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On boundary values system of sequential hybrid fractional differential equations

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Abstract

This work is devoted to study a broad system of sequential hybrid fractional differential equations (S-HFDEs). Sufficient conditions related to the existence theory for the aforementioned system are investigated by utilizing the coincidence degree theory of topology. The mentioned degree theory is a powerful instrument which can be utilized to examine a wide range of nonlinear problems for qualitative theory. In addition, for the system of boundary value problems (BVPs) of S-HFDEs under consideration, a result concerning the Ullam-Hyers (U-H) stability is also developed. A pertinent example is also given to verify our theoretical results.

Keywords: Caputo derivative, degree theory, nonlinear analysis.

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1. Introduction

Like regular calculus, fractional calculus has received a lot of recent scientific attention. This is because fractional calculus can provide better descriptions of phenomena that occur in the real world. Additionally, these derivatives have a wide range of uses in the description of systems having memory effects. The aforementioned calculus has been applied in a variety of contexts to examine various infectious disease models, such as those in [19, 25, 27]. Additionally, fractional calculus has been extensively employed in the fields of physics, engineering, and cosmology. For illustration, we quote some citations [14, 18, 38]. Here we remark that fractional calculus has significant applications in physics which have recently identified by many researchers (see some applicable results in [17, 36]). In additions, the said area have been used increasingly to investigate various viral infectious diseases including Hepatitis B, and others in [12, 13, 28, 29]. Moreover some interesting applicable in fluid mechanics, vegetable oil process, electromagnetic theory, pine wilt disease can be read in the cited articles as [1, 16, 22–24].

For existence theory and stability analysis, various issues related to the fractional calculus idea have been investigated. Investigating hybrid problems under the aforementioned notions is one of the key areas. In this regard, various boundary and initial condition problems have been researched using modern

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functional analysis tools and nonlinear analysis techniques. For instance, in [9, 35, 37, 40], the theory of hybrid fixed points has been used to the investigation of various hybrid situations. Additionally, the authors have created several results pertaining to existence theory for various BVPs of HFDEs in publications we refer to [2, 8, 10, 11, 34]. These results were developed using the aforementioned techniques. Additionally, in articles [15, 21], various writers examined integral-type FHDEs. The authors' fixed point approach was used to study a system of FHDEs of the thermostat kind. A class of FHDEs has also been studied by writers [3, 4] under integral BVPs. Similar to this, the authors of articles [6, 30–32, 39] used nonlinear functional analysis tools to investigate a variety of issues and HFDEs systems.

Additionally it is remarkable that researchers [33] designed a comprehensive survey about the computation of analytical or semi analytical solutions for double integral equations of hybrid nature. Like wise a class of HFDEs has been studied in [5]. Keeping in mind the interest of the said area and their systems, we consider the nonlinear S-HFDEs system as follows:

$$\begin{aligned}
&{}^{c}\mathbb{D}^{\wp}\Big[\frac{{}^{c}\mathbb{D}^{\upsilon}\mathbf{q}(t)-\sum_{1}^{m}\mathscr{I}^{\rho_i}\mathbb{H}_i(t,\mathbf{q}(t),{}^{c}\,\mathbb{D}^{\S}\mathbf{q}(t))}{\mathbb{L}_1(t,\mathbf{q}(t),{}^{c}\,\mathbb{D}^{\S}\mathbf{q}(t))}\Big]=\mathscr{F}_1(t,\mathbf{r}(t),\mathscr{I}^{\Upsilon}\mathbf{r}(t)),\ t\in I,\\
&{}^{c}\mathbb{D}^{\wp}\Big[\frac{{}^{c}\mathbb{D}^{\upsilon}\mathbf{r}(t)-\sum_{1}^{m}\mathscr{I}^{\rho_i}\mathbb{K}_i(t,\mathbf{r}(t),{}^{c}\,\mathbb{D}^{\S}\mathbf{r}(t))}{\mathbb{L}_2(t,\mathbf{r}(t),{}^{c}\,\mathbb{D}^{\S}\mathbf{r}(t))}\Big]=\mathscr{F}_2(t,\mathbf{q}(t),\mathscr{I}^{\Upsilon}\mathbf{q}(t)),\ t\in I,\\
&\mathbf{q}(0)=\chi_1(\mathbf{r}(\ell)),\ \mathbf{q}^{(j)}(0)=0,\mathbf{q}(1)=\chi_2(\mathbf{r}(\ell)),\\
&\mathbf{r}(0)=\chi_1(\mathbf{q}(\ell)),\ \mathbf{r}^{(j)}(0)=0,\ \mathbf{r}(1)=\chi_2(\mathbf{q}(\ell)),\ j=1,2,\dots,k-1,
\end{aligned}\tag{1.1}$$

such that $0<\wp\leqslant 1$, $\upsilon\in(k-1,k]$, $\S\in(k-2,k-1]$, with $k\geqslant 2$, $0<\ell<1$, the mappings $\mathbb{L}_j,\mathbb{H}_i,\mathbb{K}_i,\mathscr{F}_1,\mathscr{F}_2$ $\in\Big(I\times\mathbb{R}\times\mathbb{R},\mathbb{R}\Big)$ $(i=1,2,\dots,m)$, also $\mathscr{F}_1,\mathscr{F}_2$ satisfy the Caratheódory criteria, and $\mathbb{L}_j\neq 0$, $j=1,2$. In addition, χ_1, $\chi_2:\mathbb{R}\to\mathbb{R}$ are nonlinear mappings.

To create outcomes of existence and uniqueness, we employ the methods of measure of non-compactness and degree theory as a significant outcome of the qualitative theory, stability analysis. As a result, we create the framework for consistent outcomes. In this perspective, we establish a thorough examination of the difficulties under consideration using the methods indicated in articles [7, 20, 26]. Stability analysis has taken a few HFDE issues into account as U-H types. We thereby confirm the aforementioned stability finding for the problem under consideration.

Our paper is structured as follows. Section 1 is devoted to literature review and introduction. Section 2 is devoted to basic requirements. Section 3 contains first part of our main results about the existence theory. Section 4 contains stability results. Section 5 includes example. Section 6 contains a brief conclusion.

2. Fundamental results

Some basic results we need to recollect from [19, 27] for further analysis in this work as follows.

Definition 2.1. The fractional order integration of a function $\mathbf{q}\in L^1([0,1])$ is given by

$$\mathscr{I}^{r}\mathbf{q}(t)=\frac{1}{\Gamma(r)}\int_0^t(t-\zeta)^{r-1}\mathbf{q}(\zeta)d\zeta.$$

Definition 2.2. Let $\mathbf{q}\in C[0,1]$, then derivative in the Caputo sense of fractional order is given by

$${}^{c}\mathbb{D}^{r}\mathbf{q}(t)=\frac{1}{\Gamma(k-r)}\int_0^t(t-\zeta)^{k-r-1}\mathbf{q}^{(k)}(\zeta)d\zeta,$$

where $k-1=[r]$.

Theorem 2.3. *The solution of*

$$\mathscr{I}^{r}[{}^{c}\mathbb{D}^{r}\mathbf{q}(t)] = y(t),\ r \in (k-1,k],$$

is given by

$$\mathscr{I}^{r}[{}^{c}\mathbb{D}^{r}\mathbf{q}(t)] = y(t) + \mathbf{m}_0 + \mathbf{m}_1 t + \mathbf{m}_2 t^2 + \cdots + \mathbf{m}_{k-1}t^{k-1},$$

for $\mathbf{m}_i \in \mathbb{R}$.

Let $\mathbb{E}$ be the Banach space defined as $\mathbb{E} = \{\mathbf{q} \in C(I) :{}^{c} \mathbb{D}^{\upsilon-1}\mathbf{q} \in C(I)\}$ under the norm $\|\mathbf{q}\|_{\S} = \max_{0\leqslant t\leqslant 1}|\mathbf{q}(t)| + \max_{0\leqslant t\leqslant 1}|{}^{c}\mathbb{D}^{\S}\mathbf{q}|$. Also, $\mathbb{E}\times\mathbb{E}$ is a Banach space with norm $\|(\mathbf{q},\mathbf{r})\|_{\S} = \max\{\|\mathbf{q}\|_{\S}, \|\mathbf{r}\|_{\S}\}$.

3. Existence criteria

For system (1.1), we derive one of the main results in this portion.

Lemma 3.1. *The solution of* (1.1) *is given by*

$$\begin{aligned}
\mathbf{q}(t) &= \mathscr{I}^{\upsilon}\chi_1(t,\mathbf{q}(t),\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t)(t)) + \sum_{1}^{m}\mathscr{I}^{\upsilon-\rho_i}\mathbb{H}_i(t,\mathbf{q}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t)) + \chi_1(\nu(\ell))\\
&\quad + \frac{\mathscr{I}^{\upsilon}\mathbb{L}_1(t,\mathbf{q}(t),{}^{c}\mathbb{D}^{\S-1}\mathbf{q}(t))}{\mathscr{I}^{\upsilon}\mathbb{L}_1(1,\mathbf{q}(1),{}^{c}\mathbb{D}^{\S}\mathbf{q}(1))}\big(\chi_2(\nu(\ell)) - \chi_1(\nu(\ell)) - \mathscr{I}^{\upsilon}\chi_1(1,\mathbf{q}(1),\nu(1),{}^{c}\mathbb{D}^{\S}\mathbf{q}(1))\\
&\quad - \mathscr{I}^{\upsilon}\mathbb{L}_1(1,\mathbf{q}(1),{}^{c}\mathbb{D}^{\S}\mathbf{q}(1)) - \sum_{1}^{m}\mathscr{I}^{\upsilon+\rho_i}\mathbb{H}_i(1,\mathbf{q}(1),{}^{c}\mathbb{D}^{\S}\mathbf{q}(1)),\\
\mathbf{r}(t) &= \mathscr{I}^{\upsilon}\chi_2(t,\mathbf{q}(t),\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t)) + \mathbf{q}m_1^m\mathscr{I}^{\upsilon-\rho_i}\mathbb{K}_i(t,\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t)) + \chi_1(\mathbf{q}(\ell))\\
&\quad + \frac{\mathscr{I}^{\upsilon}\mathbb{L}_2(t,\mathbf{r}(t),{}^{c}\mathbb{D}^{\S-1}\mathbf{r}(t))}{\mathscr{I}^{\upsilon}\mathbb{L}_2(1,\mathbf{r}(1),{}^{c}\mathbb{D}^{\S}\mathbf{r}(1))}\big(\chi_2(\mathbf{q}(\ell)) - \chi_1(\mathbf{q}(\ell)) - \mathscr{I}^{\upsilon}\chi_2(1,\mathbf{q}(1),\mathbf{r}(1),{}^{c}\mathbb{D}^{\S}\mathbf{r}(1))\\
&\quad - \mathscr{I}^{\upsilon}\mathbb{L}_2(1,\mathbf{r}(1),{}^{c}\mathbb{D}^{\S}\mathbf{r}(1)) - \sum_{1}^{m}\mathscr{I}^{\upsilon+\rho_i}\mathbb{K}_i(1,\mathbf{r}(1),{}^{c}\mathbb{D}^{\S}\mathbf{r}(1)),
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
\chi_1(t,\mathbf{q}(t),\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t)) &= \mathbb{L}_1(t,\mathbf{q}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t))\mathscr{I}^{\wp}\mathscr{F}_1(t,\mathbf{r}(t),\mathscr{I}^{\gamma}\mathbf{r}(t)),\\
\chi_2(t,\mathbf{q}(t),\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t)) &= \mathbb{L}_2(t,\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t))\mathscr{I}^{\wp}\mathscr{F}_2(t,\mathbf{q}(t),\mathscr{I}^{\gamma}\mathbf{q}(t)),\\
&\mathscr{I}^{\upsilon}\chi_1(1,\mathbf{q}(1),\mathbf{r}(1),{}^{c}\mathbb{D}^{\S}\mathbf{q}(1)),\ \mathscr{I}^{\upsilon}\mathbb{L}_1(1,\mathbf{q}(1),{}^{c}\mathbb{D}^{\S}\mathbf{q}(1)),\\
&\mathscr{I}^{\upsilon}\chi_2(1,\mathbf{q}(1),\mathbf{r}(1),{}^{c}\mathbb{D}^{\S}\mathbf{r}(1)),\ \mathscr{I}^{\upsilon}\mathbb{L}_2(1,\mathbf{r}(1),{}^{c}\mathbb{D}^{\S}\mathbf{r}(1)),
\end{aligned}$$

stand for the values of the given integrals at $t = 1$. *In addition, the integrals*

$$\mathscr{I}^{\upsilon}\chi_j(t,\mathbf{q}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t)),\quad \mathscr{I}^{\upsilon}\mathbb{L}_j[(t,\mathbf{q}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t))],\quad j=1,2,$$

and

$$\mathscr{I}^{\upsilon+\rho_i}\mathbb{H}_i(1,\mathbf{q}(1),{}^{c}\mathbb{D}^{\S}\mathbf{q}(1)),\quad \mathscr{I}^{\upsilon+\rho_i}\mathbb{K}_i(1,\mathbf{q}(1),{}^{c}\mathbb{D}^{\S}\mathbf{q}(1)),$$

stand for value at $t=1$ *of* $\mathscr{I}^{\upsilon+\rho_i}\mathbb{H}_i(t,\mathbf{q}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t))$, $\mathscr{I}^{\upsilon+\rho_i}\mathbb{K}_i(t,\mathbf{q}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t))$.

