J. Math. Computer Sci., 33 (2024), 339–351

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Analysis of the proportional Caputo-Fabrizio derivative

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Abstract

This article examines a recently created proportional Caputo-Fabrizio derivative. We find multiple significant relationships between the beta function and this new derivative. Discreteization is applied to the new derivative. We consider stability analysis to define a stability requirement for the new derivative.

Keywords: Constant proportional Caputo-Fabrizio derivative, stability analysis, discretization.

2020 MSC: 26A33, 34A08, 34K37, 37M15.

1. Introduction

Fractional calculus, a revolutionary area of mathematics, has made significant contributions to the prediction of the dynamics of complex systems from many fields of science and engineering. Can we classify the fractional operators? is a fundamental issue that has created a massive discussion [8]. Surprisingly, there isn't a single, apparent answer to this problem; instead, numerous look to be plausible possibilities. Caputo and Fabrizio suggested a novel non-singular fractional operator, whereas Atangana and Baleanu generalized their discovery.

The study of fractional differential equations has substantially improved thanks to the Caputo-Fabrizio fractional derivative. One of the most enticing aspects of the new derivative is its nonsingular kernel. While being produced by the convolution of an ordinary derivative with an exponential function, the Caputo-Fabrizio derivative shares the same extra driving properties of heterogeneity and configuration with multiple scales as the Caputo and Riemann-Liouville fractional derivatives. Over the past two years, numerous advancements on the novel Caputo-Fabrizio fractional derivative have been made [4]. For instance, Losada et al. [14] studied the solutions to a number of linear fractional differential equations and the corresponding fractional integral. The proportional derivative with exponential-decay kernel has

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doi: 10.22436/jmcs.033.04.02

Received: 2023-05-09 *Revised:* 2023-08-16 *Accepted:* 2023-10-14

been presented in [2]. In our paper we discuss some new aspects of the operator introduced in [2]. Since the proportional derivative is an important tool especially for engineering applications, the properties of this derivative should be investigated. For more details see [1, 12, 18, 20].

Tassaddiq et al. [19] considered about the analysis of differential equations employing the Caputo-Fabrizio fractional operator and its applications to reaction-diffusion equations. Arshad et al. [5] devised an innovative numerical approach for solving the Caputo-Fabrizio fractional differential equation. Owolabi [15] has presented the computational analysis of different Pseudoplatystoma species patterns the Caputo-Fabrizio derivative. Citations [16, 17] provide an analysis and application of a new fractional Adams-Bashforth scheme with Caputo-Fabrizio derivative, as well as numerical approximation of nonlinear fractional parabolic differential equations with Caputo-Fabrizio derivative in Riemann-Liouville sense. Caputo et al. [11] have investigated the singular kernels for fractional derivatives and some applications to partial differential equations. Atangana et al. [6] have investigated the Caputo-Fabrizio derivative applied to groundwater flow within confined aquifer. Aydogan et al. [7] have presented the mathematical model of Rabies by using the fractional Caputo-Fabrizio derivative. Baleanu et al. [9] have studied on a fractional operator combining proportional and classical differintegrals. Hristov [13] have investigated the derivatives with non-singular kernels from the Caputo-Fabrizio definition and beyond.

A new fractional operator has recently been created in [2]:

$$ {}_0^{PCF}D_t^{\sigma}h(t) = \frac{M(\sigma)}{1-\sigma}\int_0^t \left(K_1(\sigma,\rho)h(t) + K_0(\sigma,\rho)h'(t)\right)\exp\left(-\frac{\sigma}{1-\sigma}(t-\rho)\right)d\rho, $$

where $K_1(\sigma,\rho) = (1-\rho)\rho^{\sigma}$, $K_0(\sigma,\rho) = \sigma\rho^{1-\sigma}c^{2\sigma}$ and the two terms in the supplied derivative use c as a constant. For more details see [3, 10]. Since the proportional derivative is an important tool especially for engineering applications, the properties of this derivative should be investigated. Our aim is to investigate the properties of this derivative in details in this work.

The organisation of our manuscript is as follows. Investigation of proportional Caputo-Fabrizio Derivative has been given in Section 2. Some main theorems have been proved in this section. Discretization of the proposed derivative has been presented in Section 3. Stability analysis has been given in Section 4. Numerical results have been given in Section 5. Some conclusions have been presented in the last section.

2. Investigation of the proportional Caputo-Fabrizio derivative

The Caputo-Fabrizio derivative is modified by the addition of a proportionate element to create the proportional Caputo-Fabrizio derivative. The proportional Caputo-Fabrizio derivative has been applied to modeling of diffusion processes, fractional control systems, and viscoelastic materials. As it retains causality and resolves the beginning value problem, it has several advantages over other fractional derivatives like the Riemann-Liouville and Caputo derivatives.

Theorem 2.1. *We arrive at the relationship shown below for the supplied derivative as ([2]):*

$$ \left|{}_0^{PCF}D_t^{\sigma}h(t)\right| < t^{\sigma+1}M(\alpha)\|h\|\Gamma(\alpha+1)E_{1,\sigma+2}\left(-\frac{\sigma}{1-\sigma}t\right) + \frac{\Gamma(2-\sigma)c^{2\sigma}\sigma M(\sigma)}{1-\sigma}t^{2-\sigma}\|h'\|_{\infty}E_{1,3-\sigma}\left(-\frac{\sigma}{1-\sigma}t\right). $$

Proof. We have

$$ \begin{aligned} &\left|{}_0^{PCF}D_t^{\sigma}h(t)\right| \\ &= \frac{M(\sigma)}{1-\sigma}\left|\int_0^t \left(K_1(\sigma,\rho)h(t) + K_0(\sigma,\rho)h'(t)\right)\exp\left(-\frac{\sigma}{1-\sigma}(t-\rho)\right)d\rho\right| \end{aligned} $$

$$
\begin{aligned}
&\leqslant \frac{M(\sigma)}{1-\sigma}\left|\int_0^t K_1(\sigma,\rho)h(t)\exp\left(-\frac{\sigma}{1-\sigma}(t-\rho)\right)d\rho\right| + \frac{M(\sigma)}{1-\sigma}\left|\int_0^t K_0(\sigma,\rho)h'(t)\exp\left(-\frac{\sigma}{1-\sigma}(t-\rho)\right)d\rho\right| \\
&\leqslant \frac{M(\sigma)}{1-\sigma}\left|\int_0^t h(t)(1-\rho)\rho^{\sigma}\exp\left(-\frac{\sigma}{1-\sigma}(t-\rho)\right)d\rho\right| + \frac{M(\sigma)}{1-\sigma}\left|\int_0^t h'(t)\sigma\rho^{1-\sigma}c^{2\sigma}\exp\left(-\frac{\sigma}{1-\sigma}(t-\rho)\right)d\rho\right| \\
&\leqslant M(\sigma)\int_0^t \rho^{\sigma}\exp\left(-\frac{\sigma}{1-\sigma}(t-\rho)\right)|h(\rho)|d\rho + \frac{\sigma M(\sigma)c^{2\sigma}}{1-\sigma}\int_0^t \rho^{1-\sigma}\exp\left(-\frac{\sigma}{1-\sigma}(t-\rho)\right)|h'(\rho)|d\rho \\
&\leqslant M(\sigma)\int_0^t \rho^{\sigma}\exp\left(-\frac{\sigma}{1-\sigma}(t-\rho)\right)\sup_{\rho\in[0,t]}|h(\rho)|d\rho + \frac{\sigma M(\sigma)c^{2\sigma}}{1-\sigma}\int_0^t \rho^{1-\sigma}\exp\left(-\frac{\sigma}{1-\sigma}(t-\rho)\right)\sup_{\rho\in[0,t]}|h'(\rho)|d\rho \\
&\leqslant M(\sigma)\|h\|_\infty\int_0^t \rho^{\sigma}\sum_{k=0}^{\infty}\frac{\left(-\frac{\sigma}{1-\sigma}(t-\rho)\right)^k}{k!}d\rho + \frac{\sigma M(\sigma)c^{2\sigma}}{1-\sigma}\|h'\|_\infty\int_0^t \rho^{1-\sigma}\sum_{k=0}^{\infty}\frac{\left(-\frac{\sigma}{1-\sigma}(t-\rho)\right)^k}{k!}d\rho \\
&\leqslant M(\sigma)\|h\|_\infty\sum_{k=0}^{\infty}\frac{\left(-\frac{\sigma}{1-\sigma}\right)^k}{k!}\int_0^t \rho^{\sigma}(t-\rho)^k d\rho + \frac{\sigma M(\sigma)c^{2\sigma}}{1-\sigma}\|h'\|_\infty\sum_{k=0}^{\infty}\frac{\left(-\frac{\sigma}{1-\sigma}\right)^k}{k!}\int_0^t \rho^{1-\sigma}(t-\rho)^k d\rho.
\end{aligned}
$$

