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Certain properties on Bell-based Apostol-type Frobenius-Genocchi polynomials of complex variable

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Abstract

In this study, we introduce sine and cosine Bell-based Frobenius-type Genocchi polynomials, and by presenting several relations and applications, we analyze certain properties. Our first step is to obtain diverse relations and formulas that cover summation formulas, addition formulas, relations with earlier polynomials in the literature, and differentiation rules. Finally, after determining the first few zero values of the Frobenius-type Genocchi polynomials, we draw graphical representations of these zero values.

Keywords: Bell polynomials, Apostol-type Frobenius-Genocchi polynomials, Bell-based Apostol-type Frobenius-Genocchi polynomials.

2020 MSC: 11B68, 33C45, 11S40, 11S80.

1. Introduction

Recently, many authors [11, 12, 23, 25] have introduced and constructed generating functions for new families of special polynomials including two parametric kinds of polynomials as Bernoulli, Euler, Genocchi, etc. They have given fundamental properties of these polynomials. Also, they have established more identities, and relations among trigonometric functions, two parametric kinds of special polynomials by using generating functions. Applying the partial derivative operator to these generating functions, some derivative formulae, and finite combinatorial sums involving the aforementioned polynomials and numbers are obtained. In addition, these special polynomials allow the derivation of different useful identities in a fairly straightforward way and help in introducing new families of special polynomials. The Apostol-type Frobenius-Genocchi polynomials appear in combinatorial mathematics and play an important role in the theory and applications of mathematics, thus many number theory and combinatorics experts have extensively studied their properties and obtained series of interesting results (see [1–8, 16, 17, 20]).

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The Apostol-type Frobenius-Euler polynomials $\mathbb{H}_j^{(\alpha)}(\xi;u;\lambda)$ of order $\alpha \in \mathbb{C}$ are defined by (see [14, 15, 18]):

$$\left(\frac{1-u}{\lambda e^z-u}\right)^{\alpha} e^{\xi z}=\sum_{j=0}^{\infty}\mathbb{H}_j^{(\alpha)}(\xi;u;\lambda)\frac{z^j}{j!}, \tag{1.1}$$

where $u\in\mathbb{C}\setminus\{1\}$, $\xi\in\mathbb{R}$ and $|z|<\left|\log\left(\frac{\lambda}{u}\right)\right|$. At the point $\xi=0$, $\mathbb{H}_j^{(\alpha)}(u;\lambda)=\mathbb{H}_j^{(\alpha)}(0;u;\lambda)$ are called the Apostol-type Frobenius-Euler numbers of order α. From (1.1), we find

$$\mathbb{H}_j^{(\alpha)}(\xi;u;\lambda)=\sum_{\nu=0}^{j}\binom{j}{\nu}\mathbb{H}_\nu^{(\alpha)}(u;\lambda)\xi^{j-\nu},$$

and

$$\mathbb{H}_j^{(\alpha)}(\xi;-1;\lambda)=\mathbb{E}_j^{(\alpha)}(\xi;\lambda),$$

where $\mathbb{E}_j^{(\alpha)}(\xi;\lambda)$ are the j^{th} Apostol-Euler polynomial of order α.

Recently, Yaşar and Özarslan [27] introduced the Apostol-type Frobenius-Genocchi polynomials defined by means of the following generating relation:

$$\frac{(1-u)z}{\lambda e^z-u}e^{\xi z}=\sum_{j=0}^{\infty}G_j^F(\xi;u;\lambda)\frac{z^j}{j!}, \tag{1.2}$$

where $u\in\mathbb{C}\setminus\{1\}$, $\xi\in\mathbb{R}$ and $|z|<\left|\log\left(\frac{\lambda}{u}\right)\right|$. Note that

$$G_j^F(\xi;-1;\lambda)=G_j(\xi;\lambda),$$

where $G_j(\xi;\lambda)$ are called the Apostol-Genocchi polynomials. For $j\geqslant 0$, the Stirling numbers of the first kind are defined by (see [21, 22]):

$$(\xi)_j=\sum_{p=0}^{j}S_1(j,p)\xi^p, \tag{1.3}$$

where $(\xi)_0=1$, and $(\xi)_j=\xi(\xi-1)\cdots(\xi-j+1), (j\geqslant 1)$. From (1.3), we get

$$\frac{1}{r!}(\log(1+z))^r=\sum_{j=r}^{\infty}S_1(j,r)\frac{z^j}{j!},\quad (r\geqslant 0).$$

For $j\geqslant 0$, the Stirling numbers of the second kind are defined by

$$\xi^j=\sum_{q=0}^{j}S_2(j,q)(\xi)_q. \tag{1.4}$$

From (1.4), we see that

$$\frac{1}{r!}(e^z-1)^r=\sum_{j=r}^{\infty}S_2(j,r)\frac{z^j}{j!}.$$

For any nonnegative integer r, the r-Stirling numbers $S_r(j,k)$ of the second kind are defined by (see [6])

$$\frac{1}{k!}e^{rz}(e^z-1)^k=\sum_{j=k}^{\infty}S_r(j+r,k+r)\frac{z^j}{j!}. \tag{1.5}$$

