_1-\sigma)^{\vartheta-1} - (t_2-\sigma)^{\vartheta-1} \big] \xi(\sigma, \aleph(\sigma)) ds \\
&\quad - \int_{t_0}^{t_1} \big[(t_1-\sigma)^{\vartheta-1} \int_{\sigma}^{t_1} \zeta_1(\rho, \sigma, \aleph(\sigma)) d\rho - (t_2-\sigma)^{\vartheta-1} \int_{\sigma}^{t_2} \zeta_1(\rho, \sigma, \aleph(\sigma)) d\rho \big] ds \\
&\quad - \int_{t_0}^{t_1} \big[(t_1-\sigma)^{\vartheta-1} \int_{\sigma}^{b} \zeta_2(\rho, \sigma, \aleph(\sigma)) d\rho - (t_2-\sigma)^{\vartheta-1} \int_{\sigma}^{b} \zeta_2(\rho, \sigma, \aleph(\sigma)) d\rho \big] ds \Bigg] \Bigg\|.
\end{aligned}
$$

As t_2 approaches t_1, we observe that the right-hand side of the preceding equation tends to zero, denoted as $\|(\Psi\aleph)(t_2) - (\Psi\aleph)(t_1)\| \to 0$. Through the combined implications of Steps 1-3 and the Arzela-Ascoli theorem, we establish the continuity and compactness of Ψ. Applying Schauder's theorem, we consequently ascertain the existence of a fixed point $\aleph$ that acts as a solution to the problem (3.4)-(3.5). Therefore, the system (3.4)-(3.5) is controllable on $\mathfrak{J} = [t_0, b]$. This completes the proof of the theorem. □

4. Neutral Volterra-Fredholm integro-differential equation

In this section, we delve into the investigation of both the existence of solutions, as well as the controllability results for neutral Volterra-Fredholm integro-differential equation.

4.1. Existence results

In this subsection, we explore into the Caputo fractional Volterra-Fredholm neutral integro-differential equation, given by:

$$ {}^{c}D^{\vartheta}\left[\aleph(t) - \xi_1(t, \aleph(t))\right] = \Upsilon(t)\aleph(t) + \xi_2(t, \aleph(t)) + \int_{t_0}^{t} \zeta_1(t, \sigma, \aleph(\sigma))ds + \int_{t_0}^{b} \zeta_2(t, \sigma, \aleph(\sigma))ds. \tag{4.1}$$

This equation is accompanied by the initial condition:

$$\aleph(t_0) = \aleph_0. \tag{4.2}$$

In the above expressions, ${}^{c}D^{\vartheta}$ denotes Caputo's fractional derivative with $0 < \vartheta \leqslant 1$, and $\aleph : \mathfrak{J} \to \mathbb{R}$, where $\mathfrak{J} = [t_0, b]$, represents the continuous function under consideration. Additionally, $g_i : \mathfrak{J} \times \mathbb{R} \to \mathbb{R}$ and $\zeta_i : \mathfrak{J} \times \mathfrak{J} \times \mathbb{R} \to \mathbb{R}$, where $i = 1, 2$, are continuous functions. Before commencing our main results and their proofs, we present the following lemma along with some essential hypotheses.

(B1) Consider continuous functions ζ_1 and $\zeta_2 : \mathfrak{J} \times \mathfrak{J} \times \mathbb{R} \to \mathbb{R}$ defined on the set $D = \{(t, \sigma) : 0 \leqslant t_0 \leqslant \sigma \leqslant t \leqslant b\}$. They satisfy the following conditions:

$$\begin{aligned} |\zeta_1(\rho, \sigma, \aleph_1(\sigma)) - \zeta_1(\rho, \sigma, \aleph_2(\sigma))| &\leqslant L_{\zeta_1}\|\aleph_1(\sigma) - \aleph_2(\sigma)\|, \\ |\zeta_2(\rho, \sigma, \aleph_1(\sigma)) - \zeta_2(\rho, \sigma, \aleph_2(\sigma))| &\leqslant L_{\zeta_2}\|\aleph_1(\sigma) - \aleph_2(\sigma)\|. \end{aligned}$$

(B2) The functions ξ_1 and $\xi_2 : \mathfrak{J} \times \mathbb{R} \to \mathbb{R}$ are continuous, and they satisfy the conditions

$$|\xi_1(t, \aleph_1) - \xi_1(t, \aleph_2)| \leqslant L_{\xi_1}\|\aleph_1 - \aleph_2\|, \quad |\xi_2(t, \aleph_1) - \xi_2(t, \aleph_2)| \leqslant L_{\xi_2}\|\aleph_1 - \aleph_2\|.$$

(B3) The function $\Upsilon : \mathfrak{J} \to \mathbb{R}$ is continuous, and the constants L_{ζ_1}, L_{ζ_2}, L_{ξ_1}, and L_{ξ_2} are all positive.

