J. Math. Computer Sci., 34 (2024), 27–34

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Some fixed point results of F-contraction in parametric S-metric spaces

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Abstract

In this paper, we introduce the concept of F-contractions in parametric S-metric space. Further, we prove a unique fixed point theorem in parametric S-metric space. Another result is also obtained as corollary.

Keywords: F-contraction, fixed points, parametric S-metric space.

2020 MSC: 47H10, 54H25.

1. Introduction

The concept of metric space has been extended to various forms viz. b-metric space [2], S-metric space [21], parametric metric space [7] parametric b-metric space [8], S_b-metric space [19], parametric S-metric space [18] etc. The main aim is to extend the Banach fixed point theorem. Banach fixed point theorem is also extended by generalizing the contraction conditions. Some of the research works in this direction can be seen in [1, 3–6, 9–17, 20, 22–25] and references there in. One of the notable works in this direction is due to Wardowski [25]. He extended Banach fixed point theorem by introducing F-contraction concept. Motivated by the works in [15, 18], in this paper we introduce the concept of F-contraction in parametric S-metric space and proved a unique fixed point theorem. For our study, we will use the notations $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$ as the set of real numbers, set of positive real numbers, and set of natural numbers, respectively. Wardowski [25] defined the following.

Definition 1.1 ([25])**.** A self mapping P in a metric space (X, d) is said to be an F-contraction if for all $x, y \in X$ and $d(Px, Py) > 0$ implies

$$\tau + F(d(Px, Py)) \leqslant F(d(x, y))), \tag{1.1}$$

where $\tau > 0$ and $F \in \mathcal{F}$. Here $\mathcal{F}$ is the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying

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doi: 10.22436/jmcs.034.01.03

Received: 2023-09-29 *Revised:* 2023-12-09 *Accepted:* 2023-12-11

(F1) F is increasing strictly;
(F2) $\lim_{n\to+\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$ for each sequence $\{\alpha_n\} \subset \mathbb{R}^+$;
(F3) for $0 < k < 1$, $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Wardowski also pointed out that by considering different types of mappings in (1.1) variety of contractions can be obtained. He also remarked that from (F1) and (1.1), it can be concluded that F-contraction mappings are contractive and hence continuous. Moreover, if F_1, F_2 satisfy the properties (F1)-(F3) in Definition 1.1 and if $F_1(\alpha) \leqslant F_2(\alpha)$ for all $\alpha > 0$ and a mapping $G = F_2 - F_1$ is decreasing, then every F_1-contraction P is F_2-contraction. The following theorem was proved by Wardowski.

Theorem 1.2 ([25])**.** *In a complete metric space (X, d), let a self mapping P be an F-contraction. Then for every $x \in X$, the sequence $\{P^n x\}$, $n \in \mathbb{N}$ converges to $x^* \in X$, where x^* is the unique fixed point of P.*

Secelean [20] replaced (F2) of Definition 1.1 by either of the property given as following:

(F2′) $\inf F = -\infty$; or
(F2″) a sequence $\{\alpha_n\}$, $n \in \mathbb{N}$ of positive real numbers exist such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

Secelean [20] also proved the following.

Lemma 1.3 ([20])**.** *Consider a sequence $\{\alpha_n\}$, $n \in \mathbb{N}$ and an increasing mapping $F : \mathbb{R}^+ \to \mathbb{R}$. Then the following conditions hold true:*

(i) $\lim_{n\to\infty} F(\alpha_n) = -\infty$ *implies* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\inf F = -\infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$, *imply* $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

Wardowski also pointed out that Banach contractions are F-contractions and converse is not true. Following definition was given by Cosentino and Verto [3].

Definition 1.4 ([3])**.** In a complete metric space (X, d), a self mapping P is said to be Hardy-Rogers type F-contraction if $F \in \mathcal{F}$ and $\tau > 0$ satisfies

$$\tau + F(d(Px, Py)) \leqslant F(a_1.d(x, y) + a_2.d(x, Px) + a_3.d(y, Py) + a_4.d(x, Py) + a_5.d(y, Px))$$

with $d(Px, Py) > 0$ for all $x, y \in X$, where a_1, a_2, a_3, a_4, and a_5 are non-negative numbers, $a_3 \neq 1$ and $a_1 + a_2 + a_3 + 2a_4 = 1$.

