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Developed analytical approach for a special kind of differential-difference equation: Exact solution

Laila F. Seddek[a,b,*], Abdelhalim Ebaid[c], Essam R. El-Zahar[a,d]

[a]Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam bin Abdulaziz University, P.O. Box 83, Al-Kharj 11942, Saudi Arabia.

[b]Department of Engineering Mathematics and Physics, Faculty of Engineering, Zagazig University, Zagazig, 44519, Egypt.

[c]Department of Mathematics, Faculty of Science, University of Tabuk, P.O. Box 741, Tabuk 71491, Saudi Arabia.

[d]Department of Basic Engineering Science, Faculty of Engineering, Menoufia University, Shebin El-Kom 32511, Egypt.

Abstract

This paper investigates a differential-difference equation with a variable coefficient of exponential order in the form$\phi'(t) = \alpha\phi(t) + \beta e^{\sigma t}\phi(-t)$. In literature, periodic solution has been obtained at the special case $\sigma = 0$. In this paper, an effective approach is developed to determine the exact solution in terms of exponential and trigonometric functions. In addition, the exact solution is expressed in terms of exponential and hyperbolic functions under specific conditions of the involved parameters. Exact solutions of several special cases are derived and found in full agreement with the corresponding results in the relevant literature. Some theoretical results are presented and proved which can be generalized to include other complex models. The behavior of the obtained shows periodicity in the absence of σ while the damped oscillations are shown graphically when σ is assigned to negative values.

Keywords: Ansatz method, differential-difference equation, exact solution.

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1. Introduction

This paper focuses on solving the functional equation:

$$\phi'(t) = \alpha\phi(t) + \beta e^{\sigma t}\phi(-t), \quad \phi(0) = \lambda, \tag{1.1}$$

where α, β, σ, and λ are real constants. At $\sigma = 0$ and $\gamma = -1$, the present model is a special case of the pantograph equation $\phi'(t) = \alpha\phi(t) + \beta\phi(\gamma t)$. The pantograph equation is of practical interest when studying the collected current in electric trains [1, 10, 14, 27–30, 34]. Another particular application is in astronomy, know as Ambartusmian-equation [3–6, 9, 12, 22, 26, 31, 32, 35], when $\alpha = -1$, $\beta = \gamma = 1/\xi$ $(\xi > 1)$ and $\sigma = 0$.

*Corresponding author
Email address: l.morad@psau.edu.sa (Laila F. Seddek)

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In the literature [2, 13, 16–18, 24, 36], the Adomian decomposition method (ADM) was widely applied to investigate several physical problems. The ADM needs to calculate the Adomian polynomials if the problem being solved is nonlinear. Beside, the homotopy perturbation method (HPM) [11, 20, 33] is also an effective tool to solve ODEs/PDEs. However, the HPM implements an auxiliary parameter. These methods, probably under certain canonical forms, lead to the same series solution when applying the standard series method (SSM) [23]. Although the ADM and the HPM can be applied to solve the current model, we prefer to develop a direct ansatz method to obtain the exact solution of in a straightforward manner.

The proposed ansatz method is based on expressing the solution in the form of an exponential function multiplied by the sum of two trigonometric functions. The coefficients of the trigonometric functions in addition to the exponent of the exponential function are to be determined in terms of the model's parameters λ, α, β, and σ. Furthermore, the solutions of several special cases are to be established in exact forms. These exact forms are valid at certain relationships between the parameters α, β, and σ. The advantage of this is that it generalizes previous results in the relevant literature.