(B4) The constants $L^*_{\zeta_1}$, $L^*_{\zeta_2}$, $L^*_{\xi_1}$, and $L^*_{\xi_2}$ such that

$$L^*_{\zeta_1} = \max_{0\leqslant t_0\leqslant\sigma\leqslant t\leqslant b}\|\zeta_1(t, \sigma, 0)\|, \; L^*_{\zeta_2} = \max_{0\leqslant t_0\leqslant\sigma\leqslant t\leqslant b}\|\zeta_2(t, \sigma, 0)\|, \; L^*_{\xi_1} = \max_{t\in\mathfrak{J}}\|\xi_1(t, 0)\|, \; L^*_{\xi_2} = \max_{t\in\mathfrak{J}}\|\xi_2(t, 0)\|.$$

Lemma 4.1. *If* $\aleph_0(t) \in C(\mathfrak{J}, \mathbb{R})$, *then* $\aleph(t) \in C(\mathfrak{J}, \mathbb{R}^+)$ *is a solution to problem* (4.1)-(4.2) *if and only if it satisfies:*

$$\begin{aligned} \aleph(t) = {} & \aleph_0 - \xi_1(t_0, \aleph_0) + \xi_1(t, \aleph(t)) + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\Upsilon(\sigma)\aleph(\sigma)ds \\ & + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\xi_2(\sigma, \aleph(\sigma))ds \\ & + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\int_{\sigma}^{t}\zeta_1(\rho, \sigma, \aleph(\sigma))d\rho ds + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho, \sigma, \aleph(\sigma))d\rho ds \end{aligned} \tag{4.3}$$

for $t \in \mathfrak{J}$.

Proof. This can be readily demonstrated by utilizing the integral operator (2.1) on both sides of equation (4.1), resulting in the integral equation (4.3). □

To establish the foundation of our investigation, we now present the following theorem that addresses the existence of solutions in the context of Volterra-Fredholm integro differential equation, as described in equations (4.1)-(4.2).

Theorem 4.2. *Given that conditions* (B1)-(B4) *hold, it can be asserted that equation* (4.1)-(4.2) *possesses at least one solution.*

Proof. Let us consider the operator $\rho : C(\mathfrak{J}, \mathbb{R}) \to C(\mathfrak{J}, \mathbb{R})$. The corresponding integral equation (4.3) can be expressed in terms of this operator as

$$(\rho\aleph)(t) = \aleph_0 - \xi_1(t_0,\aleph_0) + \xi_1(t,\aleph(t)) + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\Upsilon(\sigma)\aleph(\sigma)ds + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\xi_2(\sigma,\aleph(\sigma))ds$$
$$+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\left(\int_{\sigma}^{t}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho + \int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho\right)ds.$$

Initially, we note that the operator ρ maps into itself. To illustrate, we select arbitrary values, denoting t from the set $\mathfrak{J}$ and $\aleph$ from the set Φ_γ. It follows that:

$$\begin{aligned}
&|(\rho\aleph)(t)|\\
&\leqslant |\aleph_0| + |\xi_1(t_0,\aleph_0)| + |\xi_1(t,\aleph(t))| + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}|\Upsilon(\sigma)||\aleph(\sigma)|ds + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}|\xi_2(\sigma,\ \aleph(\sigma))|ds\\
&\quad+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\left(\int_{\sigma}^{t}|\zeta_1(\rho,\ \sigma,\ \aleph(\sigma))|d\rho + \int_{t_0}^{b}|\zeta_2(\rho,\ \sigma,\ \aleph(\sigma))|d\rho\right)ds\\
&\leqslant |\aleph_0| + |\xi_1(t_0,\aleph_0)| + |\xi_1(\sigma,\aleph(\sigma)) - \xi_1(\sigma,0)| + |\xi_1(\sigma,0)| + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\|\Upsilon\|_\infty\|\aleph\|_\infty ds\\
&\quad+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\big(|\xi_2(\sigma,\ \aleph(\sigma)) - \xi_2(\sigma,\ 0)| + |\xi_2(\sigma,\ 0)|\big)ds\\
&\quad+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\left(\int_{\sigma}^{t}\big(|\zeta_1(\rho,\ \sigma,\ \aleph(\sigma)) - \zeta_1(\rho,\ \sigma,\ 0)| + |\zeta_1(\rho,\ \sigma,\ O)|\big)d\rho\right.\\
&\quad\left.+\int_{t_0}^{b}\big(|\zeta_2(\rho,\ \sigma,\ \aleph(\sigma)) - \zeta_2(\rho,\ \sigma,\ 0)| + |\zeta_2(\rho,\ \sigma,\ 0)|\big)d\rho\right)ds\\
&\leqslant |\aleph_0| + |\xi_1(t_0,\aleph_0)| + (L_{\xi_1} + L^*_{\xi_1}\gamma) + \frac{\|\Upsilon\|_\infty b^\vartheta\gamma}{\Gamma(\vartheta+1)} + \frac{b^\vartheta}{\Gamma(\vartheta+1)}(L_{\xi_2}\gamma + L^*_{\xi_2})\\
&\quad+\frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_1}\gamma + L^*_{\zeta_1}) + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_2}\gamma + L^*_{\zeta_2})\\
&\leqslant |\aleph_0| + |\xi_1(t_0,\aleph_0)| + (L_{\xi_1}\gamma + L^*_{\xi_1}) + b^\vartheta\left(\frac{L^*_{\xi_2}}{\Gamma(\vartheta+1)} + \frac{(L^*_{\zeta_1}+L^*_{\zeta_2})b}{(\vartheta+1)\Gamma(\vartheta)}\right) + b^\vartheta\gamma\left(\frac{\|\Upsilon\|_\infty + L_{\xi_2}}{\Gamma(\vartheta+1)} + \frac{(L_{\zeta_1}+L_{\zeta_2})b}{(\vartheta+1)\Gamma(\vartheta)}\right),\\
&|(\rho\aleph)(t) - \aleph_0|\\
&\leqslant |\xi_1(t_0,\aleph_0)| + L_{\xi_1}\gamma + L^*_{\xi_1} + b^\vartheta\left(\frac{L^*_{\xi_2}}{\Gamma(\vartheta+1)} + \frac{(L^*_{\zeta_1}+L^*_{\zeta_2})b}{(\vartheta+1)\Gamma(\vartheta)}\right) + b^\vartheta\gamma\left(\frac{\|\Upsilon\|_\infty + L_{\xi_2}}{\Gamma(\vartheta+1)} + \frac{(L_{\zeta_1}+L_{\zeta_2})b}{(\vartheta+1)\Gamma(\vartheta)}\right) = \mathfrak{L}.
\end{aligned}$$