Theorem 1.5 ([3])**.** *In a complete metric space (X, d), let a self mapping P be a Hardy-Rogers-type contraction and $a_3 \neq 1$. Then P has a fixed point. Further, P has a unique fixed point if $a_1 + a_4 + a_5 \leqslant 1$.*

In Definition 1.1, the condition (F3) was replaced by Piri and Kumam [13] as follows:

(F3′) F is continuous on $(0, +\infty)$.

They defined a family of functions $\mathcal{F}$ satisfying (F1), (F2′), and (F3′) and proved the following.

Theorem 1.6 ([13])**.** *In a complete metric space (X, d), let P be a self mapping. Let $F \in \mathcal{F}$ satisfies*

$$\forall x, y \in X, [d(Px, Py) > 0 \textit{ implies } \tau + F(d(Px, Py)) \leqslant F(d(x, y))],$$

where $\tau > 0$. Then P has a unique fixed point $x^* \in X$ and the sequence $\{P^n x\}_{n\in\mathbb{N}}$ converges to x^* for each $x \in X$.

Piri and Kumam [13] showed the independence of (F3) and (F3′). The next result was proved by Popescu and Gabrial [15] by generalizing the results in [3, 25].

Theorem 1.7 ([15]). *In a complete metric space* (X, d)*, let* P *be a self mapping. For* $\tau > 0$*, let* $x, y \in X$*,* $d(Px, Py) > 0$ *implies*

$$\tau + F(d(Px, Py)) \leqslant F(a_1.d(x, y) + a_2.d(x, Px) + a_3.d(y, Py) + a_4.d(x, Py) + a_5.d(y, Px)),$$

where the mapping $F : \mathbb{R}^+ \to \mathbb{R}$ *is increasing,* a_1, a_2, a_3, a_4, a_5 *are non-negative numbers,* $a_4 < 1/2, a_3 < 1, a_1 + a_2 + a_3 + 2a_4 = 1$*,* $0 < a_1 + a_4 + a_5 \leqslant 1$*. Then* P *has a unique fixed point* $x^* \in X$*, also the sequence* $\{P^n x\}_{n \in \mathbb{N}}$ *converges to* x^* *for each* $x \in X$*.*

Sedghi et al. [21] generalized metric space by introducing S-metric space.

Definition 1.8 ([21]). Let X be a non-empty set. An S-metric on X is a function $S : X \times X \times X \to [0, +\infty)$ that satisfies the following conditions.

1. $S(x, y, z) \geqslant 0$ for all $x, y, z \in X$;
2. $S(x, y, z) = 0$ if and only if $x = y = z$ for every $x, y, z \in X$;
3. $S(x, y, z) \leqslant S(x, x, a) + S(y, y, a) + S(z, z, a)$ for every $x, y, z, a \in X$.

The pair (X, S) is called an S-metric space.

Hussain et al. [7] defined the concept of parametric metric space.

Definition 1.9 ([7]). Let X be a non-empty set and let $P : X \times X \times (0, \infty) \to [0, \infty)$ be a function. P is called a parametric metric on X if,

1. (P1) $P(a, b, t) = 0$ if and only if $a = b$;
2. (P2) $P(a, b, t) = P(b, a, t)$;
3. (P3) $P(a, b, t) \leqslant P(a, x, t) + P(x, b, t)$,

for each $a, b, x \in X$ and all $t > 0$. The pair (X, P) is called a parametric metric space.

Definition 1.10 ([7]). Let (X, P) be a parametric metric space and let $\{a_n\}$ be a sequence in X.

(i) $\{a_n\}$ converges to x if and only if there exists $n_0 \in \mathbb{N}$ such that $P(a_n, x, t) < \epsilon$, for all $n \geqslant n_0$ and all $t > 0$; that is, $\lim_{n\to\infty} P(a_n, x, t) = 0$. It is denoted by $\lim_{n\to\infty} a_n = x$.
(ii) $\{a_n\}$ is called a Cauchy sequence if, for all $t > 0$, $\lim_{n\to\infty} P(a_n, a_m, t) = 0$.
(iii) (X, P) is called complete if every Cauchy sequence is convergent.