Let $\rho = th$. Then, we obtain

$$
\begin{aligned}
&\left|{}_0^{PCF}D_t^{\sigma}h(t)\right| \\
&\leqslant M(\sigma)\|h\|_\infty\sum_{k=0}^{\infty}\frac{\left(-\frac{\sigma}{1-\sigma}\right)^k}{k!}\int_0^t (th)^{\sigma}(t-th)^k t\,dh + \frac{\sigma M(\sigma)c^{2\sigma}}{1-\sigma}\|h'\|_\infty\sum_{k=0}^{\infty}\frac{\left(-\frac{\sigma}{1-\sigma}\right)^k}{k!}\int_0^t (th)^{1-\sigma}(t-th)^k t\,dh \\
&\leqslant M(\sigma)\|h\|_\infty\sum_{k=0}^{\infty}\frac{\left(-\frac{\sigma}{1-\sigma}\right)^k}{k!}t^{\sigma+k+1}\int_0^1 h^{\sigma}(1-h)^k dh + \frac{\sigma M(\sigma)c^{2\sigma}}{1-\sigma}\|h'\|_\infty\sum_{k=0}^{\infty}\frac{\left(-\frac{\sigma}{1-\sigma}\right)^k}{k!}t^{2+k-\sigma}\int_0^1 h^{1-\sigma}(1-h)^k dh \\
&\leqslant M(\sigma)\|h\|_\infty\sum_{k=0}^{\infty}\frac{\left(-\frac{\sigma}{1-\sigma}\right)^k}{k!}t^{\sigma+k+1}\beta(\sigma+1,k+1) + \frac{\sigma M(\sigma)c^{2\sigma}}{1-\sigma}\|h'\|_\infty\sum_{k=0}^{\infty}\frac{\left(-\frac{\sigma}{1-\sigma}\right)^k}{k!}t^{2+k-\sigma}\beta(2-\sigma,k+1) \\
&\leqslant M(\sigma)\|h\|_\infty\sum_{k=0}^{\infty}\frac{\left(-\frac{\sigma}{1-\sigma}\right)^k}{k!}t^{\sigma+k+1}\frac{\Gamma(\sigma+1)\Gamma(k+1)}{\Gamma(\sigma+k+2)} + \frac{\sigma M(\sigma)c^{2\sigma}}{1-\sigma}\|h'\|_\infty\sum_{k=0}^{\infty}\frac{\left(-\frac{\sigma}{1-\sigma}\right)^k}{k!}t^{2+k-\sigma}\frac{\Gamma(2-\sigma)\Gamma(k+1)}{\Gamma(3-\sigma+k)} \\
&\leqslant M(\sigma)\|h\|_\infty\Gamma(\sigma+1)t^{\sigma+1}\sum_{k=0}^{\infty}\frac{\left(-\frac{\sigma}{1-\sigma}\right)^k}{\Gamma(\sigma+k+2)} + \frac{\sigma M(\sigma)c^{2\sigma}\Gamma(2-\sigma)}{1-\sigma}\|h'\|_\infty t^{2-\sigma}\sum_{k=0}^{\infty}\frac{\left(-\frac{\sigma}{1-\sigma}\right)^k}{\Gamma(3-\sigma+k)} \\
&\leqslant M(\sigma)\|h\|_\infty\Gamma(\sigma+1)t^{\sigma+1}E_{1,\sigma+2}\left(-\frac{\sigma t}{1-\sigma}\right) + \frac{\sigma M(\sigma)c^{2\sigma}\Gamma(2-\sigma)}{1-\sigma}\|h'\|_\infty t^{2-\sigma}E_{1,3-\sigma}\left(-\frac{\sigma t}{1-\sigma}\right).
\end{aligned}
$$

□

Theorem 2.2. *Suppose that g and h can be differentiated and are constrained. Next, we achieve ([3]):*

$$
\begin{aligned}
\left|{}_0^{PCF}D_t^{\sigma}g(t)h(t)\right| \leqslant{}& M(\sigma)\|g\|_\infty\|h\|_\infty t^{\sigma+1}\Gamma(\sigma+1)E_{(1,\sigma+2)}\left(-\frac{\sigma t}{1-\sigma}\right) \\
&+ \frac{\sigma c^{2\sigma}M(\sigma)}{1-\sigma}\|g'\|_\infty\|h\|_\infty t^{2-\sigma}\Gamma(2-\sigma)E_{(1,3-\sigma)}\left(-\frac{\sigma t}{1-\sigma}\right) \\
&+ \frac{\sigma c^{2\sigma}M(\sigma)}{1-\sigma}\|g'\|_\infty\|h'\|_\infty t^{2-\sigma}\Gamma(2-\sigma)E_{(1,3-\sigma)}\left(-\frac{\sigma t}{1-\sigma}\right).
\end{aligned}
$$

Proof. We obtain:

$$
\begin{aligned}
&\left|{}_0^{PCF}D_t^{\sigma}g(t)h(t)\right| \\
&= \frac{M(\sigma)}{1-\sigma}\int_0^t (K_1(\sigma,\rho)g(\rho)h(\rho) + K_0(\sigma,\rho)(g(\rho)h'(\rho)))\exp(-\frac{\sigma}{1-\sigma}(t-\rho))d\rho \\
&= \frac{M(\sigma)}{1-\sigma}\int_0^t (K_1(\sigma,\rho)g(\rho)h(\rho) + K_0(\sigma,\rho)(g'(\rho)h(\rho)) + g(\rho)h'(\rho)))\exp(-\frac{\sigma}{1-\sigma}(t-\rho))d\rho
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \frac{M(\sigma)}{1-\sigma}\int_0^t (K_1(\sigma,\rho)|g(\rho)||h(\rho)|\exp(-\frac{\sigma}{1-\sigma}(t-\rho))d\rho + \frac{M(\sigma)}{1-\sigma}\int_0^t (K_0(\sigma,\rho)|g^{'}(\rho)||h(\rho)|\exp(-\frac{\sigma}{1-\sigma}(t-\rho))d\rho \\
&\quad + \frac{M(\sigma)}{1-\sigma}\int_0^t (K_0(\sigma,\rho)|g(\rho)||h^{'}(\rho)|\exp(-\frac{\sigma}{1-\sigma}(t-\rho))d\rho \\
&\leqslant \frac{M(\sigma)}{1-\sigma}\int_0^t (1-\sigma)\rho^{\sigma}\sup|g(\rho)|\sup|h(\rho)|\exp(-\frac{\sigma}{1-\sigma}(t-\rho))d\rho \\
&\quad + \frac{M(\sigma)}{1-\sigma}\int_0^t \sigma\rho^{1-\sigma}c^{2\sigma}\sup|g^{'}(\rho)|\sup|h(\rho)|\exp(-\frac{\sigma}{1-\sigma}(t-\rho))d\rho \\
&\quad + \frac{M(\sigma)}{1-\sigma}\int_0^t \sigma\rho^{1-\sigma}c^{2\sigma}\sup|g(\rho)|\sup|h^{'}(\rho)|\exp(-\frac{\sigma}{1-\sigma}(t-\rho))d\rho \\
&\leqslant M(\sigma)\|g\|_\infty\|h\|_\infty\int_0^t \rho^{\sigma}\exp(-\frac{\sigma}{1-\sigma}(t-\rho))d\rho + \frac{\sigma c^{2\sigma}M(\sigma)}{1-\sigma}\|g^{'}\|_\infty\|h\|_\infty\int_0^t \rho^{1-\sigma}\exp(-\frac{\sigma}{1-\sigma}(t-\rho))d\rho \\
&\quad + \frac{\sigma c^{2\sigma}M(\sigma)}{1-\sigma}\|g\|_\infty\|h^{'}\|_\infty\int_0^t \rho^{1-\sigma}\exp(-\frac{\sigma}{1-\sigma}(t-\rho))d\rho \\
&\leqslant M(\sigma)\|g\|_\infty\|h\|_\infty\int_0^t \rho^{\sigma}\sum_{k=0}^{\infty}\frac{(-\frac{\sigma}{1-\sigma}(t-\rho))^k}{k!}d\rho + \frac{\sigma c^{2\sigma}M(\sigma)}{1-\sigma}\|g^{'}\|_\infty\|h\|_\infty\int_0^t \rho^{1-\sigma}\sum_{k=0}^{\infty}\frac{(-\frac{\sigma}{1-\sigma}(t-\rho))^k}{k!}d\rho \\
&\quad + \frac{\sigma c^{2\sigma}M(\sigma)}{1-\sigma}\|g\|_\infty\|h^{'}\|_\infty\int_0^t \rho^{1-\sigma}\sum_{k=0}^{\infty}\frac{(-\frac{\sigma}{1-\sigma}(t-\rho))^k}{k!}d\rho \\
&\leqslant M(\sigma)\|g\|_\infty\|h\|_\infty\sum_{k=0}^{\infty}\frac{(-\frac{\sigma}{1-\sigma})^k}{k!}\int_0^t \rho^{\sigma}(t-\rho)^k d\rho + \frac{\sigma c^{2\sigma}M(\sigma)}{1-\sigma}\|g^{'}\|_\infty\|h\|_\infty\sum_{k=0}^{\infty}\frac{(-\frac{\sigma}{1-\sigma})^k}{k!}\int_0^t \rho^{1-\sigma}(t-\rho)^k d\rho \\
&\quad + \frac{\sigma c^{2\sigma}M(\sigma)}{1-\sigma}\|g\|_\infty\|h^{'}\|_\infty\sum_{k=0}^{\infty}\frac{(-\frac{\sigma}{1-\sigma})^k}{k!}\int_0^t \rho^{1-\sigma}(t-\rho)^k d\rho \\
&\leqslant M(\sigma)\|g\|_\infty\|h\|_\infty\sum_{k=0}^{\infty}\frac{(-\frac{\sigma}{1-\sigma})^k}{k!}\int_0^1 (th)^{\sigma}(t-th)^k t dh + \frac{\sigma c^{2\sigma}M(\sigma)}{1-\sigma}\|g^{'}\|_\infty\|h\|_\infty\sum_{k=0}^{\infty}\frac{(-\frac{\sigma}{1-\sigma})^k}{k!}\int_0^1 (th)^{1-\sigma}(t-th)^k t dh \\
&\quad + \frac{\sigma c^{2\sigma}M(\sigma)}{1-\sigma}\|g\|_\infty\|h^{'}\|_\infty\sum_{k=0}^{\infty}\frac{(-\frac{\sigma}{1-\sigma})^k}{k!}\int_0^1 (th)^{1-\sigma}(t-th)^k t dh \\
&\leqslant M(\sigma)\|g\|_\infty\|h\|_\infty\sum_{k=0}^{\infty}\frac{(-\frac{\sigma}{1-\sigma})^k}{k!}\int_0^1 t^{\sigma}h^{\sigma}t^k(1-h)^k t dh \\
&\quad + \frac{\sigma c^{2\sigma}M(\sigma)}{1-\sigma}\|g^{'}\|_\infty\|h\|_\infty\sum_{k=0}^{\infty}\frac{(-\frac{\sigma}{1-\sigma})^k}{k!}\int_0^1 t^{1-\sigma}h^{1-\sigma}t^k(1-h)^k t dh \\
&\quad + \frac{\sigma c^{2\sigma}M(\sigma)}{1-\sigma}\|g\|_\infty\|h^{'}\|_\infty\sum_{k=0}^{\infty}\frac{(-\frac{\sigma}{1-\sigma})^k}{k!}\int_0^1 t^{1-\sigma}h^{1-\sigma}t^k(1-h)^k t dh \\
&\leqslant M(\sigma)\|g\|_\infty\|h\|_\infty\sum_{k=0}^{\infty}\frac{(-\frac{\sigma}{1-\sigma})^k}{k!}t^{\sigma+k+1}\int_0^1 h^{\sigma}(1-h)^k dh \\
&\quad + \frac{\sigma c^{2\sigma}M(\sigma)}{1-\sigma}\|g^{'}\|_\infty\|h\|_\infty\sum_{k=0}^{\infty}\frac{(-\frac{\sigma}{1-\sigma})^k}{k!}t^{2-\sigma+k}\int_0^1 h^{1-\sigma}(1-h)^k dh \\
&\quad + \frac{\sigma c^{2\sigma}M(\sigma)}{1-\sigma}\|g\|_\infty\|h^{'}\|_\infty\sum_{k=0}^{\infty}\frac{(-\frac{\sigma}{1-\sigma})^k}{k!}t^{2-\sigma+k}\int_0^1 h^{1-\sigma}(1-h)^k dh \\
&\leqslant M(\sigma)\|g\|_\infty\|h\|_\infty t^{\sigma+1}\sum_{k=0}^{\infty}\frac{(-\frac{\sigma t}{1-\sigma})^k}{k!}\beta(\sigma+1,k+1) + \frac{\sigma c^{2\sigma}M(\sigma)}{1-\sigma}\|g^{'}\|_\infty\|h\|_\infty t^{2-\sigma}\sum_{k=0}^{\infty}\frac{(-\frac{\sigma t}{1-\sigma})^k}{k!}\beta(2-\sigma,k+1) \\
&\quad + \frac{\sigma c^{2\sigma}M(\sigma)}{1-\sigma}\|g\|_\infty\|h^{'}\|_\infty t^{2-\sigma}\sum_{k=0}^{\infty}\frac{(-\frac{\sigma t}{1-\sigma})^k}{k!}\beta(2-\sigma,k+1)
\end{aligned}
$$