The Apostol-type Bernoulli polynomials $\mathbb{B}_j^{(\alpha)}(\xi;\lambda)$ of order α, the Apostol-type Euler polynomials $\mathbb{E}_j^{(\alpha)}(\xi;\lambda)$ of order α and the Apostol-type Genocchi polynomials $\mathbb{G}_j^{(\alpha)}(\xi;\lambda)$ of order α are defined by (see [10, 16, 24, 26]):

$$\left(\frac{z}{\lambda e^z-1}\right)^{\alpha} e^{\xi z}=\sum_{j=0}^{\infty}\mathbb{B}_j^{(\alpha)}(\xi;\lambda)\frac{z^j}{j!}\quad (|\,z+\log\lambda\,|)<2\pi,$$

$$\left(\frac{2}{\lambda e^z+1}\right)^{\alpha} e^{\xi z}=\sum_{j=0}^{\infty}\mathbb{E}_j^{(\alpha)}(\xi;\lambda)\frac{z^j}{j!}\quad (|\,z+\log\lambda\,|)<\pi,$$

$$\left(\frac{2z}{\lambda e^z+1}\right)^{\alpha} e^{\xi z}=\sum_{j=0}^{\infty}\mathbb{G}_j^{(\alpha)}(\xi;\lambda)\frac{z^j}{j!},(|\,z+\log\lambda\,|<\pi),$$

respectively. Clearly, we have

$$\mathbb{B}_j^{(\alpha)}(\lambda)=\mathbb{B}_j^{(\alpha)}(0;\lambda),\mathbb{E}_j^{(\alpha)}(\lambda)=\mathbb{E}_j^{(\alpha)}(0;\lambda),\mathbb{G}_j^{(\alpha)}(\lambda)=\mathbb{G}_j^{(\alpha)}(0;\lambda).$$

The Bell polynomials $\mathrm{Bel}_j(\xi)$ are defined by the generating function (see [5, 12])

$$e^{\xi(e^z-1)}=\sum_{j=0}^{\infty}\mathrm{Bel}_j(\xi)\frac{z^j}{j!}. \tag{1.6}$$

When $\xi=1$, $\mathrm{Bel}_j=\mathrm{Bel}_j(1),(j\geqslant 0)$ are called the Bell numbers. From (1.4) and (1.6), we note that

$$\mathrm{Bel}_j(\xi)=\sum_{k=0}^{j}S_2(j,k)\xi^k\quad (j\geqslant 0).$$

Recently, Duran et al. [8] introduced the generalized Bell-based Bernoulli polynomials of two variables ${}_{\mathrm{Bel}}\mathbb{B}_j^{(\alpha)}(\xi;\eta)$ defined by

$$\left(\frac{z}{e^z-1}\right)^{\alpha} e^{\xi z+\eta(e^z-1)}=\sum_{j=0}^{\infty}{}_{\mathrm{Bel}}\mathbb{B}_j^{(\alpha)}(\xi;\eta)\frac{z^j}{j!}, \tag{1.7}$$

so that

$${}_{\mathrm{Bel}}\mathbb{B}_j^{(\alpha)}(\xi;\eta)=\sum_{r=0}^{j}\binom{j}{r}\mathbb{B}_{j-r}^{(\alpha)}(\xi)\mathrm{Bel}_r(\eta).$$

Jamei et al. [9] and Kim and Ryoo [13, 19] introduced the Bernoulli and Euler polynomials of complex variable defined by

$$\frac{z}{e^z-1}e^{\xi z}\cos\eta z=\sum_{j=0}^{\infty}\frac{\mathbb{B}_j(\xi+i\eta)+\mathbb{B}_j(\xi-i\eta)}{2}\frac{z^j}{j!}=\sum_{j=0}^{\infty}\mathbb{B}_j^{(c)}(\xi,\eta)\frac{z^j}{j!},$$

$$\frac{z}{e^z-1}e^{\xi z}\sin\eta z=\sum_{j=0}^{\infty}\frac{\mathbb{B}_j(\xi+i\eta)-\mathbb{B}_j(\xi-i\eta)}{2i}\frac{z^j}{j!}=\sum_{j=0}^{\infty}\mathbb{B}_j^{(s)}(\xi,\eta)\frac{z^j}{j!},$$

and

$$\frac{2}{e^z+1}e^{\xi z}\cos\eta z=\sum_{j=0}^{\infty}\frac{\mathbb{E}_j(\xi+i\eta)+\mathbb{E}_j(\xi-i\eta)}{2}\frac{z^j}{j!}=\sum_{j=0}^{\infty}\mathbb{E}_j^{(c)}(\xi,\eta)\frac{z^j}{j!},$$

$$\frac{2}{e^z+1}e^{\xi z}\sin\eta z=\sum_{j=0}^{\infty}\frac{\mathbb{E}_j(\xi+i\eta)-\mathbb{E}_j(\xi-i\eta)}{2i}\frac{z^j}{j!}=\sum_{j=0}^{\infty}\mathbb{E}_j^{(s)}(\xi,\eta)\frac{z^j}{j!},$$

respectively. Also they have prove that

$$e^{\xi z}\cos\eta z=\sum_{r=0}^{\infty}C_r(\xi,\eta)\frac{z^r}{r!},$$

and

$$e^{\xi z}\sin\eta z=\sum_{r=0}^{\infty}S_r(\xi,\eta)\frac{z^r}{r!},$$

where

$$C_r(\xi,\eta)=\sum_{j=0}^{[\frac{r}{2}]}(-1)^j\binom{r}{2j}\xi^{r-2j}\eta^{2j},$$

and

$$S_r(\xi,\eta)=\sum_{j=0}^{[\frac{r-1}{2}]}\binom{r}{2j+1}(-1)^j\xi^{r-2j-1}\eta^{2j+1}.$$

Motivated and inspired by the definitions (1.2), (1.6), and (1.7), we first consider the Bell-based Frobenius-type Genocchi polynomials of complex variable. Here, we introduce the cosine and sine Bell-based Frobenius-type Genocchi numbers and polynomials, and then we derive several properties and identities for the above polynomials. Also, we find zero values of the Bell-based Frobenius-type Genocchi polynomials.