The concept of parametric S-metric was introduced by Rao et al. [18].

Definition 1.11 ([18, 24]). Let X be a non-empty set and let $P_s : X \times X \times X \times (0, \infty) \to [0, \infty)$ be a function. P_s is called a parametric S-metric on X if,

(PS1) $P_s(a, b, c, t) = 0$ if and only if $a = b = c$;
(PS2) $P_s(a, b, c, t) \leqslant P_s(a, a, x, t) + P_s(b, b, x, t) + P_s(c, c, x, t)$,

for each $a, b, c, x \in X$ and all $t > 0$. The pair (X, P_s) is called a parametric S-metric space.
Following examples and properties of parametric S-metric space was given by Tas and Ozgur [24].

Example 1.12 ([24]). Let $X = \big\{f | f : (0, \infty) \to \mathbb{R} \text{ be a function}\big\}$ and let the function $P_s : X \times X \times X \times (0, \infty) \to [0, \infty)$ be defined by

$$P_s(f, g, h, t) = |f(t) - h(t)| + |g(t) - h(t)|,$$

for each $f, g, h \in X$ and all $t > 0$. Then P_s is a parametric S-metric and the pair (X, P_s) is a parametric S-metric space.

Example 1.13 ([24])**.** Let $X = \mathbb{R}$ and let the function $P_s : X \times X \times X \times (0,\infty) \to [0,\infty)$ be defined by

$$Ps(a,b,c,t) = g(t)(|a-b| + |b-c| + |a-c|),$$

for each $a,b,c \in \mathbb{R}$ and all $t > 0$, where $g : (0,\infty) \to (0,\infty)$ is a continuous function. Then P_s is a parametric S-metric and the pair $(\mathbb{R}, P_s)$ is a parametric S-metric space.

Lemma 1.14 ([24])**.** *Let (X, P_s) be a parametric S-metric space. Then we have*

$$P_s(a,a,b,t) = P_s(b,b,a,t),$$

for each $a, b \in X$ and all $t > 0$.

Lemma 1.15 ([24])**.** *Let (X, P) be a parametric metric space and let the function $P_s^p : X \times X \times X \times (0,\infty) \to [0,\infty)$ be defined by*

$$P_s^p(a,b,c,t) = P(a,c,t) + P(b,c,t),$$

for each $a,b,c \in X$ and all $t > 0$. Then P_s^p is a parametric S-metric and the pair (X, P_s^p) is a parametric S-metric space.

Example 1.16 ([24])**.** Let $X = \mathbb{R}$ and let the function $P_s^p : X \times X \times X \times (0,\infty) \to [0,\infty)$ be defined by

$$P_s(a,b,c,t) = t(|a-c| + |a+c-2b|),$$

for each $a,b,c \in \mathbb{R}$ and all $t > 0$. Then P_s is a parametric S-metric and the pair $(\mathbb{R}, P_s)$ is a parametric S-metric space. We have $P_s \neq P_s^p$; that is P_S is not generated by any parametric metric P.

Example 1.17 ([24])**.** Let $X = \{f | f : (0,\infty) \to \mathbb{R}$ be a function$\}$ and let the function $P_s : X \times X \times X \times (0,\infty) \to (0,\infty)$ be defined by

$$P_s(f,g,h,t) = |e^{f(t)} - e^{h(t)}| + |e^{f(t)} + e^{h(t)} - 2e^{g(t)}|,$$

for each $f,g,h \in X$ and all $t > 0$. Then P_s is a parametric S-metric and the pair (X, P_s) is a parametric S-metric space. We have $P_s \neq P_s^p$; that is P_S is not generated by any parametric metric P.

