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Analyzing convex univalent functions on semi-infinite strip domains

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Abstract

In this paper, a new class of analytic functions called convex univalent functions is introduced. These functions are of the form

$$1+\frac{1}{a}\log\frac{1-bz}{1-z} \ \text{ for } \ a>0,\ -1<b<1,$$

and they map the open unit disk onto a horizontal semi-infinite strip domain. The paper focuses on function families for which zf'/f maps the unit disk to a subset of this strip domain. Several properties of this class of functions are discussed, including coefficient estimates, extreme points, and growth properties. The paper also explores connections to other classes of functions, such as starlike functions. There are several applications of this class of functions. They can be used in conformal mapping problems and problems related to the analysis of complex networks. The results presented in the paper can also be applied in constructing mathematical models that describe various physical phenomena, such as fluid dynamics and electromagnetism.

Keywords: Univalent functions, subordination, starlike functions, horizontal semi-infinite strip.

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1. Introduction

The Euler dilogarithm or Spence's function is defined by

$$\mathrm{Li}_2(z)=-\int_0^z\frac{\log(1-t)}{t},dt, \tag{1.1}$$

for complex argument $z\notin[1,\infty)$, where

$$\log z=\ln|z|+i,\mathrm{Arg},z \ \text{ for } \ z\neq 0,-\pi<\mathrm{Arg},z<\pi.$$

However, the principal branch of the dilogarithm is defined by the integrals in (1.1) as a single-valued analytic function in the entire z-plane, except for the points on the cut along the real axis from 1 to $+\infty$ $(-\pi<\mathrm{Arg}(1-z)<\pi)$.

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The integral (1.1) may, therefore, be used to obtain analytic continuations of the dilogarithm. A survey of the notations and definitions adopted by different authors may be found in Lewin [19, §1.10, pp. 27-29].

Using the representations (1.1), one may expand the logarithm in powers of z, obtaining the Taylor series expansion for the dilogarithm, valid for $|z| \leqslant 1$,

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

Using the former definition above, the dilogarithm function is analytic everywhere on the complex plane except at $z = 1$, where it has a logarithmic branch point. The standard choice of branch cut is along the positive real axis $(1, \infty)$. However, the function is continuous at the branch point and takes on the value $\mathrm{Li}_2(1) = \pi^2/6$.

The class $\mathcal{A}$ consists of functions f that are analytic within the open unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$ and can be represented as:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \quad \text{for} \quad z \in \mathbb{D}. \tag{1.2}$$

The subset of $\mathcal{A}$ that contains all functions f within $\mathbb{D}$ that are univalent is referred to as $\mathcal{S}$.

Assume that F and H are analytic functions within $\mathbb{D}$, the open unit disk in the complex plane $\mathbb{C}$. We say that F is subordinate to H in $\mathbb{D}$ if there exists a self-map function ω that is analytic within $\mathbb{D}$, has the property $\omega(0) = 0$, and satisfies the inequality $|\omega(z)| < 1$ for all $z \in \mathbb{D}$, such that $F(z) = H(\omega(z))$ for all $z \in \mathbb{D}$. If the function H is univalent within $\mathbb{D}$, then $F \prec H$ if and only if $F(0) = H(0)$ and $F(\mathbb{D}) \subset H(\mathbb{D})$. This notation and definition can be found, for example, in [7].

Ma and Minda [20] used subordination to introduce the class of starlike functions given by:

$$\mathcal{ST}(\phi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\}, \tag{1.3}$$

where ϕ is an analytic function whose real part is positive, and whose range in the open unit disk is symmetric about the real axis and starlike with respect to $\phi(0) = 1$ and $\phi'(0) > 0$.

