J. Math. Computer Sci., 34 (2024), 99–115

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Existence, uniqueness, and Hyers-Ulam stability of abstract neutral differential equation containing state-dependent fractional integrable impulses

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Abstract

This article focuses on the existence, uniqueness as well as regularity of solution of abstract neutral differential equation containing state-dependent fractional integrable impulses. Furthermore, we also examine Hyers-Ulam stability using the properties of analytic semigroup, abstract Grönwall lemma and fixed point technique. An example is presented at the end.

Keywords: Analytic semigroup, mild solution, impulsive differential equation.

2020 MSC: 34K30, 34K45, 35R12, 47D06.

1. Introduction

Inspired by the work done in [11], in this article we deal with the existence and uniqueness of a mild solution and strict solution as well as Hyers-Ulam stability for the following neutral differential equation with state-dependent fractional integrable impulses:

$$\begin{aligned}&\frac{d}{d\tau}\Big(\Psi(\tau)+\mathcal{U}\big(\tau,\Psi(\tau-\mathfrak{J}(\tau,\Psi(\tau)))\big)\Big)\\&\quad=\mathfrak{D}\Big(\Psi(\tau)+\mathcal{U}\big(\tau,\Psi(\tau-\mathfrak{J}(\tau,\Psi(\tau)))\big)\Big)+\Theta\big(\tau,\Psi(\tau-\mathcal{V}(\tau,\Psi(\tau)))\big),\quad \tau\in(\omega_i,\ \tau_{i+1}], i\in\mathbb{N}_0^n:=0,1,\dots,n, \qquad (1.1)\end{aligned}$$

$$\Psi(\tau)=\frac{1}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i\big(\Psi(\mathfrak{J}_i(\Psi(\tau_i)))\big)d\omega,\quad \tau\in(\tau_i,\omega_i], i\in\mathbb{N}_1^n:=1,2,\dots,n, \qquad (1.2)$$

$$\Psi_0=\mathcal{B}\in\mathbb{C}([-\lambda,0];\mathcal{Y}), \qquad (1.3)$$

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doi: 10.22436/jmcs.034.02.01

Received: 2023-10-02 *Revised:* 2023-12-11 *Accepted:* 2023-12-19

where $\alpha \in (0,1)$ and $\mathfrak{D} : \Omega(\mathfrak{D}) \subset \mathcal{Y} \mapsto \mathcal{Y}$ represents the family of an analytic semigroup $(\mathfrak{I}(\tau))_{\tau \geqslant 0}$ of bounded linear operators on a Banach space $(\mathbb{BS})$ $(\mathcal{Y}, \| \cdot \|)$, the pre-fixed points are taken such that $0 = \tau_0 = \omega_0 < \tau_1 < \omega_1 < \tau_2 < \omega_2 < \cdots < \tau_n < \omega_n < \tau_{n+1} < \omega_{n+1} = \kappa$. Furthermore, $g_i : \mathcal{Y} \mapsto \mathcal{Y}$, $\mathcal{U} : [0,\kappa] \times \mathcal{Y} \mapsto \mathcal{Y}$, $\Theta : [0,\kappa] \times \mathcal{Y} \mapsto \mathcal{Y}$, $\mathfrak{J}$, $\mathcal{V} : [0,\kappa] \mapsto [-\lambda,\kappa]$ and $\mathfrak{J}_i : \mathcal{Y} \mapsto [-\lambda,\kappa]$, $i \in \mathbb{N}_1^n$, are suitable mappings.

The philosophy of differential equations containing state-dependent argument is deep area of research because of the fact that qualitative notion of state-dependent argument is more ambiguous and different than the simple delay differential equations. In this regard, we mention [21] and the work done by Hartung et al. [10] and the references in that for related state-dependent argument in ordinary differential equations. The research on state-dependent argument abstract and partial differential equations is mainly related to non-neutral problems, see the survey by Yunfei et al. [25, 26], Rezounenko et al. [21], Baliki and Benchohra [3], Chaudhary and Panday [5], Fu and Huang [8], Ma and Liu [27], Ravichandran et al. [33] and the work done by Kosovalic et al. [19, 20] as well as the latest articles by Hernández et al. [13, 14, 17, 29]. We cite the contributions by Hernández et al. [15, 16] for the abstract neutral differential equations. The philosophy of differential equations containing state-dependent argument is deep area of research because of the fact that qualitative notion of state-dependent argument more ambiguous and different than the simple delay differential equations. In this regard, we mention the work done by Hartung et al. [10] and the references in that for related state-dependent argument in ordinary differential equations, as well as the latest articles by Hernández et al. [13, 14, 17]. We cite the contributions by Hernández et al. [15, 16] for the abstract neutral differential equations.

In 1940, at Wisconsin University, Ulam put forward a question concerning the approximate solution for the exact homomorphism [38]. Hyers positively replied to the question in case of Banach spaces [18]. This famous theory was extended by Aoki [2] as well as Rassias [32]. In these articles the authors studied the norm of differences and Cauchy differences, $g(s+t) - g(s) - g(t)$. Answers to Ulam's question, its attractions and inductions for various situations is an important area of research and is known as Ulam's stability. For detailed information about Ulam's stability, we recommend [12, 30, 31, 34, 35, 37, 41–44] and references therein.

The notion of neutral differential equations is the remarkable region in the research area of functional differential equations, with applications and mathematical contributions [6, 22]. For abstract and partial neutral differential equations, we mention the work done in [4, 9, 39, 40] and the articles [1, 7, 11, 15]. Also, we mention contributions made in the article [4], where the authors introduced and examined partial neutral differential equations with state-dependent argument emanating in populations dynamics, we also refer the reader to the papers [23, 24], where applications to populations dynamics are premised faced by studies of partial differential equations.

In this article, we utilized the central concepts of [14, 17] to establish the results examined in [15, 16] for the type of abstract neutral differential equations containing state-dependent argument with impulsive effects. Particularly, we examine the central ideas established in Theorem 2.6 by considering $\mathbb{L}_p$-Lipschitz mappings $\Theta(\cdot)$ and $\mathcal{V}(\cdot)$. The notion of $\mathbb{L}_p$-Lipschitz mapping (see assertion $\mathbf{H}^{\mathbf{W}}_{\mathbf{q},\mathbb{P}}$) was presented in the article [15] in order to prove the existence and uniqueness of solutions without taking the assumption that the nonlinear mappings are locally Lipschitz. Also, note that mappings of the form $\Psi \to \Psi(\mathcal{B}, \nu(\cdot, \Psi(\cdot)))$ are not Lipschitz, in general, on spaces of continuous mappings and we remark that

$$\|\Psi(\cdot - \mathcal{V}(\Psi(\cdot))) - \Theta(\cdot - \mathcal{V}(\Theta(\cdot)))\|_{C([0,\kappa];\mathcal{Y})} \leqslant (1 + [\Psi]_{C_{\mathrm{Lip}}([-\lambda,\kappa];\mathcal{Y})}[\mathcal{V}]_{C_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\|\Psi - \Theta\|_{C([-\lambda,\kappa];\mathcal{Y})},$$

is true, if the mappings $\Psi(\cdot)$ and $\mathcal{V}(\cdot)$ are Lipschitz.

The remaining paper is divided into three sections. The second section is devoted to the existence and uniqueness of solution for system (1.1)-(1.3). We examine our main result in Theorem 2.6 by taking that $\Theta(\cdot)$ and $\mathcal{V}(\cdot)$ are $\mathbb{L}^p$-Lipschitz and that $\mathcal{U}(\cdot)$ and $\mathfrak{J}(\cdot)$ are Lipschitz. Also, we examine the existence of a strict solution in a similar section. The third section is concerned with Hyers-Ulam stability for the proposed model. The final section is devoted to an example for the validity of our main results.

The norms $\mathbb{C}([\kappa,\rho];\mathcal{Y})$ and $\mathbb{C}_{\mathrm{Lip}}([\kappa,\rho];\mathcal{Y})$ in the present article are represented as $\|\cdot\|_{\mathbb{C}([\kappa,\rho];\mathcal{Y})}$ and $\|\cdot\|_{\mathbb{C}_{\mathrm{Lip}}([\kappa,\rho];\mathcal{Y})}$, such that $\|\xi\|_{\mathbb{C}_{\mathrm{Lip}}([\kappa,\rho];\mathcal{Y})} = \|\xi\|_{\mathbb{C}([\kappa,\rho];\mathcal{Y})} + [\xi]_{\mathbb{C}_{\mathrm{Lip}}([\kappa,\rho];\mathcal{Y})}$ and $[\xi]_{\mathbb{C}_{\mathrm{Lip}}([\kappa,\rho];\mathcal{Y})} = \sup_{\tau,\omega\in[\kappa,\rho]}$, $\tau \neq \omega \| \frac{\|\xi(\tau)-\xi(\omega)\|}{|\tau-\omega|}$. Similarly, the norm of $\mathbb{C}_{\mathrm{Lip}([c,d]\times\mathcal{Y};\mathcal{Y})}$ is defined. $\mathbb{PC}(\mathcal{Y})$ represents the space generated by all bounded mappings $\Psi : [0,\kappa] \mapsto \mathcal{Y}$, provided that $\Psi(\cdot)$ is continuous at $\tau \neq \tau_i$, $\Psi(\tau_i^-) = \Psi(\tau_i)$ and $\Psi(\tau + i)$ exists for all $i = \mathbb{N}_1^n$ such that $\|\Psi\|_{\mathbb{PC}(\mathcal{Y})} = \max_{i=\mathbb{N}_0^n} \|\Psi\|_{\mathbb{C}((\tau_i,\tau_{i+1}];\mathcal{Y})}$. Furthermore, $\mathbb{PC}_{\mathrm{Lip}}(\mathcal{Y})$ denotes the space of mappings $\Psi \in \mathbb{PC}(\mathcal{Y})$, provided $\Psi|_{(\tau_i,\tau_{i+1}]} \in \mathbb{C}_{\mathrm{Lip}}((\tau_i,\tau_{i+1}];\mathcal{Y})$ for all $i = \mathbb{N}_0^n$ with the norm $\|\Psi\|_{\mathbb{PC}_{\mathrm{Lip}}(\mathcal{Y})} = \max_{i=\mathbb{N}_0^n} \|\Psi_{|_{(\tau_i,\tau_{i+1}]}}\|_{\mathbb{C}_{\mathrm{Lip}}((\tau_i,\tau_{i+1}];\mathcal{Y})}$.

$\mathbb{BPC}(\mathcal{Y})$ represents the collection of all the mappings $\Psi : [-\lambda,\kappa] \mapsto \mathcal{Y}$, provided $\Psi_{|_{[-\lambda,\tau_1]}} \in \mathbb{C}([-\lambda,\tau_1];\mathcal{Y})$ and $\Psi_{|_{[0,\kappa]}} \in \mathbb{PC}(\mathcal{Y})$. Furthermore, $\mathbb{BPC}_{\mathrm{Lip}}(\mathcal{Y})$ represents the space of all mappings generated by $\Psi : [-\lambda,\kappa] \mapsto \mathcal{Y}$, provided $\Psi \in \mathbb{BPC}(\mathcal{Y})$, $\Psi_{|_{[-\lambda,0]}} \in \mathbb{C}_{\mathrm{Lip}}([-\lambda,0];\mathcal{Y})$ and $\Psi_{|_{[0,a]}} \in \mathbb{PC}_{\mathrm{Lip}}(\mathcal{Y})$, with the norm $\|\Psi\| \in \mathbb{BPC}_{\mathrm{Lip}}(\mathcal{Y}) = \max\{\|\Psi_{\mathcal{J}_i}\|_{\mathbb{C}_{\mathrm{Lip}}}(\mathcal{J}_i;\mathcal{Y}) : i = \mathbb{N}_{-1}^n\}$, where $\mathcal{J}_1 = [-\lambda, 0]$.

For $\Psi \in \mathbb{BPC}(\mathcal{Y})$ and $i \in \mathbb{N}_{-1}^n$, we use $\widetilde{\Psi}_i$ for the function $\widetilde{\Psi}_i \in \mathbb{C}([\tau_i,\tau_{i+1}];\mathcal{Y})$ given by $\widetilde{\Psi}_i(\tau) = \Psi(\tau_i^+)$ for $\tau \in (\tau_i,\tau_{i+1}]$ and $\Psi_i(\tau) = \Psi(\tau_i^+)$ for $\tau = \tau_i$. For $B \in \mathbb{BPC}(\mathcal{Y})$ and $i \in \mathbb{N}_{-1}^n$, $\widetilde{B}_i$ is the set $\widetilde{B}_i = \{\widetilde{\Psi}_i : \Psi \in B\}$.

Lemma 1.1. *A collection $B \subset \mathbb{BPC}(\mathcal{Y})$ is relatively compact in $\mathbb{BPC}(\mathcal{Y})$ if and only if each collection $\widetilde{B}_i$ is relatively compact in $\mathbb{C}([\tau_i,\tau_{i+1}],\mathcal{Y})$.*

As pointed above, let $\mathfrak{D} : \Omega(\mathfrak{D}) \subset \mathcal{Y} \to \mathcal{Y}$ be the collection of a $\mathbb{C}_0$-semigroup of bounded linear operators $(T(\tau))_{\tau\geqslant 0}$ on $\mathcal{Y}$, also let $\|\mathcal{I}(\tau)\| \leqslant \mathbb{C}_0$, $\forall\, \tau \in [0,\kappa]$.