Define operators $\mathbb{A}_i$, $\mathbb{B}_i, \mathbb{C}_i : \mathbb{E}\to\mathbb{E}$ by

$$\begin{aligned}
\mathbb{A}_1(\mathbf{q},\mathbf{r})(t) &= \mathscr{I}^{\upsilon}\chi_1(t,\mathbf{q}(t),\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t)) + \sum_{1}^{m}\mathscr{I}^{\upsilon+\rho_i}\mathbb{H}_i(t,\mathbf{q}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t)) + \chi_1(\mathbf{r}(\ell)),\\
\mathbb{A}_2(\mathbf{q},\mathbf{r})(t) &= \mathscr{I}^{\upsilon}\chi_2(t,\mathbf{q}(t),\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t)) + \sum_{1}^{m}\mathscr{I}^{\upsilon+\rho_i}\mathbb{K}_i(t,\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t)) + \chi_2(\mathbf{q}(\ell)),
\end{aligned}$$

$$
\begin{aligned}
\mathbb{B}_1(\mathbf{q})(t) &= \frac{\mathscr{I}^{\upsilon}\mathbb{L}_1(t,\mathbf{q}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t))}{\mathscr{I}^{\upsilon}\mathbb{L}_1(1,\mathbf{q}(1),{}^{c}\mathbb{D}^{\S}\mathbf{q}(1))},\\
\mathbb{B}_2(\mathbf{r})(t) &= \frac{\mathscr{I}^{\upsilon}\mathbb{L}_2(t,\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t))}{\mathscr{I}^{\upsilon}\mathbb{L}_2(1,\mathbf{r}(1),{}^{c}\mathbb{D}^{\S}\mathbf{r}(1))},\\
\mathbb{C}_1(\mathbf{q},\mathbf{r})(t) &= \chi_2(\mathbf{r}(\ell)) - \chi_1(\mathbf{r}(\ell) - \mathscr{I}^{\upsilon}\chi_1(1,\mathbf{q}(1),\mathbf{r}(1),{}^{c}\mathbb{D}^{\S}\mathbf{q}(1)) - \mathscr{I}^{\upsilon}\mathbb{L}_1(1,\mathbf{q}(1),{}^{c}\mathbb{D}^{\S}\mathbf{q}(1))\\
&\quad - \sum_{1}^{m}\mathscr{I}^{\upsilon+\rho_i}\mathbb{H}_i(1,\mathbf{q}(1),{}^{c}\mathbb{D}^{\S}\mathbf{q}(1))),\\
\mathbb{C}_2(\mathbf{q},\mathbf{r})(t) &= \chi_2(\mathbf{q}(\ell)) - \chi_1(\mathbf{q}(\ell) - \mathscr{I}^{\upsilon}\chi_2(1,\mathbf{q}(1),\mathbf{r}(1),{}^{c}\mathbb{D}^{\S}\mathbf{r}(1)) - \mathscr{I}^{\upsilon}\mathbb{L}_2(1,\mathbf{r}(1),{}^{c}\mathbb{D}^{\S}\mathbf{r}(1))\\
&\quad - \sum_{1}^{m}\mathscr{I}^{\upsilon+\rho_i}\mathbb{K}_i(1,\mathbf{r}(1),{}^{c}\mathbb{D}^{\S}\mathbf{r}(1))),
\end{aligned}
\tag{3.2}
$$

then (3.1) of operators is expressed as

$$
\begin{aligned}
\mathbf{q}(t) &= \mathbb{A}_1(\mathbf{q},\mathbf{r})(t) + \mathbb{B}_1(\mathbf{q})(t)\mathbb{C}_1(\mathbf{q},\mathbf{r})(t) = \mathfrak{T}_1(\mathbf{q},\mathbf{r})(t), t\in I,\\
\mathbf{r}(t) &= \mathbb{A}_2(\mathbf{q},\mathbf{r})(t) + \mathbb{B}_2(\mathbf{r})(t)\mathbb{C}_2(\mathbf{q},\mathbf{r})(t) = \mathfrak{T}_2(\mathbf{q},\mathbf{r})(t), t\in I.
\end{aligned}
\tag{3.3}
$$

Using the definition of the direct product

$$
(\mathbb{B}_1(\mathbf{q})\mathbb{C}_1(\mathbf{q},\mathbf{r}),\mathbb{B}_2(\mathbf{r})\mathbb{C}_2(\mathbf{q},\mathbf{r})) = (\mathbb{B}_1(\mathbf{q}),\mathbb{B}_2(\mathbf{r}))(\mathbb{C}_1(\mathbf{q},\mathbf{r}),\mathbb{C}_2(\mathbf{q},\mathbf{r})),
$$

we write (3.3) as an operator equation

$$
(\mathbf{q},\mathbf{r})(t) = \mathbb{A}(\mathbf{q},\mathbf{r})(t) + \mathbb{B}(\mathbf{q},\mathbf{r})(t)\mathbb{C}(\mathbf{q},\mathbf{r})(t) = \mathfrak{T}(\mathbf{q},\mathbf{r})(t), t\in I,
\tag{3.4}
$$

where $\mathbb{A} = (\mathbb{A}_1,\mathbb{A}_2)$, $\mathbb{B} = (\mathbb{B}_1,\mathbb{B}_2)$, $\mathbb{C} = (\mathbb{C}_1,\mathbb{C}_2)$. Further, (3.2) yields

$$
\begin{aligned}
{}^{c}\mathbb{D}^{\S}(\mathbb{A}_1(\mathbf{q},\mathbf{r})(t)) &= \mathscr{I}^{(\upsilon-\S)}\chi_1(t,\mathbf{q}(t),\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t)) + \sum_{1}^{m}\mathscr{I}^{(\upsilon+\rho_i-\S)}\mathbb{H}_i(t,\mathbf{q}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t)),\\
{}^{c}\mathbb{D}^{\S}(\mathbb{A}_2(\mathbf{q},\mathbf{r})(t)) &= \mathscr{I}^{(\upsilon-\S)}\chi_2(t,\mathbf{q}(t),\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t)) + \sum_{1}^{m}\mathscr{I}^{(\upsilon+\rho_i-\S)}\mathbb{K}_i(t,\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t)),\\
{}^{c}\mathbb{D}^{\S}(\mathbb{B}_1(\mathbf{q})(t) &= \frac{\mathscr{I}^{(\upsilon-\S)}\mathbb{L}_1(t,\mathbf{q}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t))}{\mathscr{I}^{\upsilon}\mathbb{L}_1(1,\mathbf{q}(1),{}^{c}\mathbb{D}^{\S}\mathbf{q}(1))},\\
{}^{c}\mathbb{D}^{\S}(\mathbb{B}_2(\mathbf{r})(t) &= \frac{\mathscr{I}^{(\upsilon-\S)}\mathbb{L}_2(t,\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t))}{\mathscr{I}^{\upsilon}\mathbb{L}_2(1,\mathbf{r}(1),{}^{c}\mathbb{D}^{\S}\mathbf{r}(1))},\\
{}^{c}\mathbb{D}^{\S}(\mathbb{C}_1(\mathbf{q},\mathbf{r})(t)) &= 0,\ {}^{c}\mathbb{D}^{\S}(\mathbb{C}_2(\mathbf{q},\mathbf{r})(t)) = 0.
\end{aligned}
\tag{3.5}
$$

Using (3.2) and (3.5), we obtain

$$
\begin{aligned}
|\mathbb{A}_1(\mathbf{q},\mathbf{r})| &\leqslant |\mathscr{I}^{\upsilon}\chi_1(t,\mathbf{q}(t),\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t))| + \sum_{1}^{m}|\mathscr{I}^{(\upsilon+\rho_i)}\mathbb{H}_i(t,\mathbf{q}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t))| + |\chi_1(\mathbf{r})|,\\
|\mathbb{A}_2(\mathbf{q},\mathbf{r})| &\leqslant |\mathscr{I}^{\upsilon}\chi_2(t,\mathbf{q}(t),\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t))| + \sum_{1}^{m}|\mathscr{I}^{(\upsilon+\rho_i)}\mathbb{K}_i(t,\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t))| + |\chi_2(\mathbf{q})|,\\
|{}^{c}\mathbb{D}^{\S}(\mathbb{A}_1(\mathbf{q},\mathbf{r}))| &\leqslant |\mathscr{I}^{(\upsilon-\S)}\chi_1(t,\mathbf{q}(t),\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t))| + \sum_{1}^{m}|\mathscr{I}^{(\upsilon+\rho_i-\S)}\mathbb{H}_i(t,\mathbf{q}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t))|,\\
|{}^{c}\mathbb{D}^{\S}(\mathbb{A}_2(\mathbf{q},\mathbf{r}))| &\leqslant |\mathscr{I}^{(\upsilon-\S)}\chi_2(t,\mathbf{q}(t),\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t))| + \sum_{1}^{m}|\mathscr{I}^{(\upsilon+\rho_i-\S)}\mathbb{K}_i(t,\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t))|,
\end{aligned}
$$

$$
\begin{aligned}
&|\mathbb{B}_1(\mathbf{q})| \leqslant \frac{|\mathscr{I}^{\upsilon}\mathbb{L}_1(t,\mathbf{q}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t))|}{|\mathscr{I}^{\upsilon}\mathbb{L}_1(1,\mathbf{q}(1),{}^{c}\mathbb{D}^{\S}\mathbf{q}(1))|},\ |{}^{c}\mathbb{D}^{\S}(\mathbb{B}_2(\mathbf{r}))| \leqslant \frac{|\mathscr{I}^{(\upsilon-\S)}\mathbb{L}_2(t,\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t))|}{|\mathscr{I}^{\upsilon}\mathbb{L}_2(1,\mathbf{r}(1),{}^{c}\mathbb{D}^{\S}\mathbf{r}(1))|},\\
&|(\mathbb{C}_1(\mathbf{q},\mathbf{r})| = |\chi_1(\mathbf{r})| + |\chi_2(\mathbf{r})| + |\mathscr{I}^{\upsilon}\chi_1(1,\mathbf{q}(1),\mathbf{r}(1),{}^{c}\mathbb{D}^{\S}\mathbf{q}(1))| + |\mathscr{I}^{\upsilon}\mathbb{L}_1(1,\mathbf{q}(1),{}^{c}\mathbb{D}^{\S}\mathbf{q}(1))|\\
&\qquad + \sum_{1}^{m} |\mathscr{I}^{\upsilon+\rho_i}\mathbb{H}_i(1,\mathbf{q}(1),{}^{c}\mathbb{D}^{\upsilon-1}\mathbf{q}(1)))|,\\
&|(\mathbb{C}_2(\mathbf{q},\mathbf{r})| = |\chi_1(\mathbf{q})| + |\chi_2(\mathbf{q})| + |\mathscr{I}^{\upsilon}\chi_2(1,\mathbf{q}(1),\mathbf{r}(1),{}^{c}\mathbb{D}^{\S}\mathbf{r}(1))| + |\mathscr{I}^{\upsilon}\mathbb{L}_2(1,\mathbf{r}(1),{}^{c}\mathbb{D}^{\S}\mathbf{r}(1))|\\
&\qquad + \sum_{1}^{m} |\mathscr{I}^{\upsilon+\rho_i}\mathbb{K}_i(1,\mathbf{r}(1),{}^{c}\mathbb{D}^{\upsilon-1}\mathbf{r}(1)))|,\\
&|{}^{c}\mathbb{D}^{\S}(\mathbb{C}_1(\mathbf{q},\mathbf{r})| = 0,\ |{}^{c}\mathbb{D}^{\S}(\mathbb{C}_2(\mathbf{q},\mathbf{r})| = 0.
\end{aligned} \tag{3.6}
$$

We describe some data dependence results.

