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Inequalities presenting error bounds for trapezoidal formula using Caputo-Fabrizio integrals via convex functions with graphical depiction

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Abstract

This paper accomplishes some new Hermite-Hadamard type inequalities for (α, s, m)-convex functions in the context of Caputo-Fabrizio integrals. We establish this new version with the aid of Hölder and power-mean inequalities. Applications to obtained results are given by special means. Furthermore, errors are estimated for the trapezoidal formula and are graphically depicted. Finally, some bounds for the expectation value of the probability density function (which is (α, s, m)-convex) are obtained using the exponential population growth model.

Keywords: Convex function, Hölder's inequality, power-mean inequality, fractional derivatives, probability density function, optimization.

2020 MSC: 26A51, 26B25, 46N10, 26A33.

1. Introduction

Fractional calculus grew quite quickly because of its applications in math and many other fields including image processing, physics, machine learning, and networking. Fractional calculus has fruitful advantages in all fields of applied sciences, see [5, 8, 13, 16, 27, 28]. The fractional derivative has rapidly piqued the interest of professionals from several scientific fields. The majority of practical issues can not be modelled using classical derivations. Fractional integral and derivative operators provide solutions that are very suitable for real-world issues (see [18, 30]). The development of fractional calculus dates back to the 17th century, when the great mathematicians Leibniz and Euler started to think about non-integer derivatives. Leibniz first introduced the derivative of order $1/2$ in his letter to L'Hospital in 1695, which marks the beginning of fractional calculus [20]. But fractional derivatives were not explicitly presented

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until the late 19th century by Liouville and Riemann. Any real or complex integer may be used as the order of the derivative or integral in fractional calculus. A fractional integral of order 3/4, on the other hand, denotes an integral that is halfway between the first integral and the second integral. There are many familiar forms of the fractional integrals which have been captivated to gain fractional inequalities. The first is the Riemann-Liouville fractional integral [9], second is Hadamard fractional integral [29], Saigo fractional integral [32] is an other form of fractional integral. The primary characteristic of Caputo-Fabrizio operator is that the Laplace transformation is used to convert a real power into an integer, as a result the precise solution to many problems is readily accessible. In 2015 Caputo and Fabrizio proposed following operator.

Definition 1.1 (Caputo-Fabrizio integral operator, [5])**.** Let $H^1(\delta, \rho)$ be the Sobolev space of order one defined as

$$H^1(\delta, \rho) = \{g \in L^2(\delta, \rho) : g' \in L^2(\delta, \rho)\},$$

where

$$L^2(\delta, \rho) = \{g(h) : \Big(\int_\delta^\rho g^2(h)dh\Big)^{\frac{1}{2}} < \infty\}.$$

Let $\tau \in H^1(\delta, \rho), \delta < \rho, \kappa \in [0, 1]$, then left derivative in the sense of Caputo-Fabrizio is defined as

$$({}_{\delta}^{CFD}D^{\kappa}\tau)(h) = \frac{T(\kappa)}{1-\kappa}\int_\delta^h \tau'(b)e^{\frac{-\kappa(h-b)^\kappa}{1-\kappa}}db,$$

$h > \kappa$, and the associated integral operator is

$$({}_{\delta}^{CF}I^{\kappa}\tau)(h) = \frac{1-\kappa}{T(\kappa)}\tau(h) + \frac{\kappa}{T(\kappa)}\int_\delta^h \tau(\nu)d\nu,$$

where $T(\kappa) > 0$ is normalization function and $T(0) = T(1) = 1$. For $\kappa = 0$ and $\kappa = 1$ the left derivative is

$$({}_{\delta}^{CFD}D^{0}\tau)(h) = \tau'(h)$$

and

$$({}_{\delta}^{CFD}I^{1}\tau)(h) = \tau(h) - \tau(\delta),$$

respectively. For the right derivative operator

$$({}_{\rho}^{CFD}D^{\kappa}\tau)(h) = \frac{-T(\kappa)}{1-\kappa}\int_h^\rho \tau'(b)e^{\frac{-\kappa(b-h)^\kappa}{1-\kappa}}db,$$

$h < \rho$ and associated integral operator is

$$({}^{CF}I_{\rho}^{\kappa}\tau)(h) = \frac{1-\kappa}{T(\kappa)}\tau(h) + \frac{\kappa}{T(\kappa)}\int_h^\rho \tau(\nu)d\nu,$$

where $T(\kappa) > 0$ is a normalization function and $T(0) = T(1) = 1$.

Convex functions are of great interest, see [2, 7, 31].

Definition 1.2 (Convex function, [11])**.** A real valued function σ is said to be convex on close interval I in $\mathbb{R}$, if

$$\sigma\Big(\tau q + (1-q)\zeta\Big) \leqslant q\sigma(\tau) + (1-q)\sigma(\zeta),$$

where τ, ζ in I and $q \in [0, 1]$.

Breckner originated the class of s-convex functions in 1978.

Definition 1.3 (s-convex function, [15]). A function $\tau : [0, \infty) \longrightarrow \mathbb{R}$ is s-convex in second sense if

$$\tau\Big(\delta\zeta + (1-\zeta)\rho\Big) \leqslant \zeta^s \tau(\delta) + (1-\zeta)^s \tau(\rho)$$

holds, where $\delta, \rho \in [0, \infty)$, $s \in (0, 1]$ and $\zeta \in [0, 1]$.

In the year 2014, Eftekhari [10] instigated (s, m)-convex function as follows.

Definition 1.4 ((s, m)-convex function). A function $\tau : [0, k] \longrightarrow \mathbb{R}$, $k > 0$ is (s, m)-convex function in the second sense with $s, m \in (0, 1]$, if

$$\tau\Big(\delta\zeta + m(1-\zeta)\rho\Big) \leqslant \zeta^s \tau(\delta) + m(1-\zeta)^s \tau(\rho)$$

holds, for all $\delta, \rho \in [0, k]$ and $\zeta \in [0, 1]$.