Lemma 1.2 (Grönwall Lemma, [36]). *For any $\tau > 0$ with*

$$u(\tau) \leqslant q(\tau) + \int_0^t p(\omega)u(\omega)d\omega + \sum_{0<\tau_k<\tau} \gamma_k u(\tau_k^-),$$

where $u, q, p \in \mathbb{PC}(R^+, R^+)$, q is nondecreasing and $\gamma > 0$. Then for $\tau \in R^+$ we have

$$u(\tau) \leqslant q(\tau)(1+\gamma_k)^k \exp\left(\int_0^\tau p(s)ds\right), \textit{ where } k \in M.$$

Remark 1.3. If we replace γ_k by $\gamma_k(\tau)$, then

$$u(\tau) \leqslant q(\tau) \prod_{0<\tau_k<\tau} \left(1+\gamma_k(\tau)\right) \exp\left(\int_0^\tau p(\omega)d\omega\right), \text{ where } k \in M.$$

2. Existence and uniqueness results

Definition 2.1. Assume that $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ is a Banach spaces, q is greater than or equal to 1, and $\mathbb{P}$ belongs to $\mathbb{C}([0,\kappa]\times\mathcal{Y};\mathcal{W})$. $\mathbb{P}$ is said to be $\mathbb{L}_q$-Lipschitz if $[\mathbb{P}](\cdot,\cdot) \leqslant \mathbb{L}_q([0,\kappa]\times[0,\kappa];\mathbb{R}^+)$ and a non-decreasing mapping $\mathbb{K} : \mathbb{R}^+ \mapsto \mathbb{R}^+$ provided $\|\mathbb{P}(\tau,y) - \mathbb{P}(\omega,z)\|_{\mathcal{W}} \leqslant [\mathbb{P}]_{(\tau,\omega)}\mathbb{K}_{\mathbb{P}}(\max\{\|y\|,\|z\|\})(|\tau-\omega| + \|y-z\|)$, $\forall$ $0 \leqslant \omega < \tau \leqslant \kappa$ and y, $z \in \mathcal{Y}$. Furthermore, $\mathbb{L}^q_{\mathrm{Lip}([0,\kappa];\mathcal{W})}$ represents the family generated by this form of mappings as below.

Remark 2.2. Concerning the above definition, we note that the function $\Theta : [0,\kappa]\times\mathcal{Y} \to \mathcal{Y}$ given by $\Theta(\tau,y) = \sqrt[\theta]{\tau}y$ is not locally Lipschitz but $\Theta \in \mathbb{L}^q_{\mathrm{Lip}}([0,\kappa];\mathcal{Y})$ with $q \in (1, \frac{\theta}{1-\theta})$, $[\Theta](\tau,\omega) = \frac{1}{\theta}\omega^{\frac{1}{\theta}-1} + \sqrt[\theta]{\omega}$ for $\tau > \omega > 0$, $[\Theta]_{(\omega,\omega)} = \sqrt[\theta]{\omega}$ for $\omega > 0$, $[\Theta](\tau,0) = \tau^{\frac{1}{\theta}-1}$ for $\tau > 0$, $[\Theta]_{(0,0)} = 0$, $[\Theta]_{(\tau,\omega)} = 0$ otherwise and $\mathbb{K}_{\Theta}(\omega) = (1+\omega)$.

From the above assume that $\Theta(\cdot)$ be the map given by $\Theta(\tau, y) = \sum_{i=1}^{n} \sqrt[\theta_i]{(\tau-\rho_i)}\mathcal{W}_{[\rho_i,\rho_{i+1}]}y_i$ such that $0 < \cdots < \rho_i < \rho_{i+1} \cdots < \kappa$, $\theta_i \in (0,1)$ and $y_i \in \mathcal{Y}$, $\forall\ i \in \mathbb{N}_1^n$, also $\mathcal{Y}_{[\rho_i,\rho_{i+1}]} : [0,\kappa] \to \mathbb{R}$ is given as $\mathcal{Y}_{[\lfloor i+1, \lfloor i]}(\tau) = 0$ for $\tau < \rho_i$ and $\mathcal{Y}_{[\rho_i,\rho_{i+1}](\tau)} = 1$ for $\tau \in [\rho_{i+1}, \kappa]$.

Notation 1. In remaining part of the article, for $\Psi \in \mathbb{C}([-\lambda, \rho]; \mathcal{Y})$, ν belongs to $\mathbb{C}([0,\kappa] \times \mathcal{Y}; [0,\lambda])$ and $\mathbb{P} \in \mathbb{C}([0,\kappa] \times \mathcal{Y}; \mathcal{W})$. In addition, we utilize $\Psi^{\nu}(\cdot)$ and $\mathbb{P}^{\nu}(\Psi)(\cdot)$ for $\Psi^{\nu} : [0,\rho] \to \mathcal{Y}$ and $\mathbb{P}^{\nu}(\Psi) : [0,\rho] \mapsto \mathcal{W}$ are the mappings defined as $\Psi^{\nu}(\omega) = \Psi(\omega - \nu(\omega, \Psi(\omega)))$ and $\mathbb{P}^{\nu}(\Psi)(\omega) = \mathbb{P}(\omega, \Psi^{\nu}(\omega))$ and $\int_{u_i}^{\omega}(\omega - u)^{\alpha-1}du \leqslant \int_{u_i}^{\omega}(\omega+u)^{\alpha-1}du$.

Definition 2.3. A mapping $\Psi \in \mathbb{BPC}([-\lambda, \rho]; \mathcal{Y})$ is known as a mild solution of the problem (1.1)-(1.3) on $[-\lambda, \rho]$ if $\Psi(\theta) = \mathcal{B}(\theta)\ \forall\ \theta \in [-\lambda, 0]$ and $\Psi(\tau) = \frac{1}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega$, $\forall\ \tau \in (\tau_i, \omega_i]$, $i = \mathbb{N}_1^n$ and

$$\begin{aligned}\Psi(\tau) &= \frac{\mathfrak{I}(\tau-\tau_i)}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega \\ &\quad - \mathcal{U}(\tau, \Psi^{\mathfrak{J}}(\tau)) + \int_{\omega_i}^{\tau}\mathfrak{I}(\tau-\sigma)\Theta(\sigma, \Psi(\sigma), \Psi(\mathcal{V}(\sigma)))d\sigma,\ \forall\ \tau \in (\omega_i, \tau_{i+1}],\ i \in \mathbb{N}_1^n, \\ \Psi(\tau) &= \mathfrak{I}(\tau)\big(\mathcal{B}(0) + \mathcal{U}(0, \Psi^{\mathfrak{J}}(0))\big) - \mathcal{U}(\tau, \Psi^{\mathfrak{J}}(\tau)) + \int_{\tau_i}^{\tau}\mathfrak{I}(\tau-\sigma)\Theta(\sigma, \Psi(\sigma), \Psi(\mathcal{V}(\sigma)))d\sigma,\ \forall\ \tau \in [0,\ \tau_1].\end{aligned}$$

Definition 2.4. A mapping $\Psi \in \mathbb{BPC}([-\lambda, \rho]; \mathcal{Y})$ $0 < \rho \leqslant \kappa$, is known as a strict solution of the problem (1.1)-(1.3) on $[-\lambda, \rho]$ if $\Psi(\theta) = \mathcal{B}(\theta)$, $\forall\ \theta \in [-\lambda, 0]$, $\Big(\Psi(\cdot) + \mathcal{U}(\cdot, \Psi^{\mathfrak{J}}(\cdot))\Big)_{|_{[0,\rho]}}$ belong to $\mathbb{PC}([0,\rho]; \mathcal{Y}_1) \cap \mathbb{PC}^1([0,\rho]; \mathcal{Y})$ and $\Psi(\cdot)$ satisfies (1.1)-(1.3) on $[0,\rho]$.

Let $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ be the Banach spaces continuously embedded in $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$, provided that $\mathfrak{D}\mathfrak{I}(\cdot) \in \mathbb{L}^{\infty}([0,\kappa]; \mathbb{L}(\mathcal{W}, \mathcal{Y}))$. Furthermore, $\mathbb{C}_0 \in \mathbb{R}$ is such that $\|\mathfrak{I}(\tau)\| \leqslant \mathbb{C}_0$, $\forall\ \tau \in [0,\kappa]$. Now, consider the following assertions.

- $\mathbf{H}_{\mathcal{V}}$: $\mathcal{V} \in \mathbb{C}_{\text{Lip}}([0,\kappa]; [-\lambda, \kappa])$, there is a mapping $k : \mathbb{N}_1^n \mapsto \mathbb{N}_{-1}^n$, provided that $\mathcal{V}(\mathcal{I}_i) \subset \mathcal{I}_{k(i)}$ and $k(i) \leqslant i$, $\forall\ i$. In addition, we use $[\mathcal{V}]_{\mathbb{C}_{\text{Lip}}}$ instead of $[\mathcal{V}]_{\mathbb{C}_{\text{Lip}}}([0,\kappa]; [-\lambda, \kappa])$.
- $\mathbf{H}_{\mathfrak{J}_1}$: There is a mapping $q : \mathbb{N}_1^n \mapsto \mathbb{N}_{-1}^n$, with $q(i) \leqslant i$ and $\mathfrak{J}_i \in \mathbb{C}(\mathcal{Y}, \mathcal{I}_{q(i)})$ for all $i \in \mathbb{N}_1^n$. Also, we use $[\mathfrak{J}_i]_{\mathbb{C}_{\text{Lip}}}$ instead of $[\mathfrak{J}_i]_{\mathbb{C}_{\text{Lip}}}(\mathcal{Y}; \mathcal{I}_{q(i)})$.
- $\mathbf{H}_{g,\mathcal{Y}}^{\mathcal{W}}$: $g_i \in \mathbb{C}_{\text{Lip}}(\mathcal{Y}; \mathcal{W})$ and $\mathbb{C}_{\mathcal{Y},\mathcal{W}}(g_i) = \|g_i\|_{\mathbb{C}(\mathcal{Y};\mathcal{W})} < \infty$, $\forall\ i \in \mathbb{N}_1^n$. Furthermore, $\mathbb{L}_{\mathcal{W},\mathcal{W}}(g_i)$ represents the Lipschitz constant of the map $g_i(\cdot)$, $\mathbb{L}_{\mathcal{W},\mathcal{W}}(g) = \max_{i\in\mathbb{N}_1^n}$, $\mathbb{L}_{\mathcal{W},\mathcal{W}}(g_i)$ and $\mathbb{C}_{\mathcal{W},\mathcal{W}}(g) = \max_{i\in\mathbb{N}_1^n}\mathbb{C}_{\mathcal{W},\mathcal{W}}(g_i)$.

Notation 1. We consider $\rho_i \leqslant \tau_{i+1} - \tau_i$, $\rho = \max_{i\in\mathbb{N}_1^n}\rho_i$, $i_c : \mathcal{W} \mapsto \mathcal{Y}$ is the inclusion map, $\Lambda_{\mathcal{Y},\mathcal{W}} = \max\{\|\mathfrak{D}\mathfrak{I}(\cdot)\|_{\mathbb{L}^{\infty}([0,\rho],\mathbb{L}(\mathcal{W},\mathcal{Y}))}, \mathbb{C}_0\|i_c\|_{\mathbb{L}(\mathcal{W},\mathcal{Y})}\}$, and $\aleph_{\mathcal{Y},\mathcal{W}} = \widetilde{\Lambda_{\mathcal{Y},\mathcal{W}}}\mathbb{C}_{\mathcal{Y},\mathcal{W}}(g)\hbar$.

From [14] we mention the following lemma to establish our main result:

Lemma 2.5. *Consider* $0 < \rho \leqslant \kappa$, $\Psi,\ \Phi \in \mathbb{BPC}([-\lambda, \rho]; \mathcal{Y})$, $\mathcal{V} \in \mathbb{L}_{\text{Lip}}^{r}([0,\kappa]; [0,\lambda])$, *and*

$$\rho = \max\{\|\Psi\|_{\mathbb{BPC}([-\lambda,\rho];\mathcal{Y})}, \|\Phi\|_{\mathbb{BPC}([-\lambda,\rho];\mathcal{Y})}\}.$$

If $\Psi = \Phi$ *on* $[-\lambda, 0]$, $\Psi_{|_{[-\lambda,0]}} \in \mathbb{C}_{\text{Lip}}([-\lambda, 0]; \mathcal{Y})$ *and* $\Psi_{|_{[0,\rho]}} \in \mathbb{PC}_{\text{Lip}}([0,\rho]; \mathcal{Y})$, *then*

$$\begin{aligned}|\ \mathcal{V}(\omega+\hbar, \Psi(\omega+\hbar)) - \mathcal{V}(\omega, \Psi(\omega))\ | &\leqslant \mathbb{K}_{\mathcal{V}}(\rho)[\mathcal{V}](\omega+\hbar, \omega)(1 + [\Psi]_{\mathbb{PC}_{\text{Lip}}([0,\rho];\mathcal{Y})})\hbar, \\ \|\Psi^{\mathcal{V}}(\omega+\hbar) - \Psi^{\mathcal{V}}(\omega\| &\leqslant [\Psi]_{\mathbb{BPC}_{\text{Lip}}([-\lambda,\rho];\mathcal{Y})}(1 + [\mathcal{V}]_{(\omega+\hbar,\omega)}\mathbb{K}_{\mathcal{V}}(\rho)(1 + [\Psi(\cdot)]_{\mathbb{PC}_{\text{Lip}}([0,\rho];\mathcal{Y})}))\hbar, \\ \|\Psi^{\mathfrak{J}}(\omega) - \Phi^{\mathfrak{J}}(\omega)\| &\leqslant (1 + [\Psi]_{\mathbb{BPC}_{\text{Lip}}([-\lambda,\rho];\mathcal{Y})}[\mathfrak{J}]_{(\omega,\omega)}\mathbb{K}_{\mathfrak{J}}(\rho))\|\Psi - \Phi\|_{\mathbb{PC}([0,\rho];\mathcal{Y})},\end{aligned}$$

for each $\omega \in [0,\rho)$ *and* $\hbar > 0$, $\omega + \hbar \in [0,\rho]$.

