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Fractional Mercer's Hermite–Hadamard type inequalities in the frame of interval analysis and its applications to matrix

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Abstract

In this paper, we aim to discuss some fractional Hermite–Hadamard (H–H)-Mercer inequality for interval-valued functions via generalized fractional integral operator (GFIO). In addition, we investigate some new variants of the H–H-Mercer inequality pertaining to GFIO. A few examples are also provided to back up our claims. The findings potentially shed fresh light on a wide range of integral inequalities for fractional fuzzy in the frame of interval analysis and the optimization challenges they present. Finally, applications involving matrices are demonstrated.

Keywords: Convex function, H–H-Mercer inequality, interval-valued function, generalized fractional integral operator.

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1. Introduction

Convex inequalities are mathematical inequalities involving convex functions. A convex inequality is similar to the definition of a convex function, but it applies to the inequalities formed by these functions. In order to design constraints that limit the viable region to convex sets, convex inequalities are crucial in optimization issues. Convexity is well known to play a significant and critical role in a range of domains such as economics, finance, optimization, game theory, statistical theory, quality management,

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and numerous sciences. Due to the wide range of uses for this idea, it has been expanded upon and generalized in several ways. This theory has been the center and driving force of remarkable mathematical study for more than a century. The topic convexity with the aid of the idea of optimization has a amazing impact on many field of applied sciences, including control systems [4], mathematical optimization for modeling [37, 51], estimation and signal processing [28], finance [19] and data analysis and computer science [14].

Inequalities have an amazing mathematical tools due to their importance in fractional calculus, traditional calculus, quantum calculus, stochastic, time-scale calculus, fractal sets, and other fields. The crucial mathematical tool that connects integrals and inequalities, integral inequalities, provides insights into the behavior of functions over particular intervals. These inequalities offer a flexible method for comprehending development patterns, convergence characteristics, and function approximations. They are used in a variety of disciplines, including physics, engineering, economics, information technology, and probability theory. For the applications, see the references [2, 7, 9, 15, 16, 32, 43]. Integral inequalities make it easier to estimate values that could be difficult to explicitly compute via bounding functions. They also prove the convergence of series and sequences, the stability of differential equation solutions, and the existence of optimal solutions in optimization problems. These mathematical tools provide a strong and elegant framework for problem-solving and analysis due to their capacity to make connections between integrals and inequalities.

Fractional calculus, which focuses on fractional integration across complex domains, has recently acquired popularity due to its practical applications and has piqued the curiosity of mathematicians. A fractional Hermite-Hadamard inequality was presented by Sarikaya et al. [46]. The research of well-known inequalities such as Ostrowski, Simpson, and Hadamard inspired the study of fractional integral inequality. Transform theory, engineering, modeling, finance, mathematical biology, fluid flow, natural phenomenon prediction, healthcare, and image processing are all domains where fractional calculus is used. The references [1, 10, 11, 24] provide further insights into this topic.

The aim and novelty of this work is to introduced the new variant of H-H-Mercer type inequality involving GFIO in the frame of interval analysis. Further, we constructed some matrix applications via GFIO.

The construction of this paper in the following ways. First of all, in Section 2, we add some recognized definitions, theorems and remarks because all these are required in upcoming subsequent sections. In Section 3, we add some notations for interval analysis as well as the background information. In Section 4, we investigate some new variants of H-H-Mercer type inequality involving convex interval-valued functions via GFIO. Next, in Section 5, we offer some applications in the manner of constructed results. Finally, in Section 6, we add a conclusion.

2. Preliminaries

In this section, it would be appropriate to concentrate and examine on a few definitions, remarks, and theorems for the readers' attention and paper quality.

Jensen first time introduced the term convexity in the following manner:

Definition 2.1 ([23, 39])**.** A function $\mathfrak{T} : [\mathfrak{a}, \mathfrak{b}] \to \mathbb{R}$ is said to be convex, if

$$\mathfrak{T}\left(\wp\pi_1 + (1-\wp)\pi_2\right) \leqslant \wp\mathfrak{T}\left(\pi_1\right) + (1-\wp)\mathfrak{T}\left(\pi_2\right),$$

where $\pi_1, \pi_2 \in [\mathfrak{a}, \mathfrak{b}]$ and $\wp \in [0, 1]$.

The Hermite–Hadamard inequality [21] states that, if $\mathfrak{T} : \mathfrak{I} \to \mathbb{R}$ is convex for $\pi_1, \pi_2 \in \mathfrak{I}$ and $\pi_2 > \pi_1$, then

$$\mathfrak{T}\left(\frac{\pi_1+\pi_2}{2}\right) \leqslant \frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\mathfrak{T}(\wp)d\wp \leqslant \frac{\mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2)}{2}. \tag{2.1}$$

In the literature, there are numerous attractive inequalities on the topic convexity, particularly Jensen's inequality standing out. This inequality, which can be demonstrated under relatively basic conditions, is often used by scholars in subjects such as inequality theory and information theory. The statement of Jensen's inequality as follows.

Assume that $0 < \zeta_1 \leqslant \zeta_2 \leqslant \ldots \leqslant \zeta_n$ and $\rho = (\rho_1, \rho_2, \ldots, \rho_n)$ be non-negative weights such that $\sum_{j=1}^{n} \rho_j = 1$. Jensen inequality asserts that if $\mathfrak{T}$ is convex on the closed interval $[\pi_1, \pi_2]$, then

$$\mathfrak{T}\left(\sum_{j=1}^{n} \rho_j\ \zeta_j\right) \leqslant \left(\sum_{j=1}^{n} \rho_j\ \mathfrak{T}\left(\zeta_j\right)\right),$$

$\forall\ \rho_j \in [0,1]\,,\ \zeta_j \in [\pi_1, \pi_2]$ and $\quad (j = 1, 2, \ldots, n)$.

This inequality have a lot of uses in information theory (see [25]).

However a lot of scholars have concentrated on Jensen's inequality, the modification proposed by Mercer is the most compelling and special among them. Mercer [33] was investigated a new version of Jensen's inequality in 2003, which is states that:

If $\mathfrak{T}$ is a convex function on the closed interval $[\pi_1, \pi_2]$, then

$$\mathfrak{T}\left(\pi_1 + \pi_2 - \sum_{j=1}^{n} \rho_j\ \zeta_j\right) \leqslant \mathfrak{T}(\pi_1) + \mathfrak{T}(\pi_2) - \sum_{j=1}^{n} \rho_j\ \mathfrak{T}\left(\zeta_j\right),$$

holds $\forall\ \rho_j \in [0,1]\,,\ \zeta_j \in [\pi_1, \pi_2]$ and $\quad (j = 1, 2, \ldots, n)$.

Pečarić et al. constructed plenty modifications on the very captivating topic Jensen-Mercer operator inequalities [30]. After Pečarić, Niezgoda [40] have presented the new variants of Mercer's type inequalities for higher dimensions. Because of its prominent characterizations, Jensen-Mercer's type inequality has recently made a substantial addition to inequality theory. Kian [26] analyzed and considered the concept of Jensen inequality pertaining to superquadratic functions.

In [27], Kian and Moslehian demonstrated the following H–H-Mercer inequality:

$$\begin{aligned}
\mathfrak{T}\left(\pi_1 + \pi_2 - \frac{u_1 + u_2}{2}\right) &\leqslant \frac{1}{u_2 - u_1}\int_{u_1}^{u_2} \mathfrak{T}(\pi_1 + \pi_2 - \wp)\, d\wp \\
&\leqslant \frac{\mathfrak{T}(\pi_1 + \pi_2 - u_1) + \mathfrak{T}(\pi_1 + \pi_2 - u_2)}{2} \\
&\leqslant \mathfrak{T}(\pi_1) + \mathfrak{T}(\pi_2) - \frac{\mathfrak{T}(u_1) + \mathfrak{T}(u_2)}{2}.
\end{aligned}$$

In above inequality $\mathfrak{T}$ represent a convex function on closed interval $[\pi_1, \pi_2]$. For the amazing literature regarding above inequality, one can refer [8, 36, 40, 54].

Even though the literature of fractional calculus is as old as that of classical calculus, it has recently received increasing attention from researchers. Because of its applications to practical issues, its applications to engineering fields, its structure that is flexible, and the additional dimensions it gives to mathematical theories, fractional analysis is constantly working to advance. Examining the new operators closely reveals a number of properties, including singularity, locality, generalization, and variations in the kernel structures. Although generalization and inference are the cornerstones of mathematical methods, the new fractional operators' many properties, particularly the time memory effect, provide additional features to problem solutions. As a result, Caputo-Fabirizio, non-conformable, Raina, conformable, Prabhakar, Hilfer, Riemann-Liouville, Grunwald Letnikov, and Katugampola are some of the renowned operators that demonstrate the promise of fractional assessment. Now we'll proceed on to the generalized fractional integral operators, which have a unique place throughout these operators.