2. Method of solution

This section proposes a direct ansatz method to exactly solve the model (1.1). The ansatz, to be developed, is assumed in the form:

$$\phi(t) = e^{\gamma t}\left[\mu_1 \cos(\theta_1 t) + \mu_2 \sin(\theta_2 t)\right], \tag{2.1}$$

where γ, μ_i, θ_i $(i = 1, 2)$ are unknowns, to be determined later. Eq. (2.1) gives

$$\begin{aligned}
\phi'(t) &= e^{\gamma t}[-\mu_1\theta_1 \sin(\theta_1 t) + \mu_2\theta_2 \cos(\theta_2 t)] + \gamma e^{\gamma t}\left[\mu_1 \cos(\theta_1 t) + \mu_2 \sin(\theta_2 t)\right], \\
&= e^{\gamma t}[(-\mu_1\theta_1 + \gamma\mu_2) \sin(\theta_1 t) + (\mu_2\theta_2 + \gamma\mu_1) \cos(\theta_2 t)], \qquad (2.2)\\
\phi(-t) &= e^{-\gamma t}\left[\mu_1 \cos(\theta_1 t) - \mu_2 \sin(\theta_2 t)\right]. \qquad (2.3)
\end{aligned}$$

Employing Eqs. (2.1)-(2.3) into Eq. (1.1), then

$$\begin{aligned}
e^{\gamma t}[(-\mu_1\theta_1 + \gamma\mu_2) \sin(\theta_1 t) + (\mu_2\theta_2 + \gamma\mu_1) \cos(\theta_2 t)] =& \alpha e^{\gamma t}\left[\mu_1 \cos(\theta_1 t) + \mu_2 \sin(\theta_2 t)\right] \\
&+ \beta e^{(\sigma-\gamma)t}\left[\mu_1 \cos(\theta_1 t) - \mu_2 \sin(\theta_2 t)\right],
\end{aligned}$$

or

$$\begin{aligned}
(-\mu_1\theta_1 + \gamma\mu_2) \sin(\theta_1 t) + (\mu_2\theta_2 + \gamma\mu_1) \cos(\theta_2 t) =& \alpha\mu_1 \cos(\theta_1 t) + \alpha\mu_2 \sin(\theta_2 t) \\
&+ \beta e^{(\sigma-2\gamma)t}\left[\mu_1 \cos(\theta_1 t) - \mu_2 \sin(\theta_2 t)\right].
\end{aligned}$$

Setting $\sigma - 2\gamma = 0$, then $\gamma = \sigma/2$ and hence

$$(-\mu_1\theta_1 + \gamma\mu_2) \sin(\theta_1 t) + (\mu_2\theta_2 + \gamma\mu_1) \cos(\theta_2 t) = (\alpha + \beta)\mu_1 \cos(\theta_1 t) + (\alpha - \beta)\mu_2 \sin(\theta_2 t). \tag{2.4}$$

Eq. (2.4) implies trivial solution for μ_i and θ_i, $i = 1, 2$. In order to avoid such situation, one can consider $\theta_1 = \theta_2 = \theta$. Accordingly, Eq. (2.4) becomes

$$\begin{aligned}
(-\mu_1\theta + \gamma\mu_2) \sin(\theta t) + (\mu_2\theta + \gamma\mu_1) \cos(\theta t) =& (\alpha + \beta)\mu_1 \cos(\theta t) \\
&+ (\alpha - \beta)\mu_2 \sin(\theta t).
\end{aligned}$$

Comparing both sides, we obtain the system:

$$\begin{aligned}
\mu_1\theta + \mu_2(\alpha - \beta - \gamma) &= 0, \qquad (2.5)\\
\mu_1(\alpha + \beta - \gamma) - \mu_2\theta &= 0. \qquad (2.6)
\end{aligned}$$

The system (2.5)-(2.6) contains three unknowns μ_1, μ_2, and θ. However, μ_1 can be directly obtained by applying the given condition $\phi(0) = \lambda$ on the ansatz (2.1), this gives $\mu_1 = \lambda$. Therefore, the system (2.5)-(2.6) reduces to

$$\lambda\theta + \mu_2(\alpha - \beta - \gamma) = 0,$$
$$\lambda(\alpha + \beta - \gamma) - \mu_2\theta = 0.$$