This establishes that ρ maps the set $\Phi_\gamma = \{\aleph \in C(\mathfrak{J},\mathbb{R}) : \|\aleph\|_\infty \leqslant \gamma\}$ onto itself. We will now demonstrate that the operator $\rho : \Phi_\gamma \to \Phi_\gamma$ fulfills all the criteria outlined in Theorem 2.7. The proof will be presented in multiple steps.

Step 1: ρ is continuous. Let $\aleph_n$ be a sequence such that $\aleph_n \to \aleph$ in Φ_γ,

$$\begin{aligned}
|(\rho\aleph_n)(t)-(\rho\aleph)(t)| &\leqslant |\xi_1(t,\aleph_n(t))-\xi_1(t,\aleph(t))| + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}|\Upsilon(\sigma)||\aleph_n(\sigma)-\aleph(\sigma)|ds\\
&\quad + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}|\xi_2(\sigma,\aleph_n(\sigma))-\xi_2(\sigma,\aleph(\sigma))|ds\\
&\quad + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\left(\int_{\sigma}^{t}|\zeta_1(\rho,\sigma,\aleph_n(\sigma))-\zeta_1(\rho,\sigma,\aleph(\sigma))|d\rho\right.\\
&\quad \left.+\int_{\sigma}^{b}|\zeta_2(\rho,\sigma,\aleph_n(\sigma))-\zeta_2(\rho,\sigma,\aleph(\sigma))|d\rho\right)ds\\
&\leqslant L_{\xi_1}\|\aleph_n(\sigma)-\aleph(\sigma)\| + \frac{\|\Upsilon\|_\infty b^\vartheta}{\Gamma(\vartheta+1)}\|\aleph_n(\sigma)-\aleph(\sigma)\| + \frac{L_{\xi_2}b^\vartheta}{\Gamma(\vartheta+1)}\|\aleph_n(\sigma)-\aleph(\sigma)\|\\
&\quad + \frac{L_{\zeta_1}b^{\vartheta+1}+L_{\zeta_2}b^{\vartheta+1}}{(\vartheta+1)\Gamma(\vartheta)}\|\aleph_n(\sigma)-\aleph(\sigma)\|\\
&= \left(L_{\xi_1}+\frac{\|\Upsilon\|_\infty+L_{\xi_2}}{\Gamma(\vartheta+1)}+\frac{(L_{\zeta_1}+L_{\zeta_2})b}{(\vartheta+1)\Gamma(\vartheta)}\right)b^\vartheta\|\aleph_n(\sigma)-\aleph(\sigma)\|.
\end{aligned}$$

Due to the continuity of ξ, ζ_1, and ζ_2, we can deduce that: $\|(\rho\aleph_n)(t)-(\rho\aleph)(t)\| \to 0$ as $n \to \infty$. As a result, ρ exhibits continuity within the set Φ_γ.

Step 2: The set $\rho(\Phi_\gamma)$ possesses a uniform bound. This is evident as $\rho(\Phi_\gamma) \subset \Phi_\gamma$, which implies boundedness.

Step 3: We demonstrate that $\rho(\Phi_\gamma)$ exhibits equicontinuity.