Currently, certain mathematicians have been fascinated with the notion of a fractional derivative. Nonlocal fractional derivatives are classified into two types: the Riemann-Liouville and Caputo derivatives have singular kernels, and the Caputo-Fabrizio and Atangana-Baleanu derivatives have non-singular kernels. Fractional derivative operators with non-singular kernels are very efficient at adressing non-locality in real-world applications.

Definition 2.2. [38] Suppose $\mathfrak{T} \in \mathcal{L}[\mathfrak{a}, \mathfrak{b}]$. The left-sided and right-sided Riemann-Liouville fractional integrals of order $\wp > 0$ defined by

$$ {}_{\mathfrak{a}}\mathfrak{J}^{\wp}\mathfrak{T}(\tau) = \frac{1}{\Gamma(\wp)} \int_{\mathfrak{a}}^{\tau} (\tau - \mu)^{\wp - 1}\mathfrak{T}(\mu)d\mu, \qquad \mathfrak{a} < \tau, \tag{2.2} $$

and

$$ \mathfrak{J}^{\wp}_{\mathfrak{b}}\mathfrak{T}(\tau) = \frac{1}{\Gamma(\wp)} \int_{\tau}^{\mathfrak{b}} (\mu - \tau)^{\wp - 1}\mathfrak{T}(\mu)d\mu, \qquad \tau < \mathfrak{b}. \tag{2.3} $$

Gamma function is defined as $\Gamma(\wp) = \int_0^\infty e^{-u}u^{\wp-1}du$.

Jarad et al. [22] investigated the following generalized fractional integral operator:

$$ {}^{\xi}_{\mathfrak{a}}\mathfrak{J}^{\wp}\mathfrak{T}(x) = \frac{1}{\Gamma(\xi)} \int_{\mathfrak{a}}^{x} \left(\frac{(x-\mathfrak{a})^{\wp} - (\wp - \mathfrak{a})^{\wp}}{\wp} \right)^{\xi - 1} \frac{\mathfrak{T}(\wp)}{(\wp - \mathfrak{a})^{1-\wp}}\, d\wp, \tag{2.4} $$

and

$$ {}^{\xi}\mathfrak{J}^{\wp}_{\mathfrak{b}}f(x) = \frac{1}{\Gamma(\xi)} \int_{x}^{\mathfrak{b}} \left(\frac{(\mathfrak{b}-x)^{\wp} - (\mathfrak{b} - \wp)^{\wp}}{\wp} \right)^{\xi - 1} \frac{\mathfrak{T}(\wp)}{(\mathfrak{b} - \wp)^{1-\wp}}\, d\wp, \tag{2.5} $$

where $\mathfrak{b} > \mathfrak{a}, \wp \in [0, 1]$.

Recently, a lot of integral inequalities have been investigated and examined via interval-valued function (IVF). For literature, see [6, 12, 13, 17, 18, 52].

3. Calculus on interval and inequalities

This section presents interval analysis symbols as well as background knowledge.

Symbol	**Meaning**
$L(\Pi_1) = \overline{\Pi}_2 - \underline{\Pi_2}$	length of the interval
$\mathcal{I}_c^+$	Sets of all closed positive intervals
$\mathcal{I}_c^-$	closed negative intervals
$\mathcal{P}_G(\Pi_1, \Pi_2)$	Pompeiu Hausdroff distance
$(\mathcal{I}_c, \mathcal{P}_G)$	complete metric space
$IR_{([\pi_1, \pi_2])}$	sets of R-L integrable interval-valued functions
$R_{([\pi_1, \pi_2])}$	real-valued functions on closed interval
CIVF	convex interval-valued function
IVF	interval-valued function
S	Riemann sum

The basic interval arithmetic and scalar multiplication operations for Π_1 and Π_2 are following,

$$ \begin{aligned} &\Pi_1 + \Pi_2 = \underline{\Pi_2} + \underline{\xi}, \overline{\Pi}_2 + \overline{\xi}. \\ &\Pi_1 - \Pi_2 = \underline{\Pi_2} - \underline{\xi}, \overline{\Pi}_2 - \overline{\xi}. \\ &\Pi_1/\Pi_2 = [\min\mathcal{Q}, \max\mathcal{Q}], \quad where \quad \mathcal{Q} = \{\underline{\Pi_2}/\underline{\xi}, \underline{\Pi_2}/\overline{\xi}, \overline{\Pi}_2/\underline{\xi}, \overline{\Pi}_2/\overline{\xi}\}; 0 \neq \Pi_2. \\ &\Pi_1.\Pi_2 = [\min\mathcal{U}, \max\mathcal{U}], \quad where \quad \mathcal{U} = \{\underline{\Pi_2}\underline{\xi}, \underline{\Pi_2}\overline{\xi}, \overline{\Pi}_2\underline{\xi}, \overline{\Pi}_2\overline{\xi}\}. \end{aligned} $$

$$\gamma\Pi_1 = \gamma[\underline{\Pi_2}, \bar{\Pi}_2] = \begin{cases} [\gamma\underline{\Pi_2}, \gamma\bar{\Pi}_2], & \gamma > 0; \\ \{0\}, & \gamma = 0; \\ [\gamma\underline{\Pi_2}, \gamma\bar{\Pi}_2], & \gamma < 0. \end{cases}$$

Definition 3.1. For $\Pi_1, \Pi_2 \in \mathcal{J}_c$, we present the G-difference of Π_1, Π_2 as $\coprod \in \mathcal{J}_c$, for which we have

$$\Pi_1 \ominus_g \Pi_2 = \coprod \Leftrightarrow \begin{cases} \text{(i)} \ \ \Pi_1 = \Pi_2 + \coprod, \\ \text{or} \\ \text{(ii)} \ \ \Pi_2 = \Pi_2 + (-\coprod). \end{cases}$$

It appears beyond controversy that

$$\Pi_1 \ominus_g \Pi_2 = \begin{cases} [\underline{\Pi_2} - \xi, \bar{\Pi}_2 - \bar{\xi}], & \text{if} \ \ L(\Pi_1) \geqslant L(\Pi_2), \\ [\bar{\Pi}_2 - \bar{\xi}, \underline{\Pi_2} - \underline{\xi}], & \text{if} \ \ L(\Pi_1) \leqslant L(\Pi_2), \end{cases}$$

where $L(\Pi_1) = \bar{\Pi}_2 - \underline{\Pi_2}$ and $L(\Pi_2) = \bar{\xi} - \underline{\xi}$. The operation specifications offer a large number of attributes, enabling $\mathcal{J}_c$ to represent a space of quasi-linear functions (see [31]).

Some of these qualities are as follows (see [29, 31, 34]):

1. (Law of associative under +) $(\Pi_1 + \Pi_2) + \Upsilon = \Pi_1 + (\Pi_2 + \Upsilon)$; for all $\Pi_1, \Pi_2, \Upsilon \in \mathcal{J}_c$.
2. (Law of associative under $\times$) $(\Pi_1.\Pi_2).\Upsilon = \Pi_1.(\Pi_2.\Upsilon)$; for all $\Pi_1, \Pi_2, \Upsilon \in \mathcal{J}_c$.
3. (Additive element) $\Pi_1 + 0 = 0 + \Pi_1 = \Pi_1$; for all $\Pi_1 \in \mathcal{J}_c$.
4. (Multiplicative element) $\Pi_1.1 = 1.\Pi_1 = \Pi_1$; for all $\Pi_1 \in \mathcal{J}_c$.
5. (Law of commutative under +) $\Pi_1 + \Pi_2 = \Pi_2 + \Pi_1$; for all $\Pi_1, \Pi_2 \in \mathcal{J}_c$.
6. (Law of commutative under $\times$) $\Pi_1.\Pi_2 = \Pi_2.\Pi_1$; for all $\Pi_1, \Pi_2 \in \mathcal{J}_c$.
7. (Law of cancellation under +) $\Pi_1 + \Upsilon = \Pi_2 + \Upsilon \Rightarrow \Pi_1 = \Pi_2$; for all $\Pi_1, \Pi_2, \Upsilon \in \mathcal{J}_c$.
8. (The first law of distributive) $\lambda(\Pi_1 + \Pi_2) = \lambda\Pi_1 + \lambda\Pi_2$; for all $\Pi_1, \Pi_2 \in \mathcal{J}_c, \lambda \in \mathbb{R}$.
9. (The second law of distributive) $(\lambda + \gamma)\Pi_1 = \lambda\Pi_1 + \gamma\Pi_1$; for all $\Pi_1 \in \mathcal{J}_c, \text{for all } \lambda, \gamma \in \mathbb{R}$.