2. Bell-based Apostol-type Frobenius-Genocchi polynomials of complex variable

In this section, we consider the Bell-based Apostol-type Frobenius-Genocchi polynomials of complex variable and deduce some identities of these polynomials. First, we begin with the following definition

$$\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha}e^{(\xi+i\eta)z}e^{\zeta(e^z-1)}=\sum_{j=0}^{\infty}{}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi+i\eta,\zeta;u;\lambda)\frac{z^j}{j!}. \tag{2.1}$$

On the other hand, we suppose that

$$e^{(\xi+i\eta)z}=e^{\xi z}e^{i\eta z}=e^{\xi z}(\cos\eta z+i\sin\eta z). \tag{2.2}$$

Thus, by (2.1) and (2.2), we have

$$\begin{aligned}\sum_{j=0}^{\infty}{}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi+i\eta,\zeta;u;\lambda)\frac{z^j}{j!}&=\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha}e^{(\xi+i\eta)z}e^{\zeta(e^z-1)}\\&=\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha}e^{\xi z}(\cos\eta z+i\sin\eta z)e^{\zeta(e^z-1)},\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\sum_{j=0}^{\infty}{}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi-i\eta,\zeta;u;\lambda)\frac{z^j}{j!}&=\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha}e^{(\xi-i\eta)z}e^{\zeta(e^z-1)}\\&=\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha}e^{\xi z}(\cos\eta z-i\sin\eta z)e^{\zeta(e^z-1)}.\end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we get

$$\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha} e^{\xi z}\cos\eta z e^{\zeta(e^z-1)}=\sum_{j=0}^{\infty}\left(\frac{{}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi+i\eta,\zeta;u;\lambda)+{}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi-i\eta,\zeta;u;\lambda)}{2}\right)\frac{z^j}{j!}, \tag{2.5}$$

and

$$\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha} e^{\xi z}\sin\eta z e^{\zeta(e^z-1)}=\sum_{j=0}^{\infty}\left(\frac{{}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi+i\eta,\zeta;u;\lambda)-{}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi-i\eta,\zeta;u;\lambda)}{2i}\right)\frac{z^j}{j!}. \tag{2.6}$$

Definition 2.1. Let $j\geqslant 0$. We define two parametric kinds of cosine Bell-based Apostol-type Frobenius-Genocchi polynomials ${}_{Bel}\mathbb{G}_j^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda)$ and sine Bell-based Apostol-type Frobenius-Genocchi polynomials ${}_{Bel}\mathbb{G}_j^{(\alpha,s)}(\xi,\eta,\zeta;u;\lambda)$, for non negative integer j are defined by

$$\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha} e^{\xi z}\cos\eta z e^{\zeta(e^z-1)}=\sum_{j=0}^{\infty}{}_{Bel}\mathbb{G}_j^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda)\frac{z^j}{j!}, \tag{2.7}$$

and

$$\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha} e^{\xi z}\sin\eta z e^{\zeta(e^z-1)}=\sum_{j=0}^{\infty}{}_{Bel}\mathbb{G}_j^{(\alpha,s)}(\xi,\eta,\zeta;u;\lambda)\frac{z^j}{j!}, \tag{2.8}$$

respectively. Note that ${}_{Bel}\mathbb{G}_j^{(\alpha,c)}(\xi,0,0;u;\lambda)=\mathbb{G}_j^{(\alpha)}(\xi;u;\lambda), {}_{Bel}\mathbb{G}_j^{(\alpha,s)}(0,0,0;u;\lambda)=0, (j\geqslant 0)$. From (2.5)-(2.8), we have

$$\begin{aligned}
{}_{Bel}\mathbb{G}_j^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda)&=\frac{{}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi+i\eta,\zeta;u;\lambda)+{}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi-i\eta,\zeta;u;\lambda)}{2},\\
{}_{Bel}\mathbb{G}_j^{(\alpha,s)}(\xi,\eta,\zeta;u;\lambda)&=\frac{{}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi+i\eta,\zeta;u;\lambda)-{}_{Bel}{}_j^{(\alpha)}(\xi-i\eta,\zeta;u;\lambda)}{2i}.
\end{aligned}$$

Remark 2.2. For $\xi=\zeta=0$ in (2.7) and (2.8), we get new type of cosine Apostol-type Frobenius-Genocchi polynomials $\mathbb{G}_j^{(\alpha,c)}(\eta;u;\lambda)$ and sine Apostol-type Frobenius-Genocchi polynomials $\mathbb{G}_j^{(\alpha,s)}(\eta;u;\lambda)$ as

$$\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha}\cos\eta z=\sum_{j=0}^{\infty}\mathbb{G}_j^{(\alpha,c)}(\eta;u;\lambda)\frac{z^j}{j!}, \tag{2.9}$$

and

$$\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha}\sin\eta z=\sum_{j=0}^{\infty}\mathbb{G}_j^{(\alpha,s)}(\eta;u;\lambda)\frac{z^j}{j!}, \tag{2.10}$$

respectively. It is clear that

$$\mathbb{G}_j^{(\alpha,c)}(0;u;\lambda)=\mathbb{G}_j^{(\alpha,c)}(u;\lambda), \mathbb{G}_j^{(\alpha,s)}(0;u;\lambda)=0, (j\geqslant 0).$$