For $-1 \leqslant B < A \leqslant 1$, the subclass $\mathcal{ST}[A, B]$ of $\mathcal{S}$ consists of Janowski starlike functions, defined as $\mathcal{ST}((1 + Az)/(1 + Bz))$. The special case $\mathcal{ST}(\beta) := \mathcal{ST}[1 - 2\beta, -1]$ is the well-known class of starlike functions of order $0 \leqslant \beta < 1$, as noted in [28]. When $\beta = 0$, i.e., $A = 1$ and $B = -1$, it results in the usual class $\mathcal{ST}$ of starlike functions. Similarly, $\mathcal{ST}_{hpl} := \mathcal{ST}((1 - z)^s)$ is the subclass of $\mathcal{ST}$ introduced by Kanas et al. [14] (see also, [22]), consisting of functions $f \in \mathcal{A}$ such that $zf(z)/f(z)$ lies in the domain bounded by a right branch of a hyperbola $\frac{1}{\rho} = \left(2 \cos \frac{\varphi}{s}\right)^s$, where $0 < s \leqslant 1$ and $|\varphi| < (\pi s)/2$. In the similar fashion, it is easily seen that is the subclass of $\mathcal{S}^*_L(\lambda) := \mathcal{ST}((1 + z)^\lambda)$ introduced by Masih et al. [21] (see also, [17, 23]), consisting of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $\rho = \left(2 \cos \frac{\varphi}{\lambda}\right)^\lambda$, where $0 < \lambda \leqslant 1$ and $|\varphi| < (\pi\lambda)/2$. Moreover, several results were obtained when $\phi(z) = q_k(z)$ which is of the form:

$$q_k(z) = \begin{cases} 1 + \frac{2}{1-k^2} \sinh^2 \left\{ \left(\frac{2}{\pi} \arccos k\right) \tanh^{-1} \sqrt{z} \right\}, & 0 \leqslant k < 1, \\ 1 + \frac{2}{\pi} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k = 1 \\ 1 + \frac{1}{k^2-1} \sin \left(\frac{\pi}{2R(t)} \int_0^{u(z)} \frac{dt}{\sqrt{1-x^2}\sqrt{1-(tx)^2}} \right), & k > 1, \end{cases}$$

where $u(z) = \frac{z-\sqrt{t}}{\sqrt{t}(1-\sqrt{tz})}$, $z \in \mathbb{D}$, and $t \in (0, 1)$ is chosen such that

$$k = \cosh\left(\pi R(\sqrt{1 - t^2})/R(t)\right),$$

$K(t)$ is the Legendre's complete elliptic integral of the first kind, see for details [15, 16] (see also [30]). The class $\mathcal{ST}(q_k)$ is called a class of k-uniformly starlike functions. Note that for $k \geqslant 0$, the region $q_k(\mathbb{D})$ is associated with the conic domain in the positive half-plane. In fact, the region $q_k(\mathbb{D})$ is the positive half plane and for $0 < k < 1$, the region $q_k(\mathbb{D})$ is associated with a parabolic domain, and is associated with a hyperbolic domain if $k > 1$.

Many subclasses of $\mathcal{ST}$ were considered in the past, for an appropriate choice of ϕ in (1.3). For instance, the interesting regions represented by the functions $1 + z - z^3/3$, $1/(1-(1+b)z+bz^2)$ $(-1 < b < 1)$, e^z, and $2/(1+e^{-z})$ were considered in place of $\varphi(z)$ by Wani and Swaminathan [33], Sokół [32], Mendiratta et al. [24], and Goel and Kumar [8] respectively, and others [3–6, 11–13, 18].

1.1. *Main contribution and motivation*

The authors of the aforementioned papers introduced and analyzed different classes of starlike functions. In this work, we focus on a subfamily of starlike functions, denoted by $\mathcal{ST}_{sis}(a, b)$, for which the function $zf'(z)/f(z)$ is contained within a domain bounded by the horizontal semi-infinite strip

$$\mathcal{D}_{a,b} = \left\{ u + iv \colon u > 1 + \frac{1}{a} \ln \frac{1+b}{2\cos av},\ |v| < \frac{\pi}{2a} \right\}.$$

In order to present the main theorem of this work, we introduce a family of univalent analytic functions, denoted by $\mathscr{L}_{a,b}(\cdot)$, in Section 2.

The rest of this work is structured as follows. In Section 2 we introduce and study the functions $\mathscr{L}_{a,b}(\cdot)$. In Section 3 we introduce the class $\mathcal{ST}_{sis}(a, b)$ and then examine the geometric properties of this class.