According to the preceding, the distributive principle usually is applicable to intervals. As an example, $\Pi_1 = [1, 2]$, $\Pi_2 = [2, 3]$ and $\Upsilon = [-2, -1]$. We have

$$\Pi_1.(\Pi_2 + \Upsilon) = [0, 4],$$

where

$$\Pi_1.\Pi_2 + \Pi_1.\Upsilon) = [-2, 5].$$

Moore [34] investigated the Riemann integral for functions in the frame of interval values. Bhurjee and Panda [3] built an approach to seek effective solutions to a huge multi-objective fractional programming problems with interval parameters in the objective functions and restrictions. Zhang et al. [50] modified the concept of invex and preinvex analysis to interval-valued function. Zhao et al. [53] investigated the interval double integral and presented integral inequalities namely Chebyshev-type for interval-valued functions. Interval analysis have potential applicable uses in beam physics, global optimization, computer graphics, signal processing, robotics, chemical engineering, economics, control circuit design and error analysis. Budak et al. [6] investigated the interval-valued right-sided R–L fractional integral and constructed a new variants of H-H-type inequalities for such integrals. Sharma et al. [48] defined preinvexity in the frame of interval analysis and presented refinements of fractional H-H-type inequalities for them.

Theorem 3.2 ([35])**.** *Let a function of interval-valued* $\mathfrak{T} : [\pi_1, \pi_2] \to \mathcal{J}_c$ *with* $\mathfrak{T}(\wp) = [\underline{\mathfrak{T}}(\wp), \bar{\mathfrak{T}}(\wp)]$. *The mapping* $\mathfrak{T} \in IR_{([\pi_1, \pi_2]} \Leftrightarrow \underline{\mathfrak{T}}(\wp), \bar{\mathfrak{T}}(\wp) \in IR_{([\pi_1, \pi_2]}$ *and*

$$(IR)\int_{\pi_1}^{\pi_2} \mathfrak{T}(\wp)\, d\wp = \left[(R)\int_{\pi_1}^{\pi_2} \underline{\mathfrak{T}}(\wp)\, d\wp, (R)\int_{\pi_1}^{\pi_2} \bar{\mathfrak{T}}(\wp)\, d\wp\right].$$

In ([52], [55]), Zhao et al. investigated CIVF:

Definition 3.3. $\forall\ \wp_1, \wp_2 \in [\pi_1, \pi_2]$ and $\nu \in (0,1)$, a mapping $\mathfrak{T} : [\pi_1, \pi_2] \to \mathfrak{J}_c^+$ is h-convex stated as

$$h(\nu)\mathfrak{T}(\wp_1) + h(1-\nu)\mathfrak{T}(\wp_2) \subseteq \mathfrak{T}(\nu\wp_1 + (1-\nu)\wp_2), \tag{3.1}$$

where $h : [c, d] \to \mathbb{R}$ is a nonnegative mapping with $h \neq 0$ and $(0, 1) \subseteq [c, d]$. We denote the set of all h-convex interval-valued functions by $SX(h, [\pi_1, \pi_2], \mathfrak{J}_c^+)$.

If $h(\nu) = \nu$ in above inequality (3.1), then we attain the standard idea of CIVF (see[45]). If $h(\nu) = \nu^s$ in inequality (3.1), then we attain the standard definition of s-CIVF (see [5]).

Zhao et al. [52] utilized the idea of the h-convexity in the frame of interval analysis and presented the following H-H inequality:

Theorem 3.4. *If* $h \in SX(h, [\pi_1, \pi_2], \mathfrak{J}_c^+)$ *and* $h(\frac{1}{2}) \neq 0$, *then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\mathfrak{T}\left(\frac{\pi_1+\pi_2}{2}\right) \supseteq \frac{1}{\pi_2-\pi_1}(IR)\int_{\pi_1}^{\pi_2} \mathfrak{T}(\wp)\, d\wp \supseteq [\mathfrak{T}(\pi_1) + \mathfrak{T}(\pi_2)] \int_0^1 h(\wp)\, d\wp. \tag{3.2}$$

Remark 3.5. If $h(\vartheta) = \vartheta$, then the inequality (3.2) becomes:

$$\mathfrak{T}\left(\frac{\pi_1+\pi_2}{2}\right) \supseteq \frac{1}{\pi_2-\pi_1}(IR)\int_{\pi_1}^{\pi_2} \mathfrak{T}(x)\, dx \supseteq \frac{\mathfrak{T}(\pi_1) + \mathfrak{T}(\pi_2)}{2}. \tag{3.3}$$

The above inequality was first time investigated by Sadowska in [45].

If $h(\vartheta) = \vartheta^s$, then the inequality (3.2) becomes:

$$2^{s-1}\mathfrak{T}\left(\frac{\pi_1+\pi_2}{2}\right) \supseteq \frac{1}{\pi_2-\pi_1}(IR)\int_{\pi_1}^{\pi_2} \mathfrak{T}(x)\, dx \supseteq \frac{\mathfrak{T}(\pi_1) + \mathfrak{T}(\pi_2)}{s+1}. \tag{3.4}$$

The above inequality was first time investigated by Osuna-Gómez et al. in [41].

Definition 3.6 ([45])**.** A real-valued function $\mathfrak{T} : [\pi_1, \pi_2] \to \mathfrak{J}_c$ is CIVF, if

$$\upsilon\mathfrak{T}(x) + (1-\nu)\mathfrak{T}(\upsilon) \subseteq \mathfrak{T}(\nu x + (1-\nu)\upsilon).$$

for all $x, \nu \in [\pi_1, \pi_2], \nu \in (0, 1)$.

Theorem 3.7. *A real-valued function* $\mathfrak{T} : [\pi_1, \pi_2] \to \mathfrak{J}_c$ *is CIVF, if and only if* $\underline{\mathfrak{T}}$ *and* $\bar{\mathfrak{T}}$ *are convex and concave functions on closed interval* $[\pi_1\pi_2]$, *respectively.*

Theorem 3.8 ([49])**.** *Let* $0 < \zeta_1 \leqslant \zeta_2 \leqslant \ldots \leqslant \zeta_n$ *and* $\mathfrak{T}$ *be a convex function IVF on an interval containing* ρ_k, *then*

$$\mathfrak{T}\left(\sum_{j=1}^{n} \rho_j\ \zeta_j\right) \supseteq \left(\sum_{j=1}^{n} \rho_j\ \mathfrak{T}\left(\zeta_j\right)\right), \tag{3.5}$$

where $\sum_{j=1}^{n}\ \rho_j = 1$, $\rho_j \in [0, 1]$.

The authors in [49] extended the above inequality (3.5) via CIVF:

Theorem 3.9 ([49])**.** *Suppose* $\mathfrak{T}$ *be a convex function IVF on* $[\pi_1\pi_2]$ *such that* $\mathcal{L}(\pi_2) \geqslant \mathcal{L}(a_o)$ *for all* $a_o \in [\pi_1, \pi_2]$, *then*

$$\mathfrak{T}\left(\pi_1 + \pi_2 - \sum_{j=1}^{n} \rho_j\ \zeta_j\right) \supseteq\ \mathfrak{T}(\pi_1) + \mathfrak{T}(\pi_2) \ominus_g \sum_{j=1}^{n} \rho_j\ \zeta_j,$$

is true.

Throughout the article, we may assume that $\Gamma(.)$ is Euler Gamma (see [42]). Also that $\wp, \xi > 0$.

4. H-H-Mercer type inequality involving convex interval-valued functions via generalized fractional integrals

Since the notion of convex function was first introduced more than a century ago, an enormous number of outstanding inequalities have been proven in the domian of the convex theory. The most widely recognized and frequently utilized inequality in the field of convex theory is the H-H inequality. Hermite and Hadamard were the ones who first suggested this inequality. Many mathematicians were motivated by the idea of this inequality to investigate and analyze the classical inequalities utilizing the many convexity senses.

The principal goal of this section is to use the CIVF via generalized fractional integral operator to derive and demonstrate the Mercer's Hermite–Hadamard type inclusion.