($\mathbb{H}_1$) $\mathbb{L}_j, \mathscr{F}_j, \mathbb{H}_i, \mathbb{K}_i, j = 1,2,\ i = 1,2,\ldots,m$ satisfy the Caratheódory conditions.

($\mathbb{H}_2$) For constants τ_1, τ_2, c_1, c_2, d_1, d_2, one has

$$
|\chi_j(\mathbf{q}_1(t)) - \chi_j(\mathbf{q}_2(t))| \leqslant \tau_j|\mathbf{q}_1 - \mathbf{q}_2|, \qquad |\chi_j(\mathbf{q})| \leqslant c_j|\mathbf{z}| + d_j,\ \mathbf{z},\mathbf{q}_1,\mathbf{q}_2 \in \mathbb{E},\ j = 1,2.
$$

($\mathbb{H}_3$) Let θ_i, θ_i^*, μ_j, $\delta_j : I \to \mathbb{R}$, such that $0 < \xi$, λ, then for $\mathbf{q}_1,\mathbf{q}_2 \in \mathbb{E}$, one has

$$
\begin{aligned}
&|\mathbb{H}_i(t,\mathbf{q}_1(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}_1(t))| \leqslant \theta_i(t)(\|\mathbf{q}_1\| + \|{}^{c}\mathbb{D}^{\S}\mathbf{q}_1\|) + \xi = \theta_i(t)\|\mathbf{q}_1\|_{\S} + \xi,\\
&|\mathbb{K}_i(t,\mathbf{q}_2(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}_2(t))| \leqslant \theta_i^*(t)(\|\mathbf{q}_2\| + \|{}^{c}\mathbb{D}^{\S}\mathbf{q}_2\|) + \xi = \theta_i^*(t)\|\mathbf{q}_2\|_{\S} + \xi,\\
&|\mathbb{L}_j(t,\mathbf{q}_1(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}_1(t))| \leqslant \mu_j(t)(\|\mathbf{q}_1\| + \|{}^{c}\mathbb{D}^{\S}\mathbf{q}_1\|) + \lambda = \mu_j(t)\|\mathbf{q}_1\|_{\S} + \lambda,\\
&|\mathscr{F}_j(t,\mathbf{q}_2(t),\mathscr{I}^{\gamma}\mathbf{q}_2(t))| \leqslant \delta_j(t),\ j = 1,2.
\end{aligned}
$$

($\mathbb{H}_4$) Let $\tau_i > 0$, such that for $\mathbf{q}_1,\mathbf{q}_2 \in \mathbb{E}$,

$$
\begin{aligned}
&|\mathbb{L}_j(t,\mathbf{q}_1(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}_1(t)) - \mathbb{L}_j(t,\mathbf{q}_2(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}_2(t))| \leqslant \mu_0\|\mathbf{q}_1 - \mathbf{q}_2\|_{\S},\\
&|\mathbb{H}_i(t,\mathbf{q}_1(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}_1(t)) - \mathbb{H}_i(t,\mathbf{q}_2(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}_2(t))| \leqslant \theta_i\|\mathbf{q}_1 - \mathbf{q}_2\|_{\S},\\
&|\mathbb{K}_i(t,\mathbf{q}_1(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}_1(t)) - \mathbb{K}_i(t,\mathbf{q}_2(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}_2(t))| \leqslant \theta_i^*\|\mathbf{q}_1 - \mathbf{q}_2\|_{\S},\\
&|\mathscr{F}_j(t,\mathbf{q}_1(t),\mathscr{I}^{\gamma}\mathbf{q}_1(t)) - \mathscr{F}_j(t,\mathbf{q}_2(t),\mathscr{I}^{\gamma}\mathbf{q}_2(t))| \leqslant \delta_0\|\mathbf{q}_1 - \mathbf{q}_2\|,\ j = 1,2,
\end{aligned}
$$

where

$$
\mu_0 = \max_{t\in I}\mu_j(t), \quad \delta_0 = \max_{t\in I}\delta_j(t), \quad \theta_i = \max_{t\in I}\theta_i(t), \quad \theta_i^* = \max_{t\in I}\theta_i^*(t),\ i = 1,2,3,\ldots,m.
$$

Inview of ($\mathbb{H}_3$), one has

$$
\begin{aligned}
|\chi_1(t,\mathbf{q}(t),\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t))| &\leqslant \frac{\delta_0}{\Gamma(\wp+1)}(\mu_0\|\mathbf{q}\|_{\S} + \lambda),\\
|\chi_2(t,\mathbf{q}(t),\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t))| &\leqslant \frac{\delta_0}{\Gamma(\wp+1)}(\mu_0\|\mathbf{r}\|_{\S} + \lambda),\\
|\mathscr{I}^{\upsilon}\chi_1(t,\mathbf{q}(t),\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t))| &\leqslant \frac{\delta_0}{\Gamma(\upsilon+1)\Gamma(\wp+1)}(\mu_0\|\mathbf{q}\|_{\S} + \lambda),\\
|\mathscr{I}^{\upsilon}\chi_2(t,\mathbf{q}(t),\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t))| &\leqslant \frac{\delta_0}{\Gamma(\upsilon+1)\Gamma(\wp+1)}(\mu_0\|\mathbf{r}\|_{\S} + \lambda),\\
|\mathscr{I}^{(\upsilon-\S)}\chi_1(t,\mathbf{q}(t),\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t))| &\leqslant \frac{\delta_0}{\Gamma(\upsilon-\S+1)\Gamma(\wp+1)}(\mu_0\|\mathbf{q}\|_{\S} + \lambda),\\
|\mathscr{I}^{(\upsilon-\S)}\chi_2(t,\mathbf{q}(t),\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t))| &\leqslant \frac{\delta_0}{\Gamma(\upsilon-\S+1)\Gamma(\wp+1)}(\mu_0\|\mathbf{r}\|_{\S} + \lambda).
\end{aligned} \tag{3.7}
$$

Similarly, one has

$$
\begin{aligned}
|\mathscr{I}^{\upsilon}\mathbb{L}_j(t,\mathbf{q}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t))| &\leqslant \frac{1}{\Gamma(\upsilon+1)}(\mu_0\|\mathbf{q}\|_{\S}+\lambda),\\
|\mathscr{I}^{(\upsilon+\rho_i)}\mathbb{H}_i(t,\mathbf{q}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t))| &\leqslant \frac{1}{\Gamma(\upsilon+\rho_i+1)}(\theta_i\|\mathbf{q}\|_{\S}+\xi),\\
|\mathscr{I}^{(\upsilon+\rho_i)}\mathbb{K}_i(t,\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t))| &\leqslant \frac{1}{\Gamma(\upsilon+\rho_i+1)}(\theta_i^*\|\mathbf{r}\|_{\S}+\xi),\\
|\mathscr{I}^{(\upsilon+\rho_i-\S)}\mathbb{H}_i(t,\mathbf{q}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t))| &\leqslant \frac{1}{\Gamma(\upsilon+\rho_i-\S+1)}(\theta_i\|\mathbf{q}\|_{\S}+\xi),\\
|\mathscr{I}^{(\upsilon+\rho_i-\S)}\mathbb{K}_i(t,\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t))| &\leqslant \frac{1}{\Gamma(\upsilon+\rho_i-\S+1)}(\theta_i^*\|\mathbf{r}\|_{\S}+\xi).
\end{aligned}
\tag{3.8}
$$

Using (3.6), (3.7), and (3.8), and $(\mathbb{H}_2)$, $(\mathbb{H}_3)$, one has

$$
\begin{aligned}
|(\mathbb{A}_1(\mathbf{q},\mathbf{r})(t)| &\leqslant \frac{\delta_0(\mu_0\|\mathbf{q}\|_{\S}+\lambda)}{\Gamma(\wp+1)\Gamma(\upsilon+1)}+\sum_1^m\frac{(\theta_i\|\mathbf{q}\|_{\S}+\xi)}{\Gamma(\upsilon+\rho_i+1)}+c_1|\mathbf{r}|+d_1,\\
|(\mathbb{A}_2(\mathbf{q},\mathbf{r})(t)| &\leqslant \frac{\delta_0(\mu_0\|\mathbf{r}\|_{\S}+\lambda)}{\Gamma(\wp+1)\Gamma(\upsilon+1)}+\sum_1^m\frac{(\theta_i^*\|\mathbf{r}\|_{\S}+\xi)}{\Gamma(\upsilon+\rho_i+1)}+c_2|\mathbf{q}|+d_2,\\
|{}^{c}\mathbb{D}^{\S}(\mathbb{A}_1(\mathbf{q},\mathbf{r})(t)| &\leqslant \frac{\delta_0(\mu_0\|\mathbf{q}\|_{\S}+\lambda)}{\Gamma(\wp+1)\Gamma(\upsilon-\S+1)}+\sum_1^m\frac{(\theta_i\|\mathbf{q}\|_{\S}+\xi)}{\Gamma(\upsilon+\rho_i-\S+1)},\\
|{}^{c}\mathbb{D}^{\S}(\mathbb{A}_2(\mathbf{q},\mathbf{r})(t)| &\leqslant \frac{\delta_0(\mu_0\|\mathbf{r}\|_{\S}+\lambda)}{\Gamma(\wp+1)\Gamma(\upsilon-\S+1)}+\sum_1^m\frac{(\theta_i^*\|\mathbf{r}\|_{\S}+\xi)}{\Gamma(\upsilon+\rho_i-\S+1)},\\
|(\mathbb{B}_1(\mathbf{q})(t)| &\leqslant \frac{(\mu_0\|\mathbf{q}\|_{\S}+\lambda)}{\Lambda_1\Gamma(\upsilon+1)},\quad |(\mathbb{B}_2(\mathbf{r})(t)| \leqslant \frac{(\mu_0\|\mathbf{r}\|_{\S}+\lambda)}{\Lambda_2\Gamma(\upsilon+1)},\\
|{}^{c}\mathbb{D}^{\S}(\mathbb{B}_1(\mathbf{q})(t)| &\leqslant \frac{(\mu_0\|\mathbf{q}\|_{\S}+\lambda)}{\Lambda_1\Gamma(\upsilon-\S+1)},\quad |{}^{c}\mathbb{D}^{\S}(\mathbb{B}_2(\mathbf{r})(t)| \leqslant \frac{(\mu_0\|\mathbf{r}\|_{\S}+\lambda)}{\Lambda_2\Gamma(\upsilon-\S+1)},\\
|(\mathbb{C}_1(\mathbf{q},\mathbf{r})(t)| &\leqslant \frac{\delta_0(\mu_0\|\mathbf{q}\|_{\S}+\lambda)}{\Gamma(\upsilon+1)\Gamma(\wp+1)}+\sum_1^m\frac{(\theta_i\|\mathbf{q}\|_{\S}+\xi)}{\Gamma(\upsilon+\rho_i+1)}+c_1|\mathbf{r}|+d_1+c_2|\mathbf{q}|+d_2,\\
|(\mathbb{C}_2(\mathbf{q},\mathbf{r})(t)| &\leqslant \frac{\delta_0(\mu_0\|\mathbf{r}\|_{\S}+\lambda)}{\Gamma(\upsilon+1)\Gamma(\wp+1)}+\sum_1^m\frac{(\theta_i^*\|\mathbf{r}\|_{\S}+\xi)}{\Gamma(\upsilon+\rho_i+1)}+c_1|\mathbf{r}|+d_1+c_2|\mathbf{q}|+d_2,\\
&|{}^{c}\mathbb{D}^{\S}(\mathbb{C}_1(\mathbf{q},\mathbf{r})(t)|=0,\ |{}^{c}\mathbb{D}^{\S}(\mathbb{C}_2(\mathbf{q},\mathbf{r})(t)|=0,
\end{aligned}
\tag{3.9}
$$

where $\Lambda_j=|\mathscr{I}^{\upsilon}\mathbb{L}_j(1,\mathbf{q}(1),{}^{c}\mathbb{D}^{\S}\mathbf{q}(1)),\ j=1,2$. Under hypothesis $(\mathbb{H}_4)$, for $j=1,2$ one has

$$
\begin{aligned}
|\mathscr{I}^{\wp}\mathscr{F}_j(t,\mathbf{q}_1(t),\mathscr{I}^{\gamma}\mathbf{q}_1(t))-\mathscr{I}^{\wp}\mathscr{F}_j(t,\mathbf{q}_2(t),\mathscr{I}^{\gamma}\mathbf{q}_2(t))| &\leqslant \frac{\S_0}{\Gamma(\wp+1)}\|\mathbf{q}_1-\mathbf{q}_2\|,\\
|\chi_j(t,\mathbf{q}_1(t),\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}_1(t))-\chi_j(t,\mathbf{q}_2(t),\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}_2(t))| &\leqslant \frac{\delta_0\mu_0}{\Gamma(\wp+1)}\|\mathbf{q}_1-\mathbf{q}_2\|_{\S}.
\end{aligned}
\tag{3.10}
$$