Lemma 1.18 ([24])**.** *Let (X, P_s) be a parametric S- metric space and let the function $P : X \times X \times X \times (0,\infty) \to [0,\infty)$ be defined by*

$$P(a,b,t) = P_s(a,a,b,t),$$

for each $a, b \in X$ and all $t > 0$. Then P is a parametric b-metric and the pair (X, P) is a parametric b-metric space.

Definition 1.19 ([24])**.** Let (X, P_s) be a parametric metric space and let $\{a_n\}$ be a sequence in X.

(1) $\{a_n\}$ converges to x if and only if there exists $n_0 \in \mathbb{N}$ such that $P_s(a_n, a_n, x, t) < \epsilon$, for all $n \geqslant n_0$ and all $t > 0$; that is, $\lim_{n\to\infty} P_s(a_n, a_n, x, t) = 0$. It is denoted by $\lim_{n\to\infty} a_n = x$.

(2) $\{a_n\}$ is called a Cauchy sequence if, for all $t > 0$, $\lim_{n,m\to\infty} P_s(a_n, a_n, a_m, t) = 0$.

(3) (X, P_s) is called complete if every Cauchy sequence is convergent.

Lemma 1.20 ([24])**.** *Let (X, P_s) be a parametric S-metric space. If $\{a_n\}$ converges to x, then x is unique.*

Lemma 1.21 ([24])**.** *Let (X, P_s) be a parametric S-metric space. If $\{a_n\}$ converges to x, then $\{a_n\}$ is Cauchy.*

Corollary 1.22 ([24])**.** *Let (X, P) be a parametric metric space and let (X, P_s^p) be a parametric S-metric space, where P_s^p is generated by parametric metric P. Then we have the following:*

(1) *$\{a_n\} \to x$ in (X, P) if and only if $\{a_n\} \to x$ in (X, P_s^p).*
(2) *$\{a_n\}$ is Cauchy in (X, P) if and only if $\{a_n\}$ is Cauchy in (X, P_s^p).*
(3) *(X, P) is complete if and only if (X, P_s^p).*

2. Main results

Now we prove the following theorem.

Theorem 2.1. *Let* f *be a self-mapping of a complete parametric* S*-Metric space* X *into itself. Suppose that there exists* $\tau > 0$ *such that for all* $x, y \in X$ *and* $t > 0$, $P_s(fx, fx, fy, t) > 0$ *implies*

$$\begin{aligned}\tau + F(P_s(fx, fx, fy, t)) \leqslant F(&a_1 P_s(x, x, y, t) + a_2 P_s(x, x, fx, t) + a_3 P_s(y, y, fy, t)\\ &+ a_4 P_s(x, x, fy, t) + a_5 P_s(y, y, fx, t)),\end{aligned}$$

where $F\colon \mathbb{R}_+ \to \mathbb{R}$ *is an increasing mapping,* a_1, a_2, a_3, a_4, a_5 *are non negative numbers,* $a_1 + a_2 + a_3 + 3a_4 + a_5 \leqslant 1$. *Then* P *has a unique fixed point* $x^* \in X$ *and for every* $x \in X$, *the sequence* $\{f^n x\}_{n\in\mathbb{N}}$ *converges to* x^*.

Proof. Let $x_0 \in X$ be an arbitrary point and we construct a sequence $\{x_n\}_{n\in\mathbb{N}} \in X$ by

$$x_1 = fx_0,\ x_2 = fx_1 = f^2 x_0, \ldots, x_n = fx_{n-1} = f^n x_0,\ \forall n \in \mathbb{N}.$$

If there exists $n \in \mathbb{N} \cup \{0\}$ such that $P_s(x_n, x_n, fx_n, t) = 0$, then x_n is a fixed point of f and the proof is complete. Hence, we assume that

$$0 < P_s(x_n, x_n, fx_n, t) = P_s(fx_{n-1}, fx_{n-1}, fx_n, t),\ \forall n \in \mathbb{N}.$$