In [34], Xi et al. introduced the concept of (α, s)-convex and (α, s, m)-convex function as follows.

Definition 1.5 ((α, s)-convex function). A function $\sigma : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be (α, s)-convex on close interval I, if

$$\sigma(\tau q + (1-q)\zeta) \leqslant q^{\alpha s} \sigma(\tau) + (1-q^\alpha)^s \sigma(\zeta)$$

holds, where $\tau, \zeta \in I, s \in [-1, 1], \alpha \in (0, 1]$ and $q \in (0, 1)$.

Definition 1.6 ((α, s, m)-convex function). A function $\sigma : [0, u] \longrightarrow \mathbb{R}$ is said to be (α, s, m)-convex with $\alpha, m \in (0, 1]$, $s \in [-1, 1]$ if

$$\sigma\Big(\tau\rho + m(1-\rho)\zeta\Big) \leqslant \rho^{\alpha s} \sigma(\tau) + m(1-\rho^\alpha)^s \sigma(\zeta) \tag{1.1}$$

holds, where $\tau, \zeta \in [0, u]$ and $\rho \in (0, 1)$.

In [21], Nosheen et al. exhibited the lemmas listed below.

Lemma 1.7. *Let $\tau : [\delta, m\rho] \to \mathbb{R}$ be a function differentiable on $(\delta, m\rho)$. If τ' is integrable on $[\delta, m\rho]$, then*

$$\begin{aligned}\frac{\tau(\delta) + \tau(m\rho)}{2} - \frac{T(\kappa)}{\kappa(m\rho - \delta)}\Big[({}^{CF}_{\delta}I^{\kappa}\tau)(x) + ({}^{CF}I^{\kappa}_{m\rho}\tau)(x)\Big] + \frac{2(1-\kappa)}{(m\rho - \delta)\kappa}\tau(x) \\ = \frac{(m\rho - \delta)}{2}\int_0^1 (1-2w)\tau'(\delta w + m(1-w)\rho)dw\end{aligned}$$

holds for $\kappa \in [0, 1]$.

Lemma 1.8. *Let $\tau : [\delta, m\rho] \to \mathbb{R}$ be a function differentiable on $(\delta, m\rho)$. If $\tau''(\cdot)$ is integrable on $[\delta, m\rho]$, then*

$$\begin{aligned}\frac{\tau(\delta) + \tau(m\rho)}{2} - \frac{T(\kappa)}{\kappa(m\rho - \delta)}\Big[({}^{CF}_{\delta}I^{\kappa}\tau)(x) + ({}^{CF}I^{\kappa}_{m\rho}\tau)(x)\Big] + \frac{2(1-\kappa)}{\kappa(m\rho - \delta)}\tau(x) \\ = \frac{(m\rho - \delta)^2}{2}\int_0^1 w(1-w)\tau''(\delta w + m(1-w)\rho)dw\end{aligned}$$

holds for $\kappa \in [0, 1]$.

Definition 1.9 (Hölder's inequality, [22]). Let $p > 1$. If real functions ξ and τ are defined on $[\delta, \rho]$ and $|\xi|^p, |\tau|^q$ are integrable on $[\delta, \rho]$, then

$$\int_{\delta}^{\rho} |\xi(\nu)\tau(\nu)| d\nu \leqslant (\int_{\delta}^{\rho} |\xi(\nu)|^p d\nu)^{\frac{1}{p}} (\int_{\delta}^{\rho} |\tau(\nu)|^q d\nu)^{\frac{1}{q}} \tag{1.2}$$

holds, where $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 1.10 (Power-mean inequality, [22]). Let $q > 1$. If ξ and τ are defined on $[\delta, \rho]$ and are real functions and $|\xi|, |\xi||\tau|^q$are integrable on $[\delta, \rho]$, then

$$\int_{\delta}^{\rho} |\xi(\nu)\tau(\nu)| d\nu \leqslant (\int_{\delta}^{\rho} |\xi(\nu)| d\nu)^{1-\frac{1}{q}} (\int_{\delta}^{\rho} |\xi(\nu)||\tau(\nu)|^q d\nu)^{\frac{1}{q}} \tag{1.3}$$

holds.

Beta function is defined as follows.

Definition 1.11 (Beta function, [26]). For $z > 0$ and $r > 0$, beta function in integral form is:

$$\beta(z, r) = \int_{0}^{1} j^{z-1}(1-j)^{r-1} dj.$$

Özdemir at el. established the following theorem.

Theorem 1.12 ([23]). *Let $\tau : [\delta, m\rho] \to \mathbb{R}$ be a differentiable function on $(\delta, m\rho)$. If $|\tau''|^q$, $q > 1$, is (α, s, m)-convex function and integrable on $[\delta, m\rho]$, then for $\alpha, m \in [0, 1]$*

$$\begin{aligned} &\left|\frac{\tau(\delta) + \tau(m\rho)}{2} - \frac{1}{m\rho - \delta} \int_{\delta}^{m\rho} \tau(w) dw\right| \\ &\leqslant \frac{(m\rho - \delta)^2}{2} (\frac{1}{6})^{1-\frac{1}{q}} \left[\frac{|\tau''(\delta)|^q}{(\alpha+3)(\alpha+2)} + m|\tau''(\rho)|^q \left(\frac{1}{6} - \frac{1}{(\alpha+3)(\alpha+2)}\right)\right]^{\frac{1}{q}}. \end{aligned}$$

If we put $\alpha = 1$ in Theorem 1.12, then following inequality (1.4) is obtained:

$$\left|\frac{\tau(\delta) + \tau(m\rho)}{2} - \frac{1}{m\rho - \delta} \int_{\delta}^{m\rho} \tau(w) dw\right| \leqslant \frac{(m\rho - \delta)^2}{2} (\frac{1}{6})^{1-\frac{1}{q}} \left(\frac{|\tau''(\delta)|^q + m|\tau''(\rho)|^q}{12}\right)^{\frac{1}{q}}. \tag{1.4}$$

In 2020, Wang et al. provided inequalities for modified h-convex functions [33]. In 2020, Butt et al. gave inequalities for exponential s-convex and exponential (s, m)-convex functions involving Caputo fractional derivative [3, 4]. In 2021, Li et al. gave inequalities for strongly convex functions involving Caputo-Fabrizio integrals [19]. In 2022, Abbasi et al. set up the generalized form of Hermite-Hadamard inequality for s-convex functions via Caputo-Fabrizio integrals [1]. Yang et al. presented in 2023, new inequalities via Caputo-Fabrizio integral operator with applications [36].