Theorem 2.6. *Let* $\mathbf{H}_{\mathcal{V}}$, $\mathbf{H}_{\mathfrak{J}_1}$, *and* $\mathbf{H}^{\mathcal{W}}_{g,\mathcal{Y}}$ *be satisfied,* $\mathcal{B} \in \mathbb{C}_{\text{Lip}}([-\lambda, 0]; \mathcal{Y})$, $\mathfrak{I}(\cdot)(\mathcal{B}(0) + \mathcal{U}(0, \mathcal{B}(-\mathfrak{J}(0, \mathcal{B})))) \in \mathbb{PC}_{\text{Lip}}([0, \kappa]; \mathcal{Y})$, $\mathfrak{J} \in \mathbb{C}_{\text{Lip}}([0, \kappa]; [0, \lambda])$, $\mathcal{U} \in \mathbb{C}_{\text{Lip}}([0, \kappa] \times \mathcal{Y}; \mathcal{Y})$, $\Theta \in \mathbb{L}^q_{\text{Lip}}([0, \kappa]; \mathcal{Y})$, $\mathcal{V} \in \mathbb{L}^r_{\text{Lip}}([0, \kappa]; [0, \lambda])$ *and* $\frac{1}{\lambda} + \frac{1}{r} \leqslant 1$. *If* $[\mathcal{U}]_{\mathbb{C}_{\text{Lip}}([0,c]\times \mathbb{B}_\rho(\mathcal{B}(-\rho(0,\mathcal{B}));\mathcal{Y});\mathcal{Y})} \to 0$ *as* $c \to 0$ *for all positive values of* ρ *and*

$$\Lambda_1(c) := \sup_{\tau,\hbar \in [0,c], \tau+\hbar \leqslant c} \int_0^\tau [\Theta]_{(\sigma+\hbar,\sigma)}(1 + [\mathcal{V}](\sigma+\hbar, \sigma))d\sigma \to 0 \text{ as } c \to 0,$$

then there exists $\Psi \in \mathbb{BPC}_{\text{Lip}}([-\lambda, \rho]; \mathcal{Y})$ *a unique mild solution of the problem* (1.1)-(1.2) *on* $[-\lambda, \rho]$ *for* $0 < \rho \leqslant \kappa$.

Proof. Assume that $\mathfrak{R} > 0$ such that

$$\mathfrak{R} > 1 + [\mathcal{B}]_{\mathbb{C}_{\text{Lip}}([-\lambda,0];\mathcal{Y})} + [\mathfrak{I}(\cdot)(\mathcal{B}(0) + \mathcal{U}(0, \mathcal{B}(-\mathfrak{J}(0, \mathcal{B}))))]_{\mathbb{PC}_{\text{Lip}}([0,\kappa];\mathcal{Y})} + \mathbb{C}_0\|\Theta(\cdot, \mathcal{B}(-\mathfrak{J}(0, \mathcal{B}))))\|_{\mathbb{PC}([0,\kappa];\mathcal{Y})},$$

$$\begin{aligned}\mathfrak{R} > 1 \quad &+ [\mathcal{B}]_{\mathbb{C}_{\text{Lip}}([-\lambda,0];\mathcal{Y})} + \frac{2^\alpha(\tau-\tau_i+\hbar)^{\alpha+1}}{\Gamma(\alpha+2)}\|\mathfrak{D}\mathfrak{I}(\cdot)\|\|g_i\big(\Psi(\mathfrak{J}_i(\Psi(\tau_i^+)))\big)\|_{\mathbb{L}^\infty([0,b];\mathbb{L}(\mathcal{W},\mathcal{Y}))} \\ &+ \mathbb{C}_0\|\Theta(\cdot, \mathcal{B}(-\mathfrak{J}(0, \mathcal{B}))))\|_{\mathbb{PC}([0,\rho];\mathcal{Y})},\end{aligned}$$

and $\rho = 1 + 2\|\mathcal{B}\|$. And $\Lambda_2 : [0, \kappa] \to \mathbb{R}^+$ is the mapping defined as $\Lambda_2(\tau) = \mathbb{C}_0 \int_0^\tau [\Theta]_{(\sigma,\sigma)}(1 + [\mathcal{V}](\sigma, 0))d\sigma$ and $\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}$, $\widetilde{\mathbb{K}}_\Theta$ and $\widetilde{\mathbb{K}}_\mathcal{V}$ be the constants $\widetilde{\mathbb{K}}_\hbar = \widetilde{\mathbb{K}}_\hbar(\rho)$ for $\hbar(\cdot) = \Theta(\cdot)$, $\mathcal{V}(\cdot)$ and $\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}} = \widetilde{\mathbb{K}}_\Theta(1 + \widetilde{\mathbb{K}}_\mathcal{V})$.

Obviously $\Lambda_i(\tau) \to 0$ for $i = 1, 2$, and set $0 < b \leqslant \kappa$ with $\rho\mathfrak{R} < 1$ such that

$$\begin{aligned}&1 + [\mathcal{B}]_{\mathbb{C}_{\text{Lip}}([-\lambda,0];\mathcal{Y})} + [\mathfrak{I}(\cdot)(\mathcal{B}(0) + \mathcal{U}(0, \mathcal{B}(-\mathfrak{J}(0, \mathcal{B}))))]_{\mathbb{PC}_{\text{Lip}}([0,\kappa];\mathcal{Y})} \\ &\quad + \mathbb{C}_0\|\Theta(\cdot, \mathcal{B}(-\mathfrak{J}(0, \mathcal{B}))))\|_{\mathbb{PC}([0,\kappa];\mathcal{Y})} \\ &\quad + (1+\mathfrak{R})^2\Big[[\mathcal{U}]_{\rho,\rho}(1 + [\mathfrak{J}]_{\mathbb{C}_{\text{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])}) + \mathbb{C}_0\widetilde{\mathbb{K}}_{\Theta,\mathfrak{J}}(\Lambda_2(\rho) + \Lambda_1(\rho))\Big] < \mathfrak{R}, \\ &1 + [\mathcal{B}]_{\mathbb{C}_{\text{Lip}}([-\lambda,0];\mathcal{Y})} + \frac{2^\alpha(\tau-\tau_i+\hbar)^{\alpha+1}}{\Gamma(\alpha+2)}\|\mathfrak{D}\mathfrak{I}(\cdot)\|\|g_i\big(\Psi(\mathfrak{J}_i(\Psi(\tau_i^+)))\big)\|_{\mathbb{L}^\infty([0,b];\mathbb{L}(\mathcal{W},\mathcal{Y}))} \\ &\quad + \mathbb{C}_0\|\Theta(\cdot, \mathcal{B}(-\mathfrak{J}(0, \mathcal{B}))))\|_{\mathbb{PC}([0,\rho];\mathcal{Y})} \\ &\quad + (1+\mathfrak{R})^2\Big[[\mathcal{U}]_{\rho,\rho}(1 + [\mathfrak{J}]_{\mathbb{C}_{\text{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])}) + \mathbb{C}_0\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}(\Lambda_2(\rho) + \Lambda_1(\rho))\Big] < \mathfrak{R},\end{aligned}$$

where, for simplicity, we use $[\mathcal{U}]_{\rho,\rho}$ instead of $[\mathcal{U}]_{\mathbb{C}_{\text{Lip}}([0,\rho]\times\mathbb{B}_\rho(\mathcal{B}(-\mathfrak{J}(0,\mathcal{B}));\mathcal{Y});\mathcal{Y})}$. Consider the space $\mathcal{S}(\rho)$ defined as

$$\mathcal{S}(\rho) = \{\Psi \in \mathbb{BPC}([-\lambda, \rho] : \mathcal{Y}) : \Psi_0 = \mathcal{B}, [\Psi]_{\mathbb{PC}_{\text{Lip}}([0,b];\mathcal{Y})} \leqslant \mathfrak{R}\},$$

with the metric norm, $d(\Psi, \Phi) = \|\Psi - \Phi\|_{\mathbb{PC}([0,\rho]:\mathcal{Y})}$ and $: \mathcal{S}(\rho) \mapsto \mathbb{BPC}([-\lambda, \rho]; \mathcal{Y})$ is the mapping defined as $(\Psi)_0 = \mathcal{B}$, $\Psi(\theta) = \mathcal{B}(\theta)$, $\forall\, \theta \in [-\lambda, 0]$ and

$$\begin{aligned}\Psi(\tau) &= \frac{1}{\Gamma(\alpha)}\int_{\tau_i}^\tau (\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega, \text{ for } \tau \in (\tau_i, \omega_i], \ \forall\, i = \mathbb{N}_1^n, \\ \Psi(\tau) &= \frac{\mathfrak{I}(\tau-\tau_i)}{\Gamma(\alpha)}\int_{\tau_i}^\tau (\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega \\ &\quad - \mathcal{U}(\tau, \Psi^{\mathfrak{J}}(\tau)) + \int_{\omega_i}^\tau \mathfrak{I}(\tau-\sigma)\Theta(\sigma, \Psi^{\mathcal{V}}(\sigma))d\sigma, \ \forall\, \tau \in (\omega_i, \tau_{i+1}], \ i \in \mathbb{N}_1^n, \\ \Psi(\tau) &= \mathfrak{I}(\tau)\big(\mathcal{B}\phi(0) + \mathcal{U}(0, \Psi^{\mathfrak{J}}(0))\big) - \mathcal{U}(\tau, \Psi^{\mathfrak{J}}(\tau)) + \int_{\tau_i}^\tau \mathfrak{I}(\tau-\sigma)\Theta(\sigma, \Psi^{\mathcal{V}}(\sigma))d\sigma, \ \forall\, \tau \in [0, \ \tau_1].\end{aligned}$$

Now, to show that $(\cdot)$ is a contraction mapping on the space $\mathcal{S}(\rho)$. Next, we consider Ψ, $\Phi \in \mathcal{S}(\rho)$. For $\tau \in [0, \rho]$ noting that

$$\|\Psi(\tau)\| \leqslant \|\Psi(\tau) - \mathcal{B}(0))\| + \|\mathcal{B}\|_{\mathbb{C}([-\lambda,0];\mathcal{Y})}$$

$$\leqslant [\Psi]_{\mathbb{BPC}_{\mathrm{Lip}}([-\lambda,\rho];\mathcal{Y})}\tau + \|\mathcal{B}\|_{\mathbb{C}([-\lambda,0];\mathcal{Y})} \leqslant \mathfrak{R}\rho + \|\mathcal{B}\|_{\mathbb{C}([-\lambda,0];\mathcal{Y})} \leqslant 1 + \|\mathcal{B}\|_{\mathbb{C}([-\lambda,0];\mathcal{Y})},$$

we infer that $\|\Psi(\omega)\| \leqslant 1 + \|\mathcal{B}\|_{\mathbb{C}([-p,0];\mathcal{Y})} \leqslant \rho$ for $\omega \in [-p,\kappa]$. It is obvious that $\|\Psi^{\mathcal{V}}\|_{\mathbb{C}([0,\rho];\mathcal{Y})} \leqslant 1 + \|\mathcal{B}\|_{\mathbb{C}([-p,0];\mathcal{Y})}$. In addition, we observe that

$$\|\Psi^{\mathfrak{J}}(\tau) - \mathcal{B}(-\mathfrak{J}(0,\mathcal{B}))\| \leqslant \|\Psi^{\mathfrak{J}}(\tau)\| + \|\mathcal{B}(-\mathfrak{J}(0,\mathcal{B}))\| \leqslant \rho = 1 + 2\|\mathcal{B}\|_{\mathbb{C}([-p,0];\mathcal{Y})}, \ \forall\, \tau \in [0,\rho].$$

Utilizing aforementioned calculations along with Lemma 2.5, for $\tau,\ \hbar$ belongs to $[0,\rho]$ and $\tau + \hbar$ belongs to $[0,\rho]$, we get

$$\begin{aligned}
&\|\Theta(\sigma+\hbar, \Psi^{\mathcal{V}}(\sigma+\hbar)) - \Theta(\sigma, \Psi^{\mathcal{V}}(\sigma))\| \\
&\quad \leqslant [\Theta]_{(\sigma+\hbar,\sigma)}\widetilde{\mathbb{K}}_{\Theta}(\hbar + \|\Psi^{\mathcal{V}}(\sigma+\hbar) - \Psi^{\mathcal{V}}(\sigma)\|) \\
&\quad \leqslant [\Theta]_{(\sigma+\hbar,\sigma)}\widetilde{\mathbb{K}}_{\Theta}(\hbar + [\Psi]_{\mathbb{BPC}_{\mathrm{Lip}}([-\lambda,\rho];\mathcal{Y})}(1 + [\mathcal{V}]_{(\omega+\hbar,\omega)}\widetilde{\mathbb{K}}_{\mathcal{V}}(1 + [\Psi(\cdot)]_{\mathbb{C}_{\mathrm{Lip}}([0,\rho];\mathcal{Y})})\hbar) \\
&\quad \leqslant [\Theta]_{(\sigma+\hbar,\sigma)}\widetilde{\mathbb{K}}_{\Theta}(1 + [\Psi]_{\mathbb{BPC}_{\mathrm{Lip}}([-\lambda,\rho];\mathcal{Y})})^2(1 + [\mathcal{V}]_{(\sigma+\hbar,\sigma)}\widetilde{\mathbb{K}}_{\mathcal{V}})\hbar \\
&\quad \leqslant [\Theta]_{(\sigma+\hbar,\sigma)}(1+\mathfrak{R})^2\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}(1 + [\mathcal{V}]_{(\sigma+\hbar,\sigma)})\hbar.
\end{aligned} \tag{2.1}$$

Using the similar approach, and noting that $\Psi^{\mathfrak{J}}(\omega) \in \mathbb{B}_{\rho}(-\mathcal{B}(-\mathfrak{J}(0,\mathcal{B}))),\ \forall\ \omega \in [0,\rho]$; for $\tau, \hbar$ belongs to $[0,\rho]$ and $\tau+\hbar$ belongs to $[0,\rho]$, one can observe

$$\begin{aligned}
\|\Theta(\sigma, \Psi^{\mathcal{V}}(\sigma)) - \Theta(\sigma, \Psi^{\mathcal{V}}(0))\| &\leqslant (1+\mathfrak{R})^2\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}[\Theta]_{(\sigma,\sigma)}(1 + [\mathcal{V}]_{(\sigma,0)})\sigma, \\
\|\mathcal{U}(\sigma+\hbar, \Psi^{\mathfrak{J}}(\sigma+\hbar)) - \mathcal{U}(\sigma, \Psi^{\mathfrak{J}}(\sigma))\| &\leqslant (1+\mathfrak{R})^2[\mathcal{U}]_{\rho,\rho}(1 + [\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\hbar.
\end{aligned}$$