Solving this system for μ_2 and θ, we obtain

$$\mu_2^2 = \lambda^2 \left(\frac{\alpha + \beta - \gamma}{\gamma - \alpha + \beta}\right),$$

i.e.,

$$\mu_2 = |\lambda| \sqrt{\frac{\alpha + \beta - \gamma}{\gamma - \alpha + \beta}} = \begin{cases} \lambda\sqrt{\frac{\alpha+\beta-\gamma}{\gamma-\alpha+\beta}}, & \lambda > 0, \\ -\lambda\sqrt{\frac{\alpha+\beta-\gamma}{\gamma-\alpha+\beta}}, & \lambda < 0, \end{cases} \tag{2.7}$$

and

$$\theta = \frac{\gamma - \alpha + \beta}{\lambda}\mu_2 = \left(\frac{\gamma - \alpha + \beta}{\lambda}\right) |\lambda| \sqrt{\frac{\alpha + \beta - \gamma}{\gamma - \alpha + \beta}},$$

which gives

$$\theta = \begin{cases} \sqrt{\beta^2 - (\alpha - \gamma)^2}, & \lambda > 0, \\ -\sqrt{\beta^2 - (\alpha - \gamma)^2}, & \lambda < 0. \end{cases} \tag{2.8}$$

Inserting (2.7) and (2.8) into Eq. (2.1), then the solution reads

$$\phi(t) = \lambda\, e^{\gamma t} \left[\cos\left(\sqrt{\beta^2 - (\alpha - \gamma)^2}\, t\right) + \sqrt{\frac{\beta + \alpha - \gamma}{\beta - \alpha + \gamma}} \sin\left(\sqrt{\beta^2 - (\alpha - \gamma)^2}\, t\right)\right], \quad \left|\frac{\alpha - \gamma}{\beta}\right| < 1. \tag{2.9}$$

Here, it is noted that the choice of the +ve or the −ve sign in (2.7) and (2.8) gives the same expression in (2.9). Using $\gamma = \sigma/2$, consequently

$$\phi(t) = \lambda\, e^{\frac{1}{2}\sigma t} \left[\cos\left(\sqrt{\beta^2 - \left(\alpha - \frac{\sigma}{2}\right)^2}\, t\right) + \sqrt{\frac{\beta + \alpha - \frac{\sigma}{2}}{\beta - \alpha + \frac{\sigma}{2}}} \sin\left(\sqrt{\beta^2 - \left(\alpha - \frac{\sigma}{2}\right)^2}\, t\right)\right], \quad \left|\frac{\alpha - \frac{\sigma}{2}}{\beta}\right| < 1. \tag{2.10}$$

It is clear that the exact solution (2.10) satisfies the initial condition. Moreover, the solution (2.10) can be easily verified by direct substitution into the governing equation (1.1).

Remark 2.1. In the absence of σ, i.e., at $\sigma = 0$, the model (1.1) takes the form:

$$\phi'(t) = \alpha\phi(t) + \beta\phi(-t), \quad \phi(0) = \lambda, \tag{2.11}$$

which has been studied in Ref. [23]. In this case, the solution (2.10) reduces to

$$\phi(t) = \lambda \left[\cos\left(\sqrt{\beta^2 - \alpha^2}\, t\right) + \sqrt{\frac{\beta + \alpha}{\beta - \alpha}} \sin\left(\sqrt{\beta^2 - \alpha^2}\, t\right)\right], \quad \left|\frac{\alpha}{\beta}\right| < 1, \tag{2.12}$$

which is the same result obtained in Ref. [23], note that the solution (2.12) is periodic.

