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An outlook on stability of implicit θ-Caputo fractional differential equation with application involving the Riemann-Stieltjes type

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Abstract

The objective of the current article is to guarantee the solvability of implicit θ-Caputo fractional differential equations with integral boundary conditions. We establish the necessary conditions to guarantee unique solutions and demonstrate Ulam-Hyers-Rassias stability. Additionally, we include examples to illustrate the key findings.

Keywords: θ-Caputo fractional operator, mild solution, Ulam stability, Riemann-Stieltjes.

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1. Introduction

Fractional calculus is a potent tool in applied mathematics, offering a way to analyze a wide range of problems in various scientific and technical fields. Fractional derivatives have yielded significant results in [5, 18–23]. The study of partial fractional differential equations, as well as ordinary differential equations, has made substantial progress in recent years. For further exploration, one could refer to the monographs by Abbas et al. [2], Baleanu et al. [17], Kilbas et al. [26], Lakshmikantham et al. [28], and numerous researchers have contributed highly useful findings in this area [6, 7, 9–12, 25].

During a lecture at Wisconsin University in 1940, Ulam first brought up the subject of stability in functional equations. He asked, "Under what circumstances does the existence of an additive mapping that is close to an essentially additive mapping hold?" (See [33] for more information.) In 1941, Hyers offered the first response to Ulam's query, focusing on the situation involving Banach spaces [29]. Thistype of

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stability is referred to as Ulam-Hyers stability. In 1978, Rassias significantly expanded the Ulam-Hyers stability by including variables [14]. The concept of stability in functional equations appears when an inequality is employed as a perturbation instead of the original equation. As a result, the difference between the solutions of the inequality and those of the functional equation that is being presented revolves around the issue of stability in functional equations. Ulam-Hyers and Ulam-Hyers-Rassias stability in different types of functional equations have received a lot of attention, as addressed in the monographs by [1, 24, 29].

The Riemann-Liouville derivative has been generalized in the classical papers to include fractional derivatives of a function with regard to function, for example, called it θ. Since the kernel is visible According to θ, In [4], Almeida recently updated his analysis of this derivative and provided a Caputo-type regularization of the previous formulation along with several intriguing features. This operator's properties can be found in [3, 27, 30].

In several references, the authors have discussed the existence, stability, or other qualitative characteristics of solutions to different kinds of integral equations, For further details, the readers can refer to some interesting research papers [8, 13, 31, 32, 35].

Motivated by these works, we study existence and Ulam-Hyers-Rassias stability of following implicit θ-Caputo fractional differential problem (IFDP).

$$ {}^{c}D^{\gamma;\theta}y(t) = f\Big(t, y(t), {}^{c}D^{\delta;\theta}y(t), \int_0^t k(t,\tau)\,{}^{c}D^{\gamma;\theta}y(\tau)d\tau\Big), \tag{1.1} $$

$$ y(0) = \frac{1}{\Gamma(\rho)}\int_0^1 \theta'(\tau)(\theta(1)-\theta(\tau))^{\rho-1}\, h_1(\tau, y(\tau))\, d\tau, \tag{1.2} $$

$$ y'(1) = \frac{1}{\Gamma(\rho)}\int_0^1 \theta'(\tau)(\theta(1)-\theta(\tau))^{\rho-1}\, h_2(\tau, y(\tau))\, d\tau, \tag{1.3} $$

where $1 < \delta < \gamma \leqslant 2$ and $0 < \rho < 1$, here $\theta(t)$ is non decreasing function with $\theta'(t) \neq 0$, $\forall\, t \in \mathbb{J} = [0,1]$, and ${}^{c}D^{\gamma,\theta}$ is the θ-Caputo fractional derivative. In this study, we discuss some results on the stability of solutions to problem (1.1)-(1.3). In order to fulfill these aims, we use the concepts of fixed-point theorem to establish the existence and uniqueness of solutions for the proposed problem and analyze some stabilities, namely, Hyers–Ulam, and Hyers-Ulam-Rassias stability. We present our results in a general platform, which covers many particular cases for specific values of θ as particular cases of our main result, we obtain implicit θ-Caputo fractional differential problem with nonlocal boundary conditions involving Riemann-Stieltjes integrals, and integral equation of Volterra-Stieltjes type

The paper has been divided under four sections. Two parts make up Section 2, which includes the primary findings. In part one, it was discussed how the integral equation (2.1) and FIDE (1.1)-(1.3) are equivalent. The results of the problem (1.1)-(1.3) are presented in Part two, one utilizing the Banach contraction principle and the other employing the Krasnosel'skii fixed point theorem. In Section 3, we will also discuss the Ulam-Hyers Russian stability of our problem. Additionally, we provide some specific cases and examples that illustrate our conclusions in Section 4. We terminate the investigation with conclusions.

2. Existence results

This section discusses existence and uniqueness of mild solutions for the Ulam stability for the problem (1.1)-(1.3).

(H_1) Functions $h_i :: \mathbb{J} \times \mathbb{R} \to \mathbb{R}, i = 1, 2$ are continuous, such that there exist constants $0 \leqslant k_i < 1$, meet

$$|h_i(t, \tau_1) - h_i(t, \tau_2)| \leqslant k_i |\tau_1 - \tau_2|,$$

note that

$$|h_i(t, \tau)| \leqslant H_i + k_i |\tau|, \quad \text{where} \quad H_i = \sup_{t \in \mathbb{J}} |h_i(t, 0)|, \ i = 1, 2.$$

(H_2) $f : \mathbb{J} \times \mathbb{R}^3 \to \mathbb{R}$ is continuous, such that there exists $\psi \in C(\mathbb{J}, \mathbb{R}_+)$, with norm $\|\psi\|$, meet

$$|f(t, \tau_1, \tau_2, \tau_3) - f(t, \sigma_1, \sigma_2, \sigma_3)| \leqslant \psi(t)\big(|\tau_1 - \sigma_1| + |\tau_2 - \sigma_2| + |\tau_3 - \sigma_3|\big),$$

note that

$$|f(t, \tau_1, \tau_2, \tau_3)| \leqslant \|\psi\|(|\tau_1| + |\tau_2| + |\tau_3|) + F, \quad \text{with} \quad F = \sup_{t \in \mathbb{J}} |f(t, 0, 0, 0)|.$$