Now, let $(P_s)_n = P_s(x_n, x_n, x_{n+1}, t)$. By the hypothesis and the monotony of F, we have for all $n \in \mathbb{N}$,

$$\begin{aligned}\tau + F((P_s)_n) &= \tau + F(P_s(x_n, x_n, x_{n+1}, t))\\ &= \tau + F(P_s(fx_{n-1}, fx_{n-1}, fx_n, t))\\ &\leqslant F(a_1 P_s(x_{n-1}, x_{n-1}, x_n, t) + a_2 P_s(x_{n-1}, x_{n-1}, fx_{n-1}, t) + a_3 P_s(x_n, x_n, fx_n, t)\\ &\quad + a_4 P_s(x_{n-1}, x_{n-1}, fx_n, t) + a_5 P_s(x_n, x_n, fx_{n-1}, t))\\ &= F(a_1 P_s(x_{n-1}, x_{n-1}, x_n, t) + a_2 P_s(x_{n-1}, x_{n-1}, x_n, t) + a_3 P_s(x_n, x_n, x_{n+1}, t)\\ &\quad + a_4 P_s(x_{n-1}, x_{n-1}, x_{n+1}, t) + a_5 P_s(x_n, x_n, x_n, t))\\ &\leqslant F(a_1 (P_s)_{n-1} + a_2 (P_s)_{n-1} + a_3 (P_s)_n + a_4 2P_s(x_{n-1}, x_{n-1}, x_n, t)\\ &\quad + a_4 P_s(x_n, x_n, x_{n+1}, t) + a_5.0))\\ &\leqslant F((a_1 + a_2 + 2a_4)(P_s)_{n-1} + (a_3 + a_4)(P_s)_n).\end{aligned}$$

It follows that

$$\begin{aligned}F((P_s)_n) &\leqslant F((a_1 + a_2 + 2a_4)(P_s)_{n-1} + (a_3 + a_4)(P_s)_n) - \tau\\ &< F((a_1 + a_2 + 2a_4)(P_s)_{n-1} + (a_3 + a_4)(P_s)_n).\end{aligned} \tag{2.1}$$

So from the monotony of F, we get

$$(P_s)_n \leqslant (a_1 + a_2 + 2a_4)(P_s)_{n-1} + (a_3 + a_4)(P_s)_n$$

and hence

$$(1 - a_3 - a_4)(P_s)_n \leqslant (a_1 + a_2 + 2a_4)(P_s)_{n-1},\ \forall n \in \mathbb{N}.$$

Since $a_1 + a_2 + a_3 + 3a_4 + a_5 \leqslant 1$,

$$(P_s)_n \leqslant \frac{a_1 + a_2 + 2a_4}{1 - a_3 - a_4}(P_s)_{n-1} < (P_s)_{n-1},\ \forall n \in \mathbb{N},$$

and for all $t > 0$. Thus, we conclude that the sequence $\{(P_s)_n\}_{n\in\mathbb{N}}$ is strictly decreasing, so there exists $\lim\limits_{n\to\infty} (P_s)_n = P_s$. Suppose that $P_s > 0$. Since F is an increasing mapping, there exists $\lim\limits_{x\to S_+} F(x) = F(P_s + 0)$, so taking the limit as $n \to \infty$ in inequality (2.1), we get $F(P_s + 0) \leqslant F(P_s + 0) - \tau$, which is a contradiction. Therefore,

$$\lim_{n\to\infty} (P_s)_n = 0. \tag{2.2}$$

Now, we claim that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Arguing by contradiction, we assume that there exists $\varepsilon > 0$ and sequences $\{p(n)\}_{n\in\mathbb{N}}$ and $\{q(n)\}_{n\in\mathbb{N}}$ of natural numbers such that $p(n) > q(n) > n$,

$$P_s(x_{p(n)}, x_{p(n)}, x_{q(n)}, t) > \varepsilon, \quad P_s(x_{p(n)-1}, x_{p(n)-1}, x_{q(n)}, t) \leqslant \varepsilon, \quad \forall n \in \mathbb{N}.$$

Then, we have

$$\begin{aligned}\varepsilon < P_s(x_{p(n)}, x_{p(n)}, x_{q(n)}, t) &\leqslant 2P_s(x_{p(n)}, x_{p(n)}, x_{p(n)-1}, t) + P_s(x_{p(n)-1}, x_{p(n)-1}, x_{q(n)}, t)\\ &\leqslant 2P_s(x_{p(n)}, x_{p(n)}, x_{p(n)-1}, t) + \varepsilon.\end{aligned}$$