To show $[\Psi]_{\mathbb{BPC}_{\mathrm{Lip}}([-p,\rho];\mathcal{Y})} \leqslant \mathfrak{R}$, for $\tau \in (\tau_i, \tau_{i+1})$, $i \in \mathbb{N}_1^n$ and $\hbar > 0$ and $\tau + \hbar \in (\tau_i, \tau_{i+1}]$, we have

$$\begin{aligned}
\|\Psi(\tau+\hbar) - \Psi(\tau)\| &\leqslant \int_{\tau-\tau_i}^{\tau-\tau_i+\hbar}\left\|\frac{\mathfrak{D}\mathfrak{I}(\omega-\omega_i)}{\Gamma(\alpha)}\int_{u_i}^{\omega}(\omega-u)^{\alpha-1}g_i\big(\Psi(\mathfrak{J}_i(\Psi(\omega_i)))\big)du\right\|d\omega \\
&\quad + \|\mathcal{U}^{\mathfrak{J}}(\Psi)(\tau+\hbar)) - \mathcal{U}^{\mathfrak{J}}(\Psi)(\tau)\| + \int_0^{\hbar}\|\mathfrak{I}(\tau+\hbar-\omega)\|\|\Theta^{\mathcal{V}}(\Psi)(\sigma) - \Theta(\sigma,\Psi^{\mathcal{V}}(0))\|d\sigma \\
&\quad + \int_0^{\hbar}\|\mathfrak{I}(\tau+\hbar-\omega)\|\|\Theta(\sigma,\Psi^{\mathcal{V}}(0))\|d\sigma + \int_0^{\tau}\|\mathfrak{I}(\tau-\omega)\|\|\Theta^{\mathcal{V}}(\Psi)(\sigma+\hbar) - \Theta^{\mathcal{V}}(\Psi)(\sigma)\|d\sigma \\
&\leqslant \frac{1}{\Gamma(\alpha)}\|\mathfrak{D}\mathfrak{I}(\omega-\omega_i)\|\|g_i\big(\Psi(\mathfrak{J}_i(\Psi(\omega_i)))\big)\|\int_{\tau-\tau_i}^{\tau-\tau_i+\hbar}\int_{u_i}^{\omega}(\omega+u)^{\alpha-1}dud\omega \\
&\quad + [\mathcal{U}]_{\rho,\rho}(1+\mathfrak{R})^2(1 + [\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\hbar \\
&\quad + \mathbb{C}_0(1+\mathfrak{R})^2\widetilde{\mathbb{K}}_{\Theta,\mathfrak{J}}\int_0^{\hbar}[\Theta]_{(\sigma,\sigma)}(1 + [\mathcal{V}]_{(\sigma,0)})\sigma d\sigma + \mathbb{C}_0\|\Theta(\cdot,\mathcal{B}(-\mathfrak{J}(0,\mathcal{B})))\|_{\mathbb{PC}([0,\rho];\mathcal{Y})}\hbar \\
&\quad + (1+\mathcal{U})^2\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}\int_0^{\tau}[\Theta](\sigma+h,\sigma)(1 + [\mathcal{V}]_{(\sigma+h,\sigma)})\hbar d\sigma \\
&\leqslant \frac{2^{\alpha}(\tau-\tau_i+\hbar)^{\alpha+1}}{(\alpha+1)\Gamma(\alpha+1)}\|\mathfrak{D}\mathfrak{I}(\cdot)\|_{\mathbb{L}^{\infty}([0,b];\mathbb{L}(\mathcal{W},\mathcal{Y}))}\|g_i\big(\Psi(\mathfrak{J}_i(\Psi(\tau_i^+)))\big)\| \\
&\quad + (1+\mathfrak{R})^2[\mathcal{U}]_{\rho,\rho}(1 + [\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\hbar \\
&\quad + \mathbb{C}_0(1+\mathfrak{R})^2\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}(\Lambda_2(\rho) + \Lambda_1(\rho))\hbar + \mathbb{C}_0\|\Theta(\cdot,\mathcal{B}(-\mathfrak{J}(0,\mathcal{B})))\|_{\mathbb{PC}([0,\rho];\mathcal{Y})}\hbar, \\
&\leqslant \frac{2^{\alpha}(\tau-\tau_i+\hbar)^{\alpha+1}}{\Gamma(\alpha+2)}\Lambda_{\mathcal{Y},\mathcal{W}}\mathbb{C}_{\mathcal{Y},\mathcal{W}}(g)\hbar + (1+\mathfrak{R})^2[\mathcal{U}]_{\rho,\rho}(1 + [\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\hbar \\
&\quad + \mathbb{C}_0(1+\mathfrak{R})^2\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}(\Lambda_2(\rho) + \Lambda_1(\rho))\hbar + \mathbb{C}_0\|\Theta(\cdot,\mathcal{B}(-\mathfrak{J}(0,\mathcal{B})))\|_{\mathbb{PC}([0,\rho];\mathcal{Y})}\hbar \\
&\leqslant \widetilde{\Lambda_{\mathcal{Y},\mathcal{W}}}\mathbb{C}_{\mathcal{Y},\mathcal{W}}(g)\hbar + (1+\mathfrak{R})^2[\mathcal{U}]_{\rho,\rho}(1 + [\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\hbar
\end{aligned}$$

$$+\mathbb{C}_0(1+\mathfrak{R})^2\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}(\Lambda_2(\rho)+\Lambda_1(\rho))\hbar$$
$$+\mathbb{C}_0\|\Theta(\cdot,\mathcal{B}(-\mathfrak{J}(0,\mathcal{B}))))\|_{\mathbb{PC}([0,\rho];\mathcal{Y})}\hbar,\ \text{where } \widetilde{\Lambda_{\mathcal{Y},\mathcal{W}}}=\frac{2^{\alpha}(\tau-\tau_i+\hbar)^{\alpha+1}}{\Gamma(\alpha+2)}\Lambda_{\mathcal{Y},\mathcal{W}},$$

which implies that

$$\|\Psi(\tau+\hbar)-\Psi(\tau)\|\leqslant \aleph_{\mathcal{Y},\mathcal{W}}+(1+\mathfrak{R})^2[\mathcal{U}]_{\rho,\rho}(1+[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\hbar$$
$$+\mathbb{C}_0(1+\mathfrak{R})^2\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}(\Lambda_2(\rho)+\Lambda_1(\rho))\hbar+\mathbb{C}_0\|\Theta(\cdot,\mathcal{B}(-\mathfrak{J}(0,\mathcal{B}))))\|_{\mathbb{PC}([0,\rho];\mathcal{Y})}\hbar.$$

Now, for $\tau\in(\tau_i,\omega_i)$, $i\in\mathbb{N}_1^n$, $\hbar>0$ and $\tau+\hbar\in(\tau_i,\omega_i]$, one can see

$$\|\Psi(\tau+\hbar)-\Psi(\tau)\|\leqslant\int_{\tau-\tau_i}^{\tau-\tau_i+\hbar}\left\|\frac{\mathfrak{D}\mathfrak{I}(\omega-\omega_i)}{\Gamma(\alpha)}\int_{u_i}^{\omega}(\omega-u)^{\alpha-1}g_i\big(\Psi(\mathfrak{J}_i(\Psi(\omega_i)))\big)du\right\|d\omega\leqslant\aleph_{\mathcal{Y},\mathcal{W}}\hbar.$$

Using the similar approach, we show that

$$[(\Psi)|_{[0,\tau_1]}]_{\mathbb{PC}_{\mathrm{Lip}}([0,\tau_1];\mathcal{Y})}\leqslant[\mathfrak{I}(\cdot)(\mathcal{B}(0)+\mathcal{U}(0,\Psi^{\mathfrak{J}}(0))]_{\mathbb{PC}_{\mathrm{Lip}}([0,\kappa];\mathcal{Y})}\hbar+(1+\mathfrak{R})^2[\mathcal{U}]_{\rho,\rho}(1+[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\hbar$$
$$+\mathbb{C}_0(1+\mathfrak{R})^2\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}(\Lambda_2(\rho)+\Lambda_1(\rho))\hbar+\mathbb{C}_0\|\Theta(\cdot,\mathcal{B}(-\mathfrak{J}(0,\mathcal{B}))))\|_{\mathbb{PC}([0,\rho];\mathcal{Y})}\hbar,$$

hence, $[\Psi]_{\mathbb{PC}_{\mathrm{Lip}}([0,\rho];\mathcal{Y})}\leqslant\mathfrak{R}$ and $[\Psi]_{\mathbb{BPC}_{\mathrm{Lip}}([-\lambda,\rho];\mathcal{Y})}\leqslant\mathfrak{R}$ because $[\mathcal{B}]_{\mathbb{C}_{\mathrm{Lip}}([-\lambda,0];\mathcal{Y})}\leqslant\mathfrak{R}$, implies that is a $\mathcal{S}(\mathfrak{R})$-valued mapping. Letting $\Psi,\ \Phi\in\mathcal{S}(\mathfrak{R}),\ i\in\mathbb{N}_1^n$, and $\tau\in(\tau_i,\tau_{i+1}]$, we have

$$\|\Psi(\tau)-\Phi(\tau)\|\leqslant\frac{(2\omega)^{\alpha}}{\Gamma(\alpha+1)}\|\mathfrak{I}(\tau-\tau_i)\|_{\mathbb{L}(\mathcal{W},\mathcal{Y})}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g)\|\Psi(\mathfrak{J}(\Psi(\tau_i)))-\Phi(\mathfrak{J}(\Phi(\tau_i)))\|$$
$$+[\mathcal{U}]_{\rho,\rho}\|\Psi^{\mathfrak{J}}(\tau)-\Phi^{\mathfrak{J}}(\tau)\|+\mathbb{C}_0\int_0^{\tau}[\Theta]_{(\sigma,\sigma)}\widetilde{\mathbb{K}}_{\Theta}\|\Psi^{\mathcal{V}}(\sigma)-\Phi^{\mathcal{V}}(\sigma)\|d\sigma$$
$$\leqslant\Big(\frac{(2\omega)^{\alpha}}{\Gamma(\alpha+1)}\mathbb{C}_0\|i_c\|_{\mathbb{L}(\mathcal{W},\mathcal{Y})}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g)(1+[\Phi]_{\mathbb{BPC}_{\mathrm{Lip}(\mathcal{Y})}}[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}})\Big)\|\Psi-\Phi\|_{\mathbb{PC}([0,\tau];\mathcal{Y})}$$
$$+[\mathcal{U}]_{\rho,\rho}(1+[\Psi]_{\mathbb{BPC}_{\mathrm{Lip}}([-\lambda,\rho];\mathcal{Y})}[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\|\Psi-\Phi\|_{\mathbb{PC}([0,\tau];\mathcal{Y})}$$
$$+\mathbb{C}_0\int_0^{\tau}[\Theta]_{(\sigma,\sigma)}\widetilde{\mathbb{K}}_{\Theta}(1+[\Psi]_{\mathbb{BPC}_{\mathrm{Lip}}([-\lambda,\rho];\mathcal{Y})}[\mathcal{V}]_{(\sigma,\sigma)}\widetilde{\mathbb{K}}_{\mathcal{V}})\|\Psi-\Phi\|_{\mathbb{PC}([0,\omega];\mathcal{Y})}d\sigma$$
$$\leqslant\Big(\frac{(2\omega)^{\alpha}}{\Gamma(\alpha+1)}\mathbb{C}_0\|i_c\|_{\mathbb{L}(\mathcal{W},\mathcal{Y})}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g)(1+[\Phi]_{\mathbb{BPC}_{\mathrm{Lip}(\mathcal{Y})}}[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}})\Big)\|\Psi-\Phi\|_{\mathbb{PC}([0,\rho];\mathcal{Y})}$$
$$+(1+\mathfrak{R})[\mathcal{U}]_{\rho,\rho}(1+[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\|\Psi-\Phi\|_{\mathbb{PC}([0,\rho];\mathcal{Y})}$$
$$+(1+\mathfrak{R})\mathbb{C}_0\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}\int_0^{\tau}[\Theta]_{(\sigma,\sigma)}(1+[\mathcal{V}]_{(\sigma,\sigma)})d\sigma\|\Psi-\Phi\|_{\mathbb{PC}([0,\rho];\mathcal{Y})}$$
$$\leqslant\Big[\Big(\frac{(2\omega)^{\alpha}}{\Gamma(\alpha+1)}\mathbb{C}_0\|i_c\|_{\mathbb{L}(\mathcal{W},\mathcal{Y})}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g)(1+[\Phi]_{\mathbb{BPC}_{\mathrm{Lip}(\mathcal{Y})}}[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}})$$
$$+(1+\mathfrak{R})[\mathcal{U}]_{\rho,\rho}(1+[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})+\mathbb{C}_0\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}(\Lambda_2(b)+\Lambda_1(b))\Big]\|\Psi-\Phi\|_{\mathbb{PC}([0,\rho];\mathcal{Y})}.$$

Now, for $\tau\in(\tau_i,\omega_i]$, $i\in\mathbb{N}_1^n$, we have that

$$\|\Psi(\tau)-\Phi(\tau)\|\leqslant\frac{(2\omega)^{\alpha}}{\Gamma(\alpha+1)}\|\mathfrak{I}(\tau-\tau_i)\|_{\mathbb{L}(\mathcal{W},\mathcal{Y})}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g)\|\Psi(\mathfrak{J}(\Psi(\tau_i)))-\Phi(\mathfrak{J}(\Phi(\tau_i)))\|$$
$$\leqslant\Big(\frac{(2\omega)^{\alpha}}{\Gamma(\alpha+1)}\mathbb{C}_0\|i_c\|_{\mathbb{L}(\mathcal{W},\mathcal{Y})}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g)(1+[\Phi]_{\mathbb{BPC}_{\mathrm{Lip}(\mathcal{Y})}}[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}})\Big)\|\Psi-\Phi\|_{\mathbb{PC}([0,\rho];\mathcal{Y})}.$$