Further $t_2>t_1$, one has

$$
\begin{aligned}
&\mathscr{I}^{\upsilon}\chi_j(t_1,\mathbf{q}(t_1),\mathbf{r}(t_1),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t_1))-\mathscr{I}^{\upsilon}\chi_j(t_2,\mathbf{q}(t_2),\mathbf{r}(t_2),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t_2))\\
&\quad=\frac{\int_0^{t_1}(t_1-s)^{\upsilon-1}\chi_j(s,\mathbf{q}(s),\mathbf{r}(s),{}^{c}\mathbb{D}^{\S}\mathbf{q}(s))ds-\int_0^{t_2}(t_2-s)^{\upsilon-1}\chi_j(s,\mathbf{q}(s),\mathbf{r}(s),{}^{c}\mathbb{D}^{\S}\mathbf{q}(s))ds}{\Gamma(\upsilon)}\\
&\quad=\frac{1}{\Gamma(\upsilon)}\Big[\int_0^{t_1}((t_1-s)^{\upsilon-1}-(t_2-s)^{\upsilon-1})\chi_j(s,\mathbf{q}(s),\mathbf{r}(s),{}^{c}\mathbb{D}^{\S}\mathbf{q}(s))ds\\
&\qquad+\int_{t_1}^{t_2}((t_2-s)^{\upsilon-1}-(t_2-s)^{\upsilon-1})\chi_j(s,\mathbf{q}(s),\mathbf{r}(s),{}^{c}\mathbb{D}^{\S}\mathbf{q}(s))ds\Big].
\end{aligned}
$$

Hence, it follows that

$$\begin{aligned}
&|\mathscr{I}^{\upsilon}\chi_j(t_1,\mathbf{q}(t_1),\mathbf{r}(t_1)^{c}\,\mathbb{D}^{\S}\mathbf{q}(t_1)) - \mathscr{I}^{\upsilon}\chi_j(t_2,\mathbf{q}(t_2),y(t_2),^{c}\,\mathbb{D}^{\S}\mathbf{q}(t_2))|\\
&\quad\leqslant \frac{\|\Psi(s,\mathbf{q}(s),\mathbf{r}(s),^{c}\,\mathbb{D}^{\S}\mathbf{q}(s)\|}{\Gamma(\upsilon+1)}(2(t_2-t_1)^{\upsilon}+t_1^{\upsilon}-t_2^{\upsilon}),
\end{aligned}$$

which in view of (3.7) implies that

$$\begin{aligned}
&|\mathscr{I}^{\upsilon}\chi_j(t_1,\mathbf{q}(t_1),\mathbf{r}(t_1),^{c}\,\mathbb{D}^{\S}\mathbf{q}(t_1)) - \mathscr{I}^{\upsilon}\chi_j(t_2,\mathbf{q}(t_2),\mathbf{r}(t_2),^{c}\,\mathbb{D}^{\S}\mathbf{q}(t_2))|\\
&\quad\leqslant \frac{\delta_0(\mu_0\|\mathbf{q}\|_{\S}+\lambda)}{\Gamma(\wp+1)\Gamma(\upsilon+1)}(2(t_2-t_1)^{\upsilon}+t_1^{\upsilon}-t_2^{\upsilon}).
\end{aligned}$$

Similarly, in view of (3.7), we obtain

$$\begin{aligned}
&|\mathscr{I}^{\upsilon-\S}\chi_j(t_1,\mathbf{q}(t_1),\mathbf{r}(t_1),^{c}\,\mathbb{D}^{\S}\mathbf{q}(t_1)) - \mathscr{I}^{\upsilon-\S}\chi_j(t_2,\mathbf{q}(t_2),\mathbf{r}(t_2),^{c}\,\mathbb{D}^{\S}\mathbf{q}(t_2))|\\
&\quad\leqslant \frac{\delta_0(\mu_0\|\mathbf{q}\|_{\S}+\lambda)}{\Gamma(\wp+1)\Gamma(\upsilon-\S)}(2(t_2-t_1)^{\upsilon-\S}+t_1^{\upsilon-\S}-t_2^{\upsilon-\S}),\\
&|\mathscr{I}^{\upsilon+\rho_i}\mathbb{H}_i(t_1,\mathbf{q}(t_1),^{c}\,\mathbb{D}^{\S}\mathbf{q}(t_1)) - \mathscr{I}^{\upsilon+\rho_i}\mathbb{H}_i(t_2,\mathbf{q}(t_2),^{c}\,\mathbb{D}^{\S}\mathbf{q}(t_2))|\\
&\quad\leqslant \frac{(\theta_i\|\mathbf{q}\|_{\S}+\xi)}{\Gamma(\upsilon+\rho_i+1)}(2(t_2-t_1)^{\upsilon+\rho_i}+t_1^{\upsilon+\rho_i}-t_2^{\upsilon+\rho_i}),\\
&|\mathscr{I}^{\upsilon+\rho_i-\S}\mathbb{H}_i(t_1,\mathbf{q}(t_1),^{c}\,\mathbb{D}^{\S}\mathbf{q}(t_1)) - \mathscr{I}^{\upsilon+\rho_i-\S}\mathbb{H}_i(t_2,\mathbf{q}(t_2),^{c}\,\mathbb{D}^{\S}\mathbf{q}(t_2))|\\
&\quad\leqslant \frac{(\theta_i\|\mathbf{q}\|_{\S}+\xi)}{\Gamma(\upsilon+\rho_i-\S+1)}(2(t_2-t_1)^{\upsilon+\rho_i-\S}+t_1^{\upsilon+\rho_i-\S}-t_2^{\upsilon+\rho_i-\S}),\\
&|\mathscr{I}^{\upsilon+\rho_i}\mathbb{K}_i(t_1,\mathbf{r}(t_1),^{c}\,\mathbb{D}^{\S}\mathbf{r}(t_1)) - \mathscr{I}^{\upsilon+\rho_i}\mathbb{K}_i(t_2,\mathbf{r}(t_2),^{c}\,\mathbb{D}^{\S}\mathbf{r}(t_2))|\\
&\quad\leqslant \frac{(\theta_i^{*}\|\mathbf{r}\|_{\S}+\xi)}{\Gamma(\upsilon+\rho_i+1)}(2(t_2-t_1)^{\upsilon+\rho_i}+t_1^{\upsilon+\rho_i}-t_2^{\upsilon+\rho_i}),\\
&|\mathscr{I}^{\upsilon+\rho_i-\S}\mathbb{K}_i(t_1,\mathbf{r}(t_1),^{c}\,\mathbb{D}^{\S}\mathbf{r}(t_1)) - \mathscr{I}^{\upsilon+\rho_i-\S}\mathbb{K}_i(t_2,\mathbf{r}(t_2),^{c}\,\mathbb{D}^{\S}\mathbf{r}(t_2))|\\
&\quad\leqslant \frac{(\theta_i^{*}\|\mathbf{r}\|_{\S}+\xi)}{\Gamma(\upsilon+\rho_i-\S+1)}(2(t_2-t_1)^{\upsilon+\rho_i-\S}+t_1^{\upsilon+\rho_i-\S}-t_2^{\upsilon+\rho_i-\S}),\\
&|\mathscr{I}^{\upsilon}\mathbb{L}_j(t_1,\mathbf{q}(t_1),^{c}\,\mathbb{D}^{\S}\mathbf{q}(t_1)) - \mathscr{I}^{\upsilon}\mathbb{L}_j(t_2,\mathbf{q}(t_2),^{c}\,\mathbb{D}^{\S}\mathbf{q}(t_2))|\\
&\quad\leqslant \frac{((\mu_0\|\mathbf{q}\|_{\S}+\lambda)}{\Gamma(\upsilon+1)}(2(t_2-t_1)^{\upsilon}+t_1^{\upsilon}-t_2^{\upsilon}), \qquad (3.11)\\
&|\mathscr{I}^{\upsilon-\S}\mathbb{L}_j(t_1,\mathbf{q}(t_1),^{c}\,\mathbb{D}^{\S}\mathbf{q}(t_1)) - \mathscr{I}^{\upsilon-\S}\mathbb{L}_j(t_2,\mathbf{q}(t_2),^{c}\,\mathbb{D}^{\S}\mathbf{q}(t_2))|\\
&\quad\leqslant \frac{((\mu_0\|\mathbf{q}\|_{\S}+\lambda)}{\Gamma(\upsilon-\S+1)}(2(t_2-t_1)^{\upsilon-\S}+t_1^{\upsilon-\S}-t_2^{\upsilon-\S}). \qquad (3.12)
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
|\mathbb{A}_1(\mathbf{q},\mathbf{r})(t)-\mathbb{A}_1(\bar{\mathbf{q}},\bar{\mathbf{r}})(t)| \leqslant{}& \frac{|\chi_1(t,\mathbf{q}(t),\mathbf{r}(t),^{c}\,\mathbb{D}^{\S}\mathbf{q}(t))-\chi_1(t,\bar{\mathbf{q}}(t),\bar{\mathbf{r}}(t),^{c}\,\mathbb{D}^{\S}\bar{\mathbf{q}}(t))|}{\Gamma(\upsilon+1)}\\
&+\sum_{1}^{m}\frac{|\mathbb{H}_i(t,\mathbf{q}(t),^{c}\,\mathbb{D}^{\S}\mathbf{q}(t))-\mathbb{H}_i(t,\bar{\mathbf{q}}(t),^{c}\,\mathbb{D}^{\S}\bar{\mathbf{q}}(t))|}{\Gamma(\upsilon+\rho_i+1)}+|\chi_1(\mathbf{r})(\ell)-\chi_1(\bar{\mathbf{r}})(\ell)|,
\end{aligned}$$

which in view of (3.10) and ($\mathbb{H}_4$) implies that

$$\begin{aligned}
|\mathbb{A}_1(\mathbf{q},\mathbf{r})(t)-\mathbb{A}_1(\bar{\mathbf{q}},\bar{\mathbf{r}})(t)| &\leqslant \frac{\delta_0\mu_0\|\mathbf{q}-\bar{\mathbf{q}}\|_{\S}}{\Gamma(\upsilon+1)\Gamma(\wp+1)}+\sum_{1}^{m}\frac{\theta_i\|\mathbf{q}-\bar{\mathbf{q}}\|_{\S}}{\Gamma(\upsilon+\rho_i+1)}+\tau_1\|\mathbf{r}-\bar{\mathbf{r}}\|\\
&\leqslant \left(\frac{\delta_0\mu_0}{\Gamma(\upsilon+1)\Gamma(\wp+1)}+\sum_{1}^{m}\frac{\theta_i}{\Gamma(\upsilon+\rho_i+1)}\right)\|\mathbf{q}-\bar{\mathbf{q}}\|_{\S}+\tau_1\|\mathbf{r}-\bar{\mathbf{r}}\|.
\end{aligned} \tag{3.13}$$