$$\leqslant M(\sigma)\|g\|_\infty\|h\|_\infty t^{\sigma+1}\sum_{k=0}^{\infty}\frac{(-\frac{\sigma t}{1-\sigma})^k}{k!}\frac{\Gamma(\sigma+1)\Gamma(k+1)}{\Gamma(\sigma+k+2)}+\frac{\sigma c^{2\sigma}M(\sigma)}{1-\sigma}\|g^{'}\|_\infty\|h\|_\infty t^{2-\sigma}\sum_{k=0}^{\infty}\frac{(-\frac{\sigma t}{1-\sigma})^k}{k!}\frac{\Gamma(2-\sigma)\Gamma(k+1)}{\Gamma(3-\sigma+k)}$$
$$+\frac{\sigma c^{2\sigma}M(\sigma)}{1-\sigma}\|g\|_\infty\|h^{'}\|_\infty t^{2-\sigma}\sum_{k=0}^{\infty}\frac{(-\frac{\sigma t}{1-\sigma})^k}{k!}\frac{\Gamma(2-\sigma)\Gamma(k+1)}{\Gamma(3-\sigma+k)}$$
$$\leqslant M(\sigma)\|g\|_\infty\|h\|_\infty t^{\sigma+1}\Gamma(\sigma+1)E_{(1,\sigma+2)}(-\frac{\sigma t}{1-\sigma})+\frac{\sigma c^{2\sigma}M(\sigma)}{1-\sigma}\|g^{'}\|_\infty\|h\|_\infty t^{2-\sigma}t^{2-\sigma}\Gamma(2-\sigma)E_{(1,3-\sigma)}(-\frac{\sigma t}{1-\sigma})$$
$$+\frac{\sigma c^{2\sigma}M(\sigma)}{1-\sigma}\|g\|_\infty\|h^{'}\|_\infty t^{2-\sigma}\Gamma(2-\sigma)E_{(1,3-\sigma)}(-\frac{\sigma t}{1-\sigma}).$$

□

Theorem 2.3. *If* g *and* h *are differentiable, then we have*

$$\|{}_0^{PCF}D_t^{\sigma}g(t)-{}_0^{PCF}D_t^{\sigma}h(t)\|_\infty<m(\sigma)\Gamma(\sigma+1)\|g-h\|_\infty E_{(1,\sigma+2)}\left(\frac{-\sigma}{1-\sigma}t\right)$$
$$+\frac{\sigma C^{2\sigma}M(\sigma)}{1-\sigma}\|g^{'}-h^{'}\|_\infty t^{2-\sigma}\Gamma(2-\sigma)E_{(1,3-\sigma)}\left(\frac{-\sigma}{1-\sigma}t\right).$$

Proof. We have

$$\|{}_0^{PCF}D_t^{\sigma}g(t)-{}_0^{PCF}D_t^{\sigma}h(t)\|_\infty$$
$$=\|\frac{M(\sigma)}{1-\sigma}\int_0^t(k_1(\sigma,\rho)g(\rho)+k_0(\sigma,\rho)g^{'}(\rho))\exp\left(\frac{-\sigma}{1-\sigma}(t-\rho)\right)d\rho$$
$$-\frac{M(\sigma)}{1-\sigma}\int_0^t(k_1(\sigma,\rho)h(\rho)+k_0(\sigma,\rho)h^{'}(\rho))\exp\left(\frac{-\sigma}{1-\sigma}(t-\rho)\right)d\rho\|$$
$$=\frac{M(\sigma)}{1-\sigma}\|\int_0^t(k_1(\sigma,\rho)(g(\rho)-h(\rho))\exp\left(\frac{-\sigma}{1-\sigma}(t-\rho)\right)+(k_0(\sigma,\rho))(g^{'}(\rho)-h^{'}(\rho))\exp\left(\frac{-\sigma}{1-\sigma}(t-\rho)\right)\|_\infty$$
$$\leqslant\frac{M(\sigma)}{1-\sigma}\|g-h\|_\infty\int_0^t(1-\sigma)\rho^\sigma\exp\left(\frac{-\sigma}{1-\sigma}(t-\rho)\right)d\rho+\frac{M(\sigma)}{1-\sigma}\|g^{'}-h'\|_\infty\int_0^t\sigma t^{1-\sigma}C^{2\sigma}\exp\left(\frac{-\sigma}{1-\sigma}(t-\rho)\right)d\rho$$
$$\leqslant\frac{M(\sigma)}{1-\sigma}\|g-h\|_\infty(1-\sigma)\int_0^t\rho^\sigma\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}(t-\rho)\right)^k}{k!}d\rho+\frac{M(\sigma)}{1-\sigma}\|g^{'}-h'\|_\infty\sigma C^{2\sigma}\int_0^t\rho^{1-\sigma}\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}(t-\rho)\right)^k}{k!}d\rho$$
$$\leqslant M(\sigma)\|g-h\|_\infty\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}\right)^k}{k!}\int_0^t\rho^\sigma(t-\rho)^k d\rho+\frac{\sigma C^{2\sigma}M(\sigma)}{1-\sigma}\|g^{'}-h'\|_\infty\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}\right)^k}{k!}\int_0^t\rho^{1-\sigma}(t-\rho)^k d\rho$$
$$\leqslant M(\sigma)\|g-h\|_\infty\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}\right)^k}{k!}\int_0^1(th)^\sigma(t-th)^k t dh+\frac{\sigma C^{2\sigma}M(\sigma)}{1-\sigma}\|g^{'}-h'\|_\infty\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}\right)^k}{k!}\int_0^1(th)^{1-\sigma}(t-th)^k t dh$$
$$\leqslant M(\sigma)\|g-h\|_\infty\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}\right)^k}{k!}t^{\sigma+k+1}\int_0^1 h^\sigma(1-h)^k dh$$
$$+\frac{\sigma C^{2\sigma}M(\sigma)}{1-\sigma}\|g^{'}-h'\|_\infty\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}\right)^k}{k!}t^{2-\sigma+k}\int_0^1 h^{1-\sigma}(1-h)^k dh$$
$$\leqslant M(\sigma)\|g-h\|_\infty\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}\right)^k}{k!}t^{\sigma+k+1}B(\sigma+1,k+1)+\frac{\sigma C^{2\sigma}M(\sigma)}{1-\sigma}\|g^{'}-h'\|_\infty\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}\right)^k}{k!}t^{2-\sigma+k}B(2-\sigma,k+1)$$
$$\leqslant M(\sigma)\|g-h\|_\infty t^{\sigma+1}\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma t}{1-\sigma}\right)^k}{\Gamma(k+1)}\frac{\Gamma(\sigma+1)\Gamma(k+1)}{\Gamma(\sigma+k+2)}+\frac{\sigma C^{2\sigma}M(\sigma)}{1-\sigma}\|g^{'}-h'\|_\infty\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma t}{1-\sigma}\right)^k}{\Gamma(k+1)}t^{2-\sigma}\frac{\Gamma(2-\sigma)\Gamma(k+1)}{\Gamma(3-\sigma+k)}$$
$$\leqslant M(\sigma)\|g-h\|_\infty t^{\sigma+1}\Gamma(\sigma+1)E_{(1,\sigma+2)}\left(\frac{-\sigma}{1-\sigma}t\right)+\frac{\sigma C^{2\sigma}M(\sigma)}{1-\sigma}\|g^{'}-h'\|_\infty t^{2-\sigma}\Gamma(2-\sigma)E_{(1,3-\sigma)}\left(\frac{-\sigma}{1-\sigma}t\right).$$