It follows from relation (2.2) and above inequality that

$$\lim_{n\to\infty} P_s(x_{p(n)}, x_{p(n)}, x_{q(n)}, t) = \varepsilon.$$

Since $P_s(x_{p(n)}, x_{p(n)}, x_{q(n)}, t) > \varepsilon > 0$, by the hypothesis and monotony of F, we have

$$\begin{aligned}
&\tau + F(P_s(x_{p(n)}, x_{p(n)}, x_{q(n)}, t))\\
&\quad = \tau + F(P_s(fx_{p(n)-1}, fx_{p(n)-1}, fx_{q(n)-1}, t))\\
&\quad \leqslant F(a_1P_s(x_{p(n)-1}, x_{p(n)-1}, x_{q(n)-1}, t) + a_2P_s(x_{p(n)-1}, x_{p(n)-1}, fx_{p(n)-1}, t)\\
&\qquad + a_3P_s(x_{q(n)-1}, x_{q(n)-1}, fx_{q(n)-1}, t) + a_4P_s(x_{p(n)-1}, x_{p(n)-1}, fx_{q(n)-1}, t)\\
&\qquad + a_5P_s(x_{q(n)-1}, x_{q(n)-1}, fx_{p(n)-1}, t))\\
&\quad = F(a_1P_s(x_{p(n)-1}, x_{p(n)-1}, x_{q(n)-1}, t) + a_2P_s(x_{p(n)-1}, x_{p(n)-1}, x_{p(n)}, t)\\
&\qquad + a_3P_s(x_{q(n)-1}, x_{q(n)-1}, x_{q(n)}, t) + a_4P_s(x_{p(n)-1}, x_{p(n)-1}, x_{q(n)}, t) + a_5P_s(x_{q(n)-1}, x_{q(n)-1}, x_{p(n)}, t))\\
&\quad \leqslant F(a_1(2P_s(x_{p(n)-1}, x_{p(n)-1}, x_{p(n)}, t) + P_s(x_{p(n)}, x_{p(n)}, x_{q(n)-1}, t)) + a_2(P_s)_{p(n)-1}\\
&\qquad + a_3(P_s)_{q(n)-1} + a_4(2P_s(x_{p(n)-1}, x_{p(n)-1}, x_{p(n)}, t) + P_s(x_{p(n)}, x_{p(n)}, x_{q(n)}, t))\\
&\qquad + a_5(2P_s(x_{q(n)-1}, x_{q(n)-1}, x_{q(n)}, t) + P_s(x_{q(n)}, x_{q(n)}, x_{p(n)}, t)))\\
&\quad \leqslant F(2a_1(P_s)_{p(n)-1} + a_1(2P_s(x_{p(n)}, x_{p(n)}, x_{q(n)}, t) + P_s(x_{q(n)}, x_{q(n)}, x_{q(n)-1}, t))\\
&\qquad + a_2(P_s)_{p(n)-1} + a_3(P_s)_{q(n)-1} + 2a_4(P_s)_{p(n)-1}\\
&\qquad + a_4P_s(x_{p(n)}, x_{p(x)}, x_{q(n)}, t) + 2a_5(P_s)_{q(n)-1} + a_5P_s(x_{p(n)}, x_{p(n)}, x_{q(n)}, t))\\
&\quad = F((2a_1 + a_4 + a_5)P_s(x_{p(n)}, x_{p(n)}, x_{q(n)}, t) + (2a_1 + a_2 + 2a_4)(P_s)_{p(n)-1} + (2a_1 + a_3 + 2a_5)(P_s)_{q(n)-1}).
\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we get