Furthermore, for $\tau\in[0,\tau_1]$, we note that

$$[(\Psi)|_{[0,\tau_1]}]_{\mathbb{PC}_{\mathrm{Lip}}([0,\tau_1];\mathcal{Y})}\leqslant\Big[(1+\mathfrak{R})^2[\mathcal{U}]_{\rho,\rho}(1+[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})+\mathbb{C}_0(1+\mathfrak{R})^2\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}(\Lambda_2(\rho)+\Lambda_1(\rho))$$

$$+\mathbb{C}_0\|\Theta(\cdot,\mathcal{B}(-\mathfrak{J}(0,\mathcal{B})))\|_{\mathbb{PC}([0,\rho];\mathcal{Y})}\Big]\|\Psi-\Phi\|_{\mathbb{PC}([0,\rho];\mathcal{Y})}.$$

From the above:

$$\begin{aligned}\|\Psi(\tau)-\Phi(\tau)\| &\leqslant \Big[\widetilde{\Lambda_{\mathcal{Y},\mathcal{W}}}^{*}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g)\Big(1+\mathfrak{R}\max_{i\in\mathbb{N}_1^n}[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}}\Big)+(1+\mathfrak{R})^2[\mathcal{U}]_{\rho,\rho}(1+[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\\ &\quad+\mathbb{C}_0(1+\mathfrak{R})^2\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}(\Lambda_2(\rho)+\Lambda_1(\rho))+\mathbb{C}_0\|\Theta(\cdot,\mathcal{B}(-\mathfrak{J}(0,\mathcal{B})))\|_{\mathbb{PC}([0,\rho];\mathcal{Y})}\Big]\\ &\quad\times\|\Psi-\Phi\|_{\mathbb{PC}([0,\rho];\mathcal{Y})}.\end{aligned}$$

where $\widetilde{\Lambda_{\mathcal{Y},\mathcal{W}}}^{*}=\frac{(2\omega)^{\alpha}}{\Gamma(\alpha+1)}\Lambda_{\mathcal{Y},\mathcal{W}}$. Hence $\Psi(\cdot)$ is clearly a contraction mapping and there exists a unique mild solution of the problem $\Psi\in\mathcal{S}(\mathfrak{R})$ of (1.1)-(1.3). □

Remark 2.7. Using the assertions in Theorem 2.6 we remark that $\mathfrak{J}(\cdot)y\in\mathbb{PC}_{\mathrm{Lip}([0,\kappa];\mathcal{Y})}$ if, e.g., $y\in\Omega(\mathfrak{D})$.

In the Proposition 2.8, we provide assertions so that the mild solution discussed in Theorem 2.6 becomes a strict solution . In the Proposition 2.8, we suppose that $(\mathfrak{J}(\tau))_{\tau\geqslant 0}$ is analytic, $\mathbb{C}_i$, $i=1,2$ are constants, and $\|(-\mathfrak{D})^i\mathfrak{J}(\tau)\|\leqslant\frac{\mathbb{C}_i}{\tau_i}$, $\forall\,\tau\in(0,\kappa]$.

Proposition 2.8. *$\Psi\in\mathbb{BPC}([-\lambda,\rho];\mathcal{Y})$ is mild solution and all the assertions used in Theorem 2.6 are fulfilled. If $\sup_{\tau\in[0,\kappa]}\|[\Theta](\tau,\cdot)\|_{\mathbb{L}_q(0,\tau)}\|[\mathcal{V}]_{(\tau,\cdot)}\|_{\mathbb{L}_r(0,\tau)}<\infty$, $\frac{1}{q}+\frac{1}{r}<1$, $\mathcal{B}(0)+\mathcal{U}(0,\mathcal{B}(-\mathfrak{J}(0,\mathcal{B})))\in\Omega(\mathfrak{D})$, and semigroup is analytic, then $\Psi(\cdot)$ is a strict solution of the problem (1.1)-(1.2) on $[-\lambda,\rho]$.*

Proof. Consider that, $\Psi_i:[0,\rho]\to\mathcal{Y}$, $i=1,2$, are the mappings defined as

$$\Psi(\tau)=\frac{1}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega,\ \text{for}\ \tau\in(\tau_i,\omega_i],\ \forall\, i=\mathbb{N}_1^n,$$

and

$$\begin{aligned}\Psi_1(\tau)&=\frac{T(\tau-\tau_i)}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega+\int_{\omega_i}^{\tau}\mathfrak{J}(\tau-\sigma)\Theta(\tau,\Psi^{\mathcal{V}}(\tau))d\tau,\ \forall\,\tau\in(\omega_i,\tau_{i+1}],\ i\in\mathbb{N}_1^n,\\ \Psi_2(\tau)&=\int_{\omega_i}^{\tau}\mathfrak{J}(\tau-\sigma)(\Theta(\sigma,\Psi^{\mathcal{V}}(\sigma))-\Theta(\tau,\Psi^{\mathcal{V}}(\tau)))d\sigma,\ \forall\,\tau\in(\omega_i,\tau_{i+1}],\ i\in\mathbb{N}_1^n.\end{aligned}$$

we note that $\Psi(\cdot)+\mathcal{U}(\cdot,\Psi^{\mathfrak{J}}(\cdot))=\Psi_1(\cdot)+\Psi_2(\cdot)$, $\forall\,\tau\in(\omega_i,\tau_{i+1}]$, $i\in\mathbb{N}_1^n$, and

$$\begin{aligned}\Psi_1(\tau)&=\mathfrak{J}(\tau)\big(\mathcal{B}\phi(0)+\mathcal{U}(0,\Psi^{\mathfrak{J}}(0))\big)+\int_{\tau_i}^{\tau}\mathfrak{J}(\tau-\sigma)\Theta(\tau,\Psi^{\mathcal{V}}(\tau))d\tau,\ \forall\,\tau\in[0,\ \tau_1],\\ \Psi_2(\tau)&=\int_{0}^{\tau}\mathfrak{J}(\tau-\sigma)(\Theta(\sigma,\Psi^{\mathcal{V}}(\sigma))-\Theta(\tau,\Psi^{\mathcal{V}}(\tau)))d\sigma,\ \forall\,\tau\in[0,\ \tau_1].\end{aligned}$$

Also, we note that $\Psi(\cdot)+\mathcal{U}(\cdot,\Psi^{\mathfrak{J}}(\cdot))=\Psi_1(\cdot)+\Psi_2(\cdot)\ \forall\,\tau\in[0,\ \tau_1]$. Furthermore, as $\mathcal{B}(0)+\mathcal{U}(0,\mathcal{B}(-\mathfrak{J}(0,\mathcal{B}))\in\Omega(\mathfrak{D})$, we get $\mathfrak{D}\Psi_1(\tau)=\mathfrak{J}(\tau)\mathfrak{D}(\mathcal{B}(0)+\mathcal{U}(0,\mathcal{B}(-\mathfrak{J}(0,\mathcal{B})))+(\mathfrak{J}(\tau)-I)\Theta(\tau,\Psi^{\mathcal{V}}(\tau)))$ for all $\tau\in[0,\tau_1]$, and

$$\mathfrak{D}\Psi_1(\tau)=\frac{\mathfrak{D}T(\tau-\tau_i)}{\Gamma(\alpha)}\int_{\omega_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega+\mathfrak{D}(\mathfrak{J}(\tau)-I)\Theta(\tau,\Psi^{\mathcal{V}}(\tau))),\ \ \forall\,\tau\in(\omega_i,\tau_{i+1}],\ i\in\mathbb{N}_1^n,$$

and

$$\mathfrak{D}\Psi(\tau)=\frac{\mathfrak{D}}{\Gamma(\alpha)}\int_{\tau_i}^{\omega}(\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega,\ \text{for}\ \omega\in(\tau_i,\omega_i],\ \forall\, i=\mathbb{N}_1^n,$$

implies $\Psi_1\in\mathbb{PC}([0,\rho],\mathcal{Y}_1)$.

Now, we study the mappings $\Psi_2(\cdot)$. Let $\rho = \|\Psi\|_{\mathbb{PC}([-\lambda,\rho];\mathcal{Y})}$, $\widetilde{\mathbb{K}}_\hbar = \widetilde{\mathbb{K}}_\hbar(\rho)$, and $\hbar(\cdot) = \Theta(\cdot)$, $\mathcal{V}(\cdot)$, also $\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}} = \widetilde{\mathbb{K}}_\Theta(1+\widetilde{\mathbb{K}}_\mathcal{V})$. Let $\mathcal{Y}(q,r) > 1$ such that $\frac{1}{q}+\frac{1}{r}+\frac{1}{\mathcal{Y}(q,r)} = 1$ and $0 < \alpha < \frac{1}{\mathcal{Y}(q,r)}$. Now, utilizing Lemma 2.5, for $\tau \in [\omega_i, \tau_{i+1}]$ we get

$$\begin{aligned}
&\int_{\omega_i}^{\tau} \|\mathfrak{D}\mathfrak{I}(\tau-\sigma)(\Theta(\sigma,\Psi^{\mathcal{V}}(\sigma)) - F(\tau,\Psi^{\mathcal{V}}(\tau)))\|d\sigma \\
&\quad \leqslant \mathbb{C}_1\widetilde{\mathbb{K}}_\Theta \int_{\omega_i}^{\tau} [\Theta]_{(\tau,\sigma)}(1+[\Psi]_{\mathbb{BPC}_{\mathrm{Lip}}([-\lambda,\rho];\mathcal{Y})}(1+[\mathcal{V}]_{(\tau,\sigma)}\widetilde{\mathbb{K}}_\mathcal{V}(1+[\Psi]_{\mathbb{BPC}_{\mathrm{Lip}}([-\lambda,\rho];\mathcal{Y})})))d\sigma \\
&\quad \leqslant \mathbb{C}_1\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}(1+[\Psi]_{\mathbb{BPC}_{\mathrm{Lip}}([-\lambda,\rho];\mathcal{Y})})^2 \int_{\omega_i}^{\tau} [\Theta]_{(\tau,\sigma)}(1+[\mathcal{V}]_{(\tau,\sigma)})d\sigma < \infty,
\end{aligned}$$

which implies $\Psi_2(\tau) \in \Omega(\mathfrak{D})$ and $\mathfrak{D}\Psi_2(\tau) = \int_0^\tau \mathfrak{D}\mathfrak{I}(\tau-\sigma)(\Theta_{(\sigma,\Psi^{\mathcal{V}}(\sigma))}) - \Theta(\tau,\Psi^{\mathcal{V}}(\tau)))d\sigma$ for $\tau \in [\omega_i, \tau_{i+1}]$. Furthermore, via the above mentioned representation and using the last estimate, for $\omega_i \leqslant \omega \leqslant \tau \leqslant \tau_{i+1}$, one can see

$$\begin{aligned}
\|\mathfrak{D}\Psi_2(\tau) - \mathfrak{D}\Psi_2(\omega)\| &\leqslant \int_{\omega}^{\tau} \|\mathfrak{D}\mathfrak{I}(\tau-\sigma)(\Theta(\sigma,\Psi^{\mathcal{V}}(\sigma)) - \Theta(\tau,\Psi^{\mathcal{V}}(\tau)))\|d\sigma \\
&\quad + \int_{\omega_i}^{\omega} \|(\mathfrak{D}\mathfrak{I}(\tau-\sigma) - \mathfrak{D}\mathfrak{I}(\omega-\sigma))(\Theta(\sigma,\Psi^{\mathcal{V}}(\sigma)) - \Theta(\omega,\Psi^{\mathcal{V}}(\omega)))\|d\sigma \\
&\quad + \|\mathfrak{D}\int_{\omega_i}^{\omega} \mathfrak{I}(\omega-\sigma))(\Theta(\tau,\Psi^{\mathcal{V}}(\tau)) - \Theta(\omega,\Psi^{\mathcal{V}}(\omega)))d\sigma\| \\
&\leqslant \mathbb{C}_1\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}(1+[\Psi]_{\mathbb{BPC}_{\mathrm{Lip}}([-\lambda,\rho];\mathcal{Y})})^2 \int_{\omega}^{\tau} [\Theta]_{(\tau,\sigma)}(1+[\mathcal{V}](\tau,\sigma))d\sigma \\
&\quad + \int_{\omega_i}^{\omega}\int_{\omega-\sigma}^{\tau-\sigma} \|\mathfrak{D}^2\mathfrak{I}(\theta)(\Theta(\sigma,\Psi^{\sigma}(\sigma)) - \Theta(\tau,\Psi^{\mathcal{V}}(\omega)))\|d\theta d\sigma \\
&\quad + \|(\mathfrak{I}(\tau) - \mathfrak{I}(\tau-\omega))(\Theta(\omega,\Psi^{\mathcal{V}}(\omega)) - \Theta(\tau,\Psi^{\mathcal{V}}(\tau)))\|,
\end{aligned}$$

and the inequality (2.1) remains true (with trivial modifications), and implies

$$\begin{aligned}
\|\mathfrak{D}\Psi_2(\tau) - \mathfrak{D}\Psi_2(\omega)\| &\leqslant \mathbb{C}_1\widetilde{\mathbb{K}}\Theta,\mathcal{V}(1+[\Psi]_{\mathbb{BPC}_{\mathrm{Lip}}([-\lambda,\rho];\mathcal{Y})})^2 \int_{\omega}^{\tau} [\Theta]_{(\tau,\sigma)}(1+[\mathcal{V}]_{(\tau,\sigma)})d\sigma \\
&\quad + \widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}\mathbb{C}_2(1+[\Psi]_{\mathbb{BPC}_{\mathrm{Lip}}([-\lambda,\rho];\mathcal{Y})})^2 \int_0^{\omega}\int_{\omega-\sigma}^{\tau-\sigma} \frac{[\Theta]_{(\omega,\sigma)}}{\theta}(1+[\mathcal{V}]_{(\omega,\sigma)})d\theta d\sigma \\
&\quad + 2\mathbb{C}_0\|\Theta(\omega,\Psi^{\mathcal{V}}(\omega)) - \Theta(\tau,\Psi^{\mathcal{V}}(\tau))\|.
\end{aligned} \tag{2.2}$$

In order to finish the proof that $\Psi_2(\cdot)$ is continuous, we examine the second term at the R.H.S of (2.2). Utilizing the inequality $\ln(1+\nu) \leqslant \frac{\nu^\beta}{\beta}$ for all $\beta > 0$ and $\nu > 0$, we get that

$$\begin{aligned}
\int_0^{\omega}\int_{\omega-\sigma}^{\tau-\sigma} \frac{[\Theta]_{(\omega,\sigma)}}{\theta}(1+[\mathcal{V}]_{(\omega,\sigma)})d\theta d\sigma &\leqslant \int_0^{\omega} \ln\left(\frac{\tau-\omega}{\omega-\sigma}+1\right)[\Theta](\omega,\sigma)(1+[\mathcal{V}]_{(\omega,\sigma)})d\sigma \\
&\leqslant \frac{1}{\alpha}\int_0^{\omega} \frac{(\tau-\omega)^\alpha}{(\omega-\sigma)^\alpha}[\Theta]_{(\omega,\sigma)}(1+[\mathcal{V}]_{(\omega,\sigma)})d\sigma \\
&\leqslant \frac{(\tau-\omega)^\alpha \kappa^{\frac{1}{\mathcal{Y}(q,r)}-1}}{\alpha[1-\alpha\mathcal{Y}(q,r)]^{\frac{1}{\mathcal{Y}(q,r)}}}\|[\Theta]_{(r,\cdot)}\|_{\mathbb{L}_q(0,r)}\|(1+[\mathcal{V}]_{(r,\cdot)})\|_{\mathbb{L}_r(0,r)}.
\end{aligned}$$

Again, utilizing Lemma 2.5 and the same approach as above for $0 \leqslant \omega \leqslant \tau \leqslant \tau_1$, $\Psi_2 \in (0,\tau_1]$, implies $\mathfrak{D}\Psi_2 \in \mathbb{PC}([0,b];\mathcal{Y})$. Using the aforementioned remarks and the general theory of regularity of mild solution s, see, e.g., Pazy [28, Chapter 4], implies that $\Psi(\cdot)$ is a strict solution.