Theorem 4.1. *Let $\wp \in (0,1)$. Suppose that $\mathfrak{T} : [\pi_1, \pi_2] \to \mathfrak{J}_c^+$ is an CIVF such that $\mathfrak{T}(\wp) = [\underline{\mathfrak{T}}(\wp), \bar{\mathfrak{T}}(\wp)]$ and $\mathcal{L}(\pi_2) \geqslant \mathcal{L}(\varpi_0)$, $\forall \varpi_0 \in [\pi_1, \pi_2]$. Then*

$$\begin{aligned}\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right) &\supseteq \frac{\wp^{\xi}\Gamma(\xi+1)}{2(\mathcal{Q}-\mathcal{W})^{\wp\xi}} \times \left\{ {}^{\xi}_{(\pi_1+\pi_2-\mathcal{Q})}\mathfrak{J}^{\wp}\mathfrak{T}(\pi_1+\pi_2-\mathcal{W}) \;+\; {}^{\xi}\mathfrak{J}^{\wp}_{(\pi_1+\pi_2-\mathcal{W})}\mathfrak{T}(\pi_1+\pi_2-\mathcal{Q}) \right\} \\ &\supseteq \frac{\mathfrak{T}(\pi_1+\pi_2-\mathcal{Q}) \;+\; \mathfrak{T}(\pi_1+\pi_2-\mathcal{W})}{2} \\ &\supseteq (\mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2)) \ominus_g \frac{\mathfrak{T}(\mathcal{W})+\mathfrak{T}(\mathcal{Q})}{2}.\end{aligned} \tag{4.1}$$

Proof. Employing the property of CIVF, then for all $u, v \in [\pi_1, \pi_2]$, we have

$$\begin{aligned}\mathfrak{T}\left(\pi_1+\pi_2-\frac{u+v}{2}\right) &= \mathfrak{T}\left(\frac{(\pi_1+\pi_2-u)+(\pi_1+\pi_2-v)}{2}\right) \\ &\supseteq \frac{1}{2}\left\{\mathfrak{T}(\pi_1+\pi_2-u)+\mathfrak{T}(\pi_1+\pi_2-v)\right\}.\end{aligned}$$

By using

$$\pi_1+\pi_2-u = \wp(\pi_1+\pi_2-\mathcal{W})+(1-\wp)(\pi_1+\pi_2-\mathcal{Q})$$

and

$$\pi_1+\pi_2-v = \wp(\pi_1+\pi_2-\mathcal{Q})+(1-\wp)(\pi_1+\pi_2-\mathcal{W}),$$

for all $\mathcal{W}, \mathcal{Q} \in [\pi_1, \pi_2]$ and $\wp \in [0,1]$, we get

$$\begin{aligned}\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right) \supseteq \frac{1}{2}\Big\{&\mathfrak{T}(\wp(\pi_1+\pi_2-\mathcal{W})+(1-\wp)(\pi_1+\pi_2-\mathcal{Q})) \\ &+\mathfrak{T}(\wp(\pi_1+\pi_2-\mathcal{Q})+(1-\wp)(\varpi_1+\pi_2-\mathcal{W}))\Big\}.\end{aligned} \tag{4.2}$$

Now, multiplying both sides of (4.2) by $\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}$ and integrating by inclusion with respect to $\wp$ over $[0,1]$, we obtain

$$\begin{aligned}&\frac{1}{\xi\wp^{\xi}}\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right) \\ &\supseteq \frac{1}{2}\Big\{\int_0^1\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}\mathfrak{T}(\wp(\pi_1+\pi_2-\mathcal{W})+(1-\wp)(\pi_1+\pi_2-\mathcal{Q}))\,d\wp \\ &\quad+\int_0^1\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}\mathfrak{T}(\wp(\pi_1+\pi_2-\mathcal{Q})+(1-\wp)(\pi_1+\pi_2-\mathcal{W}))\,d\wp\Big\}\end{aligned}$$

$$\begin{aligned}
&\supseteq\frac{1}{2}\left\{\frac{1}{(\mathfrak{Q}-\mathcal{W})^{\wp\xi}}\right.\\
&\quad\times\int_{\pi_1+\pi_2-\mathfrak{Q}}^{\pi_1+\pi_2-\mathcal{W}}\left(\frac{(\mathfrak{Q}-\mathcal{W})^{\wp}-((\pi_1+\pi_2-\mathcal{W})-z)^{\wp}}{\wp}\right)^{\xi-1}\frac{\mathfrak{T}(z)}{((\pi_1+\pi_2-\mathcal{W})-z)^{1-\wp}}\mathrm{d}z\\
&\quad+\frac{1}{(\mathfrak{Q}-\mathcal{W})^{\wp\xi}}\\
&\quad\times\left.\int_{\pi_1+\coprod_2-\mathfrak{Q}}^{\pi_1+\pi_2-\mathcal{W}}\left(\frac{(\mathfrak{Q}-\mathcal{W})^{\wp}-(z-(\pi_1+\pi_2-\mathfrak{Q}))^{\wp}}{\wp}\right)^{\xi-1}\frac{\mathfrak{T}(z)}{(z-(\coprod_1+\pi_2-\mathfrak{Q}))^{1-\wp}}\mathrm{d}z\right\}\\
&\supseteq\frac{\Gamma(\xi)}{2(\mathfrak{Q}-\mathcal{W})^{\wp\xi}}\left\{{}^{\xi}_{(\pi_1+\pi_2-\mathfrak{Q})}\mathfrak{J}^{\wp}\,\mathfrak{T}(\pi_1+\pi_2-\mathcal{W})\;+\;{}^{\xi}\mathfrak{J}^{\wp}_{(\pi_1+\pi_2-\mathcal{W})}\,\mathfrak{T}(\pi_1+\pi_2-\mathfrak{Q})\right\},
\end{aligned}\tag{4.3}$$

and the proof of the first inequality in (4.1) is attained. To prove that the second inequality in (4.1), first we mark that $\mathfrak{T}$ is an CIVF, so we have

$$\mathfrak{T}(\wp(\pi_1+\pi_2-\mathcal{W})+(1-\wp)(\pi_1+\pi_2-\mathfrak{Q}))\supseteq\;\wp\mathcal{F}(\pi_1+\pi_2-\mathcal{W})+(1-\wp)\mathfrak{T}(\varpi_1+\pi_2-\mathfrak{Q}),\tag{4.4}$$

and

$$\mathfrak{T}(\wp(\pi_1+\pi_2-\mathfrak{Q})+(1-\wp)(\pi_1+\pi_2-\mathcal{W}))\supseteq(1-\wp)\mathfrak{T}(\pi_1+\pi_2-\mathcal{W})+\wp\mathfrak{T}(\pi_1+\pi_2-\mathfrak{Q}).\tag{4.5}$$

Adding (4.4) and (4.5), we attain the following from Jensen-Mercer inequality,

$$\begin{aligned}
\mathfrak{T}(\wp(\pi_1+\pi_2-\mathcal{W})+(1-\wp)(\pi_1+\pi_2-\mathfrak{Q}))&+\mathfrak{T}(\wp(\pi_1+\pi_2-\mathfrak{Q})+(1-\wp)(\pi_1+\pi_2-\mathcal{W}))\\
&\supseteq\;\wp\mathcal{F}(\pi_1+\pi_2-\mathcal{W})+(1-\wp)\mathfrak{T}(\pi_1+\pi_2-\mathfrak{Q})\\
&\quad+\;(1-\wp)\mathfrak{T}(\pi_1+\pi_2-\mathcal{W})+\wp\mathcal{F}(\pi_1+\pi_2-\mathfrak{Q})\\
&\supseteq\mathfrak{T}(\pi_1+\pi_2-\mathcal{W})+\mathfrak{T}(\pi_1+\pi_2-\mathfrak{Q})\\
&\supseteq2\{\mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2)\}\ominus_g\{\mathfrak{T}(\mathcal{W})+\mathfrak{T}(\mathfrak{Q})\}.
\end{aligned}\tag{4.6}$$

Now, multiplying both sides of (4.6) by $\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}$ and integrating by inequality w.r.t. $\wp$ over $[0,1]$, we attain

$$\begin{aligned}
&\int_0^1\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}\mathfrak{T}(\wp(\pi_1+\pi_2-\mathcal{W})+(1-\wp)(\pi_1+\pi_2-\mathfrak{Q}))\,\mathrm{d}\wp\\
&\qquad+\int_0^1\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}\mathfrak{T}(\wp(\pi_1+\pi_2-\mathfrak{Q})+(1-\wp)(\pi_1+\pi_2-\mathcal{W}))\,\mathrm{d}\wp\\
&\qquad\supseteq\left\{\mathfrak{T}(\pi_1+\pi_2-\mathcal{W})+\mathfrak{T}(\pi_1+\pi_2-\mathfrak{Q})\right\}\int_0^1\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}\mathrm{d}\wp\\
&\qquad\supseteq\left\{2\{\mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2)\}\ominus_g\{\mathfrak{T}(\mathcal{W})+\mathfrak{T}(\mathfrak{Q})\}\right\}\int_0^1\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}\mathrm{d}\wp,
\end{aligned}$$

or

$$\begin{aligned}
&\frac{\Gamma(\xi)}{(\mathfrak{Q}-\mathcal{W})^{\wp\xi}}\left\{{}^{\xi}_{(\pi_1+\pi_2-\mathfrak{Q})}\mathfrak{J}^{\wp}\,\mathfrak{T}(\pi_1+\pi_2-\mathcal{W})\;+\;{}^{\xi}\mathfrak{J}^{\wp}_{(\pi_1+\pi_2-\mathcal{W})}\,\mathfrak{T}(\pi_1+\pi_2-\mathfrak{Q})\right\}\\
&\qquad\supseteq\frac{1}{\xi\wp^{\xi}}\left\{\mathfrak{T}(\pi_1+\pi_2-\mathcal{W})+\mathfrak{T}(\pi_1+\pi_2-\mathfrak{Q})\right\}\\
&\qquad\supseteq\frac{1}{\xi\wp^{\xi}}\left\{2\{\mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2)\}\ominus_g\{\mathfrak{T}(\mathcal{W})+\mathfrak{T}(\mathfrak{Q})\}\right\}.
\end{aligned}$$