□

Theorem 2.4. *Let* h *be analytic around 0. Thus, we reach*

$$
\begin{aligned}
{}_{0}^{PCF}D_t^{\sigma}h(t) = M(\sigma)\sum_{j=0}^{\infty} a_j t^{j+\sigma+1}\Gamma(j+\sigma+1)E_{(1,j+\sigma+2)}\left(\frac{-\sigma}{1-\sigma}t\right) \\
+ \frac{M(\sigma)\sigma}{1-\sigma}C^{2\sigma}\sum_{j=0}^{\infty} j a_j t^{j+\sigma}\Gamma(j+\sigma)E_{(1,j+\sigma+1)}\left(\frac{-\sigma}{1-\sigma}t\right).
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
&{}_{0}^{PCF}D_t^{\sigma}h(t) \\
&= \frac{M(\sigma)}{1-\sigma}\int_0^t (k_1(\sigma,\rho)h(\rho)+k_0(\sigma,\rho)h'(\rho))\exp\left(\frac{-\sigma}{1-\sigma}(t-\rho)\right)d\rho \\
&= \frac{M(\sigma)}{1-\sigma}\int_0^t ((1-\sigma)\rho^{\sigma}h(\rho)+\sigma C^{2\sigma}\rho^{1-\sigma}h'(\rho))\exp\left(\frac{-\sigma}{1-\sigma}(t-\rho)\right)d\rho \\
&= \frac{M(\sigma)}{1-\sigma}\sum_{j=0}^{\infty} a_j(1-\sigma)\int_0^t \rho^{j+\sigma}\exp\left(\frac{-\sigma}{1-\sigma}(t-\rho)\right)d\rho + \frac{M(\sigma)}{1-\sigma}\sigma C^{2\sigma}\sum_{j=0}^{\infty} a_j j\int_0^t \rho^{j+\sigma-1}\exp\left(\frac{-\sigma}{1-\sigma}(t-\rho)\right)d\rho \\
&= M(\sigma)\sum_{j=0}^{\infty} a_j\int_0^t \rho^{j+\sigma}\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}(t-\rho)\right)^k}{k!}d\rho + \frac{M(\sigma)}{1-\sigma}\sigma C^{2\sigma}\sum_{j=0}^{\infty} j a_j\int_0^t \rho^{j+\sigma-1}\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}(t-\rho)\right)^k}{k!}d\rho \\
&= M(\sigma)\sum_{j=0}^{\infty} a_j\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}\right)^k}{k!}\int_0^t \rho^{j+\sigma}(t-\rho)^k d\rho + \frac{M(\sigma)}{1-\sigma}\sigma C^{2\sigma}\sum_{j=0}^{\infty} j a_j\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}\right)^k}{k!}\int_0^t \rho^{j+\sigma-1}(t-\rho)^k d\rho \\
&= M(\sigma)\sum_{j=0}^{\infty} a_j\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}\right)^k}{k!}\int_0^1 (th)^{j+\sigma}(t-th)^k\, tdh + \frac{M(\sigma)}{1-\sigma}\sigma C^{2\sigma}\sum_{j=0}^{\infty} j a_j\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}\right)^k}{k!}\int_0^1 (th)^{j+\sigma-1}(t-th)^k\, tdh \\
&= M(\sigma)\sum_{j=0}^{\infty} a_j\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}\right)^k}{k!}\int_0^1 (th)^{j+\sigma}(t-th)^k\, tdh + \frac{M(\sigma)}{1-\sigma}\sigma C^{2\sigma}\sum_{j=0}^{\infty} j a_j\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}\right)^k}{k!}\int_0^1 (th)^{j+\sigma-1}(t-th)^k\, tdh \\
&= M(\sigma)\sum_{j=0}^{\infty} a_j\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}\right)^k}{k!}t^{j+\sigma+k+1}\int_0^1 h^{j+\sigma}(1-h)^k dh \\
&\quad + \frac{M(\sigma)}{1-\sigma}\sigma C^{2\sigma}\sum_{j=0}^{\infty} j a_j\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}\right)^k}{k!}t^{j+\sigma+k}\int_0^1 h^{j+\sigma-1}(1-h)^k dh \\
&= M(\sigma)\sum_{j=0}^{\infty} a_j\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}\right)^k}{k!}t^{j+\sigma+k+1}B(j+\sigma+1,k+1) + \frac{M(\sigma)}{1-\sigma}\sigma C^{2\sigma}\sum_{j=0}^{\infty} j a_j\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}\right)^k}{k!}t^{j+\sigma+k}B(j+\sigma,k+1) \\
&= M(\sigma)\sum_{j=0}^{\infty} a_j\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}\right)^k}{k!}t^{j+\sigma+k+1}\frac{\Gamma(j+\sigma+1)\Gamma(k+1)}{\Gamma(j+\sigma+k+2)} + \frac{M(\sigma)}{1-\sigma}\sigma C^{2\sigma}\sum_{j=0}^{\infty} j a_j\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma}{1-\sigma}\right)^k}{k!}t^{j+\sigma+k}\frac{\Gamma(j+\sigma)\Gamma(k+1)}{\Gamma(j+\sigma+k+1)} \\
&= M(\sigma)\sum_{j=0}^{\infty} a_j t^{j+\sigma+1}\Gamma(j+\sigma+1)E_{(1,j+\sigma+2)}\left(\frac{-\sigma}{1-\sigma}t\right) + \frac{M(\sigma)\sigma}{1-\sigma}C^{2\sigma}\sum_{j=0}^{\infty} j a_j t^{j+\sigma}\Gamma(j+\sigma)E_{(1,j+\sigma+1)}\left(\frac{-\sigma}{1-\sigma}t\right).
\end{aligned}
$$

□

3. Discretization of the proposed derivative

The process of transforming a continuous system or signal into a discrete system or signal is known as discretization. In other words, it entails sampling a continuous signal at predetermined time intervals to produce a series of discrete values that a computer can represent. Numerous scientific and engineering disciplines, such as digital signal processing, control theory, numerical analysis, and computer graphics, depend on discretization. We take into consideration:

$$ {}_{0}^{PCF}D_{t}^{\sigma}h(t)=\frac{M(\sigma)}{1-\sigma}\int_{0}^{t}\Big(k_1(\sigma,\rho)h(\rho)+k_1(\sigma,\rho)h^{'}(\rho)\Big)\exp\left(\frac{-\sigma}{1-\sigma}(t-\rho)\right)d\rho. $$