$\forall\ t \in \mathbb{J},\ \tau_i, \sigma_i \in \mathbb{R},\ (i = 1, 2, 3).$

(H_3) $k(t, \sigma)$ is continuous for all $(t, \sigma) \in \mathbb{J} \times \mathbb{J}$, with K is a positive constant, such that

$$\max_{t, \sigma \in \mathbb{J}} |k(t, \sigma)| = K.$$

Lemma 2.1. *The IFDP (1.1)-(1.3) has a mild solution of which it satisfies*

$$y(t) = h(t, y(t)) + \int_0^1 \mathcal{G}(t, \tau) u(\tau) d\tau, \tag{2.1}$$

here

$$u(t) = f\Big(t, h(t) + \int_0^1 \mathcal{G}(t, \tau) u(\tau) d\tau, I^{\gamma - \delta; \theta} u(t), \int_0^t k(t, \tau) u(\tau) d\tau\Big),$$

$$\mathcal{G}(t, \tau) = \begin{cases} \frac{\theta'(\tau)(\theta(t) - \theta(\tau))^{\gamma - 1}}{\Gamma(\gamma)} - \frac{\theta'(\tau)(\theta(t) - \theta(0))(\theta(1) - \theta(\tau))^{\gamma - 2}}{\theta'(1)\ \Gamma(\gamma - 1)}, & 0 \leqslant \tau \leqslant t \leqslant 1 \\[2ex] \frac{\theta'(\tau)(\theta(t) - \theta(0))(\theta(1) - \theta(\tau))^{\gamma - 2}}{\theta'(1)\ \Gamma(\gamma - 1)}, & 0 \leqslant t \leqslant \tau \leqslant 1 \end{cases} \tag{2.2}$$

$$\mathcal{G}_\circ := \max\{ \int_0^1 |\mathcal{G}(t, \tau)|\ d\tau,$$

and

$$\begin{aligned} h(t, y(t)) = &\int_0^1 \frac{\theta'(\tau)(\theta(1) - \theta(\tau))^{\rho - 1}}{\Gamma(\rho)} h_1(\tau, y(\tau)) d\tau \\ &+ \frac{(\theta(t) - \theta(0))}{\theta'(1)} \int_0^1 \frac{\theta'(\tau)(\theta(1) - \theta(\tau))^{\rho - 1}}{\Gamma(\rho)}\ h_2(\tau, y(\tau))\ d\tau. \end{aligned} \tag{2.3}$$

Proof. Let ${}^{c}D^{\gamma,\theta}y(t) = u(t)$ in (1.1), then

$$u(t) = f(t, y(t), I^{\gamma-\delta;\theta}u(t), \int_0^t k(t,\tau)u(\tau)d\tau),$$

and

$$y(t) = a_\circ + a_1(\theta(t) - \theta(0)) + \frac{1}{\Gamma(\gamma)} \int_0^t \theta'(\tau)(\theta(t) - \theta(\tau))^{\gamma-1}u(\tau)d\tau. \tag{2.4}$$

We can get the following from (1.2) and (1.3)

$$a_\circ = \int_0^1 \frac{\theta'(\tau)(\theta(1) - \theta(\tau))^{\gamma-1}}{\Gamma(\gamma)} h_1(\tau, y(\tau))d\tau,$$

differentiate (2.4) we receive

$$y'(t) = a_1\ \theta'(t) + \frac{1}{\Gamma(\gamma-1)} \int_0^t \theta'(\tau)(\theta(t) - \theta(\tau))^{\gamma-2}u(\tau)d\tau.$$

So,

$$\begin{aligned} a_1 =& \frac{1}{\theta'(1)\ \Gamma(\rho)} \int_0^1\ \theta'(\tau)(\theta(1) - \theta(\tau))^{\rho-1}\ h_2(\tau, y(\tau))\ d\tau \\ &- \frac{1}{\theta'(1)\ \Gamma(\gamma-1)} \int_0^1 \theta'(\tau)(\theta(1) - \theta(\tau))^{\gamma-2}u(\tau)d\tau. \end{aligned}$$

Consequently, the solution of (1.1)-(1.3) is outlined below:

$$\begin{aligned} y(t) =& \int_0^1 \frac{\theta'(\tau)(\theta(1) - \theta(\tau))^{\rho-1}}{\Gamma(\rho)} h_1(\tau, y(\tau))d\tau \\ &+ \frac{(\theta(t) - \theta(0))}{\theta'(1)} \int_0^1 \frac{\theta'(\tau)(\theta(1) - \theta(\tau))^{\rho-1}}{\Gamma(\rho)} h_2(\tau, y(\tau))\ d\tau \\ &- \frac{(\theta(t) - \theta(0))}{\theta'(1)\ \Gamma(\gamma-1)} \int_0^1 \theta'(\tau)(\theta(1) - \theta(\tau))^{\gamma-2}u(\tau)ds + \frac{1}{\Gamma(\gamma)} \int_0^t \theta'(\tau)(\theta(t) - \theta(\tau))^{\gamma-1}u(\tau)d\tau. \end{aligned}$$

□

Lemma 2.2. *The Lipschitzian function* $h : \mathbb{J} \times \mathbb{R} \to \mathbb{R}$, *has a Lipschitz constant* c, *with*

$$\|h(t, \mu) - h(t, \nu)\| \leqslant c\ \|\mu - \nu\|.$$

Proof. We obtain for any for all $x, y \in X$ and $t \in \mathbb{J}$

$$\begin{aligned} &\big|h(t, x(t)) - h(t, y(t))\big| \\ &\leqslant \Bigg| \int_0^1 \frac{\theta'(\tau)(\theta(1) - \theta(\tau))^{\rho-1}}{\Gamma(\rho)} h_1(\tau, x(\tau))d\tau + \frac{(\theta(t) - \theta(0))}{\theta'(1)} \int_0^1\ \frac{\theta'(\tau)(\theta(1) - \theta(\tau))^{\rho-1}}{\Gamma(\rho)}\ h_2(\tau, x(\tau))\ d\tau \\ &\quad - \int_0^1 \frac{\theta'(\tau)(\theta(1) - \theta(\tau))^{\rho-1}}{\Gamma(\rho)} h_1(\tau, y(\tau))d\tau - \frac{(\theta(t) - \theta(0))}{\theta'(1)} \int_0^1\ \frac{\theta'(\tau)(\theta(1) - \theta(\tau))^{\rho-1}}{\Gamma(\rho)}\ h_2(\tau, y(\tau))\ d\tau \Bigg| \end{aligned}$$

$$\begin{aligned}
&\leqslant \frac{1}{\Gamma(\rho+1)}\|h_1(\tau,x)-h_1(\tau,y)\| + \frac{(\theta(t)-\theta(0))}{\theta'(1)\ \Gamma(\rho+1)}\|h_2(\tau,x)-h_2(\sigma,y)\|\\
&\leqslant \frac{k_1(\theta(1)-\theta(0))^{\rho}}{\Gamma(\rho+1)}\|x-y\| + \frac{k_2\ (\theta(1)-\theta(0))^{\rho}\ (\theta(t)-\theta(0))}{\theta'(1)\ \Gamma(\rho+1)}\|x-y\|\\
&\leqslant \frac{(\theta(1)-\theta(0))^{\rho}[k_1\ \theta'(1)\ +k_2\ (\theta(t)-\theta(0))}{\theta'(1)\ \Gamma(\rho+1)]}\|x-y\|.
\end{aligned}$$

Then

$$\|h(t,x)-h(t,y)\| \leqslant c\ \|x-y\|,$$

with $c = \frac{(\theta(1)-\theta(0))^{\rho}[k_1\ \theta'(1)\ +k_2\ (\theta(1)-\theta(0))]}{\theta'(1)\ \Gamma(\rho+1)}$. □

2.1. *Exsistence of solution*

Based on the Krasnoselskii [14] fixed point Theorem, we will proved the first existence result for IFDP (1.1)-(1.3).