3. Behavior of the solution

To explore the behavior of the solution (2.10), we plotted in Figure 1 the curves of $\phi(t)$ at different values of σ ($\sigma = -3, -2, -1, 0$) when $\lambda = 1$, $\alpha = 1$ and $\beta = 3$. This figure shows the periodic property at $\sigma = 0$ and it also indicates the damped oscillations of $\phi(t)$ when $\sigma \neq 0$. The variation of the exact solution (2.10) at different values of α ($\alpha = -1/3, -1/4, 1/4, 1/3$) is depicted in Figure 2 when $\lambda = 1$, $\beta = 1$, and $\sigma = -1$. Figure 3 displays the curves of $\phi(t)$ at different values of β ($\beta = -3, -2, 2, 3$) when $\lambda = 1$, $\alpha = 1$, and $\sigma = -1$. The damped oscillations of $\phi(t)$ can be observed in Figures 2 and 3. Such behavior of damped oscillations is associated with the negative values of σ in these figures. However, another different behavior can be occurred if σ is assigned to positive values, where exponential growth or decay may be observed in this case.

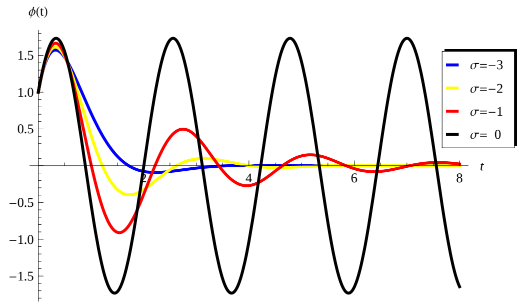

Figure 1: Plots of the exact solution $\phi(t)$, Eq. (2.10), when $\lambda = 1$, $\alpha = 1$, and $\beta = 3$ at different values of σ, $\sigma = -3, -2, -1, 0$.

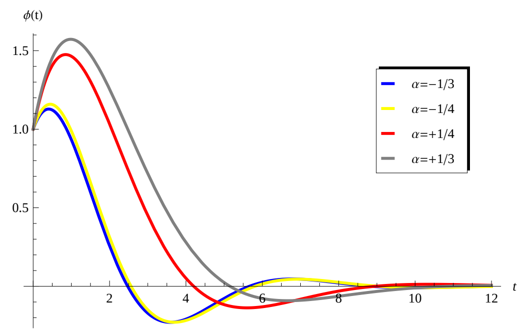

Figure 2: Plots of the exact solution $\phi(t)$, Eq. (2.10), when $\lambda = 1$, $\beta = 1$, and $\sigma = -1$ at different values of α, $\alpha = -1/3, -1/4, 1/4, 1/3$.

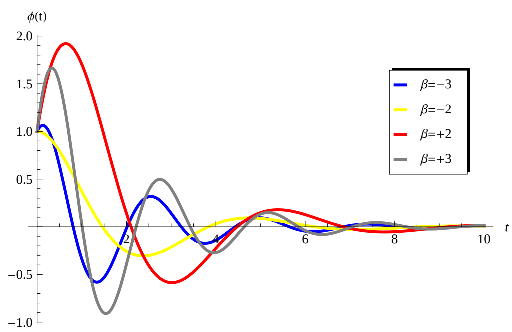

Figure 3: Plots of the exact solution $\phi(t)$, Eq. (2.10), when $\lambda = 1$, $\alpha = 1$, and $\sigma = -1$ at different values of β, $\beta = -3, -2, 2, 3$.

4. The solution in exponential/trigonometric compact form

A compact form for the exact solution (2.10) is to be derived in this section in terms of exponential and trigonometric functions. The theorem below discusses this issue and such compact form shall be compared, later, with the corresponding results in the literature at several cases.

Theorem 4.1. *A compact form for the exact solution* (2.10) *is*

$$\phi(t) = \lambda\sqrt{\frac{2\beta}{\beta-\alpha+\frac{\sigma}{2}}}\, e^{\frac{1}{2}\sigma t} \sin\left(\sqrt{\beta^2-\left(\alpha-\frac{\sigma}{2}\right)^2}\, t + \tan^{-1}\left(\sqrt{\frac{\beta-\alpha+\frac{\sigma}{2}}{\beta+\alpha-\frac{\sigma}{2}}}\right)\right),$$

provided that $\left|\frac{\alpha-\frac{\sigma}{2}}{\beta}\right| < 1$, $\beta-\alpha+\frac{\sigma}{2} \neq 0$.