Consider t_1 and t_2 belonging to the bounded set $[t_0, b]$ in $C(\mathfrak{J}, \mathbb{R})$ as in Step 2, along with $\aleph$ from Φ_γ and $t_1 < t_2$. In this context, we have:

$$\begin{aligned}
&\|(\rho\aleph)(t_2)-(\rho\aleph)(t_1)\|\\
&= \Bigg\|\aleph_0-\xi_1(t_0,\aleph_0)+\xi_1(\sigma,\aleph(\sigma))+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_2}(t_2-\sigma)^{\vartheta-1}\Upsilon(\sigma)\aleph(\sigma)ds\\
&\quad+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_2}(t_2-\sigma)^{\vartheta-1}\xi_2(\sigma,\aleph(\sigma))ds+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_2}(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{t_2}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho ds\\
&\quad+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_2}(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho ds-\aleph_0+\xi_1(t_0,\aleph_0)-\xi_1(\sigma,\aleph(\sigma))\\
&\quad-\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\Upsilon(\sigma)\aleph(\sigma)ds-\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\xi_2(\sigma,\aleph(\sigma))ds\\
&\quad-\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\int_{\sigma}^{t_1}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho ds-\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho ds\Bigg\|\\
&\leqslant \frac{1}{\Gamma(\vartheta)}\Bigg\|\int_{t_1}^{t_2}(t_2-\sigma)^{\vartheta-1}\Upsilon(\sigma)\aleph(\sigma)ds+\int_{t_1}^{t_2}(t_2-\sigma)^{\vartheta-1}\xi_2(\sigma,\aleph(\sigma))ds\\
&\quad+\int_{t_1}^{t_2}(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{t_2}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho ds+\int_{t_1}^{t_2}(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho ds\\
&\quad-\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\Upsilon(\sigma)\aleph(\sigma)ds-\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\xi_2(\sigma,\aleph(\sigma))ds\\
&\quad-\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\int_{\sigma}^{t_1}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho ds-\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_0}^{t_1}(t_2-\sigma)^{\vartheta-1}\Upsilon(\sigma)\aleph(\sigma)ds+\int_{t_0}^{t_1}(t_2-\sigma)^{\vartheta-1}\xi_2(\sigma,\aleph(\sigma))ds\\
&+\int_{t_0}^{t_1}(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{t_2}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho ds+\int_{t_0}^{t_1}(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho ds\Big\|\\
\leqslant{}&\frac{1}{\Gamma(\vartheta)}\Big\|\int_{t_1}^{t_2}(t_2-\sigma)^{\vartheta-1}\Big[\Upsilon(\sigma)\aleph(\sigma)+\xi_2(\sigma,\aleph(\sigma))+\int_{\sigma}^{t_2}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho+\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho\Big]ds\\
&-\int_{t_0}^{t_1}\Big[(t_1-\sigma)^{\vartheta-1}-(t_2-\sigma)^{\vartheta-1}\Big]\Upsilon(\sigma)\aleph(\sigma)ds-\int_{t_0}^{t_1}\Big[(t_1-\sigma)^{\vartheta-1}-(t_2-\sigma)^{\vartheta-1}\Big]\xi_2(\sigma,\aleph(\sigma))ds\\
&-\int_{t_0}^{t_1}\Big[(t_1-\sigma)^{\vartheta-1}\int_{\sigma}^{t_1}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho-(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{t_2}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho\Big]ds\\
&-\int_{t_0}^{t_1}\Big[(t_1-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho-(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho\Big]ds\Big\|.
\end{aligned}
$$

As t_2 approaches t_1, we observe that the right-hand side of the preceding equation tends to zero, denoted as $\|(\rho\aleph)(t_2)-(\rho\aleph)(t_1)\|\to 0$. Through the combined implications of Steps 1-3 and the Arzela-Ascoli theorem, we establish the continuity and compactness of ρ. Applying Schauder's theorem, we consequently ascertain the existence of a fixed point $\aleph$ that acts as a solution to the problem (4.1) - (4.2). This completes the proof of the theorem. □

4.2. *Controllability results*

In this subsection, we will establish the controllability of the Caputo fractional neutral Volterra-Fredholm integro-differential equation with a control parameter of a specific form:

$$
\begin{aligned}
{}^{c}D^{\vartheta}\Big[\aleph(t)-\xi_1(t,\aleph(t))\Big]&=\Upsilon(t)\aleph(t)+\xi_2(t,\aleph(t))+\int_{t_0}^{t}\zeta_1(t,\sigma,\aleph(\sigma))ds\\
&\quad+\int_{t_0}^{b}\zeta_2(t,\sigma,\aleph(\sigma))ds+(Bu)(t).
\end{aligned}
\tag{4.4}
$$

This equation is accompanied by the initial condition:

$$
\aleph(t_0)=\aleph_0. \tag{4.5}
$$

In the given context, the state variable $x(\cdot)$ resides in the Banach space denoted as X, while the control function $\mu(\cdot)$ is defined in the space $L^2(\mathfrak{J},U)$, representing a Banach space of permissible control functions with U as its underlying Banach space. The operator B is bounded and linear, mapping from U to X. Consequently, for the equations (4.4)-(4.5), there exists a solution in a mild form, expressed as:

$$
\begin{aligned}
\aleph(t)={}&\aleph_0-\xi_1(t_0,\aleph_0)+\xi_1(t,\aleph(t))+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\Upsilon(\sigma)\aleph(\sigma)ds\\
&+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\xi_2(\sigma,\aleph(\sigma))ds+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\int_{\sigma}^{t}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho ds\\
&+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho ds+\frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}Bu(\sigma)ds
\end{aligned}
\tag{4.6}
$$

for $t\in\mathfrak{J}$.