Theorem 2.3. *There is at least one mild solution on* I *for IFDP* (1.1)-(1.3) *under the assumptions* (H_1) *through* (H_3). *If*

$$\frac{(\theta(1)-\theta(0))^{\rho}k_1 + t(\theta(1)-\theta(0))^{\rho+1}k_2}{\Gamma(\rho+1)} + \frac{\mathcal{G}_{\circ}\ \|\psi\|}{1-\aleph} < 1.$$

Proof. Consider the operator $\mathcal{F} : \mathcal{C}(\mathbb{J},\mathbb{R}) \to \mathcal{C}(\mathbb{J},\mathbb{R})$ by:

$$\mathcal{F}y(t) = h(t,y(t)) + \int_0^1 \mathcal{G}(t,\tau)\nu(\tau)d\tau, \tag{2.5}$$

with

$$\nu(t) = f\big(t,y(t),I^{\gamma-\delta;\theta}\nu(t),\int_0^t k(t,\tau)\nu(\tau)d\tau\big).$$

Define the set

$$B_{\sigma} = \{y \in \mathcal{C}(\mathbb{J},\mathbb{R}) : \|y\| \leqslant \sigma\},$$

with

$$\sigma \geqslant \frac{\frac{(\theta(1)-\theta(0))^{\rho}H_1 + t(\theta(1)-\theta(0))^{\rho+1}H_2}{\Gamma(\rho+1)} + \frac{\mathcal{G}_{\circ}\ F}{1-\aleph}}{1-\big(\frac{(\theta(1)-\theta(0))^{\rho}k_1 + t(\theta(1)-\theta(0))^{\rho+1}k_2}{\Gamma(\rho+1)} + \frac{\mathcal{G}_{\circ}\ \|\psi\|}{1-\aleph}\big)},$$

and

$$\aleph = \frac{\|\psi\|(\theta(1)-\theta(0))^{\gamma-\delta}}{\Gamma(\gamma-\delta+1)} + \|\psi\|K.$$

Additionally, we define operators $\mathcal{F}_1$ and $\mathcal{F}_2$ on B_{σ} by

$$\begin{aligned}
\mathcal{F}_1 y(t) &= h(t,y(t)),\\
\mathcal{F}_2 y(t) &= \int_0^1 \mathcal{G}(t,\tau)\nu(\tau)d\tau.
\end{aligned}$$

Take into account that $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined on B_{σ}, and for any $y \in \mathcal{C}(\mathbb{J},\mathbb{R})$,

$$\mathcal{F}(t) = \mathcal{F}_1 y(t) + \mathcal{F}_2 y(t), \quad t \in \mathbb{J}.$$

The proof will be divided down into the following steps:

Step 1. Take $y_1, y_2 \in B_r$ and $t \in \mathbb{J}$, we obtain

$$\begin{aligned}|\mathcal{F}_1 y_1(t) + \mathcal{F}_2 y_2(t)| &\leqslant |\mathcal{F}_1 y_1(t)| + |\mathcal{F}_2 y_2(t)| \\ &\leqslant |h(t, y_1(t))| + \int_0^1 |\mathcal{G}(t,\tau)||\nu(\tau)|d\tau,\end{aligned} \tag{2.6}$$

where $\nu(t) = f\big(t, y_2(t), I^{\gamma-\delta;\theta}\nu(t), \int_0^t k(t,\tau)\nu(\tau)d\tau\big)$

$$\begin{aligned}|\nu(t)| &= \Big|f\Big(t, y_2(t), I^{\gamma-\delta;\theta}\nu(t), \int_0^t k(t,\tau)\nu(\tau)d\tau\Big)\Big| \\ &\leqslant F + \psi(t)|y_2(t)| + \psi(t)\int_0^t \frac{\theta'(\tau)(\theta(t)-\theta(\tau))^{\gamma-\delta-1}}{\Gamma(\gamma-\delta)}|\nu(\tau)|\, d\tau + \psi(t)\int_0^t |k(t,\tau)||\nu(\tau)|d\tau,\end{aligned}$$

Taking supermum for $t \in I$, we have

$$\|\nu\| \leqslant F + \|\psi\|\,\|y_2\| + \|\psi\|\frac{(\theta(1)-\theta(0))^{\gamma-\delta}}{\Gamma(\gamma-\delta+1)}\|\nu\| + \|\psi\|K\|\nu\|.$$

Then

$$\|\nu\| \leqslant \frac{F + \|\psi\|\,\sigma}{1 - \big(\frac{\|\psi\|(\theta(1)-\theta(0))^{\gamma-\delta}}{\Gamma(\gamma-\delta+1)} + \|\psi\|K\big)}.$$

And

$$\begin{aligned}|h(t, y_1(t))| &\leqslant \int_0^1 \frac{\theta'(\tau)(\theta(1)-\theta(\tau))^{\rho-1}}{\Gamma(\rho)}|h_1(\tau, y_1(\tau))|d\tau \\ &\quad + \frac{(\theta(t)-\theta(0))}{\theta'(1)}\int_0^1 \frac{\theta'(\tau)(\theta(1)-\theta(\tau))^{\rho-1}}{\Gamma(\rho)}|h_2(\tau, y_1(\tau))|d\tau,\end{aligned}$$

$$\Gamma(\rho)|[H_2 + k_2|y_2(\tau)|]d\tau \leqslant \frac{(\theta(1)-\theta(0))^{\rho}[H_1 + k_1\|y_1\|]}{\Gamma(\rho+1)} + \frac{(\theta(1)-\theta(0))^{\rho}\ (\theta(1)-\theta(0))[H_2 + k_2\|y_2\|]}{\theta'(1)\ \Gamma(\rho+1)}.$$

Hence (2.6) implying that, for each $t \in \mathbb{J}$,

$$\begin{aligned}|\mathcal{F}_1 y_1(t) + \mathcal{F}_2 y_2(t)| &\leqslant \frac{(\theta(1)-\theta(0))^{\rho}[H_1 + k_1\|y_1\|]}{\Gamma(\rho+1)} + \frac{(\theta(1)-\theta(0))^{\rho+1}\ [H_2 + k_2\|y_2\|]}{\theta'(1)\ \Gamma(\rho+1)} \\ &\quad + \frac{\mathcal{G}_\circ\ (F + \|\psi\|\ \sigma)}{1 - \ \aleph} \\ &\leqslant \ \sigma.\end{aligned}$$

Taking supremum over $t \in I$, we have

$$\|\mathcal{F}_1 y_1 + \mathcal{F}_2 y_2\| \leqslant \ \sigma.$$

This proves that $\mathcal{F}_1 y_1 + \mathcal{F}_2 y_2 \in B_\sigma, \forall\ \ y_1, y_2 \in B_\sigma$.