Proof. Suppose that

$$\phi(t) = \omega\, e^{\frac{1}{2}\sigma t} \sin\left(\sqrt{\beta^2-\left(\alpha-\frac{\sigma}{2}\right)^2}\, t + \tau\right), \tag{4.1}$$

where ω and τ are to be determined. Expanding (4.1) gives

$$\phi(t) = \omega\, e^{\frac{1}{2}\sigma t} \left[\sin\left(\sqrt{\beta^2-\left(\alpha-\frac{\sigma}{2}\right)^2}\, t\right)\cos\tau + \cos\left(\sqrt{\beta^2-\left(\alpha-\frac{\sigma}{2}\right)^2}\, t\right)\sin\tau\right]. \tag{4.2}$$

Comparing (4.2) with (2.10), we get the system:

$$\omega\cos\tau = \lambda\sqrt{\frac{\beta+\alpha-\frac{\sigma}{2}}{\beta-\alpha+\frac{\sigma}{2}}},$$
$$\omega\sin\tau = \lambda.$$

Solving this for τ and ω, we obtain

$$\omega = \lambda\sqrt{\frac{2\beta}{\beta-\alpha+\frac{\sigma}{2}}}, \quad \tau = \tan^{-1}\left(\sqrt{\frac{\beta-\alpha+\frac{\sigma}{2}}{\beta+\alpha-\frac{\sigma}{2}}}\right). \tag{4.3}$$

Inserting (4.3) into (4.1) completes the proof. □

5. Special cases

This section addresses the exact solutions at some special relationships of the parameters α, σ, and β.

5.1. $\sigma = 2\alpha$

In this case, Eq. (1.1) becomes

$$\phi'(t) = \alpha\phi(t) + \beta e^{2\alpha t}\phi(-t), \quad \phi(0) = \lambda. \tag{5.1}$$

The solution is determined from Theorem 4.1 as

$$\phi(t) = \sqrt{2}\lambda e^{\alpha t}\sin\left(\beta t + \tan^{-1} 1\right),$$

i.e.,

$$\phi(t) = \sqrt{2}\lambda e^{\alpha t}\sin\left(\beta t + \frac{\pi}{4}\right) = \lambda e^{\alpha t}\left(\cos\beta t + \sin\beta t\right).$$

Remark 5.1. When $\alpha = 0$, Eq. (5.1) reduces to $\phi'(t) = \beta\phi(-t)$. The last solution becomes $\phi(t) = \lambda\left(\cos\beta t + \sin\beta t\right)$ which coincides with the same result in Ref. [23].

5.2. $\sigma = 2(\alpha+\beta)$

Eq. (1.1) becomes

$$\phi'(t) = \alpha\phi(t) + \beta e^{2(\alpha+\beta)t}\phi(-t), \quad \phi(0) = \lambda. \tag{5.2}$$

In this case, the solution is given by

$$\phi(t) = \lambda e^{(\alpha+\beta)t} \sin\left(\tan^{-1}\infty\right) = \lambda e^{(\alpha+\beta)t}, \tag{5.3}$$

Remark 5.2. For $\beta = -\alpha$, Eq. (5.2) becomes $\phi'(t) = \alpha[\phi(t) - \phi(-t)]$. The solution (5.3) transforms to the constant function $\phi(t) = \lambda$ whatever the value of α, which agrees with Ref. [23].