Also, one has

$$|D^{\S}\mathbb{A}_1(\mathbf{q},\mathbf{r})(t)-D^{\S}\mathbb{A}_1(\bar{\mathbf{q}},\bar{\mathbf{r}})(t)|\leqslant \mathscr{I}^{(\upsilon-\S)}|\chi_1(t,\mathbf{q}(t),\mathbf{r}(t),{}^c\mathbb{D}^{\S}\mathbf{q}(t))-\chi_1(t,\mathbf{q}_2(t),\mathbf{r}_2(t),{}^c\mathbb{D}^{\S}\bar{\mathbf{q}})(t)| + \sum_1^m \mathscr{I}^{(\upsilon+\rho_i-\S)}|(\mathbb{H}_i(t,\mathbf{q}_1(t),{}^c\mathbb{D}^{\S}\mathbf{q}_1(t))-\mathbb{H}_i(t,\bar{\mathbf{q}}(t),{}^c\mathbb{D}^{\S}\bar{\mathbf{q}}(t))|,$$

which in view of (3.10) and $(\mathbb{H}_4)$ implies that

$$\begin{aligned}|{}^c\mathbb{D}^{\S}\mathbb{A}_1(\mathbf{q}_1,\mathbf{r}_1)(t)-{}^c\mathbb{D}^{\S}\mathbb{A}_1(\bar{\mathbf{q}},\bar{\mathbf{r}})(t)| &\leqslant \frac{\delta_0\mu_0\|\mathbf{q}_1-\bar{\mathbf{q}}\|_{\S}}{\Gamma(\upsilon-\S+1)\Gamma(\wp+1)}+\sum_1^m\frac{\theta_i\|\mathbf{q}_1-\bar{\mathbf{q}}\|_{\S}}{\Gamma(\upsilon+\rho_i-\S+1)}\\ &=\Big(\frac{\delta_0\mu_0}{\Gamma(\upsilon-\S+1)\Gamma(\wp+1)}+\sum_1^m\frac{\theta_i}{\Gamma(\upsilon+\rho_i-\S+1)}\Big)\|\mathbf{q}_1-\bar{\mathbf{q}}\|_{\S}.\end{aligned}\tag{3.14}$$

From (3.13) and (3.14), it follows that

$$\begin{aligned}\|\mathbb{A}_1(\mathbf{q}_1,\mathbf{r}_1)-\mathbb{A}_1(\bar{\mathbf{q}},\bar{\mathbf{r}})\|_{\S}&\leqslant \Big(\frac{\delta_0\mu_0}{\Gamma(\wp+1)}\Big(\frac{1}{\Gamma(\upsilon+1)}+\frac{1}{\Gamma(\upsilon-\S+1)}\Big)\\ &\quad+\sum_1^m\Big(\frac{\theta_i}{\Gamma(\upsilon+\rho_i+1)}+\frac{\theta_i}{\Gamma(\upsilon+\rho_i-\S+1)}\Big)\Big)\|\mathbf{q}_1-\bar{\mathbf{q}}\|_{\S}+\tau_1\|\mathbf{r}_1-\bar{\mathbf{r}}\|\\ &=\kappa_1\|\mathbf{q}_1-\bar{\mathbf{q}}\|_{\S}+\tau_1\|\mathbf{r}_1-\bar{\mathbf{r}}\|,\end{aligned}\tag{3.15}$$

where $\kappa_1=\frac{\delta_0\mu_0}{\Gamma(\wp+1)}(\frac{1}{\Gamma(\upsilon+1)}+\frac{1}{\Gamma(\upsilon-\S+1)})+\sum_1^m(\frac{\theta_i}{\Gamma(\upsilon+\rho_i+1)}+\frac{\theta_i}{\Gamma(\upsilon+\rho_i-\S+1)})$. Similarly,

$$\begin{aligned}\|\mathbb{A}_2(\mathbf{q}_1,\mathbf{r}_1)-\mathbb{A}_2(\bar{\mathbf{q}},\bar{\mathbf{r}})\|_{\S}&\leqslant \Big(\frac{\delta_0\mu_0}{\Gamma(\wp+1)}\Big(\frac{1}{\Gamma(\upsilon+1)}+\frac{1}{\Gamma(\upsilon-\S+1)}\Big)\\ &\quad+\sum_1^m\Big(\frac{\theta_i^*}{\Gamma(\upsilon+\rho_i+1)}+\frac{\theta_i}{\Gamma(\upsilon+\rho_i-\S+1)}\Big)\Big)\|\mathbf{r}_1-\bar{\mathbf{r}}\|_{\S}+\tau_2\|\mathbf{q}_1-\bar{\mathbf{q}}\|\\ &=\kappa_2\|\mathbf{r}_1-\bar{\mathbf{r}}\|_{\S}+\tau_2\|\mathbf{q}_1-\bar{\mathbf{q}}\|,\end{aligned}\tag{3.16}$$

where $\kappa_2=\frac{\delta_0\mu_0}{\Gamma(\wp+1)}(\frac{1}{\Gamma(\upsilon+1)}+\frac{1}{\Gamma(\upsilon-\S+1)})+\sum_1^m(\frac{\theta_i^*}{\Gamma(\upsilon+\rho_i+1)}+\frac{\theta_i^*}{\Gamma(\upsilon+\rho_i-\S+1)})$. Now, for $j=1,2$, we have

$$|\mathbb{B}_j(\mathbf{q}_1)-\mathbb{B}_j(\mathbf{q}_2)|\leqslant\frac{\mathscr{I}^{\upsilon}|(\mathbb{L}_j(t,\mathbf{q}_1(t),{}^c\mathbb{D}^{\S}\mathbf{q}_1(t))-\mathbb{L}_j(t,\mathbf{q}_2(t),{}^c\mathbb{D}^{\S}\mathbf{q}_2(t))|}{\Lambda_j},$$

$$|{}^c\mathbb{D}^{\S}\mathbb{B}_j(\mathbf{q}_1)-{}^c\mathbb{D}^{\S}\mathbb{B}_j(\mathbf{q}_2)|\leqslant\frac{\mathscr{I}^{(\upsilon-\S)}|(\mathbb{L}_j(t,\mathbf{q}_1(t),{}^c\mathbb{D}^{\S}\mathbf{q}_1(t))-\mathbb{L}_j(t,\mathbf{q}_2(t),{}^c\mathbb{D}^{\S}\mathbf{q}_2(t))|}{\Lambda_j},$$

which in view of $(\mathbb{H}_4)$, one has

$$|\mathbb{B}_j(\mathbf{q}_1)-\mathbb{B}_j(\mathbf{q}_2)|\leqslant\frac{\mu_0\|\mathbf{q}_1-\mathbf{q}_2\|}{\Lambda_j\Gamma(\upsilon+1)},\qquad |{}^c\mathbb{D}^{\S}\mathbb{B}_j(\mathbf{q}_1)-{}^c\mathbb{D}^{\S}\mathbb{B}_j(\mathbf{q}_2)|\leqslant\frac{\mu_0\|\mathbf{q}_1-\mathbf{q}_2\|}{\Lambda_j\Gamma(\upsilon-\S+1)}.$$

Hence, it follows that

$$\|\mathbb{B}_j\mathbf{q}_1-\mathbb{B}_j\mathbf{q}_2\|_{\S}\leqslant\frac{\mu_0}{\Lambda_j}\Big(\frac{1}{\Gamma(\upsilon+1)}+\frac{1}{\Gamma(\upsilon-\S+1)}\Big)\|\mathbf{q}_1-\mathbf{q}_2\|_{\S}=\kappa_3\|\mathbf{q}_1-\mathbf{q}_2\|_{\S},\tag{3.17}$$

where $\kappa_3=\frac{\mu_0}{\Lambda_j}(\frac{1}{\Gamma(\upsilon+1)}+\frac{1}{\Gamma(\upsilon-\S+1)})$,

$$\begin{aligned}|\mathbb{C}_1(\mathbf{q},\mathbf{r})-\mathbb{C}_1(\bar{\mathbf{q}},\bar{\mathbf{r}})|&\leqslant|\chi_2(\mathbf{r}_1)-\chi_2(\bar{\mathbf{r}})|+|\chi_1(\mathbf{r}_1)-\chi_1(\bar{\mathbf{r}})|\\ &\quad+|\mathscr{I}^{\upsilon}\chi_1(1,\mathbf{q}(1),\mathbf{r}(1),{}^c\mathbb{D}^{\S}\mathbf{q}(1))-\mathscr{I}^{\upsilon}\chi_1(1,\bar{\mathbf{q}}(1),\bar{\mathbf{r}}(1),{}^c\mathbb{D}^{\S}\bar{\mathbf{q}}(1))|\\ &\quad+|\mathscr{I}^{\upsilon}\mathbb{L}_1(1,\mathbf{q}(1),{}^c\mathbb{D}^{\S}\mathbf{q}(1))-\mathscr{I}^{\upsilon}\mathbb{L}_1(1,\bar{\mathbf{q}}(1),{}^c\mathbb{D}^{\S}\bar{\mathbf{q}}(1))|\\ &\quad+\sum_1^m|\mathscr{I}^{\upsilon+\rho_i}\mathbb{H}_i(1,\mathbf{q}_1(1),{}^c\mathbb{D}^{\S}\mathbf{q}(1))-\mathscr{I}^{\upsilon+\rho_i}\mathbb{H}_i(1,\bar{\mathbf{q}}(1),{}^c\mathbb{D}^{\S}\mathbf{q}_2(1))|.\end{aligned}$$

Using (3.10) and ($\mathbb{H}_4$), we obtain

$$\begin{aligned}|\mathbb{C}_1(\mathbf{q}_1,\mathbf{r})-\mathbb{C}_1(\bar{\mathbf{q}},\mathbf{r}_2)| &\leqslant (\tau_1+\tau_2)|\mathbf{r}_1-\bar{\mathbf{r}}|+\frac{\delta_0\mu_0\|\mathbf{q}_1-\bar{\mathbf{q}}\|_{\S}}{\Gamma(\upsilon+1)\Gamma(\wp+1)}+\frac{\mu_0\|\mathbf{q}_1-\bar{\mathbf{q}}\|_{\S}}{\Gamma(w+1)}+\sum_1^m\frac{\theta_i\|\mathbf{q}_1-\bar{\mathbf{q}}\|_{\S}}{\Gamma(\upsilon+\rho_i+1)}\\ &\leqslant \Big(\frac{\delta_0\mu_0}{\Gamma(\upsilon+1)\Gamma(\wp+1)}+\frac{\mu_0}{\Gamma(w+1)}+\sum_1^m\frac{\theta_i}{\Gamma(\upsilon+\rho_i+1)}\Big)\|\mathbf{q}_1-\bar{\mathbf{q}}\|_{\S}\\ &\quad+(\tau_1+\tau_2)|\mathbf{r}_1-\bar{\mathbf{r}}|=\kappa_4\|\mathbf{q}_1-\bar{\mathbf{q}}\|_{\S}+(\tau_1+\tau_2)|\mathbf{r}_1-\bar{\mathbf{r}}|,\end{aligned}\tag{3.18}$$

where $\kappa_4=\frac{\delta_0\mu_0}{\Gamma(\upsilon+1)\Gamma(\wp+1)}+\frac{\mu_0}{\Gamma(w+1)}+\sum_1^m\frac{\theta_i}{\Gamma(\upsilon+\rho_i+1)}$. Similarly, we have

$$\begin{aligned}|\mathbb{C}_2(\mathbf{q}_1,\mathbf{r})-\mathbb{C}_2(\bar{\mathbf{q}},\mathbf{r}_2)| &\leqslant (\tau_1+\tau_2)|\mathbf{q}_1-\bar{\mathbf{q}}|+\frac{\delta_0\mu_0\|\mathbf{r}_1-\bar{\mathbf{r}}\|_{\S}}{\Gamma(\upsilon+1)\Gamma(\wp+1)}+\frac{\mu_0\|\mathbf{r}_1-\bar{\mathbf{r}}\|_{\S}}{\Gamma(w+1)}+\sum_1^m\frac{\theta_i^*\|\mathbf{r}_1-\bar{\mathbf{r}}\|_{\S}}{\Gamma(\upsilon+\rho_i+1)}\\ &\leqslant \Big(\frac{\delta_0\mu_0}{\Gamma(\upsilon+1)\Gamma(\wp+1)}+\frac{\mu_0}{\Gamma(w+1)}+\sum_1^m\frac{\theta_i^*}{\Gamma(\upsilon+\rho_i+1)}\Big)\|\mathbf{r}_1-\bar{\mathbf{r}}\|_{\S}\\ &\quad+(\tau_1+\tau_2)|\mathbf{q}_1-\bar{\mathbf{q}}|=\kappa_5\|\mathbf{r}_1-\bar{\mathbf{r}}\|_{\S}+(\tau_1+\tau_2)|\mathbf{q}_1-\bar{\mathbf{q}}|,\end{aligned}\tag{3.19}$$

where $\kappa_5=\frac{\delta_0\mu_0}{\Gamma(\upsilon+1)\Gamma(\wp+1)}+\frac{\mu_0}{\Gamma(w+1)}+\sum_1^m\frac{\theta_i^*}{\Gamma(\upsilon+\rho_i+1)}$.