We put $t_n=n\Delta_t$, then at t_{n+1}, we have

$$ \begin{aligned}
&{}_{0}^{PCF}D_{t}^{\sigma}h(t_{n+1})\\
&=\frac{M(\sigma)}{1-\sigma}\int_{0}^{t_{n+1}}(1-\sigma)\rho^{\sigma}h(\rho)+\sigma c^{2\sigma}\rho^{1-\sigma}h^{'}(\rho)\exp\left(\frac{-\sigma}{1-\sigma}(t_{n+1}-\rho)\right)d\rho\\
&=\frac{M(\sigma)}{1-\sigma}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\left((1-\sigma)\rho^{\sigma}h^{j+1}+\sigma c^{2\sigma}\rho^{1-\sigma}\frac{h^{j+1}-h^{j}}{\Delta_t}\right)\exp\left(\frac{-\sigma}{1-\sigma}(t_{n+1}-\rho)\right)d\rho\\
&=\frac{M(\sigma)}{1-\sigma}(1-\sigma)\sum_{j=0}^{n}h^{j+1}\int_{t_j}^{t_{j+1}}\rho^{\sigma}\exp\left(\frac{-\sigma}{1-\sigma}(t_{n+1}-\rho)\right)d\rho\\
&\quad+\frac{M(\sigma)}{1-\sigma}\frac{\sigma c^{2\sigma}}{\Delta_t}\sum_{j=0}^{n}(h^{j+1}-h^{j})\int_{t_j}^{t_{j+1}}\rho^{1-\sigma}\exp\left(\frac{-\sigma}{1-\sigma}(t_{n+1}-\rho)\right)d\rho\\
&=M(\sigma)\sum_{j=0}^{n}h^{j+1}\left(\exp(\frac{-\sigma}{1-\sigma}t_{n+1})\left[E_{-\sigma}(\frac{\sigma}{\sigma-1}t_j)t_j^{\sigma+1}-E_{-\sigma}(\frac{\sigma}{\sigma-1}t_{j+1})t_{j+1}^{\sigma+1}\right]\right)\\
&\quad+\frac{M(\sigma)}{1-\sigma}\sigma\frac{c^{2\sigma}}{\Delta_t}\sum_{j=0}^{n}(h^{j+1}-h^{j})\left(\exp(\frac{-\sigma}{1-\sigma}t_{n+1})\left[E_{\sigma-1}(\frac{\sigma}{\sigma-1}t_j)t_j^{2-\sigma}-E_{\sigma-1}(\frac{\sigma}{\sigma-1}t_{j+1})t_{j+1}^{2-\sigma}\right]\right)\\
&=M(\sigma)\sum_{j=0}^{n}h^{j+1}\exp\left(\frac{-\sigma\Delta_t(n+1)}{\sigma-1}\right)\left[E_{-\sigma}(\frac{\sigma j\Delta_t}{\sigma-1})j^{\sigma+1}\Delta_t^{\sigma+1}-E_{-\sigma}(\frac{\sigma(j+1)\Delta_t}{\sigma-1})(j+1)^{\sigma+1}\Delta_t^{\sigma+1}\right]\\
&\quad+\frac{M(\sigma)}{1-\sigma}\frac{\sigma c^{2\sigma}}{\Delta_t}\sum_{j=0}^{n}(h^{j+1}-h^{j})\exp\left(\frac{-\sigma\Delta_t(n+1)}{\sigma-1}\right)\left[E_{\sigma-1}(\frac{\sigma j\Delta_t}{\sigma-1})j^{2-\sigma}\Delta_t^{2-\sigma}-E_{\sigma-1}(\frac{\sigma(j+1)\Delta_t}{\sigma-1})(j+1)^{\sigma+1}\Delta_t^{\sigma+1}\right]\\
&=M(\sigma)\exp\left(\frac{-\sigma\Delta_t(n+1)}{1-\sigma}\right)\Delta_t^{\sigma+1}\sum_{j=0}^{n}h^{j+1}\left[E_{-\sigma}(\frac{\sigma j\Delta_t}{\sigma-1})j^{\sigma+1}-E_{-\sigma}(\frac{\sigma(j+1)\Delta_t}{\sigma-1})(j+1)^{\sigma+1}\right]\\
&\quad+\frac{M(\sigma)}{1-\sigma}\sigma c^{2\sigma}\Delta_t^{1-\sigma}\exp\left(\frac{-\sigma\Delta_t(n+1)}{1-\sigma}\right)\sum_{j=0}^{n}(h^{j+1}-h^{j})\left[E_{\sigma-1}(\frac{\sigma j\Delta_t}{\sigma-1})j^{2-\sigma}-E_{\sigma-1}(\frac{\sigma(j+1)\Delta_t}{\sigma-1})(j+1)^{2-\sigma}\right],
\end{aligned} $$

where $E_n(z)$ is the exponential integral function. We define this function as,

$$ E_n(z)=\int_{1}^{\infty}\frac{\exp(-zt)}{t^n}dt. $$

We take into account the following formula:

$$ {}_{0}^{PCF}D_{t}^{\sigma}h(x,t)=A\left(x,t,h(x,t)\right). $$

Here, $h(x,0)=g(x)$, $x_m-x_{m-1}=\Delta_x$,$t_{n+1}-t_n=\Delta_t$, $t_n=n\Delta_t$, $x_m=m\Delta_x$. The above equation can be written by:

$$ \begin{aligned}
A\left(x_m,t_{n+1},h_m^{n+1}\right)=&\,\Delta_t^{\sigma+1}M(\sigma)\exp\left(\frac{-\sigma}{1-\sigma}\Delta_t(n+1)\right)\sum_{j=0}^{n}h_m^{j+1}\Bigg[E_{-\sigma}\left(\frac{\sigma j\Delta_t}{\sigma-1}\right)j^{\sigma+1}\\
&-E_{-\sigma}\left(\frac{\sigma(j+1)\Delta_t}{\sigma-1}\right)(j+1)^{\sigma+1}\Bigg]+\Delta_t^{1-\sigma}\frac{M(\sigma)}{1-\sigma}c^{2\sigma}\exp\left(\frac{-\sigma}{1-\sigma}\Delta_t(n+1)\right)\\
&\times\sum_{j=0}^{n}(h_m^{j+1}-h_m^{j})\left[E_{\sigma-1}\left(\frac{\sigma j\Delta_t}{\sigma-1}\right)j^{2-\sigma}-E_{\sigma-1}\left(\frac{\sigma(j+1)\Delta_t}{\sigma-1}\right)(j+1)^{2-\sigma}\right].
\end{aligned} $$

4. Stability analysis

We consider

$$\frac{\partial h(x,t)}{\partial t} = k\frac{\partial^2 h(x,t)}{\partial x^2}.$$

In the equation above, we modify the left side as follows:

$$ {}_{0}^{PCF}D_t^{\sigma} h(x,t) = k\frac{\partial^2 h(x,t)}{\partial x^2}.$$

Then, we obtain

$$\begin{aligned}
&\Delta t^{\sigma+1}M(\sigma)\sigma\exp\left(-\frac{\sigma}{1-\sigma}\Delta t(n+1)\right)\sum_{j=0}^{n} h_m^{j+1}\left[E_{\sigma}\left(\frac{\sigma}{\sigma-1}j\Delta t\right)j^{\sigma+1} - E_{-\sigma}\left(\frac{\sigma}{\sigma-1}(j+1)\Delta t\right)(j+1)^{\sigma+1}\right]\\
&\quad + \Delta t^{1-\sigma}\frac{M(\sigma)}{1-\sigma}c^{2\sigma}\exp\left(-\frac{\sigma}{1-\sigma}\Delta t(n+1)\right)\sum_{j=0}^{n}(h_m^{j+1}-h_m^{j})\\
&\quad \times\left[E_{\sigma-1}\left(\frac{\sigma}{\sigma-1}j\Delta t\right)j^{2-\sigma} - E_{\sigma-1}\left(\frac{\sigma}{\sigma-1}(j+1)\Delta t\right)(j+1)^{2-\sigma}\right] = k\frac{h_{m+1}^{j+1}-h_m^{j+1}+h_{m-1}^{j+1}}{(\Delta x)^2}
\end{aligned}$$

at (t_{n+1}, x_m). We put $h_m^n = \delta_n \exp(jk_m x)$. Then, we obtain

$$\begin{aligned}
&\Delta t^{\sigma+1}M(\sigma)\exp\left(-\frac{\sigma}{1-\sigma}\Delta t(n+1)\right)\sum_{j=0}^{n}\delta_{j+1}\exp(jk_m x)\\
&\quad\times\left[E_{-\sigma}\left(\frac{\sigma}{\sigma-1}j\Delta t\right)j^{\sigma+1} - E_{-\sigma}\left(\frac{\sigma}{\sigma-1}(j+1)\Delta t\right)(j+1)^{\sigma+1}\right]\\
&\quad\times\Delta t^{1-\sigma}\frac{M(\sigma)}{1-\sigma}c^{2\sigma}\left(-\frac{\sigma}{1-\sigma}(n+1)\Delta t\right)\sum_{j=0}^{n}\big(\delta_{j+1}\exp(jk_m x)-\delta_j\exp(jk_m x)\big)\\
&\quad\times\left[E_{\sigma-1}\left(\frac{\sigma}{\sigma-1}j\Delta t\right)j^{2-\sigma} - E_{\sigma-1}\left(\frac{\sigma}{\sigma-1}(j+1)\Delta t\right)(j+1)^{2-\sigma}\right]\\
&= k\frac{\delta_{n+1}\exp(jk_m(x+\Delta x)) - 2\delta_{n+1}\exp(jk_m x) + \delta_{n+1}\exp(jk_m(x-\Delta x))}{(\Delta x)^2}.
\end{aligned}$$