This work was motivated by ongoing studies from previous years on generalizations of Hermite-Hadamard type inequalities for various convexities involving certain fractional integral operators. We create new fractional versions of the Hermite-Hadamard type inequalities for functions whose absolute of the second derivative is (α, s, m)-convex and use the Caputo-Fabrizio integral operator. We examine a few intriguing applications to special means, numerical analysis and probability theory. Results are graphically elaborated for better understanding of readers. In [34], the Montgomery identity was generalized for double integrals on time scales by employing a novel analytical approach to develop the generalized Ostrowski type integral inequalities involving double integrals. In [12], inequalities involving Caputo-Fabrizio integrals with applications are established.

2. Main results

Lemma 1.7 and 1.8 are the main motivation behind the study, that demonstrate Hermite-Hadamard type inequalities via Caputo-Fabrizio integral operators. Since $\rho^\alpha \geqslant \rho$ for all $\alpha \in (0,1]$ and $\rho \in [0,1]$, we have

$$1-\rho^\alpha \leqslant 1-\rho \Rightarrow 1-\rho^\alpha \leqslant (1-\rho)^\alpha \Rightarrow (1-\rho^\alpha)^s \leqslant (1-\rho)^{\alpha s}, s \in [0,1].$$

Thus (1.1) becomes

$$\sigma(\tau\rho + m(1-\rho)\zeta) \leqslant \rho^{\alpha s}\sigma(\tau) + m(1-\rho^\alpha)^s\sigma(\zeta) \leqslant \rho^{\alpha s}\sigma(\tau) + m(1-\rho)^{\alpha s}\sigma(\zeta). \tag{2.1}$$

Theorem 2.1. *Let $\tau : [\delta, m\rho] \to \mathbb{R}$ be a function differentiable on $(\delta, m\rho)$. If $|\tau'|^q$, $q > 1$, is (α, s, m)-convex function and integrable on $[\delta, m\rho]$, then for $s \in [0,1]$ and $\kappa \in (0,1]$,*

$$\begin{aligned}&\left|\frac{\tau(\delta)+\tau(m\rho)}{2} - \frac{T(\kappa)}{\kappa(m\rho-\delta)}\left[({}_{\delta}^{CF}I^{\kappa}\tau)(x) + ({}^{CF}I^{\kappa}_{m\rho}\tau)(x)\right] + \frac{2(1-\kappa)}{\kappa(m\rho-\delta)}\tau(x)\right|\\ &\leqslant \frac{m\rho-\delta}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|\tau'(\delta)|^q + m|\tau'(\rho)|^q}{\alpha s+1}\right)^{\frac{1}{q}}\end{aligned} \tag{2.2}$$

holds, where $p^{-1} = 1 - q^{-1}$.

Proof. Using (α, s, m)-convexity of $|\tau'|^q$, (2.1), (1.2), and Lemma 1.7 we get

$$\begin{aligned}&\left|\frac{\tau(\delta)+\tau(m\rho)}{2} - \frac{T(\kappa)}{\kappa(m\rho-\delta)}\left[({}_{\delta}^{CF}I^{\kappa}\tau)(x) + ({}^{CF}I^{\kappa}_{m\rho}\tau)(x)\right] + \frac{2(1-\kappa)}{\kappa(m\rho-\delta)}\tau(x)\right|\\ &\leqslant \frac{m\rho-\delta}{2}\int_0^1 |1-2w||\tau'(\delta w + m(1-w)\rho)|dw\\ &\leqslant \frac{m\rho-\delta}{2}\left(\int_0^1 |1-2w|^p dw\right)^{\frac{1}{p}}\left(\int_0^1 |\tau'(\delta w + m(1-w)\rho)|^q dw\right)^{\frac{1}{q}}\\ &\leqslant \frac{m\rho-\delta}{2}\left(\int_0^1 |1-2w|^p dw\right)^{\frac{1}{p}}\left(\int_0^1 [w^{\alpha s}|\tau'(\delta)|^q + m(1-w)^{\alpha s}|\tau'(\rho)|^q]dw\right)^{\frac{1}{q}}\\ &= \frac{m\rho-\delta}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|\tau'(\delta)|^q + m|\tau'(\rho)|^q}{\alpha s+1}\right)^{\frac{1}{q}}.\end{aligned}$$

□

Remark 2.2. In Theorem 2.1 use the following substitutions to recapture the results existing in literature: $\alpha = m = 1$ go at [1, Theorem 3.2] and $\alpha = m = s = 1$ gives [13, Theorem 6]. Further more, $\alpha = 1$ leads towards results for (s, m)-convex and $m = 1$ gives results for (α, s)-convex function.

Theorem 2.3. *Let $\tau : [\delta, m\rho] \to \mathbb{R}$ be a function differentiable on $(\delta, m\rho)$. If $|\tau''|^q$, $q > 1$, is (α, s, m)-convex function and integrable on $[\delta, m\rho]$, then for $s \in [0,1]$ and $\kappa \in (0,1]$,*

$$\begin{aligned}&\left|\frac{\tau(\delta)+\tau(m\rho)}{2} - \frac{T(\kappa)}{\kappa(m\rho-\delta)}\left[({}_{\delta}^{CF}I^{\kappa}\tau)(x) + ({}^{CF}I^{\kappa}_{m\rho}\tau)(x)\right] + \frac{2(1-\kappa)}{\kappa(m\rho-\delta)}\tau(x)\right|\\ &\leqslant \frac{(m\rho-\delta)^2}{2}(\beta(p+1,p+1))^{\frac{1}{p}}\left(\frac{|\tau''(\delta)|^q + m|\tau''(\rho)|^q}{\alpha s+1}\right)^{\frac{1}{q}}\end{aligned} \tag{2.3}$$

holds, where $p^{-1} = 1 - q^{-1}$.