Lemma 3.2. *If* ($\mathbb{H}_1$), ($\mathbb{H}_2$), ($\mathbb{H}_3$) *hold, the operator* $\mathbb{A}$ *is* μ*-Lipschitz with constant* $\tilde{\kappa}$ *and obeys the relation*

$$\|\mathbb{A}(\mathbf{q},\mathbf{r})\|_{\S}\leqslant\Delta_1\|(\mathbf{q},\mathbf{r})\|_{\S}+\Delta_2,\tag{3.20}$$

with

$$\Delta_1=\frac{\delta_0\mu_0}{\Gamma(\wp+1)}\Big(\frac{1}{\Gamma(\upsilon+1)}+\frac{1}{\Gamma(\upsilon-\S+1)}\Big)+\sum_1^m\tilde{\theta}_i\Big(\frac{1}{\Gamma(\upsilon+\rho_i+1)}\frac{1}{\Gamma(\upsilon+\rho_i-\S+1)}\Big)+c,$$

and

$$\Delta_2=\frac{\delta_0\lambda}{\Gamma(\wp+1)}\Big(\frac{1}{\Gamma(\upsilon+1)}+\frac{1}{\Gamma(\upsilon-\S+1)}\Big)+\sum_1^m\xi\Big(\frac{1}{\Gamma(\upsilon+\rho_i+1)}\frac{1}{\Gamma(\upsilon+\rho_i-\S+1)}\Big)+d.$$

Proof. For $(\mathbf{q}_1,\mathbf{r}_1)$, $(\bar{\mathbf{q}},\bar{\mathbf{r}})$, we have

$$\|\mathbb{A}(\mathbf{q}_1,\mathbf{r}_1)-\mathbb{A}(\bar{\mathbf{q}},\bar{\mathbf{r}})\|_{\S}=\max\{\|\mathbb{A}_1(\mathbf{q}_1,\mathbf{r}_1)-\mathbb{A}_1(\bar{\mathbf{q}},\bar{\mathbf{r}})\|_{\S},\|\mathbb{A}_2(\mathbf{q}_1,\mathbf{r}_1)-\mathbb{A}_2(\bar{\mathbf{q}},\bar{\mathbf{r}})\|_{\S}\}.$$

Using (3.15) and (3.16), we obtain

$$\|\mathbb{A}(\mathbf{q}_1,\mathbf{r}_1)-\mathbb{A}(\bar{\mathbf{q}},\bar{\mathbf{r}})\|_{\S}\leqslant\kappa\max\{\|\mathbf{q}_1-\bar{\mathbf{q}}\|_{\S},\|\mathbf{r}_1-\bar{\mathbf{r}}\|_{\S}\}+\tau\max\{\|\mathbf{r}_1-\bar{\mathbf{r}}\|_{\S},\|\mathbf{q}_1-\bar{\mathbf{q}}\|_{\S}\},$$

where $\kappa=\max\{\kappa_1,\kappa_2\}$, $\tau=\max\{\tau_1,\tau_2\}$. Hence, it follows that

$$\|\mathbb{A}(\mathbf{q}_1,\mathbf{r}_1)-\mathbb{A}(\bar{\mathbf{q}},\bar{\mathbf{r}})\|_{\S}\leqslant(\kappa+\tau)\|(\mathbf{q}_1,\mathbf{r}_1)-(\bar{\mathbf{q}},\bar{\mathbf{r}})\|_{\S}=\tilde{\kappa}\|(\mathbf{q}_1,\mathbf{r}_1)-(\bar{\mathbf{q}},\bar{\mathbf{r}})\|_{\S},\tag{3.21}$$

where $\tilde{\kappa}=\kappa+\tau$.

For the growth condition, using (3.9), one has

$$\begin{aligned}\|(\mathbb{A}_1(\mathbf{q},\mathbf{r})(t)\|_{\S}&=\|(\mathbb{A}_1(\mathbf{q},\mathbf{r})(t)\|+\|({}^{c}\mathbb{D}^{\S}\mathbb{A}_1(\mathbf{q},\mathbf{r})(t)\|\\ &\leqslant\frac{\delta_0(\mu_0\|\mathbf{q}\|_{\S}+\lambda)}{\Gamma(\wp+1)}\Big(\frac{1}{\Gamma(\upsilon+1)}+\frac{1}{\Gamma(\upsilon-\S+1)}\Big)\\ &\quad+\sum_1^m(\theta_i\|\mathbf{q}\|_{\S}+\xi)\Big(\frac{1}{\Gamma(\upsilon+\rho_i+1)}\frac{1}{\Gamma(\upsilon+\rho_i-\S+1)}\Big)+c_1|\mathbf{r}|+d_1\end{aligned}$$

$$
\begin{aligned}
&+\frac{\delta_0(\mu_0\|\mathbf{r}\|_{\S}+\lambda)}{\Gamma(\wp+1)}\Big(\frac{1}{\Gamma(\upsilon+1)}+\frac{1}{\Gamma(\upsilon-\S+1)}\Big)\\
&+\sum_{1}^{m}(\theta_i^*\|\mathbf{r}\|_{\S}+\xi)\Big(\frac{1}{\Gamma(\upsilon+\rho_i+1)}\frac{1}{\Gamma(\upsilon+\rho_i-\S+1)}\Big)+c_2|\mathbf{q}|+d_2\\
\leqslant\ &\frac{\delta_0(\mu_0\|(\mathbf{q},\mathbf{r})\|_{\S}+\lambda)}{\Gamma(\wp+1)}\Big(\frac{1}{\Gamma(\upsilon+1)}+\frac{1}{\Gamma(\upsilon-\S+1)}\Big)\\
&+\sum_{1}^{m}(\tilde{\theta}_i\|(\mathbf{q},\mathbf{r})\|_{\S}+\xi)\Big(\frac{1}{\Gamma(\upsilon+\rho_i+1)}\frac{1}{\Gamma(\upsilon+\rho_i-\S+1)}\Big)+c\|(\mathbf{q},\mathbf{r})\|+d,
\end{aligned}
$$

where $\tilde{\theta}_i=\max\{\theta_i,\theta_i^*\}$, $c=\max\{c_1,c_2\}$, $d=\max\{d_1,d_2\}$. Hence, it follows that

$$\|(\mathbb{A}_1(\mathbf{q},\mathbf{r})(t)\|_{\S}\leqslant\Delta_1\|(\mathbf{q},\mathbf{r})\|_{\S}+\Delta_2, \tag{3.22}$$

where

$$
\begin{aligned}
\Delta_1&=\frac{\delta_0\mu_0}{\Gamma(\wp+1)}\Big(\frac{1}{\Gamma(\upsilon+1)}+\frac{1}{\Gamma(\upsilon-\S+1)}\Big)+\sum_{1}^{m}\tilde{\theta}_i\Big(\frac{1}{\Gamma(\upsilon+\rho_i+1)}\frac{1}{\Gamma(\upsilon+\rho_i-\S+1)}\Big)+c,\\
\Delta_2&=\frac{\delta_0\lambda}{\Gamma(\wp+1)}\Big(\frac{1}{\Gamma(\upsilon+1)}+\frac{1}{\Gamma(\upsilon-\S+1)}\Big)+\sum_{1}^{m}\xi\Big(\frac{1}{\Gamma(\upsilon+\rho_i+1)}\frac{1}{\Gamma(\upsilon+\rho_i-\S+1)}\Big)+d.
\end{aligned}
$$

□

Lemma 3.3. *Under the hypothesis* $(\mathbb{H}_1)$-$(\mathbb{H}_3)$, *the operator* $\mathbb{B}$ *is continuous and compact.*

Proof. Here $\mathbb{B}_1(\mathbf{q})(t)=\frac{\mathscr{I}^{\upsilon}\mathbb{L}_1(t,\mathbf{q}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t))}{\Lambda_1}$, $\mathbb{B}_2(\mathbf{r})(t)=\frac{\mathscr{I}^{\upsilon}\mathbb{L}_2(t,\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t))}{\Lambda_2}$, clearly, $\mathbb{B}(\mathbf{q},\mathbf{r})=(\mathbb{B}_1\mathbf{q},\mathbb{B}_2\mathbf{r})$ is continuous on $\mathbb{E}$ and is bounded as

$$|(\mathbb{B}_1\mathbf{q})(t)|=\frac{|\mathscr{I}^{\upsilon}\mathbb{L}_1(t,\mathbf{q}(t),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t))|}{\Lambda_1}\leqslant 1,\qquad |(\mathbb{B}_2\mathbf{r})(t)|=\frac{|\mathscr{I}^{\upsilon}\mathbb{L}_2(t,\mathbf{r}(t),{}^{c}\mathbb{D}^{\S}\mathbf{r}(t))|}{\Lambda_2}\leqslant 1. \tag{3.23}$$

For equi-continuity, choose $t_1<t_2\in I$, and consider

$$
\begin{aligned}
&|\mathbb{B}_1(\mathbf{q})t_2-\mathbb{B}_1(\mathbf{q})t_1|+|({}^{c}\mathbb{D}^{\S}\mathbb{B}_1\mathbf{q})t_2-({}^{c}\mathbb{D}^{\S}\mathbb{B}_1\mathbf{q})t_1|\\
&\quad\leqslant\frac{|\mathscr{I}^{\upsilon}\mathbb{L}_1(t_2,\mathbf{q}(t_2),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t_2))-\mathscr{I}^{\upsilon}\mathbb{L}_1(t_1,\mathbf{q}(t_1),{}^{c}\mathbb{D}^{\S}\mathbf{q}(t_1))|}{\Lambda_1}.
\end{aligned}
$$

In view of (3.11), we obtain

$$|\mathbb{B}_1(\mathbf{q})t_2-\mathbb{B}_1(\mathbf{q})t_1|\leqslant\frac{((\mu_0\|\mathbf{q}\|_{\S}+\lambda)}{\Gamma(\upsilon+1)\Lambda}(2(t_2-t_1)^{\upsilon}+t_1^{\upsilon}-t_2^{\upsilon}), \tag{3.24}$$

and in view of (3.12), we obtain

$$|{}^{c}\mathbb{D}^{\S}\mathbb{B}_1(\mathbf{q})t_2-{}^{c}\mathbb{D}^{\S}\mathbb{B}_1(\mathbf{q})t_1|\leqslant\frac{((\mu_0\|\mathbf{q}\|_{\S}+\lambda)}{\Gamma(\upsilon-\S+1)\Lambda}(2(t_2-t_1)^{\upsilon-\S}+t_1^{\upsilon-\S}-t_2^{\upsilon-\S}). \tag{3.25}$$