Remark 2.3. Letting $\zeta=0$ in (2.7) and (2.8), we obtain two parametric kinds of cosine Apostol-type Frobenius-Genocchi polynomials $\mathbb{G}_j^{(\alpha,c)}(\xi,\eta;u;\lambda)$ and sine Apostol-type Frobenius-Genocchi polynomials $\mathbb{G}_j^{(\alpha,s)}(\xi,\eta;u;\lambda)$ as

$$\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha} e^{\xi z}\cos\eta z=\sum_{j=0}^{\infty}\mathbb{G}_j^{(\alpha,c)}(\xi,\eta;u;\lambda)\frac{z^j}{j!},$$

and

$$\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha} e^{\xi z}\sin\eta z=\sum_{j=0}^{\infty}\mathbb{G}_j^{(\alpha,s)}(\xi,\eta;u;\lambda)\frac{z^j}{j!},$$

respectively.

Remark 2.4. On setting $\xi=0$ in (2.7) and (2.8), we get new type of cosine Bell-based Apostol-type Frobenius-Genocchi polynomials ${}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\eta,\zeta;u;\lambda)$ and sine Bell-based Apostol-type Frobenius-Genocchi polynomials as ${}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,s)}(\eta,\zeta;u;\lambda)$,

$$\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha}\cos\eta z e^{\zeta(e^z-1)}=\sum_{j=0}^{\infty}{}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\eta,\zeta;u;\lambda)\frac{z^j}{j!},$$

and

$$\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha}\sin\eta z e^{\zeta(e^z-1)}=\sum_{j=0}^{\infty}{}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,s)}(\eta,\zeta;u;\lambda)\frac{z^j}{j!},$$

respectively.

Theorem 2.5. *Let* $j\geqslant 0$. *Then*

$${}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\eta,\zeta;u;\lambda)=\sum_{\nu=0}^{[\frac{j}{2}]}\binom{j}{2\nu}(-1)^{\nu}\eta^{2\nu}{}_{\mathrm{Bel}}\mathbb{G}_{j-2\nu}^{(\alpha)}(\zeta;u;\lambda), \tag{2.11}$$

and

$${}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,s)}(\eta,\zeta;u;\lambda)=\sum_{\nu=0}^{[\frac{j-1}{2}]}\binom{j}{2\nu+1}(-1)^{\nu}\eta^{2\nu+1}{}_{\mathrm{Bel}}\mathbb{G}_{j-2\nu-1}^{(\alpha)}(\zeta;u;\lambda). \tag{2.12}$$

Proof. By (2.9) and (2.10), we have

$$\begin{aligned}\sum_{j=0}^{\infty}{}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\eta,\zeta;u;\lambda)\frac{z^j}{j!}&=\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha}\cos\eta z e^{\zeta(e^z-1)}\\&=\sum_{j=0}^{\infty}{}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\zeta;u;\lambda)\frac{z^j}{j!}\sum_{\nu=0}^{\infty}(-1)^{\nu}\eta^{2\nu}\frac{z^{\nu}}{2\nu!}\\&=\sum_{j=0}^{\infty}\left(\sum_{\nu=0}^{[\frac{j}{2}]}\binom{j}{2\nu}(-1)^{\nu}\eta^{2\nu}{}_{\mathrm{Bel}}\mathbb{G}_{j-2\nu}^{(\alpha)}(\zeta;u;\lambda)\right)\frac{z^j}{j!},\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}\sum_{j=0}^{\infty}{}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,s)}(\eta,\zeta;u;\lambda)\frac{z^j}{j!}&=\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha}\sin\eta z e^{\zeta(e^z-1)}\\&=\sum_{j=0}^{\infty}\left(\sum_{\nu=0}^{[\frac{j-1}{2}]}\binom{j}{2\nu+1}(-1)^{\nu}\eta^{2\nu+1}{}_{\mathrm{Bel}}\mathbb{G}_{j-2\nu-1}^{(\alpha)}(\zeta;u;\lambda)\right)\frac{z^j}{j!}.\end{aligned}\tag{2.14}$$

Therefore, by (2.13) and (2.14), we get (2.11). Similarly, we can easily obtain (2.12). □

Theorem 2.6. *Let* $j \geqslant 0$. *Then*

$$ {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi+i\eta,\zeta;u;\lambda)=\sum_{s=0}^{j}\binom{j}{s}(\xi+i\eta)^{j-s}{}_{Bel}\mathbb{G}_s^{(\alpha)}(\zeta;u;\lambda)=\sum_{s=0}^{j}\binom{j}{s}(i\eta)^{j-s}{}_{Bel}\mathbb{G}_s^{(\alpha)}(\xi,\zeta;u;\lambda), \tag{2.15}$$

and

$$\begin{aligned} {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi-i\eta,\zeta;u;\lambda)&=\sum_{s=0}^{j}\binom{j}{s}(\xi-i\eta)^{j-s}{}_{Bel}\mathbb{G}_s^{(\alpha)}(\zeta;u;\lambda)\\ &=\sum_{s=0}^{j}\binom{j}{s}(-1)^{j-s}(i\eta)^{j-s}{}_{Bel}\mathbb{G}_s^{(\alpha)}(\xi,\zeta;u;\lambda). \end{aligned} \tag{2.16}$$