Proof. Using Lemma 1.8, (α, s, m)-convexity of $|\tau''|^q$, (2.1), and (1.2) we get

$$\begin{aligned}
&\left|\frac{\tau(\delta)+\tau(m\rho)}{2}-\frac{T(\kappa)}{\kappa(m\rho-\delta)}\left[({}^{CF}_{\delta}I^{\kappa}\tau)(x)+({}^{CF}I^{\kappa}_{m\rho}\tau)(x)\right]+\frac{2(1-\kappa)}{\kappa(m\rho-\delta)}\tau(x)\right|\\
&\leqslant \frac{(m\rho-\delta)^2}{2}\int_0^1 (w-w^2)|\tau''(\delta w+m(1-w)\rho)|dw\\
&\leqslant \frac{(m\rho-\delta)^2}{2}\Big(\int_0^1 (w-w^2)^p dw\Big)^{\frac{1}{p}}\Big(\int_0^1 |\tau''(\delta w+m(1-w)\rho)|^q dw\Big)^{\frac{1}{q}}\\
&\leqslant \frac{(m\rho-\delta)^2}{2}\Big(\int_0^1 w^p(1-w)^p dw\Big)^{\frac{1}{p}}\Big(\int_0^1 [w^{\alpha s}|\tau''(\delta)|^q+m(1-w^{\alpha})^s|\tau''(\rho)|^q]dw\Big)^{\frac{1}{q}}\\
&= \frac{(m\rho-\delta)^2}{2}(\beta(p+1,p+1))^{\frac{1}{p}}\Big(\frac{|\tau''(\delta)|^q+m|\tau''(\rho)|^q}{\alpha s+1}\Big)^{\frac{1}{q}}.
\end{aligned}$$ □

Remark 2.4. $\alpha = 1$ leads towards results for (s, m)-convex and $m = 1$ gives results for (α, s)-convex function.

Remark 2.5. In [24], Özdemir et al. established the following inequality:

$$\begin{aligned}
&\left|\frac{\tau(\delta)+\tau(\rho)}{2}-\frac{1}{\rho-\delta}\int_\delta^\rho \tau(w)dw\right|\\
&\leqslant \frac{(\rho-\delta)^2}{2}\Big(\beta\Big(\frac{2q-1}{q-1},\frac{2q-1}{q-1}\Big)\Big)^{1-\frac{1}{q}}\Big(\frac{|\tau''(\delta)|^q}{\alpha s+1}+\frac{m}{\alpha}|\tau''(\frac{\rho}{m})|^q\beta(\frac{1}{\alpha},s+1)\Big)^{\frac{1}{q}}.
\end{aligned}\tag{2.4}$$

For $\alpha = m = 1$, (2.4) becomes

$$\left|\frac{\tau(\delta)+\tau(\rho)}{2}-\frac{1}{\rho-\delta}\int_\delta^\rho \tau(w)dw\right|\leqslant \frac{(\rho-\delta)^2}{2}(\beta(p+1,p+1))^{\frac{1}{p}}\Big(\frac{|\tau''(\delta)|^q+|\tau''(\rho)|^q}{s+1}\Big)^{\frac{1}{q}}. \tag{2.5}$$

For $\kappa = 1$ and $\alpha = m = 1$, Theorem 2.3 reduces to (2.5).

Note: It is interesting to observe that by making different substitutions in inequality obtained by Özdemir and in our Theorem 2.3 we have the same inequality (2.5).

Theorem 2.6. *Let $\tau : [\delta, m\rho] \to \mathbb{R}$ be a function differentiable on $(\delta, m\rho)$. If $|\tau'|^q$, $q > 1$, is (α, s, m)-convex function and integrable on $[\delta, m\rho]$, then*

$$\begin{aligned}
&\left|\frac{\tau(\delta)+\tau(m\rho)}{2}-\frac{T(\kappa)}{\kappa(m\rho-\delta)}\left[({}^{CF}_{\delta}I^{\kappa}\tau)(x)+({}^{CF}I^{\kappa}_{m\rho}\tau)(x)\right]+\frac{2(1-\kappa)}{\kappa(m\rho-\delta)}\tau(x)\right|\\
&\leqslant \frac{m\rho-\delta}{2}(\frac{1}{2})^{1-\frac{1}{q}}\Big(\frac{[|\tau'(\delta)|^q+m|\tau'(\rho)|^q](\alpha s+2^{-\alpha s})}{(\alpha s+1)(\alpha s+2)}\Big)^{\frac{1}{q}}
\end{aligned}\tag{2.6}$$

holds for $s \in [0, 1]$ and $\kappa \in (0, 1]$.