$$\tau + F(\varepsilon + 0) \leqslant F(\varepsilon + 0),$$

which is a Cauchy sequence. Since (X, P_s) is a complete parametric S-metric space, we have that $\{x_n\}_{n\in\mathbb{N}}$ converges to some point x^* in X. If there exists a sequence $\{p(n)\}_{n\in\mathbb{N}}$ of natural numbers such that $x_{p(n)+1} = fx_{p(n)} = fx^*$, then $\lim_{n\to\infty} x_{p(n)+1} = x^*$, so $fx^* = x^*$. Otherwise, there exists $N \in \mathbb{N}$ such that $x_{n+1} = fx_n \neq fx^*$, $\forall n \geqslant N$. Assume that $fx^* \neq x^*$. By the hypothesis, we have

$$\begin{aligned}\tau + F(P_s(fx_n, fx_n, fx^*, t)) \leqslant F(&a_1P_sx_n, x_n, x^*, t) + a_2P_s(x_n, x_n, fx_n, t) + a_3S(x^*, x^*, fx^*, t)\\ &+ a_4P_s(x_n, x_n, Px^*, t) + a_5P_s(x^*, x^*, fx^*, t)),\end{aligned}$$

so

$$\tau + F(P_s(x_{n+1}, x_{n+1}, fx^*, t)) \leqslant F(a_1P_s(x_n, x_n, x^*, t) + a_2P_s(x_n, x_n, x_{n+1}, t) + a_3P_s(x^*, x^*, fx^*, t)$$

$$+ a_4 P_s(x_n, x_n, fx^*, t) + a_5 P_s(x^*, x^*, x_{n+1}, t)).$$

Since F is increasing, we deduce that

$$\begin{aligned} P_s(x_{n+1}, x_{n+1}, fx^*, t) &\leqslant a_1 P_s(x_n, x_n, x^*, t) + a_2 P_s(x_n, x_n, x_{n+1}, t) + a_3 P_s(x^*, x^*, fx^*, t) \\ &\leqslant +a_4 P_s(x_n, x_n, fx^*, t) + a_5 P_s(x^*, x^*, x_{n+1}, t), \end{aligned}$$

so letting $n \to \infty$, we get

$$P_s(x^*, x^*, fx^*, t) \leqslant a_3 P_s(x^*, x^*, fx^*, t) + a_4 P_s(x^*, x^*, fx^*, t) = (a_3 + a_4) S(x^*, x^*, fx^*, t) < P_s(x^*, x^*, fx^*, t).$$

This is a contradiction. Therefore, $fx^* = x^*$. Now, we will show that f has a unique fixed point. Let $x, y \in X$ be two distinct fixed points of f. Thus, $fx = x \neq y = fy$. Hence, $P_s(fx, fx, fy, t) = P_s(x, x, y, t) > 0$. By the hypothesis, since $a_1 + a_2 + a_3 + 3a_4 + a_5 \leqslant 1$, we have

$$\begin{aligned} \tau + F(P_s(x, x, y, t)) &= \tau + F(P_s(fx, fy, fy, t)) \\ &\leqslant F(a_1 P_s(x, x, y, t) + a_2 P_s(x, x, fx, t,) + a_3 P_s(y, y, fy, t) \\ &\quad + a_4 P_s(x, x, fy, t) + a_5 P_s(y, y, fx, t)) \\ &= F(a_1 P_s(x, x, y, t) + a_3 P_s(x, x, y, t) + a_5 P_s(y, y, x, t)) \\ &= F((a_1 + a_3 + a_5) P_s(x, x, y, t)) \leqslant F(P_s(x, x, y, t)). \end{aligned}$$

This is a contradiction. Therefore, f has a unique fixed point. □

Corollary 2.2. *Let (X, P_s) be a complete parametric S-metric space and let f be a self mapping on X. Assume that there exists an increasing $F : \mathbb{R}^+ \to \mathbb{R}$, and $\tau > 0$ such that*

$$\tau + F(P_s(fx, fx, fy, t)) \leqslant F(a_1 P_s(x, x, y, t) + a_2 P_s(x, x, fx, t) + a_3 P_s(y, y, fy, t)), \ \forall x, y \in X, fx \neq fy,$$

where $a_1 + a_2 + a_3 \leqslant 1$. Then, f has a unique fixed point in X.

Acknowledgment

The authors thank the anonymous reviewers for the positive comments on the manuscript and recommending for publication in the journal.

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