Step 2. lemma 2.2 makes it evident that A_1 represents a contration mapping when $c < 1$.

Step 3. First, establish the continuity of operator $\mathcal{F}_2$ is continuous.
Let $\{y_n\}_{n\in N}$ be a sequence with $y_n \to y$ as $n \to \infty$ in $C(\mathbb{J}, \mathbb{R})$.
Afterward, for each $t \in \mathbb{J}$

$$|\mathcal{F}_2 y_n - \mathcal{F}_2 y| \leqslant \int_0^1 |\mathcal{G}(t,\tau)||\nu_n(\tau) - \nu(\tau)|d\tau, \tag{2.7}$$

where $\nu_n, \nu \in C(\mathbb{J}, \mathbb{R})$

$$\nu_n(t) = f\big(t, y_n(t), I^{\gamma-\delta;\theta}\nu_n(t), \int_0^t k(t,\tau)\nu_n(\tau)d\tau\big)$$
$$\nu(t) = f\big(t, y(t), I^{\gamma-\delta;\theta}\nu(t), \int_0^t k(t,\tau)\nu(\tau)d\tau\big),$$

so that by (H_2)

$$\begin{aligned}
&|\nu_n(t) - \nu(t)| \\
&\leqslant \psi(t)\left(|y_n(t) - y(t)| + \int_0^t \frac{\theta'(\tau)(\theta(t)-\theta(\tau))^{\gamma-\delta-1}}{\Gamma(\gamma-\delta)}|\nu_n(\tau) - \nu(\tau)|d\tau + \int_0^t k(t,\tau)|\nu_n(\tau)-\nu(\tau)|d\tau\right) \\
&\leqslant \|\psi\|\big(\|y_n - y\| + \frac{\|\nu_n - \nu\|(\theta(1)-\theta(\tau))^{\gamma-\delta}}{\Gamma(\gamma-\delta+1)} + K\,\|\nu_n - \nu\|\,\big).
\end{aligned}$$

Thus

$$\|\nu_n - \nu\| \leqslant \frac{\|\psi\|}{1 - \|\psi\|\big(K + \frac{(\theta(t)-\theta(\tau))^{\gamma-\delta}}{\Gamma(\gamma-\delta+1)}\big)}\|y_n - y\|.$$

For $y_n \to y$, thus, we have $\nu_n(t) \to \nu(t)$ as $n \to \infty$ for each $t \in \mathbb{J}$. And then let $\varepsilon > 0$ be shall ensure, for each $t \in \mathbb{J}$, we get $|\nu_n(\tau)| \leqslant \varepsilon$, and $|\nu(\tau)| \leqslant \varepsilon$. Then

$$\begin{aligned}
|\mathcal{G}(t,\tau)||\nu_n(\tau) - \nu(\tau)| &\leqslant |\mathcal{G}(t,\tau)|[|\nu_n(\tau)| + |\nu(\tau)|] \\
&\leqslant 2\varepsilon|\mathcal{G}(t,\tau)|,
\end{aligned}$$

$\forall\ t \in \mathbb{J}$, the function $\tau \to 2\varepsilon|\mathcal{G}(t,\tau)|$ is integrable on $\mathbb{J}$. Then applying Lebesgue Dominated Convergence Theorem, and (2.7), we may conclude that

$$\|\mathcal{F}_2 y_n - \mathcal{F}_2 y\| \to 0 \ \text{ as } \ n \to \infty.$$

As a result, $\mathcal{F}_2$ is continuous. Furthermore, it is simple to confirm that

$$\|\mathcal{F}_2 y\| \leqslant \mathcal{G}_0\,\big(\frac{F + \|\psi\|\ \sigma}{1 - \aleph}\big)\ \leqslant \sigma,$$

according to the definitions of $\aleph$. This demonstrate that $\mathcal{F}_2$ is uniformly bounded on B_σ.

In the end, we demonstrate that $\mathcal{F}_2$ maps bounded sets into equicontinuous sets of $\mathcal{C}(\mathbb{J},\mathbb{R})$, i.e., B_σ is equicontinuous.

Next, suppose that $\forall\ \epsilon>0$, $\exists\ \delta>0$ and $t_1,t_2\in I$, $t_1<t_2$, $|t_2-t_1|<\delta$, Then we get

$$\begin{aligned}|\mathcal{F}_2y(t_2)-\mathcal{F}_2y(t_1)| &\leqslant \int_0^1 |\mathcal{G}(t_2,\tau)-\mathcal{G}(t_1,\tau)|\ |\nu(\tau)|d\tau\\ &\leqslant \frac{F+\|\psi\|\ \sigma}{1-\aleph}\int_0^1 |\mathcal{G}(t_2,\tau)-\mathcal{G}(t_1,\tau)|\ d\tau.\end{aligned}$$

As $t_1\to t_2$, he right-hand side of the above mentioned inequality tends to zero and is independent of y . Consequently,

$$|\mathcal{F}_2y(t_2)-\mathcal{F}_2y(t_1)|\to 0,\ \ \forall\ |t_2-t_1|\to 0.$$

As a result, $\{\mathcal{F}y\}$ is equi-continuous on B_σ. according to the Arzela-Ascoli Theorem [16], and $\mathcal{F}$ is a compact operator, we conclusion that $\mathcal{F}:\mathcal{C}(\mathbb{J},\mathbb{R})\to\mathcal{C}(\mathbb{J},\mathbb{R})$ is completely continuous.

As a result, all assumptions of Krasnoselskii's fixed point theorem are met and demonstrates that $\mathcal{F}_1+\mathcal{F}_2$ has a fixed point on B_σ. Hence, there is a mild solution to the IFDP (1.1)-(1.3). □

Our second conclusion proves using Banach's fixed point theorem the uniqueness of solution to the IFDP (1.1)-(1.3).