Proof. By using (2.3) and (2.4), we obtain (2.15) and (2.16). So we omit the proof. □

Theorem 2.7. *Let* $j \geqslant 0$. *Then*

$$ {}_{Bel}\mathbb{G}_j^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda)=\sum_{s=0}^{j}\binom{j}{s}{}_{Bel}\mathbb{G}_{j-s}^{(\alpha)}(\zeta;u;\lambda)C_s(\xi,\eta), \tag{2.17}$$

and

$$ {}_{Bel}\mathbb{G}_j^{(\alpha,k)}(\xi,\eta,\zeta;u;\lambda)=\sum_{k=0}^{j}\binom{j}{k}{}_{Bel}\mathbb{G}_{j-k}^{(\alpha)}(\zeta;u;\lambda)S_k(\xi,\eta). \tag{2.18}$$

Proof. Consider

$$\left(\sum_{j=0}^{\infty}a_j\frac{z^j}{j!}\right)\left(\sum_{\nu=0}^{\infty}b_\nu\frac{z^\nu}{\nu!}\right)=\sum_{j=0}^{\infty}\left(\sum_{\nu=0}^{j}a_{j-\nu}b_\nu\right)\frac{z^j}{j!}.$$

Now

$$\begin{aligned}\sum_{j=0}^{\infty}{}_{Bel}\mathbb{G}_j^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda)\frac{z^j}{j!}&=\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha}e^{\xi z}\cos\eta z e^{\zeta(e^z-1)}\\ &=\left(\sum_{j=0}^{\infty}{}_{Bel}\mathbb{G}_j^{(\alpha)}(\zeta;u;\lambda)\frac{z^j}{j!}\right)\left(\sum_{\nu=0}^{\infty}C_\nu(\xi,\eta)\frac{z^j}{j!}\right)\\ &=\sum_{j=0}^{\infty}\left(\sum_{\nu=0}^{j}\binom{j}{\nu}{}_{Bel}\mathbb{G}_{j-\nu}^{(\alpha)}(\zeta;u;\lambda)C_\nu(\xi,\eta)\right)\frac{z^j}{j!},\end{aligned}$$

which proves (2.17). The proof of (2.18) is similar. □

Theorem 2.8. *Let* $j \geqslant 0$. *Then*

$$ {}_{Bel}\mathbb{G}_j^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda)=\sum_{k=0}^{j}\mathbb{G}_k^{(\alpha,c)}(\xi,\eta;u;\lambda)\mathrm{Bel}_{j-k}(\zeta), \tag{2.19}$$

$$ {}_{Bel}\mathbb{G}_j^{(\alpha,s)}(\xi,\eta,\zeta;u;\lambda)=\sum_{k=0}^{j}\mathbb{G}_k^{(\alpha,s)}(\xi,\eta;u;\lambda)\mathrm{Bel}_{j-k}(\zeta), \tag{2.20}$$

$$ {}_{Bel}\mathbb{G}_j^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda)=\sum_{k=0}^{j}{}_{Bel}\mathbb{G}_k^{(\alpha,c)}(\eta,\zeta;u;\lambda)\xi^{j-k}, \tag{2.21}$$

and

$$ {}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,s)}(\xi,\eta,\zeta;u;\lambda) = \sum_{k=0}^{j} {}_{\mathrm{Bel}}\mathbb{G}_k^{(\alpha,s)}(\eta,\zeta;u;\lambda)\xi^{j-k}. \tag{2.22} $$

Proof. Using (2.7) and (2.8), we obtain (2.19)-(2.22). Here, we omit the proof of the theorem. □

Theorem 2.9. *Let* $j \geqslant 0$. *Then*

$$ {}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\xi+s,\eta,\zeta;u;\lambda) = \sum_{k=0}^{j} \binom{j}{k} {}_{\mathrm{Bel}}\mathbb{G}_k^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda)s^{j-k}, \tag{2.23} $$

and

$$ {}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,s)}(\xi+s,\eta,\zeta;u;\lambda) = \sum_{k=0}^{j} \binom{j}{k} {}_{\mathrm{Bel}}\mathbb{G}_k^{(\alpha,s)}(\xi,\eta,\zeta;u;\lambda)s^{j-k}. \tag{2.24} $$