Definition 4.3. The fractional system described by equations (4.4)-(4.5) is considered controllable over the interval $\mathfrak{J}$ if, for any given initial states $\aleph_0$ and $\aleph_1$ in the Banach space X, there exists a control function $\mu(t)$ in the space $L^2(\mathfrak{J},U)$. This control function ensures that the mild solution $\aleph(t)$ of equations (4.4)-(4.5) satisfies the conditions $\aleph(t_0)=\aleph_0$ and $\aleph(b)=\aleph_1$.

To establish the foundation of our investigation, we now present the following theorem that addresses the controllability results in the context of the Volterra-Fredholm integro differential equation (4.4)-(4.5).

Theorem 4.4. *Assuming that hypotheses* (B1)-(B4) *are satisfied, we further postulate the following.*

(B5) *The bounded linear operator* $W : L^2(\mathfrak{J}, U) \to X$ *defined as*

$$Wu = \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{b} (b-\sigma)^{\vartheta-1} Bu(\sigma) ds$$

possesses an induced inverse operator W^{-1} *that operates within* $\frac{L^2(\mathfrak{J},U)}{kerW}$.

Additionally, there exist positive constants M_1 *and* M_2 *such that* $\|B\| \leqslant M_1$ *and* $\|W^{-1}\| \leqslant M_2$. *Given these conditions, the system described by* (4.4)-(4.5) *is controllable over the interval* $[t_0, b]$.

Proof. Consider the set of functions Φ_{γ_1}, which consists of all continuous functions denoted by $\aleph$ defined on an interval $\mathfrak{J}$ and taking real values, with the condition that $\|\aleph\|_\infty \leqslant \gamma_1$. With the assumptions provided in hypothesis (B5), we can establish a control based on the properties of an arbitrary function $\aleph(\cdot)$,

$$\mu(t) = W^{-1}\Bigg[\aleph_1 - \aleph_0 + \xi_1(t_0, \aleph_0) - \xi_1(t, \aleph(t)) - \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{b} (b-\sigma)^{\vartheta-1} \Big[\Upsilon(\sigma)\aleph(\sigma) + \xi_2(\sigma, \aleph(\sigma)) \\ + \int_{\sigma}^{t} \zeta_1(\rho, \sigma, \aleph(\sigma)) d\rho + \int_{\sigma}^{b} \zeta_2(\rho, \sigma, \aleph(\sigma)) d\rho\Big] ds\Bigg](t).$$

With the defined control, we will demonstrate that the operator Ψ mapping from the set Φ_{γ_1} to itself, which is defined as follows:

$$\Psi\aleph(t) = \aleph_0 - \xi_1(t_0, \aleph_0) + \xi_1(\sigma, \aleph(\sigma)) + \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{t} (t-\sigma)^{\vartheta-1} \Big[\Upsilon(\sigma)\aleph(\sigma) + \xi_2(\sigma, \aleph(\sigma)) \\ + \int_{\sigma}^{t} \zeta_1(\rho, \sigma, \aleph(\sigma)) d\rho + \int_{\sigma}^{b} \zeta_2(\rho, \sigma, \aleph(\sigma)) d\rho\Big] ds + (B\mu)(\sigma). \tag{4.7}$$

We can conclude that there exists a fixed point for the operator Ψ, where $\mu(t)$ is defined as per equation (4.7). This fixed point serves as the mild solution to the control problem described by equations (4.4)-(4.5). Specifically, it's evident that $\Psi\aleph(b) = \aleph_1$, and this observation implies that the system represented by equations (4.4)-(4.5) is controllable over the interval $[t_0, b]$.

By applying the definition of the control function μ, we obtain the following expression,

$$\Psi\aleph(t) = \aleph_0 - \xi_1(t_0, \aleph_0) + \xi_1(\sigma, \aleph(\sigma)) + \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{t} (t-\sigma)^{\vartheta-1} \Big[\Upsilon(\sigma)\aleph(\sigma) + \xi_2(\sigma, \aleph(\sigma)) \\ + \int_{\sigma}^{t} \zeta_1(\rho, \sigma, \aleph(\sigma)) d\rho + \int_{\sigma}^{b} \zeta_2(\rho, \sigma, \aleph(\sigma)) d\rho\Big] ds + BW^{-1}\Bigg[\aleph_1 - \aleph_0 + \xi_1(t_0, \aleph_0) \\ - \xi_1(t, \aleph(t)) - \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{b} (b-\sigma)^{\vartheta-1} \Big[\Upsilon(\sigma)\aleph(\sigma) \\ + \xi_2(\sigma, \aleph(\sigma)) + \int_{\sigma}^{t} \zeta_1(\rho, \sigma, \aleph(\sigma)) d\rho + \int_{\sigma}^{b} \zeta_2(\rho, \sigma, \aleph(\sigma)) d\rho\Big] ds\Bigg]. \tag{4.8}$$

Given that all the functions included in the operator's definition are continuous, we can conclude that the operator Ψ exhibits continuity.