Theorem 2.4. *Suppose that hypotheses of Theorem 2.3 meet, with*

$$c+\frac{\mathcal{G}_\circ\ \|\psi\|}{1-\|\psi\|\left(K\ +\frac{(\theta(1)-\theta(0))^{\gamma-\delta}}{\Gamma(\gamma-\delta+1)}\right)}<1, \tag{2.8}$$

Then, a unique mild solution on $\mathbb{J}$ provided by the IFDP (1.1)-(1.3)*.*

Proof. Therefore, IFDP (1.1)-(1.3) has at least one solution, according to Theorem 2.3. Thus, all that is required of us is to demonstrate that $\mathcal{F}$ mentioned in (2.9) is a contraction operator.

Next, select $x,y\in\mathcal{C}(\mathbb{J},\mathbb{R})$. After that, for $t\in\mathbb{J}$, we obtain

$$\mathcal{F}x(t)-\mathcal{F}y(t)=h(t,x(t))+\int_0^1\mathcal{G}(t,\tau)u(\tau)d\tau-h(t,y(t))-\int_0^1\mathcal{G}(t,\tau)\nu(\tau)d\tau, \tag{2.9}$$

here $u,\nu\in\mathcal{C}(\mathbb{J},\mathbb{R})$, with

$$\begin{aligned}u(t)&=f\Big(t,x(t),I^{\gamma-\delta;\theta}u(t),\int_0^t k(t,\tau)u(\tau)d\tau\Big).\\ \nu(t)&=f\Big(t,y(t),I^{\gamma-\delta;\theta}\nu(t),\int_0^t k(t,\tau)\nu(\tau)d\tau\Big).\end{aligned}$$

Then, for $t\in\mathbb{J}$

$$|\mathcal{F}x(t)-\mathcal{F}y(t)|\leqslant\ |h(t,x(t))-h(t,y(t))|+\int_0^1\mathcal{G}(t,\tau)\ |u(\tau)-\nu(\tau)|d\tau, \tag{2.10}$$

however, given the condition (H_2), we obtain

$$
\begin{aligned}
&|u(t)-v(t)| \\
&\leqslant \psi(t)\left(|x(t)-y(t)|+\int_0^t \frac{\theta'(\tau)(\theta(t)-\theta(\tau))^{\gamma-\delta-1}}{\Gamma(\gamma-\delta)}|u(\tau)-v(\tau)|\, d\tau+\int_0^t k(t,\tau)|u(\tau)-v(\tau)|d\tau\right) \\
&\leqslant \|\psi\| \left(\|x-y\|+\frac{(\theta(t)-\theta(0))^{\gamma-\delta}}{\Gamma(\gamma-\delta+1)}\|u-v\|+K\|u-v\|\ \right).
\end{aligned}
$$

Thus

$$
\|u-v\| \leqslant \frac{\|\psi\|}{1-\|\psi\|\left(K\ +\frac{(\theta(1)-\theta(0))^{\gamma-\delta}}{\Gamma(\gamma-\delta+1)}\right)}\|x-y\|.
$$

Back to (2.10) and using lemma 2.2, we get

$$
\begin{aligned}
&\|\mathcal{F}x-\mathcal{F}y\| \\
&\qquad\leqslant \left(c+\frac{\mathcal{G}_\circ\ \|\psi\|}{1-\|\psi\|\left(K\ +\frac{(\theta(1)-\theta(0))^{\gamma-\delta}}{\Gamma(\gamma-\delta+1)}\right)}\right)\|x-y\|.
\end{aligned}
$$

By $\left(c+\frac{\mathcal{G}_\circ\ \|\psi\|}{1-\|\psi\|\left(K\ +\frac{(\theta(1)-\theta(0))^{\gamma-\delta}}{\Gamma(\gamma-\delta+1)}\right)}\right)<1$, we deduce that $\mathcal{F}$ is a contraction. As a result, according to Banach's contraction principle, $\mathcal{F}$ has a unique fixed point on $\mathbb{J}$, and that is a mild solution of the IFDP (1.1)-(1.3). □

3. Ulam-Hyers Stability

We now investigate the Ulam stability for IFDP (1.1)-(1.3). Make $\Phi:\mathbb{J}\to\mathbb{R}_+$ be a continuous function and suppose that $\epsilon>0$. This kind of inequality is considered:

$$
|{}^cD^{\gamma;\theta}y(t)-f\big(t,y(t),{}^cD^{\delta;\theta}y(t),\int_0^t k(t,\tau){}^cD^{\gamma;\theta}y(\tau)d\tau\big)|\leqslant\epsilon,\quad t\in\mathbb{J} \tag{3.1}
$$

$$
|{}^cD^{\gamma;\theta}y(t)-f\big(t,y(t),{}^cD^{\delta;\theta}y(t),\int_0^t k(t,\tau){}^cD^{\gamma;\theta}y(\tau)d\tau\big)|\leqslant\theta(t),\ t\in\mathbb{J} \tag{3.2}
$$

$$
|{}^cD^{\gamma;\theta}y(t)-f\big(t,y(t),{}^cD^{\delta;\theta}y(t),\int_0^t k(t,\tau){}^cD^{\alpha;\phi}y(\tau)d\tau\big)|\leqslant\epsilon\ \theta(t),\quad t\in\mathbb{J}. \tag{3.3}
$$

Theorem 3.1. *Assume that the premises of Theorem 2.4 meet. Ulam–Hyers is thus stable for IFDP (1.1)-(1.3).*

Proof. Take $\epsilon>0$ and let the function $z\in\mathcal{C}(\mathbb{J},\mathbb{R})$ be fulfills inequality (3.1), i.e.,

$$
|{}^cD^{\gamma;\theta}z(t)-f\big(t,z(t),{}^cD^{\delta;\theta}z(t),\int_0^t k(t,\tau){}^cD^{\gamma;\theta}z(\tau)ds\big)|\leqslant\epsilon,\quad t\in\mathbb{J}
$$

and permit FIDE (1.1)-(1.3) to have a unique solution $y\in\mathcal{C}(\mathbb{J},\mathbb{R})$, consequently, lemma 2.1 gives the equivalence betwwen IFDP (1.1)-(1.3) and integral equation

$$y(t) = h(t, y(t)) + \int_0^1 G(t,\tau)u(\tau)d\tau,$$

$$u(t) = f\big(t, h(t, y(t)) + \frac{1}{\Gamma(\gamma)}\int_0^1 \mathcal{G}(t,\tau)u(\tau)d\tau, I^{\gamma-\delta;\theta}u(t), \int_0^t k(t,\tau)u(\tau)d\tau\big).$$