Proof. By changing ξ with $\xi+s$ in (2.7), we have

$$ \begin{aligned} \sum_{j=0}^{\infty} {}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\xi+s,\eta,\zeta;u;\lambda)\frac{z^j}{j!} &= \left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha} e^{\xi z}\cos\eta z e^{sz} e^{\zeta(e^z-1)} \\ &= \left(\sum_{j=0}^{\infty} {}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda)\frac{z^j}{j!}\right)\left(\sum_{k=0}^{\infty} s^k\frac{z^k}{k!}\right) \\ &= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\binom{j}{k} {}_{\mathrm{Bel}}\mathbb{G}_k^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda)s^{j-k}\right)\frac{z^j}{j!}, \end{aligned} $$

which completes the proof (2.23). The result (2.24) can be similarly proved. □

Theorem 2.10. *Let* $j \geqslant 1$. *Then*

$$ \frac{\partial}{\partial\xi} {}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda) = j\,{}_{\mathrm{Bel}}\mathbb{G}_{j-1}^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda), \tag{2.25} $$

$$ \frac{\partial}{\partial\eta} {}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda) = -j\,{}_{\mathrm{Bel}}\mathbb{G}_{j-1}^{(\alpha,s)}(\xi,\eta,\zeta;u;\lambda), \tag{2.26} $$

and

$$ \frac{\partial}{\partial\xi} {}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,s)}(\xi,\eta,\zeta;u;\lambda) = j\,{}_{\mathrm{Bel}}\mathbb{G}_{j-1}^{(\alpha,s)}(\xi,\eta,\zeta;u;\lambda), \tag{2.27} $$

$$ \frac{\partial}{\partial\eta} {}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,s)}(\xi,\eta,\zeta;u;\lambda) = j\,{}_{\mathrm{Bel}}\mathbb{G}_{j-1}^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda). \tag{2.28} $$

Proof. Equation (2.7) yields

$$ \begin{aligned} \sum_{j=1}^{\infty} \frac{\partial}{\partial\xi} {}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda)\frac{z^j}{j!} &= \left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha} z e^{\xi z}\cos\eta z e^{\zeta(e^z-1)} \\ &= \sum_{j=0}^{\infty} {}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda)\frac{z^{j+1}}{j!} \\ &= \sum_{j=1}^{\infty} {}_{\mathrm{Bel}}\mathbb{G}_{j-1}^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda)\frac{z^j}{(j-1)!} = \sum_{j=1}^{\infty} j\,{}_{\mathrm{Bel}}\mathbb{G}_{j-1}^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda)\frac{z^j}{j!}, \end{aligned} $$

proving (2.25). (2.26), (2.27), and (2.28) can be similarly derived. □

Theorem 2.11. *Let* $j \geqslant 0$. *Then*

$$ {}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda) = \sum_{k=0}^{j}\sum_{m=0}^{k}\binom{j}{k}{}_{\mathrm{Bel}}\mathbb{G}_{j-k}^{(\alpha,c)}(\eta,\zeta;u;\lambda)(\xi)_m S_2(k,m), \tag{2.29} $$

and

$$ {}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,s)}(\xi,\eta,\zeta;u;\lambda) = \sum_{k=0}^{j}\sum_{m=0}^{k}\binom{j}{k}{}_{\mathrm{Bel}}\mathbb{G}_{j-k}^{(\alpha,s)}(\eta,\zeta;u;\lambda)(\xi)_m S_2(k,m). \tag{2.30} $$

Proof. Using (1.4) and (2.7), we find

$$ \begin{aligned} \sum_{j=0}^{\infty}{}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda)\frac{z^j}{j!} &= \left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha}\cos\eta z e^{\zeta(e^z-1)}(e^z-1+1)^{\xi} \\ &= \left(\frac{(1-u)z}{e^z-u}\right)^{\alpha}\cos\eta z e^{\zeta(e^z-1)}\sum_{m=0}^{\infty}(\xi)_m\frac{(e^z-1)^m}{m!} \\ &= \left(\frac{(1-u)z}{e^z-u}\right)^{\alpha}\cos\eta z e^{\zeta(e^z-1)}\sum_{m=0}^{\infty}(\xi)_m\sum_{k=m}^{\infty}S_2(k,m)\frac{z^k}{k!} \\ &= \sum_{j=0}^{\infty}{}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\eta;\zeta;u;\lambda)\frac{z^j}{j!}\sum_{k=0}^{\infty}\sum_{m=0}^{k}(\xi)_m S_2(k,m)\frac{z^k}{k!} \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{m=0}^{k}\binom{j}{k}{}_{\mathrm{Bel}}\mathbb{G}_{j-k}^{(\alpha,c)}(\eta,\zeta;u;\lambda)(\xi)_m S_2(k,m)\frac{z^j}{j!}. \end{aligned} \tag{2.31} $$

In view of (2.7) and (2.31), we get (2.29). Similarly, we can easily obtain (2.30). □

Theorem 2.12. *Let* $j \geqslant 0$. *Then*

$$ {}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\xi+r,\eta,\zeta;u;\lambda) = \sum_{k=0}^{j}\sum_{m=0}^{k}\binom{j}{k}{}_{\mathrm{Bel}}\mathbb{G}_{j-k}^{(\alpha,c)}(\eta,\zeta;u;\lambda)(\xi)_m S_2(k+r,m+r), \tag{2.32} $$

and

$$ {}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,s)}(\xi+r,\eta,\zeta;u;\lambda) = \sum_{k=0}^{j}\sum_{m=0}^{k}\binom{j}{k}{}_{\mathrm{Bel}}\mathbb{G}_{j-k}^{(\alpha,s)}(\eta,\zeta;u;\lambda)(\xi)_m S_2(k+r,m+r). \tag{2.33} $$