We obtain Corollary 2.9 by utilizing Theorem 2.6 along with Proposition 2.8. We skip the proof.

Corollary 2.9. *Let* $\mathcal{B}(0)+\mathcal{U}(0,\mathcal{B}(-\mathfrak{J}(0,\mathcal{B}))) \in \Omega(\mathfrak{D})$ *and* $\mathcal{B}(\cdot)$, $\Theta(\cdot)$, $\mathcal{U}(\cdot)$, $\mathfrak{J}_i(\cdot)$, *and* $\mathfrak{J}(\cdot)$ *be Lipschitz mappings. If* $[\mathcal{U}]_{\mathbb{C}_{\mathrm{Lip}}([0,c]\times\mathbb{B}_r(\mathcal{B}(-\mathfrak{J}(0,\mathcal{B}));\mathcal{Y});\mathcal{Y})} \to 0$ *or* $[\mathcal{U}]_{\mathbb{C}_{\mathrm{Lip}}([0,c]\times\mathcal{Y};\mathcal{Y})} \to 0$ *as* $c \to 0$, *then there exists* $\Psi \in \mathbb{BPC}_{\mathrm{Lip}}([-\lambda,\rho];\mathcal{Y})$ *a mild solution of the problem* (1.1)-(1.2) *for* $0<\rho\leqslant\kappa$. *Furthermore, if the semigroup is analytic, then* $\Psi(\cdot)$ *is a strict solution on* $[-\lambda,\rho]$.

In the next Corollary 2.9 we suppose $\mathcal{U}(\cdot)$, $\Theta(\cdot)$, $\mathcal{V}(\cdot)$, $\mathfrak{J}_i(\cdot)$, and $\mathfrak{J}(\cdot)$ are Lipschitz mappings.

Corollary 2.10. *Let* $\mathcal{B} \in \mathbb{C}_{\mathrm{Lip}}([-\lambda,0];\mathcal{Y})$, $\mathfrak{I}(\cdot)(\mathcal{B}(0)+\mathcal{U}(0,\mathcal{B}(-\mathfrak{J}(0,\mathcal{B})))) \in \mathbb{C}_{\mathrm{Lip}}([0,\kappa];\mathcal{Y})$; $\mathfrak{J}$, $\mathfrak{J}_i, \mathcal{V} \in \mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])$ *and* Θ, $\mathcal{U} \in \mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};\mathcal{Y})$. *If the mapping* $\mathbb{P}:\mathbb{R}\to\mathbb{R}$ *defined as*

$$\begin{aligned}\mathbb{P}(y) = 1 &+ [\mathcal{B}]_{\mathbb{C}_{\mathrm{Lip}}([-\lambda,0];\mathcal{Y})} + \aleph_{\mathcal{Y},\mathcal{W}} + \widetilde{\Lambda_{\mathcal{Y},\mathcal{W}}}^{*}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g)\Big(1+\mathfrak{R}\max_{i\in\mathbb{N}_1^n}[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}}\Big)\\ &+ \limsup_{c\downarrow 0}[\mathfrak{I}(\cdot)(\mathcal{B}(0)+\mathcal{U}(0,\mathcal{B}(-\mathfrak{J}(0,\mathcal{B}))))]_{\mathbb{PC}_{\mathrm{Lip}}([0,c];\mathcal{Y})}\\ &+ \mathbb{C}_0\|\Theta(0,\mathcal{B}(-\mathfrak{J}(0,\mathcal{B})))\| + (1+y)^2[\mathcal{U}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};\mathcal{Y})}(1+[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])}) - y,\end{aligned}$$

has a positive root, then there exists a unique mild solution on $[-\lambda,\rho]$ *for* $0<\rho\leqslant\kappa$.

□

Proof. To initiate, in this case we remark, the mappings $\Lambda_i(\cdot)$ in the proof of Theorem 2.6 are defined as $\Lambda_1(\tau) = [\Theta]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};\mathcal{Y})}(1+[\mathcal{V}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\tau$ and $\Lambda_2(\tau) = \mathbb{C}_0[\Theta]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};\mathcal{Y})}(1+[\mathcal{V}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\tau$. Via the assertion, there exists $\mathfrak{R}>0$ such that $0>\mathbb{P}(\mathfrak{R})$. Furthermore, using the definition of $\mathbb{P}(\cdot)$, one can set $0<\rho\leqslant\kappa$ with $\mathfrak{R}\rho<1$ and

$$\begin{aligned}1 &+ [\mathcal{B}]_{\mathbb{C}_{\mathrm{Lip}}([-\lambda,0];\mathcal{Y})} + \aleph_{\mathcal{Y},\mathcal{W}} + \widetilde{\Lambda_{\mathcal{Y},\mathcal{W}}}^{*}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g)\Big(1+\mathfrak{R}\max_{i\in\mathbb{N}_1^n}[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}}\Big)\\ &+ [\mathfrak{I}(\cdot)(\mathcal{B}(0)+\mathcal{U}(0,\mathcal{B}(-\mathfrak{J}(0,\mathcal{B}))))]_{\mathbb{PC}_{\mathrm{Lip}}([0,\rho];\mathcal{Y})} + \mathbb{C}_0\|\Theta(\cdot,\mathcal{B}(-\mathcal{V}(0,\mathcal{B})))\|_{\mathbb{C}([0,\rho];\mathcal{Y})}\\ &+ (1+\mathfrak{R})^2[\mathcal{U}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};\mathcal{Y})}(1+[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\\ &+ (1+\mathfrak{R})^2 2\mathbb{C}_0[\Theta]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};\mathcal{Y})}(1+[\mathcal{V}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\rho < \mathfrak{R}.\end{aligned}$$

Utilizing the last inequality instead of inequality (2.1) and utilizing the approach used in Theorem 2.6, it is obvious that there exists $\Psi \in \mathbb{BPC}_{\mathrm{Lip}}([-\lambda,\rho];\mathcal{Y})$ a mild solution of (1.1)-(1.3). □

Corollary 2.11. *Let* $\mathcal{B} \in \mathbb{C}_{\mathrm{Lip}}([-\lambda,0];\mathcal{Y})$, $\mathfrak{I}(\cdot)(\mathcal{B}(0)+\mathcal{U}(0,\mathcal{B}(-\mathfrak{J}(0,\mathcal{B})))) \in \mathbb{PC}_{\mathrm{Lip}}([0,\kappa];\mathcal{Y})$; $\mathfrak{J}$, $\mathfrak{J}_i, \mathcal{V} \in \mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])$ *and* Θ, $\mathcal{U} \in \mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};\mathcal{Y})$. *If the mapping*

$$\begin{aligned}\mathbb{P}(y) = 1 &+ [\mathcal{B}]_{\mathbb{C}_{\mathrm{Lip}}([-\lambda,0];\mathcal{Y})} + \aleph_{\mathcal{Y},\mathcal{W}} + \widetilde{\Lambda_{\mathcal{Y},\mathcal{W}}}^{*}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g)\Big(1+\mathfrak{R}\max_{i\in\mathbb{N}_1^n}[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}}\Big)\\ &+ [\mathfrak{I}(\cdot)(\mathcal{B}(0)+\mathcal{U}(0,\mathcal{B}(-\mathfrak{J}(0,\mathcal{B}))))]_{\mathbb{PC}_{\mathrm{Lip}}([0,c];\mathcal{Y})} + \mathbb{C}_0\|\Theta(0,\mathcal{B}(-\mathfrak{J}(0,\mathcal{B})))\|\\ &+ (1+y)^2[\mathcal{U}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};\mathcal{Y})}(1+[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\\ &+ (1+y)^2 2\mathbb{C}_0[\Theta]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};\mathcal{Y})}(1+[\mathcal{V}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\kappa - y,\end{aligned}$$

has a positive root, then there exists $\Psi \in \mathbb{BPC}_{\mathrm{Lip}}([-\lambda,\kappa];X)$ *a unique mild solution of* (1.1)-(1.3).

3. Hyers-Ulam Stability

In order to establish the Hyers-Ulam stability of Eq. (1.1)-(1.3), we introduce the following assumptions.

U_1:

$$\left(\frac{\|\mathfrak{I}(\tau-\tau_i)\|}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}\|i_c\|_{\mathbb{L}(\mathcal{W},\mathcal{Y})}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g)(1+[\Theta]_{\mathbb{BPC}_{Lip(\mathcal{Y})}}[\mathfrak{J}]_{\mathbb{C}_{Lip}})d\sigma\right.$$
$$\left.+(1+\mathfrak{R})[\mathcal{U}]_{\rho,\rho}(1+[\mathfrak{J}]_{\mathbb{C}_{Lip}([0,\kappa]\times\mathcal{Y};[0,\lambda])})+(1+\mathfrak{R})\mathbb{C}_0\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}\int_0^{\tau}[\Theta]_{(\sigma,\sigma)}(1+[\mathcal{V}]_{(\sigma,\sigma)})d\sigma\right)<1.$$

U_2:

$$\delta=\left(\jmath+\frac{(1+\mathfrak{R})}{\epsilon}[\mathcal{U}]_{\rho,\rho}(1+[\mathfrak{J}]_{\mathbb{C}_{Lip}([0,\kappa]\times\mathcal{Y};[0,\lambda])})g^*(\tau)\right)>0.$$

U_3:

$$p=(1+\mathfrak{R})\mathbb{C}_0\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}[\Theta]_{(\sigma,\sigma)}(1+[\mathcal{V}]_{(\sigma,\sigma)}).$$

U_4:

$$r=\|i_c\|_{\mathbb{L}(\mathcal{W},\mathcal{Y})}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g^*)(1+[\Theta]_{\mathbb{BPC}_{Lip(\mathcal{Y})}}[\mathfrak{J}]_{\mathbb{C}_{Lip}}).$$

Assume the following inequalities

$$\|\Psi(\tau)-\frac{1}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega\|\leqslant\epsilon,\ \forall\,\tau\in(\tau_i,\omega_i],\ i=\mathbb{N}_1^n, \tag{3.1}$$

$$\begin{aligned}&\|\Psi(\tau)-\frac{\mathfrak{I}(\tau-\tau_i)}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega+\mathcal{U}(\tau,\Psi^{\mathfrak{J}}(\tau))\\&\quad-\int_{\omega_i}^{\tau}\mathfrak{I}(\tau-\sigma)\Theta(\sigma,\Psi(\sigma),\Psi(\mathcal{V}(\sigma)))d\sigma\|\leqslant\epsilon,\ \forall\,\tau\in(\omega_i,\tau_{i+1}],\ i\in\mathbb{N}_1^n,\end{aligned} \tag{3.2}$$

$$\begin{aligned}&\|\Psi(\tau)-\mathfrak{I}(\tau)\big(\mathcal{B}(0)-\mathcal{U}(0,\Psi^{\mathfrak{J}}(0))\big)+\mathcal{U}(\tau,\Psi^{\mathfrak{J}}(\tau))\\&\quad-\int_{\tau_i}^{\tau}\mathfrak{I}(\tau-\sigma)\Theta(\sigma,\Psi(\sigma),\Psi(\mathcal{V}(\sigma)))d\sigma\|\leqslant\epsilon,\ \forall\,\tau\in[0,\ \tau_1].\end{aligned} \tag{3.3}$$

Before proceeding to our main theorem of Hyers-Ulam stability, we introduce the following definition.