5.3. $\sigma = 2(\alpha-\beta)$

Setting $\sigma = 2(\alpha-\beta)$ into Eq. (1.1) gives

$$\phi'(t) = \alpha\phi(t) + \beta e^{2(\alpha-\beta)t}\phi(-t), \quad \phi(0) = \lambda. \tag{5.4}$$

In this case, the solution given by Theorem 4.1 becomes undetermined. Thus, the solution can be obtained via calculating as $\sigma \to 2(\alpha-\beta)$. To achieve this target, we assume that $\delta = \sigma - 2(\alpha-\beta)$. So, $\delta \to 0$ as $\sigma \to 2(\alpha-\beta)$. From Theorem 1, one can substitute $\sigma = \delta + 2(\alpha-\beta)$ and then calculate the limit as $\delta \to 0$, hence

$$\phi(t) = \lambda \lim_{\delta\to0} \sqrt{\frac{4\beta}{\delta}}\, e^{(\alpha-\beta+\delta/2)t} \,\sin\left(\sqrt{\beta\delta-\delta^2/4}\,t + \tan^{-1}\sqrt{\frac{\delta/2}{2\beta-\delta/2}}\right),$$

or

$$\phi(t) = 2\lambda\sqrt{\beta}\, e^{(\alpha-\beta)t} \lim_{\delta\to0} \frac{\sin\left(\sqrt{\beta\delta-\delta^2/4}\,t + \tan^{-1}\sqrt{\frac{\delta/2}{2\beta-\delta/2}}\right)}{\sqrt{\delta}},$$

i.e.,

$$\phi(t) = 2\lambda\sqrt{\beta}\, e^{(\alpha-\beta)t} \lim_{\delta\to0} \frac{\cos\left(\sqrt{\beta\delta-\delta^2/4}\,t + \tan^{-1}\sqrt{\frac{\delta/2}{2\beta-\delta/2}}\right)\frac{d}{d\delta}\left[\sqrt{\beta\delta-\delta^2/4}\,t + \tan^{-1}\sqrt{\frac{\delta/2}{2\beta-\delta/2}}\right]}{1/(2\sqrt{\delta})}.$$

The limit of the cosine term tends to one, then

$$\phi(t) = 2\lambda\sqrt{\beta}\, e^{(\alpha-\beta)t} \lim_{\delta\to0} \frac{\frac{d}{d\delta}\left[\sqrt{\beta\delta-\delta^2/4}\,t + \tan^{-1}\sqrt{\frac{\delta/2}{2\beta-\delta/2}}\right]}{1/(2\sqrt{\delta})}. \tag{5.5}$$

Eq. (5.5) is equivalent to

$$\phi(t) = 4\lambda\sqrt{\beta}\, e^{(\alpha-\beta)t} \lim_{\delta\to0} \sqrt{\delta}\frac{d}{d\delta}\left[\sqrt{\beta\delta-\delta^2/4}\,t + \tan^{-1}\sqrt{\frac{\delta/2}{2\beta-\delta/2}}\right].$$

Hence

$$\phi(t) = 4\lambda\sqrt{\beta}\, e^{(\alpha-\beta)t} \lim_{\delta\to0} \sqrt{\delta}\frac{d}{d\delta}\left[\sqrt{\beta\delta-\delta^2/4}\,t + \tan^{-1}\sqrt{\frac{\delta/2}{2\beta-\delta/2}}\right].$$

Thus

$$\phi(t) = 4\lambda\sqrt{\beta}\, e^{(\alpha-\beta)t} \lim_{\delta\to0} \sqrt{\delta}\left(\frac{(\beta-\delta/2)t}{2\sqrt{\beta\delta-\delta^2/4}} + \frac{1}{1+\frac{\delta/2}{2\beta-\delta/2}}\frac{d}{d\delta}\left[\sqrt{\frac{\delta/2}{2\beta-\delta/2}}\right]\right),$$

or

$$\phi(t) = 4\lambda\sqrt{\beta}\, e^{(\alpha-\beta)t} \lim_{\delta\to0} \sqrt{\delta}\left(\frac{(\beta-\delta/2)t}{2\sqrt{\beta\delta-\delta^2/4}} + \frac{2\beta-\delta/2}{2\beta} \times \frac{1}{2}\sqrt{\frac{2\beta-\delta/2}{\delta/2}}\frac{d}{d\delta}\left[\frac{\delta/2}{2\beta-\delta/2}\right]\right),$$

which is

$$\phi(t) = 4\lambda\sqrt{\beta}\, e^{(\alpha-\beta)t} \lim_{\delta\to 0} \sqrt{\delta} \left(\frac{(\beta-\delta/2)t}{2\sqrt{\beta\delta-\delta^2/4}} + \frac{\sqrt{2}(2\beta-\delta/2)}{4\beta} \times \sqrt{\frac{2\beta-\delta/2}{\delta}} \left[\frac{\beta}{(2\beta-\delta/2)^2} \right] \right).$$