Proof. Replacing ξ by $\xi+r$ in (2.7) and using (1.5), we find

$$ \begin{aligned} \sum_{j=0}^{\infty}{}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\xi+r,\eta,\zeta;u;\lambda)\frac{z^j}{j!} &= \left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha}\cos\eta z e^{\zeta(e^z-1)}e^{rz}(e^z-1+1)^{\xi} \\ &= \left(\frac{(1-u)z}{e^z-u}\right)^{\alpha}\cos\eta z e^{\zeta(e^z-1)}e^{rz}\sum_{m=0}^{\infty}(\xi)_m\frac{(e^z-1)^m}{m!} \\ &= \left(\frac{(1-u)z}{e^z-u}\right)^{\alpha}\cos\eta z e^{\zeta(e^z-1)}e^{rz}\sum_{m=0}^{\infty}(\xi)_m\sum_{k=m}^{\infty}S_2(k,m)\frac{z^k}{k!} \\ &= \sum_{j=0}^{\infty}{}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\eta;\zeta;u;\lambda)\frac{z^j}{j!}\sum_{k=0}^{\infty}\sum_{m=0}^{k}(\xi)_m S_2(k+r,m+r)\frac{z^k}{k!} \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{m=0}^{k}\binom{j}{k}{}_{\mathrm{Bel}}\mathbb{G}_{j-k}^{(\alpha,c)}(\eta,\zeta;u;\lambda)(\xi)_m S_2(k+r,m+r)\frac{z^j}{j!}. \end{aligned} \tag{2.34} $$

In view of (2.7) and (2.34), we get (2.32). Similarly, we can easily obtain (2.33). □

3. Some values with graphical representations and zeros of Bell-based Apostol-type Frobenius-Genocchi polynomials of complex variable

In this section, computational values and graphical representations of cosine Bell-based Apostol-type Frobenius-Genocchi polynomials ${}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda)$ are shown. A few of them are

$$
\begin{aligned}
{}_{\mathrm{Bel}}\mathbb{G}_0^{(2,c)}(\xi,\eta,\zeta;u;\lambda) &= 0,\\
{}_{\mathrm{Bel}}\mathbb{G}_1^{(2,c)}(\xi,\eta,\zeta;u;\lambda) &= 0,\\
{}_{\mathrm{Bel}}\mathbb{G}_2^{(2,c)}(\xi,\eta,\zeta;u;\lambda) &= \frac{2(-1+u)^2}{(u-\lambda)^2},\\
{}_{\mathrm{Bel}}\mathbb{G}_3^{(2,c)}(\xi,\eta,\zeta;u;\lambda) &= \frac{6(-1+u)^2(-\lambda(-2+\zeta+\xi)+u(\zeta+\xi))}{(u-\lambda)^3},\\
{}_{\mathrm{Bel}}\mathbb{G}_4^{(2,c)}(\xi,\eta,\zeta;u;\lambda) &= \frac{12(-1+u)^2(-\eta^2(u-\lambda)^2+2\lambda(u+2\lambda)-4\lambda(-u+\lambda)(\zeta+\xi))}{(u-\lambda)^4}\\
&\quad+\frac{12(-1+u)^2(u-\lambda)^2(\zeta+(\zeta+\xi)^2)}{(u-\lambda)^4},\\
{}_{\mathrm{Bel}}\mathbb{G}_5^{(2,c)}(\xi,\eta,\zeta;u;\lambda) &= \frac{40(-1+u)^2\lambda(u^2+7u\lambda+4\lambda^2)}{(-u+\lambda)^5}\\
&\quad+\frac{20(-1+u)^2(6(u-\lambda)\lambda(u+2\lambda)(\zeta+\xi)-3\eta^2(u-\lambda)^2(2\lambda+(u-\lambda)(\zeta+\xi)))}{(-u+\lambda)^5}\\
&\quad+\frac{120(-1+u)^2(u-\lambda)^2\lambda(\zeta+(\zeta+\xi)^2)}{(-u+\lambda)^5}\\
&\quad+\frac{20(-1+u)^2(-u+\lambda)^3(\zeta+2\zeta(\zeta+\xi)+(\zeta+\xi)(\zeta+(\zeta+\xi)^2))}{(-u+\lambda)^5}.
\end{aligned}
$$

We plot the zeros of the cosine Bell-based Apostol-type Frobenius-Genocchi polynomials ${}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda)=0$ for $j=20$ in Figure 1.

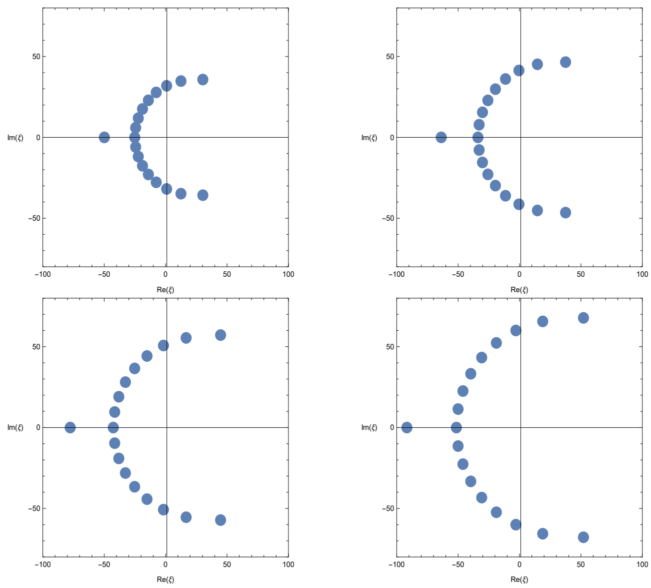

Figure 1: Zeros of ${}_{\mathrm{Bel}}\mathbb{G}_j^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda)=0$.