Definition 3.1. Eq. (1.1)-(1.3) is said to be Hyers-Ulam stable if there exists $\wp>0$ provided that for every $\epsilon>0$ and for every Ψ, the solution of inequalities (3.1)-(3.3), there exists mild solution Φ of Eq. (1.1)-(1.3) such that

$$\|\Psi(\tau)-\Phi(\tau)\|\leqslant\wp\epsilon,\ \forall\,\tau\in[0,\rho].$$

Remark 3.2. $\Psi\in\mathbb{BPC}([-\lambda,\rho];\mathcal{Y})$ satisfies the above inequalities (3.1)-(3.3) if and only if there exists $f\in\mathbb{BPC}([-\lambda,\rho];\mathcal{Y})$ and $\{f_k:k\in\mathbb{N}_1^m\}$ depending on Ψ with $\|f(\tau)\|\leqslant\epsilon$ and $\|f_k\|\leqslant\epsilon$, such that

(i) $\Psi(\tau)=\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega+f_k,\ \forall\,\tau\in(\tau_i,\omega_i],\ i=\mathbb{N}_1^n,$

(ii) $$\begin{aligned}\Psi(\tau)=&\frac{\mathfrak{I}(\tau-\tau_i)}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega-\mathcal{U}(\tau,\Psi^{\mathfrak{J}}(\tau))\\&+\int_{\omega_i}^{\tau}\mathfrak{I}(\tau-\sigma)\Theta(\sigma,\Psi(\sigma),\Psi(\mathcal{V}(\sigma)))d\sigma+f(\tau),\ \forall\,\tau\in(\omega_i,\tau_{i+1}],\ i\in\mathbb{N}_1^n,\end{aligned}$$

(iii) $\Psi(\tau)=\mathfrak{I}(\tau)\big(\mathcal{B}(0)-\mathcal{U}(0,\Psi^{\mathfrak{J}}(0))\big)-\mathcal{U}(\tau,\Psi^{\mathfrak{J}}(\tau))+\displaystyle\int_{\tau_i}^{\tau}\mathfrak{I}(\tau-\sigma)\Theta(\sigma,\Psi(\sigma),\Psi(\mathcal{V}(\sigma)))d\sigma+f(\tau),\ \forall\,\tau\in[0,\ \tau_1].$

Lemma 3.3. *If $\Psi \in \mathbb{BPC}([-\lambda, \rho]; \mathcal{Y})$ satisfies the above inequalities (3.1)-(3.3), then $\Psi(0) = \mathcal{B}(0)$ also satisfies the following inequalities*

$$\|\Psi(\tau) - \frac{1}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega\| \leqslant \epsilon, \ \forall\, \tau \in (\tau_i, \omega_i],\ i = \mathbb{N}_1^n,$$
$$\|\Psi(\tau) - \frac{\mathfrak{I}(\tau-\tau_i)}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega + \mathcal{U}(\tau, \Psi^{\mathfrak{J}}(\tau))$$
$$- \int_{\omega_i}^{\tau}\mathfrak{I}(\tau-\sigma)\Theta(\sigma, \Psi(\sigma), \Psi(\mathcal{V}(\sigma)))d\sigma\| \leqslant \jmath\epsilon, \ \forall\, \tau \in (\omega_i, \tau_{i+1}],\ i \in \mathbb{N}_1^n,$$
$$\|\Psi(\tau) - \mathfrak{I}(\tau)\big(\mathcal{B}(0) - \mathcal{U}(0, \Psi^{\mathfrak{J}}(0))\big) + \mathcal{U}(\tau, \Psi^{\mathfrak{J}}(\tau))$$
$$- \int_{\tau_i}^{\tau}\mathfrak{I}(\tau-\sigma)\Theta(\sigma, \Psi(\sigma), \Psi(\mathcal{V}(\sigma)))d\sigma\| \leqslant \tilde{\jmath}\epsilon, \ \forall\, \tau \in [0,\ \tau_1],$$

for $\tau \in (\tau_i, \tau_{i+1}]$, where $\|\mathfrak{I}(\tau-\sigma)\| \leqslant \mathbb{C}_0$, $\jmath = \mathbb{C}_0(\tau_{i+1} - \omega_i) + m$ and $\tilde{\jmath} = \mathbb{C}_0(\tau_{i+1} - \omega_i)$.

Proof. If $\Psi \in \mathbb{BPC}([-\lambda, \rho]; \mathcal{Y})$ satisfies the above inequalities (3.1)-(3.3), then using the above Remark 3.2 we have

$$\Psi(\tau) = \frac{1}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega + f_k, \ \forall\, \tau \in (\tau_i, \omega_i],\ i = \mathbb{N}_1^n,$$
$$\Psi(\tau) = \frac{\mathfrak{I}(\tau-\tau_i)}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega - \mathcal{U}(\tau, \Psi^{\mathfrak{J}}(\tau))$$
$$+ \int_{\omega_i}^{\tau}\mathfrak{I}(\tau-\sigma)\Theta(\sigma, \Psi(\sigma), \Psi(\mathcal{V}(\sigma)))d\sigma + f(\tau), \ \forall\, \tau \in (\omega_i, \tau_{i+1}],\ i \in \mathbb{N}_1^n,$$
$$\Psi(\tau) = \mathfrak{I}(\tau)\big(\mathcal{B}(0) - \mathcal{U}(0, \Psi^{\mathfrak{J}}(0))\big) - \mathcal{U}(\tau, \Psi^{\mathfrak{J}}(\tau)) + \int_{0}^{\tau}\mathfrak{I}(\tau-\sigma)\Theta(\sigma, \Psi(\sigma), \Psi(\mathcal{V}(\sigma)))d\sigma + f(\tau), \ \forall\, \tau \in [0,\ \tau_1].$$

For $\tau \in (\tau_i, \omega_i],\ i = \mathbb{N}_1^n$, we can see

$$\|\Psi(\tau) - \frac{1}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega\| \leqslant \epsilon.$$

And, for $\tau \in (\omega_i, \tau_{i+1}],\ i \in \mathbb{N}_1^n$, we have

$$\|\Psi(\tau) - \frac{\mathfrak{I}(\tau-\tau_i)}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega + \mathcal{U}(\tau, \Psi^{\mathfrak{J}}(\tau)) - \int_{\omega_i}^{\tau}\mathfrak{I}(\tau-\sigma)\Theta(\sigma, \Psi(\sigma), \Psi(\mathcal{V}(\sigma)))d\sigma\|$$
$$\leqslant \int_{\omega_i}^{\tau}\|\mathfrak{I}(\tau-\sigma)\|\|f(\sigma)\|d\sigma + \sum_{k=1}^{m}\|f_k\| \leqslant (\mathbb{C}_0(\tau-\omega_i) + m)\epsilon \leqslant (\mathbb{C}_0(\tau_{i+1}-\omega_i) + m)\epsilon \leqslant \jmath\epsilon.$$

Also, for $\tau \in (0, \tau_1]$, we obtain

$$\|\Psi(\tau) - \mathfrak{I}(\tau)\big(\mathcal{B}(0) - \mathcal{U}(0, \Psi^{\mathfrak{J}}(0))\big) + \mathcal{U}(\tau, \Psi^{\mathfrak{J}}(\tau)) - \int_{\tau_i}^{\tau}\mathfrak{I}(\tau-\sigma)\Theta(\sigma, \Psi(\sigma), \Psi(\mathcal{V}(\sigma)))d\sigma\|$$
$$\leqslant \int_{\omega_i}^{\tau}\|\mathfrak{I}(\tau-\sigma)\|\|f(\sigma)\|d\sigma \leqslant \tilde{\jmath}\epsilon.$$

Theorem 3.4. *If all the assertions of Theorem 2.6 along with* $\mathbf{U_1}$-$\mathbf{U_4}$ *are satisfied, then Eq. (1.1)-(1.3) is Hyers-Ulam stable.*

Let $\Psi(\tau)$ be a solution of the inequalities (3.1)-(3.3) and $\Psi(\tau)$ be a unique mild solution of the Eq. (1.1)-(1.3), which is given by

$$\Psi(\tau) = \frac{1}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega, \ \forall\, \tau \in (\tau_i, \omega_i],\ i = \mathbb{N}_1^n,$$

$$\Psi(\tau) = \frac{\mathfrak{I}(\tau-\tau_i)}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega - \mathcal{U}(\tau,\Psi^{\mathfrak{J}}(\tau))$$
$$+\int_{\omega_i}^{\tau}\mathfrak{I}(\tau-\sigma)\Theta(\sigma,\Psi(\sigma),\Psi(\mathcal{V}(\sigma)))d\sigma,\ \forall\,\tau\in(\omega_i,\tau_{i+1}],\ i\in\mathbb{N}_1^n,$$
$$\Psi(\tau) = \mathfrak{I}(\tau)\big(\mathcal{B}(0)-\mathcal{U}(0,\Psi^{\mathfrak{J}}(0))\big) - \mathcal{U}(\tau,\Psi^{\mathfrak{J}}(\tau)) + \int_0^{\tau}\mathfrak{I}(\tau-\sigma)\Theta(\sigma,\Psi(\sigma),\Psi(\mathcal{V}(\sigma)))d\sigma,\ \forall\,\tau\in[0,\ \tau_1].$$

Now, for $\tau\in(\omega_i,\tau_{i+1}],\ i=\mathbb{N}_1^n$, we have,

$$\begin{aligned}
\|\Psi(\tau)-\Phi(\tau)\| \leqslant{}& \|\frac{\mathfrak{I}(\tau-\tau_i)}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega \\
&+\mathcal{U}(\tau,\Psi^{\mathfrak{J}}(\tau)) - \int_{\omega_i}^{\tau}\mathfrak{I}(\tau-\sigma)\Theta(\sigma,\Psi(\sigma),\Psi(\mathcal{V}(\sigma)))d\sigma\| \\
&+\|\frac{\mathfrak{I}(\tau-\tau_i)}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i(\Psi(\mathfrak{J}(\Psi(\tau_i))))d\omega \\
&-\frac{\mathfrak{I}(\tau-\tau_i)}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}g_i(\Phi(\mathfrak{J}(\Phi(\tau_i))))d\omega\| + \|\mathcal{U}(\tau,\Psi^{\mathfrak{J}}(\tau))-\mathcal{U}(\tau,\Phi^{\mathfrak{J}}(\tau))\| \\
&+\|\int_{\omega_i}^{\tau}\mathfrak{I}(\tau-\sigma)\Theta(\sigma,\Psi(\sigma),\Psi(\mathcal{V}(\sigma)))d\sigma - \int_{\omega_i}^{\tau}\mathfrak{I}(\tau-\sigma)\Theta(\sigma,\Phi(\sigma),\Phi(\mathcal{V}(\sigma)))d\sigma\| \\
\leqslant{}& (\jmath\epsilon) + \frac{\|\mathfrak{I}(\tau-\tau_i)\|}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}\|g_i(\Psi(\mathfrak{J}(\Psi(\tau_i)))) - g_i(\Phi(\mathfrak{J}(\Phi(\tau_i))))\|d\omega \\
&+[\mathcal{U}]_{\rho,\rho}\|\Psi^{\mathfrak{J}}(\tau)-\Phi^{\mathfrak{J}}(\tau)\| + \int_{\omega_i}^{\tau}\|\mathfrak{I}(\tau-\sigma)\|\|\Theta(\sigma,\Psi(\sigma),\Psi(\mathcal{V}(\sigma)))-\Theta(\sigma,\Phi(\sigma),\Phi(\mathcal{V}(\sigma)))\|d\sigma \\
\leqslant{}& (\jmath\epsilon) + \frac{\mathbb{C}_0}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}\|i_c\|_{\mathbb{L}(\mathcal{W},\mathcal{Y})}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g)(1+[\Theta]_{\mathbb{BPC}_{\mathrm{Lip}(\mathcal{Y})}}[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}})\|\Psi(\sigma)-\Phi(\sigma)\|d\sigma \\
&+[\mathcal{U}]_{\rho,\rho}(1+[\Psi]_{\mathbb{BPC}_{\mathrm{Lip}}([-\lambda,\rho];\mathcal{Y})}[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\|\Psi(\tau)-\Phi(\tau)\| \\
&+\mathbb{C}_0\int_0^{\tau}[\Theta]_{(\sigma,\sigma)}\widetilde{\mathbb{K}}_{\Theta}(1+[\Psi]_{\mathbb{BPC}_{\mathrm{Lip}}([-\lambda,\rho];\mathcal{Y})}[\mathcal{V}]_{(\sigma,\sigma)}\widetilde{\mathbb{K}}_{\mathcal{V}})\|\Psi(\sigma)-\Phi(\sigma)\|d\sigma \\
\leqslant{}& (\jmath\epsilon) + \frac{\mathbb{C}_0}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}\|i_c\|_{\mathbb{L}(\mathcal{W},\mathcal{Y})}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g)(1+[\Theta]_{\mathbb{BPC}_{\mathrm{Lip}(\mathcal{Y})}}[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}})\|\Psi(\sigma)-\Phi(\sigma)\|d\sigma \\
&+(1+\mathfrak{R})[\mathcal{U}]_{\rho,\rho}(1+[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\|\Psi(\tau)-\Phi(\tau)\| \\
&+(1+\mathfrak{R})\mathbb{C}_0\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}\int_0^{\tau}[\Theta]_{(\sigma,\sigma)}(1+[\mathcal{V}]_{(\sigma,\sigma)})\|\Psi(\sigma)-\Phi(\sigma)\|d\sigma.
\end{aligned}$$

Next, we prove that $\Xi:\mathbb{BPC}([-\lambda,\rho];\mathcal{Y})\to\mathbb{BPC}([-\lambda,\rho];\mathcal{Y})$ is an increasing Picard operator:

$$(\Xi g(\tau)) = (\jmath\epsilon) + \frac{\mathbb{C}_0}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}\|i_c\|_{\mathbb{L}(\mathcal{W},\mathcal{Y})}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g)(1+[\Theta]_{\mathbb{BPC}_{\mathrm{Lip}(\mathcal{Y})}}[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}})g(\sigma)d\sigma$$

$$+(1+\mathfrak{R})[\mathcal{U}]_{\rho,\rho}(1+[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})g(\tau) + (1+\mathfrak{R})\mathbb{C}_0\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}\int_0^{\tau}[\Theta]_{(\sigma,\sigma)}(1+[\mathcal{V}]_{(\sigma,\sigma)})g(\sigma)d\sigma.$$