Dividing by 2 in above inclusion.

$$\begin{aligned}\frac{\Gamma(\xi)}{2(\mathcal{Q}-\mathcal{W})^{\wp\xi}}&\left\{{}_{(\pi_1+\pi_2-\mathcal{Q})}^{\xi}\mathfrak{J}^{\wp}\,\mathfrak{T}(\pi_1+\pi_2-\mathcal{W}) \;+\; {}^{\xi}\mathfrak{J}^{\wp}_{(\pi_1+\pi_2-\mathcal{W})}\,\mathfrak{T}(\pi_1+\pi_2-\mathcal{Q})\right\}\\ &\supseteq \frac{1}{2\xi\wp^{\xi}}\left\{\mathfrak{T}(\pi_1+\pi_2-\mathcal{W})+\mathfrak{T}(\pi_1+\pi_2-\mathcal{Q})\right\}\\ &\supseteq \frac{1}{2\xi\wp^{\xi}}\left\{2\{\mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2)\}\ominus_g\{\mathfrak{T}(\mathcal{W})+\mathfrak{T}(\mathcal{Q})\}\right\}.\end{aligned}\tag{4.7}$$

Concatenating the equations (4.3) and (4.7), we can get (4.1). □

Corollary 4.2. *If we suppose* $\underline{\mathfrak{T}}(\wp)=\bar{\mathfrak{T}}(\wp)$ *and employing identical procedure in Theorem 4.1, we have*

$$\begin{aligned}\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right)&\leqslant\frac{\wp^{\xi}\Gamma(\xi+1)}{2(\mathcal{Q}-\mathcal{W})^{\wp\xi}}\times\left\{{}_{(\pi_1+\pi_2-\mathcal{Q})}^{\xi}\mathfrak{J}^{\wp}\mathfrak{T}(\pi_1+\pi_2-\mathcal{W}) \;+\; {}^{\xi}\mathfrak{J}^{\wp}_{(\pi_1+\pi_2-\mathcal{W})}\mathfrak{T}(\pi_1+\pi_2-\mathcal{Q})\right\}\\ &\leqslant\frac{\mathfrak{T}(\pi_1+\pi_2-\mathcal{W})\;+\;\mathfrak{T}(\pi_1+\pi_2-\mathcal{Q})}{2}\\ &\leqslant(\mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2))\;-\;\frac{\mathfrak{T}(\mathcal{W})+\mathfrak{T}(\mathcal{Q})}{2}.\end{aligned}$$

Corollary 4.3. *If we suppose* $\mathcal{W}=\pi_1$ *and* $\mathcal{Q}=\pi_2$ *and employing identical procedure in Theorem 4.1, we have*

$$\begin{aligned}\mathfrak{T}\left(\frac{\pi_1+\pi_2}{2}\right)&\supseteq\frac{\wp^{\xi}\Gamma(\xi+1)}{2(\pi_2-\pi_1)^{\wp\xi}}\left\{{}_{\pi_1}^{\xi}\mathfrak{J}^{\wp}\mathfrak{T}(\pi_2)\;+\;{}^{\xi}\mathfrak{J}^{\wp}_{\pi_2}\mathfrak{T}(\pi_1)\right\}\\ &\supseteq\frac{\mathfrak{T}(\pi_1)\;+\;\mathfrak{T}(\pi_2)}{2}.\end{aligned}$$

Theorem 4.4. *Let* $\wp\in(0,1)$*. Suppose that* $\mathfrak{T}:[\pi_1,\pi_2]\to\mathfrak{J}_c^+$ *is CIVF such that* $\mathfrak{T}(\wp)=[\underline{\mathfrak{T}}(\wp),\bar{\mathfrak{T}}(\wp)]$ *and* $\mathcal{L}(\pi_2)\geqslant\mathcal{L}(\varpi_0)$, $\forall\varpi_0\in[\pi_1,\pi_2]$*. Then*

$$\begin{aligned}\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right)\supseteq\frac{1}{2}\left(\frac{2}{\mathcal{Q}-\mathcal{W}}\right)^{\wp\xi}\wp^{\xi}\Gamma(\xi+1)&\left\{{}_{(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2})}^{\xi}\mathfrak{J}^{\wp}\,\mathfrak{T}(\pi_1+\pi_2-\mathcal{W})\right.\\ &\left.+\;{}^{\xi}\mathfrak{J}^{\wp}_{(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2})}\,\mathfrak{T}(\pi_1+\pi_2-\mathcal{Q})\right\}\\ \supseteq\;(\mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2))\;\ominus_g&\frac{\mathfrak{T}(\mathcal{W})+\mathfrak{T}(\mathcal{Q})}{2}.\end{aligned}\tag{4.8}$$

Proof. Employing CIVF, we have $\forall\, u,v\in[\pi_1,\pi_2]$,

$$\begin{aligned}\mathfrak{T}\left(\pi_1+\pi_2-\frac{u+v}{2}\right)&=\mathfrak{T}\left(\frac{(\pi_1+\pi_2-u)+(\pi_1+\pi_2-v)}{2}\right)\\ &\supseteq\frac{1}{2}\left\{\mathfrak{T}(\pi_1+\pi_2-u)+\mathfrak{T}(\pi_1+\pi_2-v)\right\}.\end{aligned}$$

By using

$$u=\frac{\wp}{2}\mathcal{W}+\frac{2-\wp}{2}\mathcal{Q},$$

and

$$v=\frac{2-\wp}{2}\mathcal{W}+\frac{\wp}{2}\mathcal{Q}$$

for all $\mathcal{W}, \mathcal{Q} \in [\pi_1, \pi_2]$ and $\wp \in [0,1]$, we get

$$\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right) \supseteq \frac{1}{2}\left\{ \mathfrak{T}\left(\pi_1+\pi_2-[\frac{\wp}{2}\mathcal{W}+\frac{2-\wp}{2}\mathcal{Q}]\right)+\mathfrak{T}\left(\pi_1+\pi_2-[\frac{2-\wp}{2}\mathcal{W}+\frac{\wp}{2}\mathcal{Q}]\right)\right\}.$$

Now, multiplying both sides of above by $\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}$ and integrating by inclusion with respect to $\wp$ over $[0,1]$, we obtain

$$\begin{aligned}
\frac{1}{\xi\wp^{\xi}}\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right) &\supseteq \frac{1}{2}\left\{\int_0^1\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}\mathfrak{T}\left(\pi_1+\pi_2-[\frac{\wp}{2}\mathcal{W}+\frac{2-\wp}{2}\mathcal{Q}]\right)d\wp\right.\\
&\quad\left.+\int_0^1\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}\mathfrak{T}\left(\pi_1+\pi_2-[\frac{2-\wp}{2}\mathcal{W}+\frac{\wp}{2}\mathcal{Q}]\right)d\wp\right\},\\
\frac{1}{\xi\wp^{\xi}}\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right) &\supseteq \frac{1}{2}\left(\frac{2}{\mathcal{Q}-\mathcal{W}}\right)^{\wp\xi}\Gamma(\xi)\left\{ {}^{\xi}_{(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2})}\mathfrak{J}^{\wp}\,\mathfrak{T}((\pi_1+\pi_2-\mathcal{W}))\right.\\
&\quad\left.+\ {}^{\xi}\mathfrak{J}^{\wp}_{(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2})}\ \mathfrak{T}((\pi_1+\pi_2-\mathcal{Q}))\right\},
\end{aligned}\tag{4.9}$$

and the proof of the first inequality (4.8) is completed. To prove the second inequality in (4.8), first we note that $\mathfrak{T}$ is CIVF, we have

$$\mathfrak{T}\left(\pi_1+\pi_2-[\frac{\wp}{2}\mathcal{W}+\frac{2-\wp}{2}\mathcal{Q}]\right) \supseteq \mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2)\ \ominus_g [\frac{\wp}{2}\mathfrak{T}(\mathcal{W})+\frac{2-\wp}{2}\mathfrak{T}(\mathcal{Q})],$$

and

$$\mathfrak{T}\left(\pi_1+\pi_2-[\frac{2-\wp}{2}\mathcal{W}+\frac{\wp}{2}\mathcal{Q}]\right) \supseteq \mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2)\ \ominus_g [\frac{2-\wp}{2}\mathfrak{T}(\mathcal{W})+\frac{\wp}{2}\mathfrak{T}(\mathcal{Q})].$$