Similarly,

$$|\mathbb{B}_2(\mathbf{r})t_2-\mathbb{B}_2(\mathbf{r})t_1|\leqslant\frac{((\mu_0\|\mathbf{r}\|_{\S}+\lambda)}{\Gamma(\upsilon+1)\Lambda_2}(2(t_2-t_1)^{\upsilon}+t_1^{\upsilon}-t_2^{\upsilon}),$$

and in view of (3.12), we obtain

$$|{}^{c}\mathbb{D}^{\S}\mathbb{B}_2(\mathbf{r})t_2-{}^{c}\mathbb{D}^{\S}\mathbb{B}_2(\mathbf{r})t_1|\leqslant\frac{((\mu_0\|\mathbf{r}\|_{\S}+\lambda)}{\Gamma(\upsilon-\S+1)\Lambda_2}(2(t_2-t_1)^{\upsilon-\S}+t_1^{\upsilon-\S}-t_2^{\upsilon-\S}).$$

From (3.24) and (3.25), it follows that

$$\|\mathbb{B}_1(\mathbf{q})t_2-\mathbb{B}(\mathbf{q})t_1\|_{\S}=\|\mathbb{B}_1(\mathbf{q})t_2-\mathbb{B}_1(\mathbf{q})t_1\|+\|({}^c\mathbb{D}^{\S}\mathbb{B}_1\mathbf{q})t_2-({}^c\mathbb{D}^{\S}\mathbb{B}\mathbf{q})t_1\|\to 0 \text{ as } t_1\to t_2,$$
$$\|\mathbb{B}_2(\mathbf{r})t_2-\mathbb{B}_2(\mathbf{r})t_1\|_{\S}=\|\mathbb{B}_2(\mathbf{r})t_2-\mathbb{B}_2(\mathbf{r})t_1\|+\|({}^c\mathbb{D}^{\S}\mathbb{B}_2\mathbf{r})t_2-({}^c\mathbb{D}^{\S}\mathbb{B}_2\mathbf{r})t_1\|\to 0 \text{ as } t_1\to t_2.$$

Thus $\mathbb{B}$ in view of Arzelá-Ascoli theorem is compact. □

Lemma 3.4. *Under the hypothesis* ($\mathbb{H}_1$)-($\mathbb{H}_3$), *the operator* $\mathbb{C}$ *is Lipschitz with constant* $\tau_1+\tau_2+\bar{\kappa}$, *and obeys*

$$\|\bar{\mathbb{C}}\mathbf{q}\|_{\S}\leqslant\Delta_3\|\mathbf{q}\|_{\S}+\Delta_4, \tag{3.26}$$

where

$$\Delta_3=\tilde{c}+\frac{\delta_0\mu_0}{\Gamma(\upsilon+1)\Gamma(\wp+1)}+\frac{\mu_0}{\Gamma(\upsilon+1)}+\sum_1^m(\frac{\tilde{\theta}_i}{\Gamma\upsilon+\rho_i+1})$$

and

$$\Delta_4=\tilde{d}+\frac{\lambda\delta_0}{\Gamma(\upsilon+1)\Gamma(\wp+1)}+\frac{\lambda}{\Gamma(\upsilon+1)}+\sum_1^m\frac{\xi}{\Gamma(\upsilon+\rho_i+1)}.$$

Proof. For $(\mathbf{q},\mathbf{r})$, $(\bar{\mathbf{q}},\bar{\mathbf{r}})\in\mathsf{E}$, we have

$$\|\mathbb{C}(\mathbf{q},\mathbf{r})-\mathbb{C}(\bar{\mathbf{q}},\bar{\mathbf{r}})\|_{\S}=\max\{\|\mathbb{C}_1(\mathbf{q},\mathbf{r})-\mathbb{C}_1(\bar{\mathbf{q}},\bar{\mathbf{r}})\|_{\S},\|\mathbb{C}_2(\mathbf{q},\mathbf{r})-\mathbb{C}_2(\bar{\mathbf{q}},\bar{\mathbf{r}})\|_{\S}\}.$$

Using (3.18) and (3.19), we obtain

$$\begin{aligned}\|\mathbb{C}(\mathbf{q},\mathbf{r})-\mathbb{C}(\bar{\mathbf{q}},\bar{\mathbf{r}})\|_{\S}&\leqslant(\tau_1+\tau_2)\|(\mathbf{q},\mathbf{r})-(u_2,\bar{\mathbf{r}})\|_{\S}+\bar{\kappa}\|(\mathbf{q},\mathbf{r})-(\bar{\mathbf{q}},\bar{\mathbf{r}})\|_{\S}\\&=(\tau_1+\tau_2+\bar{\kappa})\|(\mathbf{q},\mathbf{r})-(\bar{\mathbf{q}},\bar{\mathbf{r}})\|_{\S},\end{aligned}\tag{3.27}$$

where $\bar{\kappa}=\max\{\kappa_4,\kappa_5\}=\frac{\delta_0\mu_0}{\Gamma(\upsilon+1)\Gamma(\wp+1)}+\frac{\mu_0}{\Gamma(w+1)}+\sum_1^m\frac{\tilde{\theta}_i}{\Gamma(\upsilon+\rho_i+1)}$. For the growth condition, we have for $(\mathbf{q},\mathbf{r})\in\mathbb{E}$,

$$\begin{aligned}|\mathbb{C}_1(\mathbf{q},\mathbf{r})(t)|&\leqslant|\chi_2(\mathbf{r}(\ell))|+|\chi_1(\mathbf{r}(\ell))|+|\mathscr{I}^{\upsilon}\chi_1(1,\mathbf{q}(1),\mathbf{r}(1),{}^c\mathbb{D}^{\S}\mathbf{q}(1))|\\&\quad+|\mathscr{I}^{\upsilon}\mathbb{L}_1(1,\mathbf{q}(1),\mathbf{r}(1),{}^c\mathbb{D}^{\S}\mathbf{q}(1))|+\sum_1^m|\mathscr{I}^{\upsilon+\rho_i}\mathbb{H}_i(1,\mathbf{q}(1),{}^c\mathbb{D}^{\S}\mathbf{q}(1)))|,\end{aligned}$$

which is in view of ($\mathbb{H}_2$) and (3.7) implies that

$$|\mathbb{C}_1(\mathbf{q},\mathbf{r})(t)|\leqslant\tilde{c}|\mathbf{r}(\ell)|+\tilde{d}+\frac{\delta_0(\mu_0\|\mathbf{q}\|_{\S}+\lambda)}{\Gamma(\upsilon+1)\Gamma(\wp+1)}+\frac{(\mu_0\|\mathbf{q}\|_{\S}+\lambda)}{\Gamma(\upsilon+1)}+\sum_1^m\frac{(\theta_i\|\mathbf{q}\|_{\S}+\xi)}{\Gamma(\upsilon+\rho_i+1)}.$$

Similarly

$$|\mathbb{C}_2(\mathbf{q},\mathbf{r})(t)|\leqslant\tilde{c}|\mathbf{q}(\ell)|+\tilde{d}+\frac{\delta_0(\mu_0\|\mathbf{r}\|_{\S}+\lambda)}{\Gamma(\upsilon+1)\Gamma(\wp+1)}+\frac{(\mu_0\|\mathbf{r}\|_{\S}+\lambda)}{\Gamma(\upsilon+1)}+\sum_1^m\frac{(\theta_i^*\|\mathbf{r}\|_{\S}+\xi)}{\Gamma(\upsilon+\rho_i+1)}.$$

Thus it yields

$$\begin{aligned}\|\mathbb{C}(\mathbf{q},\mathbf{r})\|_{\S}&\leqslant\tilde{c}|(\mathbf{q},\mathbf{r})|+\tilde{d}+\frac{\delta_0(\mu_0\|(\mathbf{q},\mathbf{r})\|_{\S}+\lambda)}{\Gamma(\upsilon+1)\Gamma(\wp+1)}+\frac{1}{\Gamma(\upsilon+1)}(\mu_0\|(\mathbf{q},\mathbf{r})\|_{\S}+\lambda)\\&\quad+\sum_1^m\frac{(\tilde{\theta}_i\|(\mathbf{q},\mathbf{r})\|_{\S}+\xi)}{\Gamma(\upsilon+\rho_i+1)}\leqslant\Delta_3\|(\mathbf{q},\mathbf{r})\|+\Delta_4.\end{aligned}$$

□

Choose the parameters such that $\Delta_1 + \Delta_3 < 1$. Choose $R \geqslant \max\{\tau_1 + \tau_2 + \bar{\kappa}, \tilde{\kappa}, \frac{\Delta_2+\Delta_4}{1-(\Delta_1+\Delta_3)}\}$. Define $\mathbf{S} = \{(\mathbf{q}, \mathbf{r}) \in \mathbb{E} \times \mathbb{E} : \|(\mathbf{q}, \mathbf{r})\|_{\S} \leqslant \mathbb{R}\}$, then $\mathbf{S}$ is closed, convex, and bounded subset of $\mathbb{E} \times \mathbb{E}$.

Theorem 3.5. *If* $((\mathbb{H}_1)$-$(\mathbb{H}_3)$ *and* $\Delta_1 + \Delta_3 < 1$ *hold, then system* (3.4) *has at least one solution.*

Proof. By Lemma 3.2, the operator $\mathbb{A}$ is μ-Lipschitz with constant $\tilde{\kappa}$ and by Lemma 3.4, the operator $\mathbb{C}$ is μ-Lipschitz with constant $\tau_1 + \tau_2 + \tilde{\kappa}$. By Lemma 3.3, the operator $\mathbb{B}$ is compact. Now for $(\mathbf{q}, \mathbf{r}) \in \mathbf{S}$, take $(\mathbf{q}, \mathbf{r}) = \mathbb{A}(\mathbf{q}, \mathbf{r}) + \mathbb{B}(\mathbf{q}, \mathbf{r})\mathbb{C}(\mathbf{q}, \mathbf{r})$, which implies that

$$\|(\mathbf{q}, \mathbf{r})\|_{\S} \leqslant \|\mathbb{A}(\mathbf{q}, \mathbf{r})\|_{\S} + \|\mathbb{B}(\mathbf{q}, \mathbf{r})\|_{\S}\|\mathbb{C}(\mathbf{q}, \mathbf{r})\|_{\S}.$$

In view of the growth conditions (3.20), (3.23), and (3.26), one has

$$\|(\mathbf{q}, \mathbf{r})\|_{\S} \leqslant (\Delta_1 + \Delta_3)\|(\mathbf{q}, \mathbf{r})\|_{\S} + \Delta_2 + \Delta_4.$$

That implies

$$(1 - (\Delta_1 + \Delta_3))\|(\mathbf{q}, \mathbf{r})\|_{\S} \leqslant (\Delta_2 + \Delta_4).$$

Hence, it follows that

$$\|(\mathbf{q}, \mathbf{r})\|_{\S} \leqslant \frac{\Delta_2 + \Delta_4}{(1 - (\Delta_1 + \Delta_3)} \leqslant R,$$

which implies that $(\mathbf{q}, \mathbf{r}) \in \mathbf{S}$. Further we have $M = \|\mathbb{B}(\mathbf{q}, \mathbf{r})\|_{\S} = 1$ and $R \geqslant \tau_1 + \tau_2 + \bar{\kappa}, \tilde{\kappa}$. Thus all conditions are satisfied. Hence, (3.4) has at least one solution. □

Theorem 3.6. *If* $(\mathbb{H}_1)$-$(\mathbb{H}_4)$ *and* $\tilde{\kappa} + \kappa_2(\Delta_3 R + \Delta_4) + (\tau_1 + \tau_2 + \bar{\kappa}) < 1$ *hold, then* (3.4) *has a unique solution.*