Simplifying the last equation leads to

$$\phi(t) = 4\lambda\sqrt{\beta}\, e^{(\alpha-\beta)t} \lim_{\delta\to 0} \sqrt{\delta} \left(\frac{(\beta-\delta/2)t}{2\sqrt{\beta\delta-\delta^2/4}} + \frac{\sqrt{2}}{4\sqrt{\delta}(2\beta-\delta/2)^{1/2}} \right),$$

which finally gives

$$\phi(t) = \lambda\, e^{(\alpha-\beta)t}\,(2\beta t+1)\,.$$

This is the exact solution of the model (5.4) which can be easily verified by direct substitution. Moreover, the case $\beta = \alpha$ implies $\phi(t) = \lambda\,(2\alpha t+1)$ which has been derived in Ref. [23] for the corresponding equation: $\phi'(t) = \alpha\,[\phi(t)+\phi(-t)]$.

6. The solution in exponential/hyperbolic form

This section finds an explicit form for the exact solution in terms of exponential and hyperbolic functions.

Theorem 6.1. *For $0 < \beta < \alpha - \frac{\sigma}{2}$, the solution takes the form:*

$$\phi(t) = \lambda\, e^{\frac{1}{2}\sigma t} \left[\cosh\left(\sqrt{\left(\alpha-\frac{\sigma}{2}\right)^2 - \beta^2}\; t \right) + \sqrt{\frac{\beta+\alpha-\frac{\sigma}{2}}{\alpha-\beta-\frac{\sigma}{2}}}\, \sinh\left(\sqrt{\left(\alpha-\frac{\sigma}{2}\right)^2 - \beta^2}\; t \right) \right],$$

provided that $\alpha-\beta-\frac{\sigma}{2} \neq 0$.

Proof. The assumption $0 < \beta < \alpha - \frac{\sigma}{2}$ implies that $\beta^2 - \left(\alpha-\frac{\sigma}{2}\right)^2 < 0$, $\beta+\alpha-\frac{\sigma}{2} > 0$, and $\beta-\alpha+\frac{\sigma}{2} < 0$. This yields $\sqrt{\beta^2-\left(\alpha-\frac{\sigma}{2}\right)^2} = i\sqrt{\left(\alpha-\frac{\sigma}{2}\right)^2-\beta^2}$ and $\sqrt{\frac{\beta+\alpha-\frac{\sigma}{2}}{\beta-\alpha+\frac{\sigma}{2}}} = -i\sqrt{\frac{\beta+\alpha-\frac{\sigma}{2}}{\alpha-\beta-\frac{\sigma}{2}}}$, where $i = \sqrt{-1}$. Hence, one can rewrite Eq. (2.10) as

$$\phi(t) = \lambda\, e^{\frac{1}{2}\sigma t} \left[\cos\left(i\sqrt{\left(\alpha-\frac{\sigma}{2}\right)^2 - \beta^2}\; t \right) - i\sqrt{\frac{\beta+\alpha-\frac{\sigma}{2}}{\alpha-\beta-\frac{\sigma}{2}}}\, \sin\left(i\sqrt{\left(\alpha-\frac{\sigma}{2}\right)^2 - \beta^2}\; t \right) \right].$$