Adding above equations, we get

$$\mathfrak{T}\left(\pi_1+\pi_2-[\frac{\wp}{2}\mathcal{W}+\frac{2-\wp}{2}\mathcal{Q}]\right)+\mathfrak{T}\left(\pi_1+\pi_2-[\frac{2-\wp}{2}\mathcal{W}+\frac{\wp}{2}\mathcal{Q}]\right) \supseteq 2\{\mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2)\}\ominus_g\{\mathfrak{T}(\mathcal{W})+\mathfrak{T}(\mathcal{Q})\}. \tag{4.10}$$

Multiplying both sides of above by $\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}$ and integrating by inclusion with respect to $\wp$ over $[0,1]$, we obtain

$$\begin{aligned}
&\left\{\int_0^1\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}\mathfrak{T}\left(\pi_1+\pi_2-[\frac{\wp}{2}\mathcal{W}+\frac{2-\wp}{2}\mathcal{Q}]\right)d\wp\right.\\
&\quad\left.+\int_0^1\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}\mathfrak{T}\left(\pi_1+\pi_2-[\frac{2-\wp}{2}\mathcal{W}+\frac{\wp}{2}\mathcal{Q}]\right)d\wp\right\}\\
&\supseteq (2\{\mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2)\}\ominus_g\{\mathfrak{T}(\mathcal{W})+\mathfrak{T}(\mathcal{Q})\})\int_0^1\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}d\wp.
\end{aligned}$$

Dividing by 2 in above inclusion.

$$\begin{aligned}
&\frac{1}{2}\left(\frac{2}{\mathcal{Q}-\mathcal{W}}\right)^{\wp\xi}\Gamma(\xi)\left\{ {}^{\xi}_{(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2})}\mathfrak{J}^{\wp}\,\mathfrak{T}((\pi_1+\pi_2-\mathcal{W}))\ +\ {}^{\xi}\mathfrak{J}^{\wp}_{(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2})}\ \mathfrak{T}((\pi_1+\pi_2-\mathcal{Q}))\right\}\\
&\qquad\supseteq \frac{1}{2\xi\wp^{\xi}}\left(2\{\mathfrak{T}(\pi_1)+\mathfrak{T}(\varpi_2)\}\ominus_g\{\mathfrak{T}(\mathcal{W})+\mathfrak{T}(\mathcal{Q})\}\right).
\end{aligned}\tag{4.11}$$

Concatenating the equations (4.9) and (4.11), we can get (4.8). This is the required proof. □

Corollary 4.5. *If we suppose $\underline{\mathfrak{T}}(\wp) = \bar{\mathfrak{T}}(\wp)$ and employing identical procedure in Theorem 4.4, we have*

$$\begin{aligned}\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right) &\leqslant \frac{1}{2}\left(\frac{2}{\mathcal{Q}-\mathcal{W}}\right)^{\wp\xi}\wp^{\xi}\Gamma(\xi+1)\left\{{}^{\xi}_{\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right)}\mathfrak{J}^{\wp}\,\mathfrak{T}(\pi_1+\pi_2-\mathcal{W})\right.\\ &\quad\left.+{}^{\xi}\mathfrak{J}^{\wp}_{\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right)}\,\mathfrak{T}(\pi_1+\pi_2-\mathcal{Q})\right\}\\ &\leqslant (\mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2)) - \frac{\mathfrak{T}(\mathcal{W})+\mathfrak{T}(\mathcal{Q})}{2}.\end{aligned}$$

Remark 4.6. If we letting $\mathcal{W} = \pi_1$ and $\mathcal{Q} = \pi_2$ in Corollary 4.5, it reduces to [20, Theorem 2.1].

Remark 4.7. If we letting $\mathcal{W} = \pi_1$ and $\mathcal{Q} = \pi_2$ and $\wp = 1$ in Corollary 4.5, it reduces to [47, Theorem 4].

Theorem 4.8. *If $\mathfrak{T} : [\pi_1, \pi_2] \to \mathfrak{I}_c^+$ is an CIVF such that $\mathfrak{T}(\wp) = [\underline{\mathfrak{T}}(\wp), \bar{\mathfrak{T}}(\wp)]$ and $\mathcal{L}(\pi_2) \geqslant \mathcal{L}(\varpi_0), \forall \varpi_0 \in [\pi_1, \pi_2]$, then*

$$\begin{aligned}\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right) &\supseteq \frac{\wp^{\xi}}{2}\left(\frac{2}{\mathcal{Q}-\mathcal{W}}\right)^{\wp\xi}\Gamma(\xi+1)\left\{{}^{\xi}_{(\pi_1+\pi_2-\mathcal{Q})}\mathfrak{J}^{\wp}\,\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right)\right.\\ &\quad\left.+\,{}^{\xi}\mathfrak{J}^{\wp}_{(\pi_1+\pi_2-\mathcal{W})}\,\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right)\right\}\\ &\supseteq (\,\mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2))\ominus_g\frac{\mathfrak{T}(\mathcal{W})+\mathfrak{T}(\mathcal{Q})}{2}.\end{aligned}\tag{4.12}$$

Proof. Employ the property of CIVF, so we have for all $u, v \in [\pi_1, \pi_2]$,

$$\begin{aligned}\mathfrak{T}\left(\pi_1+\pi_2-\frac{u+v}{2}\right) &= \mathfrak{T}\left(\frac{(\pi_1+\pi_2-u)+(\pi_1+\pi_2-v)}{2}\right)\\ &\supseteq \frac{1}{2}\left\{\mathfrak{T}(\pi_1+\pi_2-u)+\mathfrak{T}(\pi_1+\pi_2-v)\right\}.\end{aligned}$$

Let

$$u = \frac{1-\wp}{2}\mathcal{W} + \frac{1+\wp}{2}\mathcal{Q},$$

and

$$v = \frac{1+\wp}{2}\mathcal{W} + \frac{1-\wp}{2}\mathcal{Q}$$

for all $\mathcal{W}, \mathcal{Q} \in [\pi_1, \pi_2]$ and $\wp \in [0, 1]$, we get

$$\begin{aligned}\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right) &\supseteq \frac{1}{2}\left\{\mathfrak{T}\left(\pi_1+\pi_2-\left[\frac{1-\wp}{2}\mathcal{W}+\frac{1+\wp}{2}\mathcal{Q}\right]\right)\right.\\ &\quad\left.+\,\mathfrak{T}\left(\pi_1+\pi_2-\left[\frac{1+\wp}{2}\mathcal{W}+\frac{1-\wp}{2}\mathcal{Q}\right]\right)\right\}.\end{aligned}$$

Now, multiplying both sides of above by $\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}$ and integrating by inclusion w.r.t. $\wp$ over $[0, 1]$, we obtain

$$\begin{aligned}
&\frac{1}{\xi\wp^{\xi}}\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right)\\
&\supseteq \frac{1}{2}\left\{\int_0^1\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}\mathfrak{T}\left(\pi_1+\pi_2-[\frac{1-\wp}{2}\mathcal{W}+\frac{1+\wp}{2}\mathcal{Q}]\right)d\wp\right.\\
&\left.+\int_0^1\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}\mathfrak{T}\left(\pi_1+\pi_2-[\frac{1+\wp}{2}\mathcal{W}+\frac{1-\wp}{2}\mathcal{Q}]\right)d\wp\right\}\\
&\supseteq \frac{1}{2}\left(\frac{2}{\mathcal{Q}-\mathcal{W}}\right)^{\wp\xi}\\
&\times\left\{\int_{\pi_1+\pi_2-\mathcal{Q}}^{\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}}\left(\frac{\left(\frac{2}{\mathcal{Q}-\mathcal{W}}\right)^{\wp}-(z-(\coprod_1+\pi_2-\mathcal{Q}))^{\wp}}{\wp}\right)^{\xi-1}\frac{\mathfrak{T}(z)}{(z-(\pi_1+\pi_2-\mathcal{Q}))^{1-\wp}}dz\right.\\
&\left.+\int_{\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}}^{\pi_1+\coprod_2-\mathcal{W}}\left(\frac{\left(\frac{2}{\mathcal{Q}-\mathcal{W}}\right)^{\wp}-((\pi_1+\pi_2-\mathcal{W})-z)^{\wp}}{\wp}\right)^{\xi-1}\frac{\mathfrak{T}(z)}{((\pi_1+\coprod_2-\mathcal{W})-z)^{1-\wp}}dz\right\},
\end{aligned}\tag{4.13}$$

$$\begin{aligned}
&\frac{1}{\xi\wp^{\xi}}\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right)\\
&\supseteq \frac{\Gamma(\xi)}{2}\left(\frac{2}{\mathcal{Q}-\mathcal{W}}\right)^{\wp\xi}\\
&\times\left\{{}^{\xi}_{(\pi_1+\pi_2-\mathcal{Q})}\mathfrak{J}^{\wp}\,\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right)+{}^{\xi}\mathfrak{J}^{\wp}_{(\pi_1+\pi_2-\mathcal{W})}\,\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right)\right\}.
\end{aligned}$$