Operating by $I^{\gamma,\theta}$ on (3.1), and by integrating, we obtain

$$\left|z(t) - h(t, z(t)) - \int_0^1 \mathcal{G}(t,\tau)v(\tau)d\tau\right| \leqslant \frac{\epsilon\ (\theta(1)-\theta(0))^{\gamma}}{\Gamma(\gamma+1)}. \tag{3.4}$$

For each $t \in \mathbb{J}$, we have

$$\begin{aligned}
|z(t) - y(t)| = &\Big|z(t) - h(t, y(t) - \int_0^1 \mathcal{G}(t,\tau)u(\tau)d\tau\Big| \\
\leqslant &\Big|z(t) - h(t, z(t) + \int_0^1 \mathcal{G}(t,\tau)v(\tau)d\tau\Big| \\
&+ \Big|h(t, z(t) + \int_0^1 \mathcal{G}(t,\tau)v(\tau)ds - h(t, y(t) - \int_0^1 \mathcal{G}(t,\tau)u(\tau)d\tau\Big| \\
\leqslant &\frac{\epsilon\ (\theta(1)-\theta(0))^{\gamma}}{\Gamma(\gamma+1)} + \big|h(t, z(t)) - h(t, y(t))\big| + \int_0^1 \mathcal{G}(t,\tau)|v(\tau) - u(\tau)|d\tau \\
\leqslant &\frac{\epsilon\ (\theta(1)-\theta(0))^{\gamma}}{\Gamma(\gamma+1)} + c\ \|z-y\| + \mathcal{G}_\circ\|u-v\|\ .
\end{aligned}$$

Actuality, demonstration of Theorem 2.4 gives

$$\|u - v\| \leqslant \frac{\|\psi\|}{1 - \|\psi\|\big(K\ + \frac{(\theta(1)-\theta(0))^{\gamma-\delta}}{\Gamma(\gamma-\delta+1)}\big)}\|z-y\|.$$

Then, $\forall\ \ t \in \mathbb{J}$

$$\|z-y\| \leqslant \frac{\epsilon\ (\theta(1)-\theta(0))^{\gamma}}{\Gamma(\gamma+1)} + c\ \|z-y\| + \frac{\mathcal{G}_\circ\ \|\psi\|}{1 - \|\psi\|\big(K\ + \frac{(\theta(1)-\theta(0))^{\gamma-\delta}}{\Gamma(\gamma-\delta+1)}\big)}\|z-y\|.$$

Thus

$$\|z-y\| \leqslant \frac{\epsilon\ (\theta(1)-\theta(0))^{\gamma}}{\Gamma(\gamma+1)}\left[1 - \Big(c\ + \frac{\mathcal{G}_\circ\ \|\psi\|}{1 - \|\psi\|\big(K\ + \frac{(\theta(1)-\theta(0))^{\gamma-\delta}}{\Gamma(\gamma-\delta+1)}\big)}\Big)\right]^{-1} = \chi\ \epsilon,$$

for let $\chi = \frac{(\theta(1)-\theta(0))^{\gamma}}{\Gamma(\gamma+1)}\left[1 - \Big(c\ + \frac{\mathcal{G}_\circ\ \|\psi\|}{1-\|\psi\|\big(K\ + \frac{(\theta(1)-\theta(0))^{\gamma-\delta}}{\Gamma(\gamma-\delta+1)}\big)}\Big)\right]^{-1}$. So, the FIDE (1.1)-(1.3) is Ulam-Hyers stable. □

By setting $\Phi(\epsilon) = \chi\ \epsilon$, $\Phi(0) = 0$. It can be demonstrated that the the FIDE (1.1)-(1.3) is generalized Ulam-Hyers stable.

3.1. Ulam-Hyers-Rassias Stability.

Next, we present the Ulam−Hyers −Rassias stable result.

Theorem 3.2. *Assume assumptions* $(H_1)-(H_3)$ *and*

(H_4) *The function* $\Phi \in \mathcal{C}(\mathbb{J}, \mathbb{R}_+)$ *is increasing and there exists* $\lambda_\Phi > 0$ *with,* $\forall\ t \in \mathbb{J}$*, we get*

$$I^\gamma\ \Phi(\tau) \leqslant \lambda_\Phi\ \Phi(\tau).$$

are satisfied. Then IFDP (1.1)-(1.3) is Ulam−Hyers−Rassias stable with regard to to Φ*.*

Proof. Consider $z \in C(\mathbb{J}, \mathbb{R})$ to be the solution of Eq.(3.3), that is,

$$\left|{}^c D^{\gamma;\theta} z(t) - f\big(t, z(t), {}^c D^{\delta;\theta} z(t), \int_0^t k(t,\tau){}^c D^{\gamma;\theta} z(\tau) d\tau\big)\right| \leqslant \epsilon\ \Phi,\quad t \in \mathbb{J}$$

and take y is a solution of the problem (1.1)−(1.3). As a result, we get

$$y(t) = h(t, y(t)) + \int_0^1 \mathcal{G}(t,\tau) u(\tau) d\tau,$$

with $u \in C(\mathbb{I}, \mathbb{R})$

$$u(t) = f\big(t, y(t), I^{\gamma-\delta;\theta} u(t), \int_0^t k(t,\tau) u(\tau) d\tau\big).$$

Applying I^γ on each sides of the inequality (3.3) and then integrating, we get

$$\begin{aligned}\left|z(t) - h(t, z(t)) - \int_0^1 \mathcal{G}(t,\tau) v(\tau) d\tau\right| &\leqslant \frac{\epsilon}{\Gamma(\gamma)} \int_0^t \theta'(\tau)(\theta(t) - \theta(\tau))^{\gamma-1}\ \Phi(\tau) d\tau \\ &\leqslant \epsilon \lambda_\Phi \Phi(t),\end{aligned}$$

with $v \in \mathcal{C}(\mathbb{J}, \mathbb{R})$

$$v(t) = f\big(t, z(t), I^{\gamma-\delta;\theta} v(t), \int_0^t k(t,\tau) v(\tau) d\tau\big).$$

For each $t \in \mathbb{J}$, we obtain

$$\begin{aligned}|z(t) - y(t)| =& \left|z(t) - h(t, y(t)) + \int_0^1 \mathcal{G}(t,\tau) u(\tau) d\tau\right| \\ \leqslant& \left|z(t) - h(t, z(t)) + \int_0^1 \mathcal{G}(t,\tau) v(\tau) d\tau\right| \\ &+ \left|h(t, z(t)) + \int_0^1 \mathcal{G}(t,\tau) v(\tau) ds - h(t, y(t)) - \int_0^1 \mathcal{G}(t,\tau) u(\tau) d\tau\right| \\ \leqslant& \epsilon\ \lambda_\Phi\ \Phi(t) + c\ \|z - y\| + \mathcal{G}_\circ \|v - u\|.\end{aligned}$$