Proof. Using Lemma 1.7, (α, s, m)-convexity of $|\tau'|^q$, (2.1), and (1.3) we get

$$\left|\frac{\tau(\delta)+\tau(m\rho)}{2}-\frac{T(\kappa)}{\kappa(m\rho-\delta)}\left[({}^{CF}_{\delta}I^{\kappa}\tau)(x)+({}^{CF}I^{\kappa}_{m\rho}\tau)(x)\right]+\frac{2(1-\kappa)}{\kappa(m\rho-\delta)}\tau(x)\right|$$

$$\leqslant \frac{m\rho-\delta}{2}\int_0^1 |(1-2w)\tau'(\delta w+m(1-w)\rho)|dw$$
$$\leqslant \frac{m\rho-\delta}{2}\Big(\int_0^1 |1-2w|dw\Big)^{1-\frac{1}{q}}\Big(\int_0^1 |(1-2w)||(\tau'(\delta w+m(1-w)\rho))|^q dw\Big)^{\frac{1}{q}}$$
$$\leqslant \frac{m\rho-\delta}{2}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\Big(\int_0^1 (1-2w)[w^{\alpha s}|\tau'(\delta)|^q+m(1-w^{\alpha})^s|\tau'(\rho)|^q]dw\Big)^{\frac{1}{q}}$$
$$=\frac{m\rho-\delta}{2}(\frac{1}{2})^{1-\frac{1}{q}}\Big(\frac{[|\tau'(\delta)|^q+m|\tau'(\rho)|^q](\alpha s+2^{-\alpha s})}{(\alpha s+1)(\alpha s+2)}\Big)^{\frac{1}{q}}. \qquad \square$$

Theorem 2.7. *Let* $\tau:[\delta,m\rho]\to\mathbb{R}$ *be a differentiable function on* $(\delta,m\rho)$*. If* $|\tau''|^q$*,* $q>1$*, is* (α,s,m)*-convex function and integrable on* $[\delta,m\rho]$*, then for* $s\in[0,1]$ *and* $\kappa\in(0,1]$*,*

$$\Big|\frac{\tau(\delta)+\tau(m\rho)}{2}-\frac{T(\kappa)}{\kappa(m\rho-\delta)}\Big[({}^{CF}_{\delta}I^{\kappa}\tau)(x)+({}^{CF}I^{\kappa}_{m\rho}\tau)(x)\Big]+\frac{2(1-\kappa)}{\kappa(m\rho-\delta)}\tau(x)\Big|$$
$$\leqslant \frac{(m\rho-\delta)^2}{2}(\frac{1}{6})^{1-\frac{1}{q}}\Big(\frac{|\tau''(\delta)|^q+m|\tau''(\rho)|^q}{(\alpha s+3)(\alpha s+2)}\Big)^{\frac{1}{q}}.$$

Proof. Using Lemma 1.8, (α,s,m)-convexity of $|\tau''|^q$, (2.1), and (1.3) we get

$$\Big|\frac{\tau(\delta)+\tau(m\rho)}{2}-\frac{T(\kappa)}{\kappa(m\rho-\delta)}\Big[({}^{CF}_{\delta}I^{\kappa}\tau)(x)+({}^{CF}I^{\kappa}_{m\rho}\tau)(x)\Big]+\frac{2(1-\kappa)}{\kappa(m\rho-\delta)}\tau(x)\Big|$$
$$\leqslant \frac{(m\rho-\delta)^2}{2}\int_0^1 |(w-w^2)\tau''(\delta w+m(1-w)\rho)|dw$$
$$\leqslant \frac{(m\rho-\delta)^2}{2}\Big(\int_0^1 (w-w^2)dw\Big)^{1-\frac{1}{q}}\Big(\int_0^1 |w-w^2||\tau''(\delta w+m(1-w)\rho)|^q dw\Big)^{\frac{1}{q}}$$
$$\leqslant \frac{(m\rho-\delta)^2}{2}\Big(\frac{1}{6}\Big)^{1-\frac{1}{q}}\Big(\int_0^1 (w-w^2)[w^{\alpha s}|\tau''(\delta)|^q+m(1-w^{\alpha})^s|\tau''(\rho)|^q]dw\Big)^{\frac{1}{q}}$$
$$\leqslant \frac{(m\rho-\delta)^2}{2}\Big(\frac{1}{6}\Big)^{1-\frac{1}{q}}\Big(\int_0^1 [w^{\alpha s+1}(1-w)|\tau''(\delta)|^q+mw(1-w)^{\alpha s+1}|\tau''(\rho)|^q]dw\Big)^{\frac{1}{q}}$$
$$=\frac{(m\rho-\delta)^2}{2}(\frac{1}{6})^{1-\frac{1}{q}}\Big(\frac{|\tau''(\delta)|^q+m|\tau''(\rho)|^q}{(\alpha s+3)(\alpha s+2)}\Big)^{\frac{1}{q}}. \qquad \square$$

Remark 2.8.

(a) For $\kappa=1$ and $\alpha=s=1$, Theorem 2.7 gives (1.4).

3. Applications

This section presents applications of inequalities obtained in main findings and their graphical depiction.

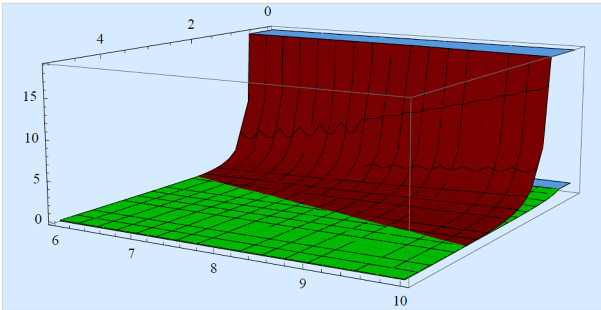
3.1. Means

Let $0 < \delta < \rho$, and $\delta \neq \rho$. Then Arithmetic mean and Logarithmic mean are defined as $A(\delta, \rho) = \frac{\delta+\rho}{2}$ and $L(\delta, \rho) = \left(\frac{\delta-\rho}{\ln(\delta)-\ln(\rho)}\right)$, respectively [6, 35].