The proof of the first inequality is completed. To prove that the second inequality in (4.12), using Jensen-Mercer inequality, we have

$$\mathfrak{T}\left(\pi_1+\pi_2-[\frac{1-\wp}{2}\mathcal{W}+\frac{1+\wp}{2}\mathcal{Q}]\right)\supseteq\mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2)\ominus_g[\frac{1-\wp}{2}\mathfrak{T}(\mathcal{W})+\frac{1+\wp}{2}\mathfrak{T}(\mathcal{Q})],$$

and

$$\mathfrak{T}\left(\pi_1+\pi_2-[\frac{1+\wp}{2}\mathcal{W}+\frac{1-\wp}{2}\mathcal{Q}]\right)\supseteq\mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2)\ominus_g[\frac{1+\wp}{2}\mathfrak{T}(\mathcal{W})+\frac{1-\wp}{2}\mathfrak{T}(\mathcal{Q})].$$

Adding above equations, we get

$$\begin{aligned}
&\mathfrak{T}\left(\pi_1+\pi_2-[\frac{1-\wp}{2}\mathcal{W}+\frac{1+\wp}{2}\mathcal{Q}]\right)+\mathfrak{T}\left(\pi_1+\pi_2-[\frac{1+\wp}{2}\mathcal{W}+\frac{1-\wp}{2}\mathcal{Q}]\right)\\
&\supseteq\mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2)\ominus_g(\mathfrak{T}(\mathcal{W})+\mathfrak{T}(\mathcal{Q})).
\end{aligned}\tag{4.14}$$

Multiplying both sides of (4.14) by $\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}$ and integrating by inclusion with respect to $\wp$ over $[0,1]$, we obtain

$$\begin{aligned}
&\left\{\int_0^1\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}\mathfrak{T}\left(\pi_1+\pi_2-[\frac{1-\wp}{2}\mathcal{W}+\frac{1+\wp}{2}\mathcal{Q}]\right)d\wp\right.\\
&\left.+\int_0^1\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}\mathfrak{T}\left(\pi_1+\pi_2-[\frac{1+\wp}{2}\mathcal{W}+\frac{1-\wp}{2}\mathcal{Q}]\right)d\wp\right\}\\
&\supseteq\left(2\{\mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2)\}\ominus_g\{\mathfrak{T}(\mathcal{W})+\mathfrak{T}(\mathcal{Q})\}\right)\int_0^1\left(\frac{1-(1-\wp)^{\wp}}{\wp}\right)^{\xi-1}(1-\wp)^{\wp-1}d\wp,
\end{aligned}$$

$$\begin{aligned}
&\Gamma(\xi)\left(\frac{2}{\mathcal{Q}-\mathcal{W}}\right)^{\wp\xi}\times\left\{{}^{\xi}_{(\pi_1+\pi_2-\mathcal{Q})}\mathfrak{J}^{\wp}\,\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right)+{}^{\xi}\mathfrak{J}^{\wp}_{(\pi_1+\pi_2-\mathcal{W})}\,\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right)\right\}\\
&\supseteq\frac{1}{\xi\wp^{\xi}}\left(2\{\mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2)\}\ominus_g\{\mathfrak{T}(\mathcal{W})+\mathfrak{T}(\mathcal{Q})\}\right).
\end{aligned}$$

Dividing by 2, we get

$$\begin{aligned} \frac{\Gamma(\xi)}{2}\left(\frac{2}{\mathfrak{Q}-\mathcal{W}}\right)^{\wp\xi} &\times\left\{ {}^{\xi}_{(\pi_1+\pi_2-\mathfrak{Q})}\mathfrak{J}^{\wp}\,\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathfrak{Q}}{2}\right)+{}^{\xi}\mathfrak{J}^{\wp}_{(\pi_1+\pi_2-\mathcal{W})}\,\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathfrak{Q}}{2}\right)\right\} \\ &\supseteq \frac{1}{2\xi\wp^{\xi}}\Big(2\{\mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2)\}\ominus_g\{\mathfrak{T}(\mathcal{W})+\mathfrak{T}(\mathfrak{Q})\}\Big). \end{aligned} \tag{4.15}$$

Concatenating the equations (4.13) and (4.15), we can get (4.12). This is the required. □

Corollary 4.9. *If we suppose $\underline{\mathfrak{T}}(\wp)=\bar{\mathfrak{T}}(\wp)$ and employing same procedure in Theorem 4.8, we have*

$$\begin{aligned} \mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathfrak{Q}}{2}\right) &\leqslant \frac{\wp^{\xi}}{2}\left(\frac{2}{\mathfrak{Q}-\mathcal{W}}\right)^{\wp\xi}\Gamma(\xi+1)\left\{ {}^{\xi}_{(\pi_1+\pi_2-\mathfrak{Q})}\mathfrak{J}^{\wp}\,\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathfrak{Q}}{2}\right)\right. \\ &\quad\left. +\,{}^{\xi}\mathfrak{J}^{\wp}_{(\pi_1+\pi_2-\mathcal{W})}\,\mathfrak{T}\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathfrak{Q}}{2}\right)\right\} \\ &\leqslant (\mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2))-\frac{\mathfrak{T}(\mathcal{W})+\mathfrak{T}(\mathfrak{Q})}{2}. \end{aligned}$$

Corollary 4.10. *If we suppose $\mathcal{W}=\pi_1$ and $\mathfrak{Q}=\pi_2$ and employing same procedure in Theorem 4.8, we have*

$$\begin{aligned} \mathfrak{T}\left(\frac{\pi_1+\pi_2}{2}\right) &\supseteq \frac{\wp^{\xi}}{2}\left(\frac{2}{\pi_2-\pi_1}\right)^{\wp\xi}\Gamma(\xi+1)\left\{ {}^{\xi}_{\pi_1}\mathfrak{J}^{\wp}\,\mathfrak{T}\left(\frac{\pi_1+\pi_2}{2}\right)+{}^{\xi}\mathfrak{J}^{\wp}_{\pi_2}\,\mathfrak{T}\left(\frac{\pi_1+\pi_2}{2}\right)\right\} \\ &\supseteq \frac{\mathfrak{T}(\pi_1)+\mathfrak{T}(\pi_2)}{2}. \end{aligned}$$

5. Applications to matrix

The subject convex analysis and fractional mathematics are both utilized in applied sciences. The literature makes it clear that these ideas have a broad spectrum of potential uses in multiple fields of research, from fluid dynamics to optimization. In order to be more precise, we are going to add matrices-related applications.

In [44], Sababheh presented that the function $\psi(\Upsilon)=\left\|\mathcal{G}^{\Upsilon}\mathcal{T}\mathcal{O}^{1-\Upsilon}+\mathcal{G}^{1-\Upsilon}\mathcal{T}\mathcal{O}^{\Upsilon}\right\|, \mathcal{G},\mathcal{O}\in M_n^{+}, \mathcal{T}\in M_n$, is convex for all $\Upsilon\in[0,1]$.