In actuality, from evidence of Theorem 2.4,

$$\|u-v\| \leqslant \frac{\|\psi\|}{1-\|\psi\|\big(K+\frac{(\theta(1)-\theta(0))^{\gamma-\delta}}{\Gamma(\gamma-\delta+1)}\big)}\|z-y\|.$$

Hence, for each $t \in \mathbb{J}$

$$\|z-y\| \leqslant \epsilon\,\lambda_\Phi\,\Phi(t)+c\,\|z-y\|+\frac{\mathcal{G}_\circ\,\|\psi\|}{1-\|\psi\|\big(K+\frac{(\theta(1)-\theta(0))^{\gamma-\delta}}{\Gamma(\gamma-\delta+1)}\big)}\|z-y\|.$$

Thus

$$\|z-y\| \leqslant \Big[1-\Big(c+\frac{\mathcal{G}_\circ\,\|\psi\|}{1-\|\psi\|\big(K+\frac{(\theta(1)-\theta(0))^{\gamma-\delta}}{\Gamma(\gamma-\delta+1)}\big)}\Big)\Big]\,\epsilon\,\lambda_\Phi\,\Phi(t)=c_\Phi\epsilon\,\Phi(t),$$

where

$$c_\Phi=\Big[1-\Big(c+\frac{\mathcal{G}_\circ\,\|\psi\|}{1-\|\psi\|\big(K+\frac{(\theta(1)-\theta(0))^{\gamma-\delta}}{\Gamma(\gamma-\delta+1)}\big)}\Big)\Big]\,\lambda_\Phi.$$

Therefore, the IFDP (1.1)-(1.3) is Ulam-Hyers-Rassias stable considering Φ.

□

4. Discussion and Illustrations

In this part, we offer specific existence results for a few boundary value problems that serve as special illustrations of our fundamental finding.

- Assume $\Upsilon(\tau)=\frac{(\theta(1)-\theta(\tau))^\rho}{\Gamma(\rho+1)}$, Then, using Riemann-Stieltjes integrals, we build

$$^cD^{\gamma,\theta}y(t)=f\big(t,y(t),{}^cD^{\delta,\theta}y(t),\int_0^t k(t,\tau)\,{}^cD^{\gamma,\theta}y(\tau)d\tau\big),\quad t\in(0,1).$$

  With the Riemann-Stieltjes boundary condition

$$y(0)=\int_0^1 h_1(\tau,y(\tau))d\Upsilon(\tau),$$
$$y'(1)=\int_0^1 h_2(\tau,y(\tau))\,d\Upsilon(\tau).$$

  Several investigations have looked into this type of boundary condition, for instance [34].

- For $\rho\to 1$, $\theta(t)=t$, $f(t,y(t),u(t),v(t))=-\chi\,n(t)\,\phi(y(t))$, $h_1(t,y(t))=h_2(t,y(t))=y(t)$, a nonlocal value problem with an integral condition is obtained.

$$^cD^{\gamma,\theta}y(t)+\chi\,n(t)\phi(y(t))=0,\quad t\in(0,1)$$
$$y(0)=\int_0^1 y(\tau)d\tau \text{ and } y'(1)=\int_0^1 y(\tau)d\tau,$$

  which this kind of problem is investigated in [15].

- For $\gamma \to 0$, $\theta(t) = t$ $f(t, y(t), u(t), v(t)) = p(t) + f_1(t, v(t))$, and $k(t,\tau) = \frac{t}{t+\tau}$. Then problem (1.1)-(1.3) represents Chandrasekhar's integral equations with integral boundary condition as follows

$$y(t) = p(t) + f_1\big(t, \int_0^t \frac{t}{t+\tau} y(\tau) d\tau\big), \tag{4.1}$$

$$y(0) = \frac{1}{\Gamma(\rho)} \int_0^1 (1-\tau)^{\rho-1}\, h_1(\tau, y(\tau))\, d\tau, \tag{4.2}$$

$$y'(1) = \frac{1}{\Gamma(\rho)} \int_0^1 (1-\tau)^{\rho-1}\, h_2(\tau, y(\tau))\, d\tau, \tag{4.3}$$

let us analyze the function $g : R \times R \to R$, as defined by the formula

$$g(t,\tau) = \begin{cases} t\ \ln \frac{t+\tau}{t}, & \text{for } t > 0 \text{ and } \tau \geqslant 0, \\ 0 & \text{for } t = 0 \text{ and } \tau \geqslant 0. \end{cases}$$

We observe that the problem (4.1)-(4.3) can be expressed as Volterra-Stieltjes type

$$y(t) = p(t) + f\big(t, \int_0^t y(\tau) d_\tau g(t,\tau)\big), \tag{4.4}$$

with integral bounday conditions (4.2)-(4.3). Recently, the problem of such a type was intensively investigated in [19].

Example 1. Take the following IFDP into account:

$$ {}^cD^{\frac{4}{3};\ln(1+t)} = \frac{e^{-t}}{e^t+8}\Big(\frac{|y(t)|}{1+|y(t)|} - \frac{|{}^cD^{\frac{5}{4};\ln(1+t)}y(t)|}{1+|{}^cD^{\frac{5}{4};\ln(1+t)}y(t)|} - \frac{|\int_0^1 \ln(t+\tau)\,{}^cD^{\frac{5}{4};\ln(1+t)}y(t)|}{1+|\int_0^1 \ln(t+\tau)\,{}^cD^{\frac{5}{4};\ln(1+t)}y(t)|}\Big) \tag{4.5}$$

$$y(0) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^1 (\ln(2) - \ln(1+\tau))^{\frac{1}{2}}\, \frac{\sin y(\tau)}{20(1+\tau)}\, d\tau, \tag{4.6}$$

$$y'(1) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^1 (\ln(2) - \ln(1+\tau))^{\frac{1}{2}}\, \frac{e^{-y(\tau)}}{30(1+\tau)}\, d\tau. \tag{4.7}$$

Let

$$f(t,\mu,\nu,\omega) = \frac{e^{-t}}{e^t+8}\Big(\frac{|\mu(t)|}{1+|\mu(t)|} - \frac{|\nu(t)|}{1+|\nu(t)|} - \frac{|\omega(t)|}{1+|\omega(t)|}\Big).$$