Proposition 3.1. *Let* $\delta, \rho \in \mathbb{R}^+$ *and* $\delta < m\rho$. *Then*

$$\left|A(\delta^{-1}, (m\rho)^{-1}) - L^{-1}(\delta, m\rho)\right| \leqslant \frac{(m\rho-\delta)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left[\frac{|\frac{2}{\delta^3}|^q + m|\frac{2}{\rho^3}|^q}{(\alpha s+2)(\alpha s+3)}\right]^{\frac{1}{q}}. \tag{3.1}$$

Proof. By putting $\tau(x) = \frac{1}{x}$ and $\kappa = 1$ in Theorem 2.7, we get (3.1). □



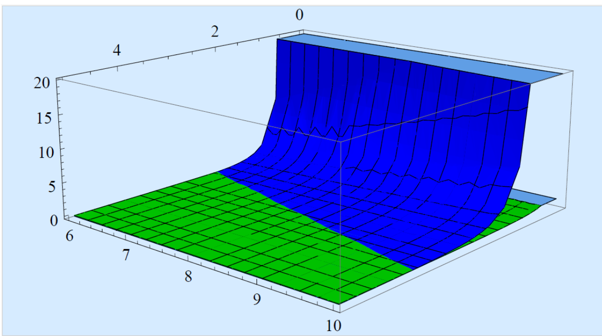
(a)

Figure 1: Graphical description of inequality obtained in Proposition 3.1 for $\alpha = s = m = 1$, where the left side of inequality is shown in green color and the right side of that inequality is shown in red color.

Proposition 3.2. *Let* $\delta, \rho \in \mathbb{R}^+$ *and* $\delta < m\rho$. *Then*

$$\left|A(\delta^{-1}, (m\rho)^{-1}) - L^{-1}(\delta, m\rho)\right| \leqslant \frac{(m\rho-\delta)^2}{2}(\beta(p+1, p+1))^{\frac{1}{p}}\left[\frac{|\frac{2}{\delta^3}|^q + m|\frac{2}{\rho^3}|^q}{(\alpha s+1)}\right]^{\frac{1}{q}}. \tag{3.2}$$

Proof. By putting $\tau(x) = \frac{1}{x}$ and $\kappa = 1$ in Theorem 2.3, we get (3.2). □



(a)

Figure 2: Graphical description of inequality obtained in Proposition 3.2 for $\alpha = s = m = 1$, where the left side of inequality is shown in blue color and the right side of that inequality is shown in green color.

3.2. Numerical applications

Suppose b is from set of natural numbers and $B : \delta = z_0 < z_1 < z_2 < \cdots < z_b = m\rho$ is a partition of the interval $[\delta, m\rho]$ and consider the quadrature formula

$$\int_{\delta}^{m\rho} \tau(q)dq = Y(\tau, B) + S(\tau, B), \tag{3.3}$$

where

$$Y(\tau, B) = \sum_{l=0}^{b-1} \left(\frac{\tau(z_l) + \tau(z_{l+1})}{2} \right) (z_{l+1} - z_l)$$

for the trapezoidal version and $S(\tau, B)$ denotes the associated approximation error [17].

Theorem 3.3. *Consider the assumptions of Theorem 2.6, for each partition B of $[\delta, m\rho]$, the trapezoidal error estimate satisfies*

$$|S(\tau, B)| \leqslant \sum_{l=0}^{b-1} \frac{(z_{l+1} - z_l)^2}{2} \Big(\frac{1}{2}\Big)^{1-\frac{1}{q}} \left(\frac{[|\tau'(z_l)|^q + m|\tau'(z_{l+1})|^q](\alpha s + 2^{-\alpha s})}{(\alpha s + 1)(\alpha s + 2)} \right)^{\frac{1}{q}}.$$

Proof. Consider Theorem 2.6 for $[z_l, z_{l+1}]$ $(l = 0, 1, \ldots, b-1)$ and substituting $\kappa = 1$,

$$\left| \frac{\tau(z_l) + \tau(z_{l+1})}{2} - \frac{1}{(z_{l+1} - z_l)} \int_{z_l}^{z_{l+1}} \tau(c)dc \right| \leqslant \frac{(z_{l+1} - z_l)}{2} \Big(\frac{1}{2}\Big)^{1-\frac{1}{q}} \left(\frac{[|\tau'(z_l)|^q + m|\tau'(z_{l+1})|^q](\alpha s + 2^{-\alpha s})}{(\alpha s + 1)(\alpha s + 2)} \right)^{\frac{1}{q}}.$$

Hence in (3.3) we have

$$\begin{aligned}
\left| \int_{\delta}^{\rho} \tau(c)dc - Y(\tau, B) \right| &= \left| \sum_{l=0}^{b-1} \int_{z_l}^{z_{l+1}} \tau(u)du - \sum_{l=0}^{b-1} \frac{\tau(z_l) + \tau(z_{l+1})}{2} (z_{l+1} - z_l) \right| \\
&\leqslant \sum_{l=0}^{b-1} \left| \int_{z_l}^{z_{l+1}} \tau(u)du - \frac{\tau(z_l) + \tau(z_{l+1})}{2} (z_{l+1} - z_l) \right| \\
&\leqslant \sum_{l=0}^{b-1} \frac{(z_{l+1} - z_l)^2}{2} \Big(\frac{1}{2}\Big)^{1-\frac{1}{q}} \left(\frac{[|\tau'(z_l)|^q + m|\tau'(z_{l+1})|^q](\alpha s + 2^{-\alpha s})}{(\alpha s + 1)(\alpha s + 2)} \right)^{\frac{1}{q}},
\end{aligned}$$

which completes the proof. □

Remark 3.4. For each partition B of $[\delta, m\rho]$ and $\kappa = 1$, the following error estimates are satisfied.