Example 5.1. *Employing Theorem 4.1, we have*

$$\begin{aligned} &\left\|\mathcal{G}^{\pi_1+\pi_2-\frac{\mathcal{W}+\mathfrak{Q}}{2}}\mathcal{T}\mathcal{O}^{1-(\pi_1+\pi_2-\frac{\mathcal{W}+\mathfrak{Q}}{2})}+\mathcal{G}^{1-(\pi_1+\pi_2-\frac{\mathcal{W}+\mathfrak{Q}}{2})}\mathcal{T}\mathcal{O}^{\pi_1+\pi_2-\frac{\mathcal{W}+\mathfrak{Q}}{2}}\right\| \\ &\supseteq \frac{\wp^{\xi}\Gamma(\xi+1)}{2(\mathfrak{Q}-\mathcal{W})^{\wp\xi}}\times\left[{}^{\xi}_{\pi_1+\pi_2-\mathfrak{Q}}\mathfrak{J}^{\wp}\left\|\mathcal{G}^{\pi_1+\pi_2-\mathcal{W}}\mathcal{T}\mathcal{O}^{1-(\pi_1+\pi_2-\mathcal{W})}+\mathcal{G}^{1-(\pi_1+\pi_2-\mathcal{W})}\mathcal{T}\mathcal{O}^{\pi_1+\pi_2-\mathcal{W}}\right\|\right. \\ &\quad\left. +\,{}^{\xi}\mathfrak{J}^{\wp}_{\pi_1+\pi_2-\mathcal{W}}\left\|\mathcal{G}^{\pi_1+\pi_2-\mathfrak{Q}}\mathcal{T}\mathcal{O}^{1-(\pi_1+\pi_2-\mathfrak{Q})}+\mathcal{G}^{1-(\pi_1+\pi_2-\mathfrak{Q})}\mathcal{T}\mathcal{O}^{\pi_1+\pi_2-\mathfrak{Q}}\right\|\right] \\ &\supseteq \frac{1}{2}\left\{\left\|\mathcal{G}^{\pi_1+\pi_2-\mathcal{W}}\mathcal{T}\mathcal{O}^{1-(\pi_1+\pi_2-\mathcal{W})}+\mathcal{G}^{1-(\pi_1+\pi_2-\mathcal{W})}\mathcal{T}\mathcal{O}^{\pi_1+\pi_2-\mathcal{W}}\right\|\right. \\ &\quad\left. +\left\|\mathcal{G}^{\pi_1+\pi_2-\mathfrak{Q}}\mathcal{T}\mathcal{O}^{1-(\pi_1+\pi_2-\mathfrak{Q})}+\mathcal{G}^{1-(\pi_1+\pi_2-\mathfrak{Q})}\mathcal{T}\mathcal{O}^{\pi_1+\pi_2-\mathfrak{Q}}\right\|\right\} \\ &\supseteq \left\|\mathcal{G}^{\pi_1}\mathcal{T}\mathcal{O}^{1-\pi_1}+\mathcal{G}^{1-\pi_1}\mathcal{T}\mathcal{O}^{\pi_1}\right\|+\left\|\mathcal{G}^{\pi_2}\mathcal{T}\mathcal{O}^{1-\pi_2}+\mathcal{G}^{1-\pi_2}\mathcal{T}\mathcal{O}^{\pi_2}\right\| \\ &\quad\ominus_g\frac{1}{2}\left\{\left\|\mathcal{G}^{\mathcal{W}}\mathcal{T}\mathcal{O}^{1-\mathcal{W}}+\mathcal{G}^{1-\mathcal{W}}\mathcal{T}\mathcal{O}^{\mathcal{W}}\right\|+\left\|\mathcal{G}^{\mathfrak{Q}}\mathcal{T}\mathcal{O}^{1-\mathfrak{Q}}+\mathcal{G}^{1-\mathfrak{Q}}\mathcal{T}\mathcal{O}^{\mathfrak{Q}}\right\|\right\}. \end{aligned}$$

Example 5.2. *If we suppose Theorem 4.4 and employing same procedure of Example 5.1, we have*

$$\begin{aligned}
&\left\|\mathcal{G}^{\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}}\mathcal{T}\mathcal{O}^{1-\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right)}+\mathcal{G}^{1-\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right)}\mathcal{T}\mathcal{O}^{\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}}\right\| \\
&\supseteq \frac{1}{2}\left(\frac{2}{\mathcal{Q}-\mathcal{W}}\right)^{\wp\xi}\wp^{\xi}\Gamma(\xi+1)\times\left[{}^{\xi}\mathfrak{J}^{\wp}_{\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}}\left\|\mathcal{G}^{\pi_1+\pi_2-\mathcal{Q}}\mathcal{T}\mathcal{O}^{1-(\pi_1+\pi_2-\mathcal{Q})}+A^{1-(\pi_1+\pi_2-\mathcal{Q})}XB^{\pi_1+\pi_2-\mathcal{Q}}\right\|\right. \\
&\quad\left.+{}^{\xi}_{\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}}\mathfrak{J}^{\wp}\left\|\mathcal{G}^{\pi_1+\pi_2-\mathcal{W}}\mathcal{T}\mathcal{O}^{1-(\pi_1+\pi_2-\mathcal{W})}+\mathcal{G}^{1-(\pi_1+\pi_2-\mathcal{W})}\mathcal{T}\mathcal{O}^{\pi_1+\pi_2-\mathcal{W}}\right\|\right] \\
&\supseteq\left\{\left\|\mathcal{G}^{\pi_1}\mathcal{T}\mathcal{O}^{1-\pi_1}+\mathcal{G}^{1-\pi_1}\mathcal{T}\mathcal{O}^{\pi_1}\right\|+\left\|\mathcal{G}^{\pi_2}\mathcal{T}\mathcal{O}^{1-\pi_2}+\mathcal{G}^{1-\pi_2}\mathcal{T}\mathcal{O}^{\pi_2}\right\|\right\} \\
&\quad\ominus_g\frac{1}{2}\left\{\left\|\mathcal{G}^{\mathcal{W}}\mathcal{T}\mathcal{O}^{1-\mathcal{W}}+\mathcal{G}^{1-\mathcal{W}}\mathcal{T}\mathcal{O}^{\mathcal{W}}\right\|+\left\|\mathcal{G}^{\mathcal{Q}}\mathcal{T}\mathcal{O}^{1-\mathcal{Q}}+\mathcal{G}^{1-\mathcal{Q}}\mathcal{T}\mathcal{O}^{\mathcal{Q}}\right\|\right\}.
\end{aligned}$$

Example 5.3. *If we suppose Theorem 4.8 and employing same procedure of Example 5.1, we have*

$$\begin{aligned}
&\left\|\mathcal{G}^{\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}}\mathcal{T}\mathcal{O}^{1-\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right)}+\mathcal{G}^{1-\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right)}\mathcal{T}\mathcal{O}^{\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}}\right\| \\
&\supseteq\frac{\wp^{\xi}}{2}\left(\frac{2}{\mathcal{Q}-\mathcal{W}}\right)^{\wp\xi}\Gamma(\xi+1)\times{}^{\xi}_{\pi_1+\pi_2-\mathcal{Q}}\mathfrak{J}^{\wp} \\
&\quad\times\left\|\mathcal{G}^{\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}}\mathcal{T}\mathcal{O}^{1-\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right)}+\mathcal{G}^{1-\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right)}\mathcal{T}\mathcal{O}^{\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}}\right\| \\
&\quad+{}^{\xi}\mathfrak{J}^{\wp}_{\pi_1+\pi_2-\mathcal{W}} \\
&\quad\times\left\|\mathcal{G}^{\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}}\mathcal{T}\mathcal{O}^{1-\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right)}+\mathcal{G}^{1-\left(\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}\right)}\mathcal{T}\mathcal{O}^{\pi_1+\pi_2-\frac{\mathcal{W}+\mathcal{Q}}{2}}\right\| \\
&\supseteq\left\{\left\|\mathcal{G}^{\pi_1}\mathcal{T}\mathcal{O}^{1-\pi_2}+\mathcal{G}^{1-\pi_1}\mathcal{T}\mathcal{O}^{\pi_1}\right\|+\left\|\mathcal{G}^{\pi_2}\mathcal{T}\mathcal{O}^{1-\pi_2}+\mathcal{G}^{1-\pi_2}\mathcal{T}\mathcal{O}^{\pi_2}\right\|\right\} \\
&\quad\ominus_g\frac{1}{2}\left\{\left\|\mathcal{G}^{\mathcal{W}}\mathcal{T}\mathcal{O}^{1-\mathcal{W}}+\mathcal{G}^{1-\mathcal{W}}\mathcal{T}\mathcal{O}^{\mathcal{W}}\right\|+\left\|\mathcal{G}^{\mathcal{Q}}\mathcal{T}\mathcal{O}^{1-\mathcal{Q}}+\mathcal{G}^{1-\mathcal{Q}}\mathcal{T}\mathcal{O}^{\mathcal{Q}}\right\|\right\}.
\end{aligned}$$

6. Conclusions

Fractional calculus has a greater influence and provides more precise results when examining computer models. Fractional calculus is widely utilized in applied mathematics, mathematical biology, engineering, simulation, and inequality theory. Numerous researchers across multiple scientific domains have expressed a keen interest in fractional calculus. In this paper:

1) First, we added some related definitions, theorems and remarks because all these are necessary in upcoming subsequent sections.

2) We added some notations for interval analysis as well as the background information.

3) We explored some novel variants of H-H-Mercer inclusions for convex interval-valued functions pertaining to generalized fractional integrals.

4) To improve the reader's interest and overall quality, we added some corollaries and remarks.

5) Finally, some meaningful applications to matrix are explored.

It is an intriguing and novel problem in which aspiring researchers can attain identical inequalities involving variant type of of convexities in the frame of fractional integrals. The theory of convexity can be used to achieve an assortment of conclusions in quantum mechanics and special functions, associated optimization theory, and mathematical inequalities, as well as to motivate further research in a multitude of pure and applied sciences.

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