On applying the identities $\cos(ix) = \cosh(x)$ and $\sin(ix) = i\sinh(x)$, then

$$\phi(t) = \lambda\, e^{\frac{1}{2}\sigma t} \left[\cosh\left(\sqrt{\left(\alpha-\frac{\sigma}{2}\right)^2 - \beta^2}\; t \right) + \sqrt{\frac{\beta+\alpha-\frac{\sigma}{2}}{\alpha-\beta-\frac{\sigma}{2}}}\, \sinh\left(\sqrt{\left(\alpha-\frac{\sigma}{2}\right)^2 - \beta^2}\; t \right) \right],$$

which completes the proof. □

Lemma 6.2. *For $0 < \beta < \frac{\sigma}{2} - \alpha$, the solution takes the form:*

$$\phi(t) = \lambda\, e^{\frac{1}{2}\sigma t} \left[\cosh\left(\sqrt{\left(\frac{\sigma}{2}-\alpha\right)^2 - \beta^2}\; t \right) - \sqrt{\frac{\frac{\sigma}{2}-\alpha-\beta}{\frac{\sigma}{2}-\alpha+\beta}}\, \sinh\left(\sqrt{\left(\frac{\sigma}{2}-\alpha\right)^2 - \beta^2}\; t \right) \right],$$

provided that $\frac{\sigma}{2}-\alpha+\beta \neq 0$.

Proof. Let $0 < \beta < \frac{\sigma}{2} - \alpha$, then $\beta^2 - \left(\frac{\sigma}{2} - \alpha\right)^2 < 0$, $\beta + \alpha - \frac{\sigma}{2} < 0$, and $\beta - \alpha + \frac{\sigma}{2} > 0$. This yields $\sqrt{\beta^2 - \left(\alpha - \frac{\sigma}{2}\right)^2} = i\sqrt{\left(\alpha - \frac{\sigma}{2}\right)^2 - \beta^2} = i\sqrt{\left(\frac{\sigma}{2} - \alpha\right)^2 - \beta^2}$ and $\sqrt{\frac{\beta+\alpha-\frac{\sigma}{2}}{\beta-\alpha+\frac{\sigma}{2}}} = i\sqrt{\frac{\frac{\sigma}{2}-\alpha-\beta}{\frac{\sigma}{2}-\alpha+\beta}}$. Therefore, Eq. (2.10) gives

$$\phi(t) = \lambda\, e^{\frac{1}{2}\sigma t}\left[\cos\left(i\sqrt{\left(\frac{\sigma}{2} - \alpha\right)^2 - \beta^2}\; t\right) + i\sqrt{\frac{\frac{\sigma}{2} - \alpha - \beta}{\frac{\sigma}{2} - \alpha + \beta}}\sin\left(i\sqrt{\left(\frac{\sigma}{2} - \alpha\right)^2 - \beta^2}\; t\right)\right].$$

which directly gives the result of this lemma. □

Remark 6.3. Applying Theorem 6.1 when $\sigma = 0$, we obtain

$$\phi(t) = \lambda\left[\cosh\left(\sqrt{\alpha^2 - \beta^2}t\right) + \sqrt{\frac{\beta + \alpha}{\alpha - \beta}}\sinh\left(\sqrt{\alpha^2 - \beta^2}t\right)\right], \quad \beta < \alpha,$$

which is the same hyperbolic form obtained in Ref. [23].

7. Conclusions

In this work, the differential-difference equation with a variable coefficient in the form $\phi'(t) = \alpha\phi(t) + \beta e^{\sigma t}\phi(-t)$ was solved utilizing a developed effective approach. The exact solution was obtained in terms of exponential and trigonometric functions which was also re-expressed in terms of exponential and hyperbolic functions under specific conditions of the involved parameters. In absence of the exponential term, i.e., at $\sigma = 0$, the periodic solution in the literature was recovered as a special case of the present results. Moreover, the solutions of several special cases in the relevant literature were derived through the current analysis. Theoretical theorems were proved which describes the nature of the obtained exact solutions. Damped oscillations are shown graphically when σ is assigned to negative values. The simplicity of our approach allows possible generalization to include other complex models. Finally, the proposed method is applicable to other kinds of differential difference equations that follow the same structure of the present equation.

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