In Figure 1 (top-left), we choose $\eta = 4, \zeta = 3, u = 4$, and $\lambda = 3$. In Figure 1 (top-right), we choose $\eta = 5, \zeta = 5, u = 5$, and $\lambda = 4$. In Figure 1 (bottom-left), we choose $\eta = 6, \zeta = 7, u = 6$, and $\lambda = 5$. In Figure 1 (bottom-right), we choose $\eta = 7, \zeta = 9, u = 7$, and $\lambda = 6$.

Stacks of zeros of the cosine Bell-based Apostol-type Frobenius-Genocchi polynomials ${}_{Bel}\mathbb{G}_j^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda) = 0$ for $3 \leqslant j \leqslant 20$, forming a 3D structure, are presented in Figure 2.

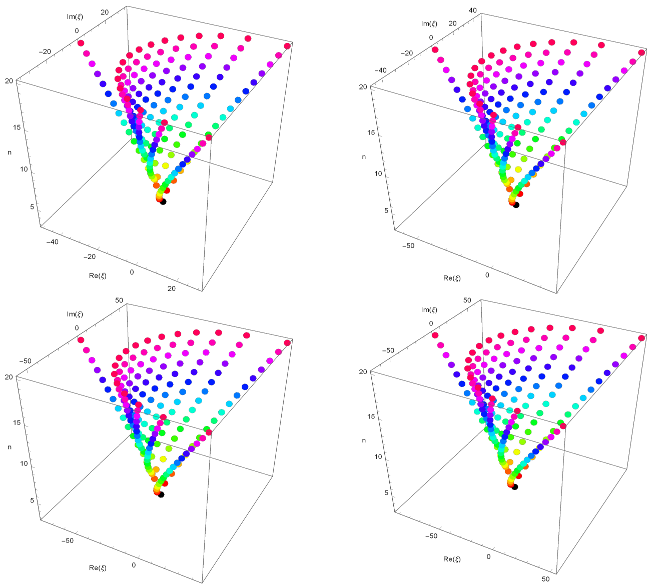

Figure 2: Zeros of ${}_{Bel}\mathbb{G}_j^{(\alpha,c)}(\xi,\eta,\zeta;u;\lambda) = 0$.

In Figure 2 (top-left), we choose $\eta = 4, \zeta = 3, u = 4$, and $\lambda = 3$. In Figure 2 (top-right), we choose $\eta = 5, \zeta = 5, u = 5$, and $\lambda = 4$. In Figure 2 (bottom-left), we choose $\eta = 6, \zeta = 7, u = 6$, and $\lambda = 5$. In Figure 2 (bottom-right), we choose $\eta = 7, \zeta = 9, u = 7$, and $\lambda = 6$,.

Next, we calculated an approximate solution satisfying the cosine Bell-based Apostol-type Frobenius-Genocchi polynomials ${}_{Bel}\mathbb{G}_j^{(2,c)}(\xi,7,9;7;6) = 0$. The results are given in Table 1.

Table 1: Approximate solutions of ${}_{Bel}\mathbb{G}_j^{(2,c)}(\xi,7,9;7;6) = 0$.

degree j	ξ
3	-21.000
4	$-21.000 - 6.633i,\ -21.000 + 6.633i$
5	$-27.377,\ -17.812 - 12.747i,\ -17.812 + 12.747i$
6	$-27.704 - 3.567i,\ -27.704 + 3.567i,$
	$-14.296 - 18.471i,\ -14.296 + 18.471i$
7	$-34.538,\ -24.745 - 10.044i,\ -24.745 + 10.044,$
	$-10.485 - 23.566i,\ -10.485 + 23.566$
8	$-37.686,\ -30.059,\ -22.718 - 16.169i,,$
	$-22.718 + 16.169i,\ -6.409 - 28.190i,\ -6.409 + 28.190i$
9	$-42.796,\ -29.582 - 8.328i,\ -29.582 + 8.328i,\ -20.402 - 21.664i,$
	$-20.402 + 21.664i,\ -2.118 - 32.456i,\ -2.118 + 32.456i,$
10	$-47.050,\ -33.043,\ -28.536 - 14.626i,$
	$-28.536 + 14.626i,\ -17.763 - 26.725i,\ -17.763 + 26.725i,$
	$2.345 - 36.441i,\ 2.345 + 36.441i$
11	$-51.590,\ -33.721 - 7.309i,\ -33.721 + 7.309i,$
	$-27.088 - 20.277i,\ -27.088 + 20.277i,\ -14.845 - 31.456i,$
	$-14.845 + 31.456i,\ 6.949 - 40.196i,\ 6.949 + 40.196i$

4. Concluding remarks

Our paper introduced sine and cosine Bell-based Apostol-type Frobenius-Genocchi polynomials and analyzed their properties by providing several relations and applications. Also, various formulas and properties including differentiation rules, addition formulas, relations, and summation formulas have been investigated. Moreover, after determining the first few zero values of the Apostol-type Frobenius-Genocchi polynomials, we have drawn graphical representations of these zero values. It is possible that this papers idea can be applied to polynomials that are similar and these polynomials have potential applications in other fields of science in addition to the applications at the end of the article. We will continue to explore this opinion in various directions in our next scientific works to advance the purpose of this article.

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