For all $g_1,g_2\in\mathbb{BPC}([-\lambda,\rho];\mathcal{Y}),\ \tau\in(\omega_i,\tau_{i+1}],\ i=\mathbb{N}_1^n$, we get

$$\|(\Xi g_1)(\tau)-(\Xi g_2)(\tau)\| \leqslant \frac{\mathbb{C}_0}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}\|i_c\|_{\mathbb{L}(\mathcal{W},\mathcal{Y})}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g)(1+[\Theta]_{\mathbb{BPC}_{\mathrm{Lip}(\mathcal{Y})}}[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}})\|g_1(\sigma)-g_2(\sigma)\|d\sigma$$

$$+(1+\mathfrak{R})[\mathcal{U}]_{\rho,\rho}(1+[\mathfrak{J}]_{\mathbb{C}_{\mathrm{Lip}}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\|g_1(\tau)-g_2(\tau)\|$$

$$
\begin{aligned}
&+(1+\mathfrak{R})\mathbb{C}_0\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}\int_0^{\tau}[\Theta]_{(\sigma,\sigma)}(1+[\mathcal{V}]_{(\sigma,\sigma)})d\sigma\\
\leqslant&\left(\frac{\mathbb{C}_0}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}\|i_c\|_{\mathbb{L}(\mathcal{W},\mathcal{Y})}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g)(1+[\Theta]_{\mathbb{BPC}_{Lip(\mathcal{Y})}}[\mathfrak{J}]_{\mathbb{C}_{Lip}})d\sigma\right.\\
&+(1+\mathfrak{R})[\mathcal{U}]_{\rho,\rho}(1+[\mathfrak{J}]_{\mathbb{C}_{Lip}([0,\kappa]\times\mathcal{Y};[0,\lambda])})\\
&\left.+(1+\mathfrak{R})\mathbb{C}_0\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}\int_0^{\tau}[\Theta]_{(\sigma,\sigma)}(1+[\mathcal{V}]_{(\sigma,\sigma)})d\sigma\right)\|g_1-g_2\|_{\mathbb{PC}([0,\rho];\mathcal{Y})}.
\end{aligned}
$$

Using $\mathbf{U_1}$, the above operator Ξ is clearly strictly contractive on $(\omega_i,\tau_{i+1}]$, $i=\mathbb{N}_1^n$, so it is a Picard operator on $\mathbb{BPC}([-\lambda,\rho];\mathcal{Y})$. Hence, we conclude that $g* \in \mathbb{BPC}([-\lambda,\rho];\mathcal{Y})$ is the only one fixed point of Ξ, that is,

$$
\begin{aligned}
(\Xi g^*(\tau))=(\jmath\epsilon)&+\frac{\mathbb{C}_0}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}\|i_c\|_{\mathbb{L}(\mathcal{W},\mathcal{Y})}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g^*)(1+[\Theta]_{\mathbb{BPC}_{Lip(\mathcal{Y})}}[\mathfrak{J}]_{\mathbb{C}_{Lip}})g^*(\sigma)d\sigma\\
&+(1+\mathfrak{R})[\mathcal{U}]_{\rho,\rho}(1+[\mathfrak{J}]_{\mathbb{C}_{Lip}([0,\kappa]\times\mathcal{Y};[0,\lambda])})g^*(\tau)+(1+\mathfrak{R})\mathbb{C}_0\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}\int_0^{\tau}[\Theta]_{(\sigma,\sigma)}(1+[\mathcal{V}]_{(\sigma,\sigma)})g^*(\sigma)d\sigma.
\end{aligned}
$$

By simple calculation we can conclude that

$$
\begin{aligned}
\Xi g^*(\tau)&=(\jmath\epsilon)+\frac{\mathbb{C}_0}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}\|i_c\|_{\mathbb{L}(\mathcal{W},\mathcal{Y})}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g^*)(1+[\Theta]_{\mathbb{BPC}_{Lip(\mathcal{Y})}}[\mathfrak{J}]_{\mathbb{C}_{Lip}})g^*(\sigma)d\sigma\\
&\quad+(1+\mathfrak{R})[\mathcal{U}]_{\rho,\rho}(1+[\mathfrak{J}]_{\mathbb{C}_{Lip}([0,\kappa]\times\mathcal{Y};[0,\lambda])})g^*(\tau)+(1+\mathfrak{R})\mathbb{C}_0\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}\int_0^{\tau}[\Theta]_{(\sigma,\sigma)}(1+[\mathcal{V}]_{(\sigma,\sigma)})g^*(\sigma)d\sigma\\
&=(\jmath\epsilon)+(1+\mathfrak{R})[\mathcal{U}]_{\rho,\rho}(1+[\mathfrak{J}]_{\mathbb{C}_{Lip}([0,\kappa]\times\mathcal{Y};[0,\lambda])})g^*(\tau)\\
&\quad+\frac{\mathbb{C}_0}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}\|i_c\|_{\mathbb{L}(\mathcal{W},\mathcal{Y})}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g^*)(1+[\Theta]_{\mathbb{BPC}_{Lip(\mathcal{Y})}}[\mathfrak{J}]_{\mathbb{C}_{Lip}})g^*(\sigma)d\sigma\\
&\quad+(1+\mathfrak{R})\mathbb{C}_0\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}\int_0^{\tau}[\Theta]_{(\sigma,\sigma)}(1+[\mathcal{V}]_{(\sigma,\sigma)})g^*(\sigma)d\sigma\\
&=\left(\jmath+\frac{(1+\mathfrak{R})}{\epsilon}[\mathcal{U}]_{\rho,\rho}(1+[\mathfrak{J}]_{\mathbb{C}_{Lip}([0,\kappa]\times\mathcal{Y};[0,\lambda])})g^*(\tau)\right)\epsilon\\
&\quad+\frac{\mathbb{C}_0}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}\|i_c\|_{\mathbb{L}(\mathcal{W},\mathcal{Y})}\mathbb{L}_{\mathcal{Y},\mathcal{W}}(g^*)(1+[\Theta]_{\mathbb{BPC}_{Lip(\mathcal{Y})}}[\mathfrak{J}]_{\mathbb{C}_{Lip}})g^*(\sigma)d\sigma\\
&\quad+(1+\mathfrak{R})\mathbb{C}_0\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}\int_0^{\tau}[\Theta]_{(\sigma,\sigma)}(1+[\mathcal{V}]_{(\sigma,\sigma)})g^*(\sigma)d\sigma\\
&\leqslant\delta\epsilon+\frac{\mathbb{C}_0}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}r(\sigma)g^*(\sigma)d\sigma+\int_0^{\tau}p(\sigma)g^*(\sigma)d\sigma,
\end{aligned}
$$

where $\delta>0$, for $i\in\mathbb{N}_1^m$ we have

$$
g^*(\tau)\leqslant\delta\epsilon+\sum_{0<\omega_i<\tau}\frac{\mathbb{C}_0}{m\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}r(\sigma)g^*(\sigma)d\sigma+\int_0^{\tau}p(\sigma)g^*(\sigma)d\sigma.
$$

Utilizing Grönwall Lemma, we conclude that

$$
g^*(\tau)\leqslant\delta\epsilon\prod_{\omega_i<\omega<\tau}\left(1+\frac{\mathbb{C}_0}{m\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}r(\sigma)d\sigma\right)\exp\left(\int_0^{\tau}p(\sigma)d\sigma\right),
$$

where $q=(1+\mathfrak{R})\mathbb{C}_0\widetilde{\mathbb{K}}_{\Theta,\mathcal{V}}$. If we substitute $g=\|\Psi-\Phi\|_{\mathbb{PC}([0,\rho];\mathcal{Y})}$,

$$
\|\Psi-\Phi\|_{\mathbb{PC}([0,\rho];\mathcal{Y})}\leqslant\delta\epsilon\prod_{\omega_i<\omega<\tau}\left(1+\frac{\mathbb{C}_0}{m\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}r(\sigma)d\sigma\right)\exp\left(\int_0^{\tau}p(\sigma)d\sigma\right),
$$

which implies that the proposed equations are Hyers-Ulam stable, which completes the proof. □

4. An example

Let us consider the following abstract neutral differential equations containing state-dependent fractional integrable impulses

$$\begin{aligned} &\frac{d}{d\tau}[\Psi(\tau,\chi)+\chi_1(\tau)h(\Psi(\tau-\frac{\|\Psi(\tau)\|}{1+\|\Psi(\tau)\|},\chi))] \qquad (4.1)\\ &=B(\Psi(\tau,\chi)+\chi_1(\tau)h(\Psi(\tau-\frac{\|\Psi(\tau)\|}{1+\|\Psi(\tau)\|},\chi)))+\chi_2(\tau)g(\Psi(\tau-\frac{\|\Psi(\tau)\|}{1+\|\Psi(t)\|},\chi)),\tau\in[0,2]\setminus(1.2,1.6],\\ &\Psi(\tau)=\frac{1}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}h(\Psi(\tau-\frac{\|\Psi(\tau)\|}{1+\|\Psi(\tau)\|},\chi))d\omega,\tau\in(1.2,1.6], \qquad (4.2)\\ &\Psi(0)=\mathcal{B}\in\mathbb{C}([-1,0];\mathcal{Y}), \qquad (4.3)\end{aligned}$$

where $(\tau,\chi)\in[0,2]\times\mathfrak{D}\subset\mathbb{R}^N$, $\mathfrak{D}$ is compact $\mathfrak{D}\subset\mathbb{R}^N$ and B is a square matrix. We set $\mathcal{Y}=C(\mathfrak{D};\mathbb{R}^N)$, g,h belongs to $\mathbb{C}_{\text{Lip}}(\mathbb{R}^N;\mathbb{R}^N)$, and $\chi_1\in\mathbb{C}^1([0,2];\mathbb{R})$ and $\chi_2\in\mathbb{C}([0,2];\mathbb{R})$. Furthermore, we take $\chi_1(0)=\chi_1'(0)=0$, $\chi_2(\cdot)$ is differentiable a.e. on the provided domain, $\chi_2'(\cdot)$ belongs to $L^q([0,2];\mathbb{R})$ for $q=1$ and ξ is defined from $[0,2]\times[0,2]$ to $\mathbb{R}^+$ provided $|\chi_2(\tau)-\chi_2(\omega)|\leqslant\chi_2'(\zeta(\tau,\omega))|\tau-\omega|$ and $\omega\leqslant\chi_{(\tau,\omega)}\leqslant\tau$, $\forall$ $0<\omega<\tau<2$, and $[\chi_2]_{(\cdot,\cdot)}=\chi_2'(\zeta_{(\cdot,\cdot)})$ is integrable on $U=\{(\tau,\omega)\in[0,2]\times[0,2],\omega\leqslant\tau\}$. Consider the associated inequalities

$$\begin{aligned} &\|\frac{d}{d\tau}[\Psi(\tau,\chi)+\chi_1(\tau)h(\Psi(\tau-\frac{\|\Psi(\tau)\|}{1+\|\Psi(\tau)\|},\chi))]-B(\Psi(\tau,\chi)+\chi_1(\tau)h(\Psi(\tau-\frac{\|\Psi(\tau)\|}{1+\|\Psi(\tau)\|},\chi)))\\ &\quad-\chi_2(\tau)g(\Psi(\tau-\frac{\|\Psi(\tau)\|}{1+\|\Psi(t)\|},\chi))\|\leqslant\epsilon,\tau\in[0,2]\setminus(1.2,1.6],\\ &\|\Psi(\tau)-\frac{1}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}h(\Psi(\tau-\frac{\|\Psi(\tau)\|}{1+\|\Psi(\tau)\|},\chi))d\omega\|\leqslant\epsilon,\ \tau\in(1.2,1.6],\\ &\|\Psi(0)-\mathcal{B}\|\leqslant\epsilon,\mathcal{B}\in\mathbb{C}([-1,0];\mathcal{Y}),\end{aligned}$$

substitute $\epsilon=1$. If $\Psi\in\mathbb{C}_{\text{Lip}}(\mathbb{R}^N;\mathbb{R}^N)$ holds the above inequalities, then there exist $h\in\mathbb{C}_{\text{Lip}}(\mathbb{R}^N;\mathbb{R}^N)$ and $h_0\in\mathbb{R}^N$ such that $|h(\tau)|\leqslant 1$. The equation becomes

$$\begin{aligned} &\frac{d}{d\tau}[\Psi(\tau,\chi)+\chi_1(\tau)h(\Psi(\tau-\frac{\|\Psi(\tau)\|}{1+\|\Psi(\tau)\|},\chi))]\\ &=B(\Psi(\tau,\chi))+\chi_2(\tau)g(\Psi(\tau-\frac{\|\Psi(\tau)\|}{1+\|\Psi(t)\|},\chi))+h(\tau),\tau\in[0,2]\setminus(1.2,1.6],\\ &\Psi(\tau)=\frac{1}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}h(\Psi(\tau-\frac{\|\Psi(\tau)\|}{1+\|\Psi(\tau)\|},\chi))d\omega+h_0,\tau\in(1.2,1.6],\\ &\Psi(0)=\mathcal{B}\in\mathbb{C}([-1,0];\mathcal{Y}).\end{aligned}$$

So the solution of above Eq. (4.1)-(4.3) is

$$\begin{aligned} \Psi(\tau)=&\frac{\mathfrak{I}(\tau-\tau_i)}{\Gamma(\alpha)}\int_{\tau_i}^{\tau}(\tau-\omega)^{\alpha-1}h(\Psi(\tau-\frac{\|\Psi(\tau)\|}{1+\|\Psi(\tau)\|},\chi))d\omega-\chi_1(\tau)h(\Psi(\tau-\frac{\|\Psi(\tau)\|}{1+\|\Psi(\tau)\|},\chi))\\ &+\int_{\omega_i}^{\tau}\mathfrak{I}(\tau-\sigma)\chi_1(\tau)\Big(h(\Psi(\tau-\frac{\|\Psi(\tau)\|}{1+\|\Psi(\tau)\|},\chi))\\ &-\chi_2(\tau)g(\Psi(\tau-\frac{\|\Psi(\tau)\|}{1+\|\Psi(t)\|},\chi))\|\Big)d\sigma,\ \forall\tau\in[0,2]\setminus(1.2,1.6].\end{aligned}$$

According to our results, Eq. (4.1)-(4.3) has a unique solution $\Psi\in\mathbb{C}_{\text{Lip}}(\mathbb{R}^N;\mathbb{R}^N)$ and is Hyers-Ulam stable on aforementioned domain.

5. Conclusion

In this article, we proved the existence, uniqueness and Hyers-Ulam stability of Eq. (1.1)-(1.3), using the properties of analytic semigroup and fixed point approach. In addition, we developed our main results by using the abstract Grönwall lemma. Hyers-Ulam stability provides the bound between exact and approximate solutions. Therefore, our results are important in approximation theory.

Conflict of interest

This work does not have any conflicts of interest.

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