Following simplification, we arrive at:

$$\begin{aligned}
&\Delta t^{\sigma+1}M(\sigma)\exp\left(-\frac{\sigma}{1-\sigma}\Delta t(n+1)\right)\sum_{j=0}^{n}\delta_{j+1}\left[E_{-\sigma}\left(\frac{\sigma}{\sigma-1}j\Delta t\right)j^{\sigma+1} - E_{-\sigma}\left(\frac{\sigma}{\sigma-1}(j+1)\Delta t\right)(j+1)^{\sigma+1}\right]\\
&\times\Delta t^{1-\sigma}\frac{M(\sigma)}{1-\sigma}c^{2\sigma}\left(-\frac{\sigma}{1-\sigma}(n+1)\Delta t\right)\sum_{j=0}^{n}(\delta_{j+1}-\delta_j)\left[E_{\sigma-1}\left(\frac{\sigma}{\sigma-1}j\Delta t\right)j^{2-\sigma} - E_{\sigma-1}\left(\frac{\sigma}{\sigma-1}(j+1)\Delta t\right)(j+1)^{2-\sigma}\right]\\
&= k\delta_{n+1}\frac{\exp(jk_m\delta x) - 2 + \exp(jk_m\delta x)}{(\Delta x)^2},
\end{aligned}$$

for simplicity, we take

$$\begin{aligned}
A_{n,\sigma} &= \Delta t^{\sigma+1}M(\sigma)\exp\left(-\frac{\sigma}{1-\sigma}\Delta t(n+1)\right),\\
B_{n,\sigma} &= \Delta t^{1-\sigma}\frac{M(\sigma)}{1-\sigma}c^{2\sigma}\left(-\frac{\sigma}{1-\sigma}(n+1)\Delta t\right),
\end{aligned}$$

$$C_{j,\sigma} = \left[E_{-\sigma}\left(\frac{\sigma}{\sigma-1}j\Delta t\right)j^{\sigma+1} - E_{-\sigma}\left(\frac{\sigma}{\sigma-1}(j+1)\Delta t\right)(j+1)^{\sigma+1}\right],$$
$$D_{j,\sigma} = \left[E_{\sigma-1}\left(\frac{\sigma}{\sigma-1}j\Delta t\right)j^{2-\sigma} - E_{\sigma-1}\left(\frac{\sigma}{\sigma-1}(j+1)\Delta t\right)(j+1)^{2-\sigma}\right],$$
$$M = \frac{k}{(\Delta x)^2}.$$

Then, we get

$$A_{n,\sigma}\sum_{j=0}^{n} C_{j,\sigma}\delta_{j+1} + B_{n,\sigma}\sum_{j=0}^{n}(\delta_{j+1}-\delta j)D_{j,\sigma} = M\delta_{n+1}(\exp(jk_m\delta x) - 2 + \exp(-jk_m\delta x)).$$

We make use of the connection between exponential and trigonometric functions. Then, we reach

$$\sum_{j=0}^{n}\left[A_{n,\sigma}C_{j,\sigma}\delta_{j+1} + B_{n,\sigma}(\delta_{j+1}-\delta_j)D_{j,\sigma}\right] = -4M\delta_{j+1}\sin^2\left(\frac{k_m\Delta x}{2}\right).$$

For n=0, we get

$$A_{0,\sigma}C_{0,\sigma}\delta_1 + B_{0,\sigma}\delta_1 D_{0,\sigma} - \delta_0 B_{0,\sigma}D_{0,\sigma} = -4M\delta_1\sin^2\left(\frac{k_m\delta x}{2}\right),$$
$$\delta_1\sin^2\left(\frac{k_m\delta x}{2}\right)\delta_1\left(A_{0,\sigma}C_{0,\sigma} + B_{0,\sigma}D_{0,\sigma} + 4M\sin^2\left(\frac{k_m\delta x}{2}\right)\right) = B_{0,\sigma}D_{0,\sigma}\delta_0.$$

Here $\left|\frac{\delta_1}{\delta_0}\right| < 1$ implies:

$$\left|\frac{B_{0,\sigma}D_{0,\sigma}\delta_0}{A_{0,\sigma}C_{0,\sigma} + B_{0,\sigma}D_{0,\sigma} + 4M\sin^2\left(\frac{k_m\delta x}{2}\right)}\right| < 1.$$

This is true for $\forall_m$. Thus, we reach

$$\left|\frac{B_{0,\sigma}D_{0,\sigma}}{A_{0,\sigma}C_{0,\sigma} + B_{0,\sigma}D_{0,\sigma} + 4M}\right| < 1.$$

We assume that $\left|\frac{\delta_n}{\delta_0}\right| < 1$. We need to show that $\left|\frac{\delta_{n+1}}{\delta_0}\right| < 1$. We know that:

$$\begin{aligned}
&\sum_{j=0}^{n}[A_{n,\sigma}C_{j,\sigma}\delta_{j+1} + B_{n,\sigma}(\delta_{j+1}-\delta_j)D_{j,\sigma}] \\
&\quad = -4M\delta_{n+1}\sin^2\left(\frac{km\Delta x}{2}\right)A_{n,\sigma}C_{n,\sigma}\delta_{n+1} + B_{n,\sigma}(\delta_{n+1}-\delta_n)D_{n,\sigma} \\
&\qquad + \sum_{j=0}^{n-1}[A_{n,\sigma}C_{j,\sigma}\delta_{j+1} + B_{n,\sigma}(\delta_{j+1}-\delta_j)D_{j,\sigma}] \\
&\quad = -4M\delta_{n+1}\sin^2\left(\frac{km\Delta x}{2}\right)\delta_{j+1}\left(A_{n,\sigma}C_{n,\sigma} + B_{n,\sigma}D_{n,\sigma} + 4M\sin^2\left(\frac{km\Delta x}{2}\right)\right) \\
&\quad = D_{n,\sigma}\delta_n - \sum_{j=0}^{n-1}[A_{n,\sigma}C_{j,\sigma}\delta_{j+1} + B_{n,\sigma}(\delta_{j+1}-\delta_j)D_{j,\sigma}] \\
&\qquad \times \delta_{n+1}\left|A_{n,\sigma}C_{n,\sigma} + B_{n,\sigma}D_{n,\sigma} + 4M\sin^2\left(\frac{km\Delta x}{2}\right)\right| < \delta_0\left|D_{n,\sigma} - \sum_{j=0}^{n-1}A_{n,\sigma}C_{j,\sigma}\right|.
\end{aligned}$$

Here, $\left|\frac{\delta_{n+1}}{\delta_0}\right| < 1$ implies

$$\frac{\left|D_{n,\sigma} - \sum_{j=0}^{n} A_{n,\sigma}C_{j,\sigma}\right|}{\left|A_{n,\sigma}C_{n,\sigma} + B_{n,\sigma}D_{n,\sigma} + 4M\sin^2\left(\frac{km\Delta x}{2}\right)\right|} < 1.$$

This is true for $\forall_{m.}$. Thus, we obtain

$$\frac{\left|D_{n,\sigma} - \sum_{j=0}^{n} A_{n,\sigma}C_{j,\sigma}\right|}{|A_{n,\sigma}C_{n,\sigma} + B_{n,\sigma}D_{n,\sigma} + 4M|} < 1.$$

Therefore, the method is stable if

$$\min\left(\left|\frac{B_{0,\sigma}D_{0,\sigma}}{A_{0,\sigma}C_{0,\sigma} + B_{0,\sigma}D_{0,\sigma} + 4M}\right|, \frac{\left|D_{n,\sigma} - \sum_{j=0}^{n} A_{n,\sigma}C_{j,\sigma}\right|}{|A_{n,\sigma}C_{n,\sigma} + B_{n,\sigma}D_{n,\sigma} + 4M|}\right) < 1.$$