(I) Applying Theorem 2.7 for $[z_l, z_{l+1}]$, we have

$$|S(\tau, B)| \leqslant \sum_{l=0}^{b-1} \frac{(z_{l+1} - z_l)^3}{2} \Big(\frac{1}{6}\Big)^{1-\frac{1}{q}} \left(\frac{[|\tau''(z_l)|^q + m|\tau''(z_{l+1})|^q]}{(\alpha s + 3)(\alpha s + 2)} \right)^{\frac{1}{q}}. \tag{3.4}$$

Example 3.5. For $\tau(x) = e^x$, $b = 2$, $z_0 = \delta = 0, z_1 = 0.5, z_2 = m\rho = 1$, $\alpha = \frac{1}{3}$, and $s = \frac{1}{7}$ in (3.4) we have

$$|S(\tau, B)| = \left| \int_0^1 \tau(c)dc - \sum_{l=0}^{1} \frac{\tau(z_l) + \tau(z_{l+1})}{2} (z_{l+1} - z_l) \right| = \left| \int_0^1 e^x dx - 0.5\Big(\frac{1 + e^{0.5}}{2} + \frac{e^{0.5} + e^1}{2}\Big) \right| = |-0.03564|$$

and

$$\begin{aligned}\sum_{l=0}^{1}\frac{(z_{l+1}-z_l)^3}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}&\left(\frac{[|e^{z_l}|^q+m|e^{z_{l+1}}|^q]}{(\alpha s+3)(\alpha s+2)}\right)^{\frac{1}{q}}\\ &=\frac{0.5^3}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\left(\frac{441(1+0.9e^{0.5q})}{2752}\right)^{\frac{1}{q}}+\left(\frac{441(e^{0.5q}+0.9e^{q})}{2752}\right)^{\frac{1}{q}}\right)=h(q).\end{aligned}$$

Hence $|S(\tau,B)|\leqslant h(q)$.

(II) Applying Theorem 2.1 for $[z_l,z_{l+1}]$, we have

$$|S(\tau,B)|\leqslant\sum_{l=0}^{b-1}\frac{(z_{l+1}-z_l)^2}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|\tau'(z_l)|^q+m|\tau'(z_{l+1})|^q}{\alpha s+1}\right)^{\frac{1}{q}}. \tag{3.5}$$

Example 3.6. Putting $\tau(x)=e^x$ in (3.5) with same substitution of Example 3.5, we have

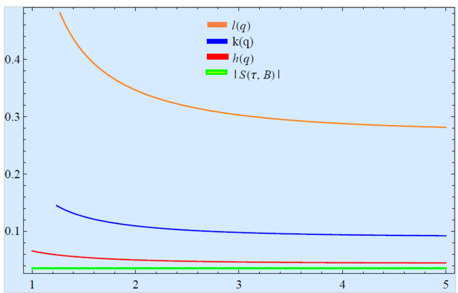
$$|S(\tau,B)|=|-0.03564|\leqslant\frac{0.5^2}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\left(\frac{21(1+0.9e^{0.5q})}{22}\right)^{\frac{1}{q}}+\left(\frac{21(e^{0.5q}+0.9e^{q})}{22}\right)^{\frac{1}{q}}\right)=l(q).$$

(III) Using equation (2.3) for $[z_l,z_{l+1}]$ we get

$$|S(\tau,B)|\leqslant\sum_{l=0}^{b-1}\frac{(z_{l+1}-z_l)^3}{2}(\beta(p+1,p+1))^{\frac{1}{p}}\left(\frac{|\tau''(z_l)|^q+m|\tau''(z_{l+1})|^q}{\alpha s+1}\right)^{\frac{1}{q}}. \tag{3.6}$$

Example 3.7. Putting $\tau(x)=e^x$ in (3.6) with same substitution of Example 3.5, we have

$$\begin{aligned}|S(\tau,B)|&=|-0.03564|\\ &\leqslant\frac{0.5^3}{2}\left(\beta\left(\frac{2q-1}{q-1},\frac{2q-1}{q-1}\right)\right)^{1-\frac{1}{q}}\left(\left(\frac{21(1+0.9e^{0.5q})}{22}\right)^{\frac{1}{q}}+\left(\frac{21(e^{0.5q}+0.9e^{q})}{22}\right)^{\frac{1}{q}}\right)=k(q).\end{aligned}$$



(a)

Figure 3: Graphs show that Theorem 2.7 provides the best error estimate for the Trapezoidal formula.

3.3. *Probability theory*

Suppose Z is a continuous random variable choosing values in finite interval $[\delta,m\rho]$ with the probability density function $\tau:[\delta,m\rho]\to[0,1]$ and with the cumulative distribution function

$$N(z)=P(Z\leqslant z)=\int_{\delta}^{z}\tau(r)dr,$$

where $z\in[\delta,m\rho]$, $N(\delta)=0$ and $N(m\rho)=1$. The expectation of Z is:

$$E(Z)=\int_{\delta}^{m\rho}l\tau(l)dl=m\rho-\int_{\delta}^{m\rho}N(l)dl. \tag{3.7}$$

Proposition 3.8. *Consider the assumptions of Theorem 2.1, then*

$$\left|\frac{1}{2}-\frac{m\rho-E(Z)}{m\rho-\delta}\right| \leqslant \frac{m\rho-\delta}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|\tau(\delta)|^q+m|\tau(\rho)|^q}{\alpha s+1}\right)^{\frac{1}{q}}. \tag{3.8}$$

Proof. Choose $\tau = N$ and $\kappa = 1$ in (2.2), then applying (3.7) we obtain (3.8). □

Proposition 3.9. *Consider the assumptions of Theorem 2.3, then*

$$\left|\frac{1}{2}-\frac{m\rho-E(Z)}{m\rho-\delta}\right| \leqslant \frac{(m\rho-\delta)^2}{2}(\beta(p+1,p+1))^{\frac{1}{p}}\left(\frac{|\tau'(\delta)|^q+m|\tau'(\rho)|^q}{\alpha s+1}\right)^{\frac{1}{q}}. \tag{3.9}$$

Proof. Choose $\tau = N$ and $\kappa = 1$ in (2.3), then applying (3.7) we get (3.9). □

Proposition 3.10. *Consider the assumptions of Theorem 2.6, then*

$$\left|\frac{1}{2}-\frac{m\rho-E(Z)}{m\rho-\delta}\right| \leqslant \frac{m\rho-\delta}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{[|\tau(\delta)|^q+m|\tau(\rho)|^q](\alpha s+2^{-\alpha s})}{(\alpha s+1)(\alpha s+2)}\right)^{\frac{1}{q}}. \tag{3.10}$$