Expanding upon equation (4.6), for any function $\aleph$ belonging to the set Φ_{γ_1} and for all values of t within the interval $[t_0, b]$, we can establish the following relationship:

$$\begin{aligned}\|\mu(t)\| &\leqslant \|W^{-1}\|\Bigg[\|\aleph_1\| + \|\aleph_0\| + \|\xi_1(t_0,\aleph_0)\| + (L_{\xi_1} + L^*_{\xi_1}\gamma_1) + \frac{\|\Upsilon\|_\infty b^\vartheta\gamma_1}{\Gamma(\vartheta+1)} + \frac{b^\vartheta}{\Gamma(\vartheta+1)}(L_{\xi_2}\gamma_1 + L^*_{\xi_2})\\ &\quad + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_1}\gamma_1 + L^*_{\zeta_1}) + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_2}\gamma_1 + L^*_{\zeta_2})\Bigg]\\ \|\mu(t)\| &\leqslant M_2\Bigg[\|\aleph_1\| + \|\aleph_0\| + \|\xi_1(t_0,\aleph_0)\| + (L_{\xi_1} + L^*_{\xi_1}\gamma_1) + \frac{\|\Upsilon\|_\infty b^\vartheta\gamma_1}{\Gamma(\vartheta+1)} + \frac{b^\vartheta}{\Gamma(\vartheta+1)}(L_{\xi}\gamma_1 + L^*_{\xi})\\ &\quad + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_1}\gamma_1 + L^*_{\zeta_1}) + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_2}\gamma_1 + L^*_{\zeta_2})\Bigg].\end{aligned} \tag{4.9}$$

Applying equations (4.8) and (4.9), we can derive the subsequent result:

$$\begin{aligned}\|\Psi\aleph(t)\| &\leqslant \|\aleph_0\| + \|\xi_1(t_0,\aleph_0)\| + (L_{\xi_1} + L^*_{\xi_1}\gamma_1) + \frac{\|\Upsilon\|_\infty b^\vartheta\gamma_1}{\Gamma(\vartheta+1)} + \frac{b^\vartheta}{\Gamma(\vartheta+1)}(L_{\xi_2}\gamma_1 + L^*_{\xi_2})\\ &\quad + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_1}\gamma_1 + L^*_{\zeta_1}) + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_2}\gamma_1 + L^*_{\zeta_2}) + \frac{b^\vartheta}{\Gamma(\vartheta+1)}M_1\|\mu(t)\|\\ &\leqslant \|\aleph_0\| + \|\xi_1(t_0,\aleph_0)\| + (L_{\xi_1} + L^*_{\xi_1}\gamma_1) + \frac{\|\Upsilon\|_\infty b^\vartheta\gamma_1}{\Gamma(\vartheta+1)} + \frac{b^\vartheta}{\Gamma(\vartheta+1)}(L_{\xi_2}\gamma_1 + L^*_{\xi_2})\\ &\quad + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_1}\gamma_1 + L^*_{\zeta_1}) + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_2}\gamma_1 + L^*_{\zeta_2})\\ &\quad + \frac{b^\vartheta}{\Gamma(\vartheta+1)}M_1\Bigg[M_2\|\aleph_1\| + M_2\|\aleph_0\| + M_2\|\xi_1(t_0,\aleph_0)\| + M_2(L_{\xi_1} + L^*_{\xi_1}\gamma_1) + M_2\frac{\|\Upsilon\|_\infty b^\vartheta\gamma_1}{\Gamma(\vartheta+1)}\\ &\quad + M_2\frac{b^\vartheta}{\Gamma(\vartheta+1)}(L_{\xi_2}\gamma_1 + L^*_{\xi_2}) + M_2\frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_1}\gamma_1 + L^*_{\zeta_1}) + M_2\frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_2}\gamma_1 + L^*_{\zeta_2})\Bigg]\\ &\leqslant \frac{b^\vartheta}{\Gamma(\vartheta+1)}M_1M_2\|\aleph_1\| + \left(1 + \frac{b^\vartheta}{\Gamma(\vartheta+1)}M_1M_2\right)\|\aleph_0\| + \|\xi_1(t_0,\aleph_0)\|(1 + M_1M_2)\\ &\quad + (1 + M_1M_2)(L_{\xi_1} + L^*_{\xi_1}\gamma_1) + \frac{\|\Upsilon\|_\infty b^\vartheta\gamma_1}{\Gamma(\vartheta+1)}\left(1 + \frac{b^\vartheta}{\Gamma(\vartheta+1)}M_1M_2\right)\\ &\quad + \frac{b^\vartheta}{\Gamma(\vartheta+1)}(L_{\xi_2}\gamma_1 + L^*_{\xi_2})\left(1 + \frac{b^\vartheta}{\Gamma(\vartheta+1)}M_1M_2\right) + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_1}\gamma_1 + L^*_{\zeta_1})\left(1 + \frac{b^\vartheta}{\Gamma(\vartheta+1)}M_1M_2\right)\\ &\quad + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_2}\gamma_1 + L^*_{\zeta_2})\left(1 + \frac{b^\vartheta}{\Gamma(\vartheta+1)}M_1M_2\right)\\ &\leqslant \frac{b^\vartheta}{\Gamma(\vartheta+1)}M_1M_2\|\aleph_1\| + \left(1 + \frac{b^\vartheta}{\Gamma(\vartheta+1)}M_1M_2\right)\Bigg[\|\aleph_0\| + \frac{\|\Upsilon\|_\infty b^\vartheta\gamma_1}{\Gamma(\vartheta+1)} + \frac{b^\vartheta}{\Gamma(\vartheta+1)}(L_{\xi_2}\gamma_1 + L^*_{\xi_2})\\ &\quad + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_1}\gamma_1 + L^*_{\zeta_1}) + \frac{b^{(\vartheta+1)}}{(\vartheta+1)\Gamma(\vartheta)}(L_{\zeta_2}\gamma_1 + L^*_{\zeta_2})\Bigg]\\ &\quad + \Big(\|\xi_1(t_0,\aleph_0)\| + (L_{\xi_1} + L^*_{\xi_1}\gamma_1)\Big)(1 + M_1M_2) \leqslant \mathfrak{L}_1.\end{aligned}$$