Note that, f is continuous function. For each $\mu_i, \nu_i, \omega_i \in \mathbb{R}$, and $t \in [0,1]$, $(i = 1,2)$

$$\begin{aligned}|f(t,\mu_1,\nu_1,\omega_1) - f(t,\mu_2,\nu_2,\omega_2)| &\leqslant \frac{e^{-t}}{e^t+8}\big(|\mu_1-\mu_2| + |\nu_1-\nu_2| + |\omega_1-\omega_2|\big)\\ &\leqslant \frac{1}{9}\big(|\mu_1-\mu_2| + |\nu_1-\nu_2| + |\omega_1-\omega_2|\big).\end{aligned}$$

Thus, (H_2) is verified with $\|\psi\| = \frac{1}{9}$, and

$$\begin{aligned}|h(t,x(t)) - h(t,y(t))| \leqslant& \frac{1}{1.772453}\int_0^1 (\ln(2) - \ln(1+\tau))^{\frac{1}{2}} \frac{|\sin x(\tau) - \sin y(\tau)|}{20(1+\tau)}\, d\tau\\ &+ \frac{1}{1.772453}\int_0^1 (\ln(2) - \ln(1+\tau))^{\frac{1}{2}} \frac{|e^{-x(\tau)} - e^{-y(\tau)}|}{30(1+\tau)}\, d\tau.\end{aligned}$$

And we are going to be meeting the lemma 2.2, so, with constant $c = 0.031348808$ the function h is lipschitz. We shall show that condition (2.8) holds.

$$c + \frac{\mathcal{G}_\circ \ \|\psi\|}{1 - \|\psi\|\big(K \ + \frac{(\phi(1)-\phi(0))^{\gamma-\delta}}{\Gamma(\gamma-\delta+1)}\big)} \simeq 0.4297118717 < 1,$$

Hence $\gamma = \frac{4}{3}$, $\delta = \frac{5}{4}$, $c = 0.03134$, $\|\psi\| = \frac{1}{9}$, $K = \ln(2)$ and $\mathcal{G}_0 < 3$. We can see that all hypotheses of Theorem 2.4 are fulfilled. Consequently, problem IFDP (4.5)-(4.7) has a unique mild solution defined on I. Moreover, problem IFDP (4.5)-(4.7) is Ulam-Hyers-Rassias stable.

Example 2. Take the following IFDP into account:

$$^{c}D^{\frac{3}{2};t}y(t) = \frac{2 + y(t) +^{c} D^{\frac{4}{3};t}y(t) + \int_0^1 e^{t-\tau}\ {}^{c}D^{\frac{3}{2};t}y(\tau)d\tau}{2e^{t+1}\big(1 + y(t) +^{c} D^{\frac{4}{3};t}y(t) + \int_0^1 e^{t-\tau}\ {}^{c}D^{\frac{3}{2};t}y(\tau)d\tau\big)}, \qquad t \in [0,1] \tag{4.8}$$

$$y(0) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^1 (1-\tau)^{\frac{1}{2}} \frac{y}{10e^{-\tau+2}(1+y)}\, d\tau, \tag{4.9}$$

$$y'(1) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^1 (1-\tau)^{\frac{1}{2}} \frac{\cos y}{30(\tau+2)}\, d\tau. \tag{4.10}$$

Let

$$f(t,\mu,\nu,\omega) = \frac{2 + |\mu| + |\nu| + |\omega|}{2e^{t+1}\big(1 + |\mu| + |\nu| + |\omega|\big)},$$

Note that, f is continuous function. For each $\mu_i, \nu_i, \omega_i \in \mathbb{R}$, and $t \in [0,1]$, $(i = 1,2)$

$$|f(t,\mu_1,\nu_1,\omega_1) - f(t,\mu_2,\nu_2,\omega_2)| \leqslant \frac{1}{2e}\big(|\mu_1-\mu_2| + |\nu_1-\nu_2| + |\omega_1-\omega_2|\big).$$

Thus, (H_2) is verified with $\psi(t) = \frac{1}{2e^{t+1}}$. In addition, we obtain

$$|f(t,\mu,\nu,\omega)| = \frac{1}{2e^{t+1}}\left(2+|\mu|+|\nu|+|\omega|\right),$$

with $f(t,0,0,0) = \frac{1}{e^{t+1}}$, and $\|\psi\| = \frac{1}{2e}$.

Set $h_1(t,x(t)) = \frac{\cos x}{30(t+2)}$ and $h_2(t,x(t)) = \frac{x}{10e^{-s+2}(1+x)}$

$$\begin{aligned}|h_1(t,x(t))-h_1(t,y(t))| &\leqslant \left|\frac{\cos x}{30(t+2)}-\frac{\cos y}{30(t+2)}\right|\\ &\leqslant \frac{1}{30}\,|x(t)-y(t)|,\end{aligned}$$

and

$$\begin{aligned}|h_2(t,x(t))-h_2(t,y(t))| &\leqslant \left|\frac{x}{10e^{-t+2}(1+x)}-\frac{y}{10e^{-t+2}(1+y)}\right|\\ &\leqslant \frac{1}{10e}\,|x(t)-y(t)|.\end{aligned}$$

Hence, the condition (H_1) is satisfied with $k_1 = \frac{1}{30}$ and $k_2 = \frac{1}{10e}$. From (2.2) clearly for $\alpha = \frac{3}{2}$, then $\mathcal{G}_0 < 1$. Thus condition

$$\frac{k_1+tk_2}{\Gamma(\gamma+1)} + \frac{\mathcal{G}_\circ\,\|\psi\|}{1-\aleph} \simeq 0.4103216862 < 1,$$

where $\aleph = \frac{\|\psi\|}{\Gamma(\gamma-\delta+1)} + \|\psi\|K = 0.2568803025$ is satisfied with $\gamma = \frac{3}{2}$, $\delta = \frac{4}{3}$, $F = \frac{1}{e}$, $\|\psi\| = \frac{1}{2e}$ $k_1 = \frac{1}{30}$, $k_2 = \frac{1}{10e}$ and $K = e$. Theorem 2.3 implies that the IFDP (4.8)-(4.10) has at least one mild solution on I.

5. Conclusion

Our purpose in this paper is to study the existence and uniqueness of mild solutions for boundary value problems of implicit $\theta-$Caputo fractional differential equation (1.1)-(1.3) our established based on Krasnoselskii's fixed point theorem and Banach contraction principle. In addition, stability analysis in the Ulam–Hyers sense of a given implicit $\theta-$Caputo differential equation of fractional order, supplemented with fractional integral type boundary conditions was considered. Finally, we end the article with illustrations were provided to confirm the results' applicability. In the future, the concept presented here can be extended to the system of fractional integral equations of n-product type. New results can also be obtained by considering more generalized kernels. Interested researchers can subsequently extend this concept to two-dimensional integral equations of fractional order.

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