5. Numerical results

We consider ${}_{0}^{CPCF}D_t^{\sigma}h(t) = \sin t$. We solve the given problem using the Laplace transform to get:

$$\mathcal{L}\left\{{}_{0}^{CPCF}D_t^{\sigma}h(t)\right\} = \mathcal{L}\{\sin t\}, \quad \mathcal{L}\{h(t)\}\left[\frac{M(\sigma)K_1(\sigma)}{\sigma + s(1-\sigma)} + \frac{sM(\sigma)K_0(\sigma)}{\sigma + s(1-\sigma)}\right] - \frac{M(\sigma)K_0}{\sigma + s(1-\sigma)}h(0) = \frac{1}{1+s^2}.$$

After simplification, we get

$$\mathcal{L}\{h(t)\} = \frac{\sigma + s(1-\sigma) + (1+s^2)K_0(\sigma)h(0)}{(1+S^2)(M(\sigma)K_1(\sigma) + sM(\sigma)K_0(\sigma))}.$$

Applying the inverse Laplace transform yields:

$$\begin{aligned} h(t) = {} & \frac{1}{(K_0(\sigma)^2 + K_1(\sigma)^2)\,M(\sigma)} \exp\left(-\frac{K_1(\sigma)}{K_0(\sigma)}t\right)\left[-K_1(\sigma) + \sigma(K_0(\sigma) + K_1(\sigma)) + h(0)(K_0(\sigma)^2 + K_1(\sigma)^2)\right. \\ & \left. \times \exp\left(\frac{K_1(\sigma)}{K_0(\sigma)}\right) + (K_1(\sigma) - \sigma(K_0(\sigma) + K_1(\sigma))\cos(t))\ (K_0(\sigma) - \sigma K_0(\sigma) + (\sigma K_1(\sigma))\sin(t))\right]. \end{aligned}$$

We demonstrate the simulations of the solution for different values of σ in Figures 1-6. In these figures, we choose $K_0(\sigma) = \sigma w^{1-\sigma}$, $K_1(\sigma) = (1-\sigma)w^{\sigma}$, and $w = 0.5$.

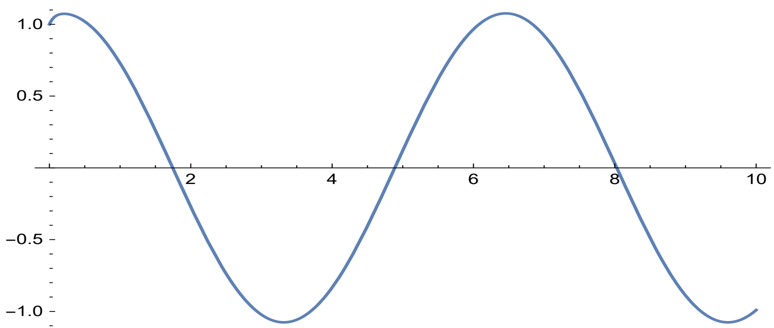

Figure 1: Simulation of solution for $\sigma = 0.1$.

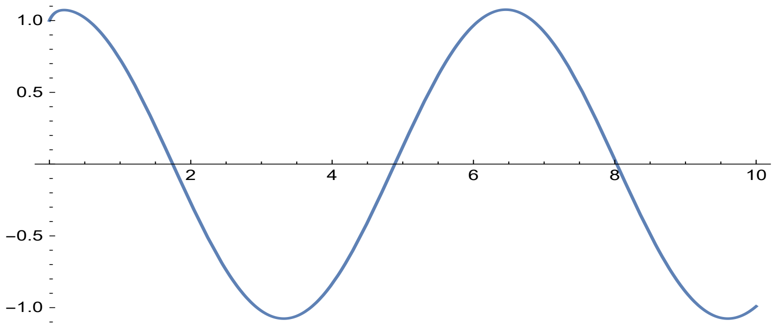

Figure 2: Simulation of solution for $\sigma = 0.2$.

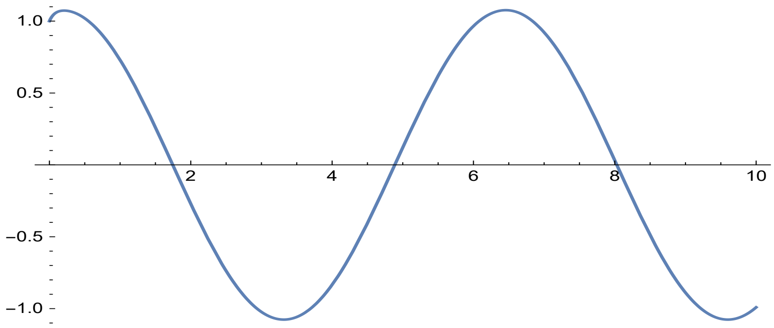

Figure 3: Simulation of solution for $\sigma = 0.3$.

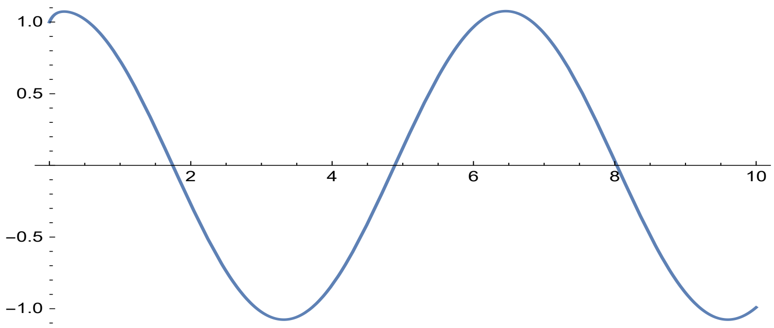

Figure 4: Simulation of solution for $\sigma = 0.65$.

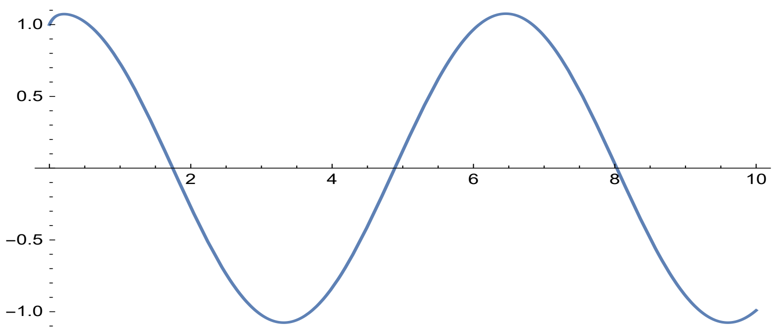

Figure 5: Simulation of solution for $\sigma = 0.9$.

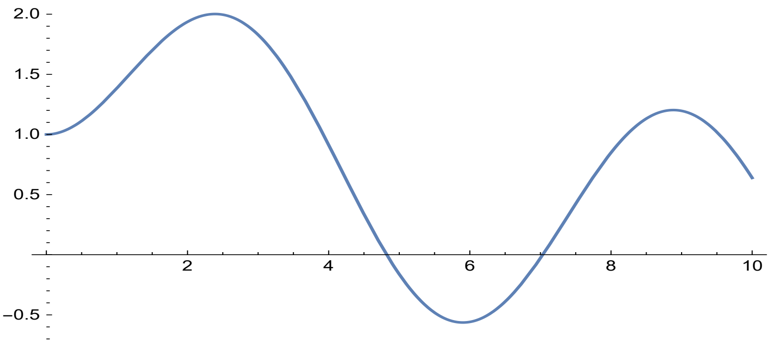

Figure 6: Simulation of solution for $\sigma = 0.8$.

6. Conclusion

In this paper, we covered the study of the proportional Caputo-Fabrizio derivative. We offered some theoretical backing for this unique derivative. The new derivative was discretized. The experiments and stability analyses were topics of discussion. We determined the stability requirement of a problem using the new derivative. We considered a problem involving the constant proportional Caputo-Fabrizio derivative. To solve the problem, we applied the Laplace transform. To demonstrate how the calculations worked in numerical form, we selected a few figures.

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