Proof. Choose $\tau = N$ and $\kappa = 1$ in (2.6), then applying (3.7) we obtain (3.10). □

Proposition 3.11. *Consider the assumptions of Theorem 2.7, then*

$$\left|\frac{1}{2}-\frac{m\rho-E(Z)}{m\rho-\delta}\right| \leqslant \frac{(m\rho-\delta)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{|\tau'(\delta)|^q+m|\tau'(\rho)|^q}{(\alpha s+3)(\alpha s+2)}\right)^{\frac{1}{q}}. \tag{3.11}$$

Proof. Choose $\tau = N$ and $\kappa = 1$ in Theorem 2.7, then applying (3.7) we obtain (3.11). □

Example 3.12. From two arbitrary sample points, e.g., 1960 and 2009, where the world population was 3.0402 and 6.8158 billion, respectively, Dean Hathout presented the exponential growth population model [14], which can be written as:

$$\tau(x) = 3.0402e^{0.016476(x-1960)}.$$

We have normalized this model for the year 1960 to 2009 and make the following probability density function:

$$P(x) = 0.013226676e^{0.016476(x-1960)}.$$

For $\alpha = \frac{1}{2} = s$ and $m = 1$, $\tau(x) = 0.013226676e^{0.016476(x-1960)}$ is (α, s, m)-convex function on $[1960, 2009]$. Substitute $\alpha = \frac{1}{2} = s$, $m = 1$, $\delta = 1960$, and $\rho = 2009$ in the definition of (α, s, m)-convex function to get:

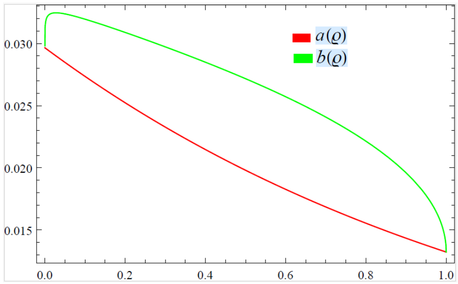
$$a(\rho) = 0.013226676e^{0.016476(1960\rho+(1-\rho)2009-1960)} \leqslant 0.0132267\rho^{\frac{1}{4}} + 0.02965(1-\rho^{\frac{1}{2}})^{\frac{1}{2}} = b(\rho).$$

Correspondingly, the cumulative distribution function is

$$N(z) = 0.013226676\int_{1960}^{z} e^{0.016476(x-1960)}dx = 0.80278(e^{0.016476(z-1960)}-1)$$

and the expectation of Z is:

$$E(Z) = 2009 - 0.80278\int_{1960}^{2009} r(e^{0.016476(r-1960)}-1)dr = 1987.82562. \tag{3.12}$$



(a)

Figure 4: (α, s, m)-convexity of $\tau(x) = 0.013226676e^{0.016476(x-1960)}$.

From (3.12) and other substitutions in (3.8), (3.9), (3.10), and (3.11), we get

$$|L| = |0.06786| \leqslant \frac{49}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\frac{0.01323^q}{1.25}(1+e^{0.807324q})\right)^{\frac{1}{q}} = A(q), \tag{3.13}$$

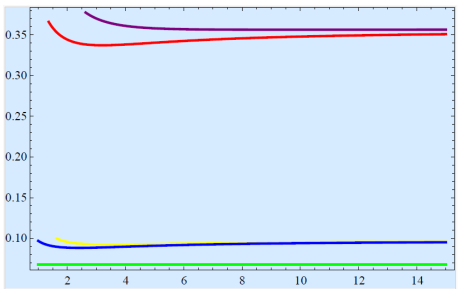
$$|L| = |0.06786| \leqslant \frac{2401}{2}\left(\beta\left(\frac{2q-1}{q-1}, \frac{2q-1}{q-1}\right)\right)^{1-\frac{1}{q}}\left(\frac{0.0002179^q}{1.25}(1+e^{0.807324q})\right)^{\frac{1}{q}} = K(q), \tag{3.14}$$

$$|L| = |0.06786| \leqslant \frac{49}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(0.38787(1+e^{0.807324q})0.01323^q\right)^{\frac{1}{q}} = Z(q), \tag{3.15}$$

and

$$|L| = |0.06786| \leqslant \frac{2401}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{0.0002179^q}{7.3125}(1+e^{0.807324q})\right)^{\frac{1}{q}} = Y(q), \tag{3.16}$$

respectively. (3.13), (3.14), (3.15), and (3.16) estimate the bounds for the average population in [1960, 2009]. In Figure 5 $|L|$ is represented by green line. Furthermore, $A(q), K(q), Z(q)$ and $Y(q)$ are shown by purple, yellow, red and blue respectively. It is cleared from Figure 5 that difference of bounds for inequalities (3.13) and (3.15) is shorter, hence these inequalities give the best estimation.



(a)

Figure 5: Bounds for estimation of population.

4. Conclusion

With the use of the Caputo-Fabrizio integral operators, the primary results provide a generalization of Hermite-Hadamard-type inequalities for the class of (α, s, m)-convex functions. Lemmas 1.7 and 1.8 are used to generate some innovative inequalities involving the Caputo-Fabrizio integral operator, which are then used to provide some special means inequalities. These lemmas are also appropriate to obtain new error estimates for trapezoidal formula. Some novel inequalities are constructed in Examples 3.5, 3.6, and 3.7 and are presented graphically, which show the validity of our theorems. Left hand side of these inequalities are same, by comparing we find that Theorem 2.7 provides the best error estimate for trapezoidal formula. Moreover, the novel study of this article, that is discussed in Example 3.12, is bounds for expectation value of random variable and these bounds are justified in Figure 5, which provide a better understanding and validity of main results. In the future, scholars may explore inequalities using different type of convexities.

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