This demonstrates that the operator Ψ maps the set $\Phi_{\gamma_1} = \{\aleph \in C(\mathfrak{J},\mathbb{R}) : \|\aleph\|_\infty \leqslant \gamma_1\}$ onto itself. We will now proceed to demonstrate that the operator $\Psi : \Phi\gamma_1 \to \Phi_{\gamma_1}$ fulfills all the requirements of the Schauder theorem (Theorem 2.7). The proof will be presented in multiple steps.

Step 1: Ψ is continuous. Let $\aleph_n$ be a sequence such that $\aleph_n \to \aleph$ in Φ_{γ_1},

$$|(\Psi\aleph_n)(t) - (\Psi\aleph)(t)|$$

$$
\begin{aligned}
&\leqslant |\xi_1(t,\aleph_n(t))-\xi_1(t,\aleph(t))| + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}|\Upsilon(\sigma)||\aleph_n(\sigma)-\aleph(\sigma)|ds\\
&\quad + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}|\xi_2(\sigma,\aleph_n(\sigma))-\xi_2(\sigma,\aleph(\sigma))|ds\\
&\quad + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}\left(\int_{\sigma}^{t}|\zeta_1(\rho,\sigma,\aleph_n(\sigma))-\zeta_1(\rho,\sigma,\aleph(\sigma))|d\rho\right.\\
&\quad \left.+\int_{\sigma}^{b}|\zeta_2(\rho,\sigma,\aleph_n(\sigma))-\zeta_2(\rho,\sigma,\aleph(\sigma))|d\rho\right)ds + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t}(t-\sigma)^{\vartheta-1}BW^{-1}\Big[|\xi_1(t,\aleph_n(t))-\xi_1(t,\aleph(t))|\\
&\quad - \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{b}(b-\sigma)^{\vartheta-1}\Big[\Upsilon(\sigma)|\aleph_n(\sigma)-\aleph(\sigma)| + |\xi_2(\sigma,\aleph_n(\sigma))-\xi_2(\sigma,\aleph(\sigma))|\\
&\quad + \int_{\sigma}^{t}|\zeta_1(\rho,\sigma,\aleph_n(\sigma))-\zeta_1(\rho,\sigma,\aleph(\sigma))|d\rho + \int_{\sigma}^{b}|\zeta_2(\rho,\sigma,\aleph_n(\sigma))-\zeta_2(\rho,\sigma,\aleph(\sigma))|d\rho\Big]ds\Big].
\end{aligned}
$$

Due to the continuity of ξ, ζ_1, and ζ_2, we can deduce that: $\|(\Psi\aleph_n)(t)-(\Psi\aleph)(t)\| \to 0$ as $n\to\infty$. As a result, ρ exhibits continuity within the set Φ_{γ_1}.

Step 2: The set $\Psi(\Phi_{\gamma_1})$ possesses a uniform bound. This is evident as $\Psi(\Phi_{\gamma_1}) \subset \Phi_{\gamma_1}$, which implies boundedness.

Step 3: We demonstrate that $\Psi(\Phi_{\gamma_1})$ exhibits equicontinuity.

Consider t_1 and t_2 belonging to the bounded set $[t_0,b]$ in $C(\mathfrak{J},\mathbb{R})$ as in Step 2, along with $\aleph$ from Φ_{γ_1} and $t_1<t_2$. In this context, we have:

$$
\begin{aligned}
&\|(\Psi\aleph)(t_2)-(\Psi\aleph)(t_1)\|\\
&\quad = \Bigg\|\aleph_0 - \xi_1(t_0,\aleph_0) + \xi_1(\sigma,\aleph(\sigma)) + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_2}(t_2-\sigma)^{\vartheta-1}\Upsilon(\sigma)\aleph(\sigma)ds\\
&\qquad + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_2}(t_2-\sigma)^{\vartheta-1}\xi_2(\sigma,\aleph(\sigma))ds + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_2}(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{t_2}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho ds\\
&\qquad + \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_2}(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho ds\\
&\qquad + BW^{-1}\Big[\aleph_1 - \aleph_0 + \xi_1(t_0,\aleph_0) - \xi_1(\sigma,\aleph(\sigma)) - \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_2}(t_2-\sigma)^{\vartheta-1}\Big[\Upsilon(\sigma)\aleph(\sigma)\\
&\qquad + \xi_2(\sigma,\aleph(\sigma)) + \int_{\sigma}^{t_2}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho + \int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho\Big]ds\Big] - \aleph_0 + \xi_1(t_0,\aleph_0)\\
&\qquad - \xi_1(\sigma,\aleph(\sigma)) - \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\Upsilon(\sigma)\aleph(\sigma)ds - \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\xi_2(\sigma,\aleph(\sigma))ds\\
&\qquad - \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\int_{\sigma}^{t_1}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho ds\\
&\qquad - \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho ds - BW^{-1}\Big[\aleph_1 - \aleph_0 + \xi_1(t_0,\aleph_0)\\
&\qquad - \xi_1(\sigma,\aleph(\sigma)) - \frac{1}{\Gamma(\vartheta)}\int_{t_0}^{t_1}(t_1-\sigma)^{\vartheta-1}\Big[\Upsilon(\sigma)\aleph(\sigma) + \xi_2(\sigma,\aleph(\sigma)) + \int_{\sigma}^{t_1}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho\\
&\qquad + \int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho\Big]ds\Big]\Bigg\|\\
&\quad \leqslant \frac{1}{\Gamma(\vartheta)}\Bigg\|\int_{t_1}^{t_2}(t_2-\sigma)^{\vartheta-1}\big[\Upsilon(\sigma)\aleph(\sigma) + \xi_2(\sigma,\aleph(\sigma)) + \int_{\sigma}^{t_2}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho + \int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho\big]ds
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{t_0}^{t_1}\left[(t_1-\sigma)^{\vartheta-1}-(t_2-\sigma)^{\vartheta-1}\right]\Upsilon(\sigma)\aleph(\sigma)ds-\int_{t_0}^{t_1}\left[(t_1-\sigma)^{\vartheta-1}-(t_2-\sigma)^{\vartheta-1}\right]\xi_2(\sigma,\aleph(\sigma))ds\\
&-\int_{t_0}^{t_1}\left[(t_1-\sigma)^{\vartheta-1}\int_{\sigma}^{t_1}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho-(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{t_2}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho\right]ds\\
&-\int_{t_0}^{t_1}\left[(t_1-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho-(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho\right]ds\\
&+BW^{-1}\left[\frac{1}{\Gamma(\vartheta)}\int_{t_1}^{t_2}(t_2-\sigma)^{\vartheta-1}\left[\Upsilon(\sigma)\aleph(\sigma)+\xi_2(\sigma,\aleph(\sigma))+\int_{\sigma}^{t_2}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho\right.\right.\\
&\left.+\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho\right]ds-\int_{t_0}^{t_1}\left[(t_1-\sigma)^{\vartheta-1}-(t_2-\sigma)^{\vartheta-1}\right]\Upsilon(\sigma)\aleph(\sigma)ds\\
&-\int_{t_0}^{t_1}\left[(t_1-\sigma)^{\vartheta-1}-(t_2-\sigma)^{\vartheta-1}\right]\xi_2(\sigma,\aleph(\sigma))ds\\
&-\int_{t_0}^{t_1}\left[(t_1-\sigma)^{\vartheta-1}\int_{\sigma}^{t_1}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho-(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{t_2}\zeta_1(\rho,\sigma,\aleph(\sigma))d\rho\right]ds\\
&\left.\left.-\int_{t_0}^{t_1}\left[(t_1-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho-(t_2-\sigma)^{\vartheta-1}\int_{\sigma}^{b}\zeta_2(\rho,\sigma,\aleph(\sigma))d\rho\right]ds\right]\right\|.
\end{aligned}
$$

As t_2 approaches t_1, we observe that the right-hand side of the preceding equation tends to zero, denoted as $\|(\Psi\aleph)(t_2)-(\Psi\aleph)(t_1)\|\to 0$. Through the combined implications of Steps 1-3 and the Arzela-Ascoli theorem, we establish the continuity and compactness of Ψ. Applying Schauder's theorem, we consequently ascertain the existence of a fixed point $\aleph$ that acts as a solution to the problem (4.4)-(4.5). Therefore, the system (4.4)-(4.5) is controllable on $\mathfrak{J}=[t_0,b]$. This completes the proof of the theorem. □

5. Conclusion

In this paper, the investigation delves into a Volterra-Fredholm integro-differential equation enhanced with fractional Caputo derivatives under specific order conditions. The study rigorously established the existence of solutions, employing the Schauder fixed-point theorem. Moreover, it extended its domain to encompass neutral Volterra-Fredholm integro-differential equations, significantly broadening the scope of the findings. Additionally, the concept of controllability for the obtained solutions was explored, yielding valuable insights into their long-term behavior. The comprehensive approach taken in this study, combining theoretical rigor with practical demonstration, strengthens the significance of the research, contributing to a deeper understanding of these complex mathematical structures.

Acknowledgement

Researchers would like to thank the Deanship of Scientific Research, Qassim University for funding publication of this project.

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