Proof. For $(\mathbf{q}, \mathbf{r})$, $(\bar{\mathbf{q}}, \bar{\mathbf{r}}) \in \mathbf{S}$, consider

$$\begin{aligned} \|\mathfrak{T}(\mathbf{q}, \mathbf{r}) - \mathfrak{T}(\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S} &= \|\mathbb{A}(\mathbf{q}, \mathbf{r}) + \mathbb{B}(\mathbf{q}, \mathbf{r})\mathbb{C}(\mathbf{q}, \mathbf{r}) - (\mathbb{A}(\bar{\mathbf{q}}, \bar{\mathbf{r}}) + \mathbb{B}(\bar{\mathbf{q}}, \bar{\mathbf{r}})\mathbb{C}(\bar{\mathbf{q}}, \bar{\mathbf{r}}))\|_{\S} \\ &\leqslant \|\mathbb{A}(\mathbf{q}, \mathbf{r}) - \mathbb{A}(\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S} + \|\mathbb{C}(\mathbf{q}, \mathbf{r})\|_{\S}\|\mathbb{B}(\mathbf{q}, \mathbf{r}) - \mathbb{B}(\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S} \\ &\quad + \|\mathbb{B}(\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S}\|\mathbb{C}(\mathbf{q}, \mathbf{r}) - \mathbb{C}(\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S}. \end{aligned} \tag{3.28}$$

By (3.17), (3.21), and (3.27), we have

$$\begin{aligned} \|\mathbb{A}(\mathbf{q}, \mathbf{r}) - \mathbb{A}(\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S} &\leqslant \tilde{\kappa}\|(\mathbf{q}, \mathbf{r}) - (\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S}, \\ \|\mathbb{B}(\mathbf{q}, \mathbf{r}) - \mathbb{B}(\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S} &\leqslant \kappa_2\|(\mathbf{q}, \mathbf{r}) - (\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S}, \\ \|\mathbb{C}(\mathbf{q}, \mathbf{r}) - \mathbb{C}(\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S} &\leqslant (\tau_1 + \tau_2 + \bar{\kappa})\|((\mathbf{q}, \mathbf{r}) - (\bar{\mathbf{q}}, \bar{\mathbf{r}}))\|_{\S}. \end{aligned} \tag{3.29}$$

Using (3.29) in (3.28), we obtain

$$\begin{aligned} \|\mathfrak{T}(\mathbf{q}, \mathbf{r}) - \mathfrak{T}(\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S} \leqslant \tilde{\kappa}\|(\mathbf{q}, \mathbf{r}) - (\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S} &+ \|\mathbb{C}(\mathbf{q}, \mathbf{r})\|_{\S}\kappa_2\|(\mathbf{q}, \mathbf{r}) - (\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S} \\ &+ (\tau_1 + \tau_2 + \bar{\kappa})\|(\mathbf{q}, \mathbf{r}) - (\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S}\|\mathbb{B}(\mathbf{q}, \mathbf{r})\|_{\S}, \end{aligned}$$

which further implies that

$$\begin{aligned} \|\mathfrak{T}(\mathbf{q}, \mathbf{r}) - \mathfrak{T}(\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S} &\leqslant \big(\tilde{\kappa} + \kappa_2(\Delta_3\|(\mathbf{q}, \mathbf{r})\|_{\S} + \Delta_4) + (\tau_1 + \tau_2 + \bar{\kappa})\big)\|(\mathbf{q}, \mathbf{r}) - (\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S} \\ &\leqslant \big(\tilde{\kappa} + \kappa_2(\Delta_3 R + \Delta_4) + (\tau_1 + \tau_2 + \bar{\kappa})\big)\|(\mathbf{q}, \mathbf{r}) - (\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S}. \end{aligned}$$

Hence the result is received. □

4. U-H stability

Here we construct second important result devoted to U-H stability.

Definition 4.1. The system (3.4) will be U-H stable, for positive $\zeta > 0$, $\varphi > 0$ corresponding to every solution $(\mathbf{q}, \mathbf{r})$ of

$$\|(\mathbf{q}, \mathbf{r}) - (\mathbb{A}(\mathbf{q}, \mathbf{r}) + \mathbb{B}(\mathbf{q}, \mathbf{r})\mathbb{C}(\mathbf{q}, \mathbf{r}))\|_{\S} < \varphi_1,$$

if one has unique solution $(\bar{\mathbf{q}}, \bar{\mathbf{r}})$ of (3.4), which fulfils the following result,

$$\|(\mathbf{q}, \mathbf{r}) - (\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S} < \varphi\zeta.$$

Theorem 4.2. *If* $(\mathbb{H}_2)$ *and* $(\mathbb{H}_4)$*, and* $\tilde{\kappa} + \kappa_2(\Delta_3 R + \Delta_4) + (\tau_1 + \tau_2 + \bar{\kappa}) < 1$ *hold, then* (1.1) *is U-H stable.*

Proof. If $(\mathbf{q}, \mathbf{r})$ be any solution and $(\bar{\mathbf{q}}, \bar{\mathbf{r}})$ be a unique solution of (1.1), then lets take

$$\begin{aligned}
\|(\mathbf{q}, \mathbf{r}) - (\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S} &= \|(\mathbf{q}, \mathbf{r}) - (\mathbb{A}(\bar{\mathbf{q}}, \bar{\mathbf{r}}) + \mathbb{B}(\bar{\mathbf{q}}, \bar{\mathbf{r}})\mathbb{C}(\bar{\mathbf{q}}, \bar{\mathbf{r}}))\|_{\S} \\
&\leqslant \|(\mathbf{q}, \mathbf{r}) - (\mathbb{A}(\mathbf{q}, \mathbf{r}) + \mathbb{B}(\mathbf{q}, \mathbf{r})\mathbb{C}(\mathbf{q}, \mathbf{r}))\|_{\S} \\
&\quad + \|(\mathbb{A}(\mathbf{q}, \mathbf{r}) + \mathbb{B}(\mathbf{q}, \mathbf{r})\mathbb{C}(\mathbf{q}, \mathbf{r})) - (\mathbb{A}(\bar{\mathbf{q}}, \bar{\mathbf{r}}) + \mathbb{B}(\bar{\mathbf{q}}, \bar{\mathbf{r}})\mathbb{C}(\bar{\mathbf{q}}, \bar{\mathbf{r}}))\|_{\S} \\
&< \varphi + \|\mathfrak{T}(\mathbf{q}, \mathbf{r}) - \mathfrak{T}(\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S},
\end{aligned}$$

which further means

$$\|(\mathbf{q}, \mathbf{r}) - (\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S} \leqslant \varphi + \big(\tilde{\kappa} + \kappa_2(\Delta_3 R + \Delta_4) + (\tau_1 + \tau_2 + \bar{\kappa})\big)\|(\mathbf{q}, \mathbf{r}) - (\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S} = \varphi + K\|(\mathbf{q}, \mathbf{r}) - (\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S},$$

with $K = \tilde{\kappa} + \kappa_2(\Delta_3 R + \Delta_4) + (\tau_1 + \tau_2 + \bar{\kappa})$, which implies

$$\|(\mathbf{q}, \mathbf{r}) - (\bar{\mathbf{q}}, \bar{\mathbf{r}})\|_{\S} < \varphi\zeta, \text{ where } \zeta = \frac{1}{1-K}.$$

□

5. Example

Consider a test example as follows.

Example 5.1.

$$\begin{aligned}
&{}^{c}\mathbb{D}^{0.5}\left[\frac{{}^{c}\mathbb{D}^{1.5}\mathbf{q}(t) - \sum_1^m \mathscr{I}^{\rho_i}\mathbb{H}_i(t, \mathbf{q}(t), {}^{c}\mathbb{D}^{0.5}\mathbf{q}(t))}{\mathbb{L}_1(t, \mathbf{q}(t), {}^{c}\mathbb{D}^{0.5}\mathbf{q}(t))}\right] = \mathscr{F}_1(t, \mathbf{r}(t), \mathscr{I}^{1.5}\mathbf{r}(t)),\ t \in I, \\
&{}^{c}\mathbb{D}^{0.5}\left[\frac{{}^{c}\mathbb{D}^{1.5}\mathbf{r}(t) - \sum_1^m \mathscr{I}^{\rho_i}\mathbb{K}_i(t, \mathbf{r}(t), {}^{c}\mathbb{D}^{\S}\mathbf{r}(t))}{\mathbb{L}_2(t, \mathbf{r}(t), {}^{c}\mathbb{D}^{0.5}\mathbf{r}(t))}\right] = \mathscr{F}_2(t, \mathbf{q}(t), \mathscr{I}^{1.5}\mathbf{q}(t)),\ t \in I, \\
&\mathbf{q}(0) = \sin(|\mathbf{r}(0.5)|),\ \mathbf{q}^{(1)}(0) = 0, \mathbf{q}(1) = \tan(|\mathbf{r}(0.5)|), \\
&\mathbf{r}(0) = \sin(|\mathbf{q}(0.5)|),\ \mathbf{r}^{(1)}(0) = 0,\ \mathbf{r}(1) = \tan(|\mathbf{q}(0.5)|).
\end{aligned} \tag{5.1}$$

Here we see in (5.1) that $\wp = \S = \ell = 0.5$, $\gamma = \upsilon = 1.5$, and $\chi_1 = \sin(|\mathbf{r}(0.5)|)$, $\chi_2 = \tan(|\mathbf{r}(0.5)|)$, and consider

$$\begin{aligned}
\mathbb{H}_i(t, \mathbf{q}(t), {}^{c}\mathbb{D}^{\S}\mathbf{q}(t)) &= \frac{(|\mathbf{q}(t)| + {}^{c}\mathbb{D}^{0.5}|\mathbf{r}(t)|)^i}{50 + t^2}, \quad i = 1, 2, \ldots, 5, \\
\mathbb{K}_i(t, \mathbf{r}(t), {}^{c}\mathbb{D}^{\S}\mathbf{r}(t)) &= \frac{(|\mathbf{q}(t)| + {}^{c}\mathbb{D}^{0.5}|\mathbf{r}(t)|)^i}{50 + t^2}, \quad i = 1, 2, \ldots, 5.
\end{aligned}$$

Also,

$$\mathscr{F}_1(t,\mathbf{r}(t),\mathscr{I}^{1.5}\mathbf{r}(t)) = \frac{\sqrt{|\mathbf{r}(t)|}+\mathscr{I}^{1.5}|\mathbf{r}(t)|}{100+t^3}, \quad \mathscr{F}_2(t,\mathbf{q}(t),\mathscr{I}^{1.5}\mathbf{q}(t)) = \frac{\sqrt{|\mathbf{q}(t)|}+\mathscr{I}^{1.5}|\mathbf{q}(t)|}{100+t^3}.$$

Furthermore,

$$\mathbb{L}_1(t,\mathbf{q}(t),{}^c\mathbb{D}^{0.5}\mathbf{q}(t)) = \frac{\sin|\mathbf{q}(t)|+{}^c\mathbb{D}^{1.5}[\sin|\mathbf{q}(t)|]}{100+t^3}, \quad \mathbb{L}_2(t,\mathbf{r}(t),{}^c\mathbb{D}^{0.5}\mathbf{r}(t)) = \frac{\sin|\mathbf{r}(t)|+{}^c\mathbb{D}^{1.5}[\sin|\mathbf{r}(t)|]}{100+t^3}.$$

Now it is easy to derive the results of Theorems 3.5, 3.6, 4.2, respectively. Because ($\mathbb{H}_1$)-($\mathbb{H}_3$) hold and $\Delta_1+\Delta_3 \approx 0.675 < 1$, thus the given example has at least one solution. Also, on calculation, we see

$$\tilde{\kappa}+\kappa_2(\Delta_3 R+\Delta_4)+(\tau_1+\tau_2+\bar{\kappa}) \approx 0.0345 < 1.$$

Hence the uniqueness condition holds. Finally, we see that $K \approx 0.0345 < 1$, which grantees the condition of U-H stability.

6. Conclusion

Some results devoted to existence theory and stability have been constructed for a highly nonlinear system of S-HFDEs. By using the degree theory with a measure of non-compactness, we have established the required results. Using usual mathematical analysis tools, we have derived results for U-H stability which has never been established for such a problem. In the future, the idea will be extended to the system of S-HFDEs with a nonsingular derivative of fractional order. Also many biological processes when formulated we get hybrid type mathematical models. For the existence of such problems, our analysis will provide base to the young researchers.

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