2. A family of functions

In this section, we consider the family of analytic functions $\mathscr{L}_{a,b}(z)$ defined as:

$$\mathscr{L}_{a,b}(z) = 1 + \frac{1}{a} \log \frac{1-bz}{1-z} = 1 + \sum_{n=1}^{\infty} B_n z^n \quad \text{for} \quad a > 0,\ -1 < b < 1, \tag{2.1}$$

where $B_n = (1-b^n)/(an)$. Here, the branch of the logarithm is chosen such that $\log 1 := 0$ and maps the unit circle $\partial\mathbb{D}$ onto a domain $\partial\mathcal{D}_{a,b}$ defined by

$$\partial\mathcal{D}_{a,b} = \left\{ u + iv \colon u = 1 + \frac{1}{a} \ln \frac{1+b}{2\cos av},\ -\frac{\pi}{2a} < v < \frac{\pi}{2a} \right\}. \tag{2.2}$$

We assert that the function $\mathscr{L}_{a,b}(z)$ defined by equation (2.1) maps the open unit disk $\mathbb{D}$ onto a region bounded by the curve $\partial\mathcal{D}_{a,b}$ given by (2.2). To see why, suppose $w = u + iv$ is a point on $\partial\mathcal{D}_{a,b}$ and consider the pre-image z under the function $\mathscr{L}_{a,b}(z)$ such that $\mathscr{L}_{a,b}(z) = w$. Then, we have

$$a(w-1) = \log \frac{1-bz}{1-z} \quad \text{for} \quad z \in \mathbb{D}.$$

For $z \in \mathbb{D}$ and $-1 < b < 1$, we have $\Re\{(1-bz)/(1-z)\} > 0$, which is equivalent to $a|v| = |\Im\{a(w-1)\}| < \pi/2$. Since the exponential function e^t is univalent in the strip $|\Im\, t| < \pi/2$, we can conclude that $\left|1 - e^{a(w-1)}\right| = \left|b - e^{a(w-1)}\right|$ $(|z| = 1,\ z \neq 1)$, and hence $\Re\, e^{a(w-1)} = (1+b)/2$, which further implies that $\mathscr{L}_{a,b}(\partial\mathbb{D}) = \partial\mathcal{D}_{a,b}$.

In the rest of the section, we use $z = e^{i\theta}$ $(0 < \theta < 2\pi)$, and we have the following expression for $\mathscr{L}_{a,b}\left(e^{i\theta}\right)$:

$$\begin{aligned} \mathscr{L}_{a,b}\left(e^{i\theta}\right) &= 1 + \frac{1}{a} \ln \left| \frac{1-be^{i\theta}}{1-e^{i\theta}} \right| + \frac{i}{a} \operatorname{Arg} \frac{1-be^{i\theta}}{1-e^{i\theta}} \\ &= 1 + \frac{1}{2a} \ln \frac{1+b^2-2b\cos\theta}{2(1-\cos\theta)} + \frac{i}{a} \tan^{-1}\left(\frac{1-b}{1+b} \cos \frac{\theta}{2} \right). \end{aligned} \tag{2.3}$$

Moreover, using equation (2.3), we can obtain more precise inequalities for the real and imaginary parts of $\mathscr{L}_{a,b}\left(e^{i\theta}\right)$ such as:

$$\Re\left\{\mathscr{L}_{a,b}\left(e^{i\theta}\right)\right\} \geqslant 1+\frac{1}{a}\ln\frac{1+b}{2} \quad \text{and} \quad -\frac{\pi}{2a} < \Im\left\{\mathscr{L}_{a,b}\left(e^{i\theta}\right)\right\} < \frac{\pi}{2a}.$$

Let

$$g(z) := \frac{a}{1-b}\left(\mathscr{L}_{a,b}(z)-1\right) = \sum_{n=1}^{\infty}\frac{1-b^n}{1-b}\cdot\frac{z^n}{n} \quad \text{for} \quad z\in\mathbb{D}.$$

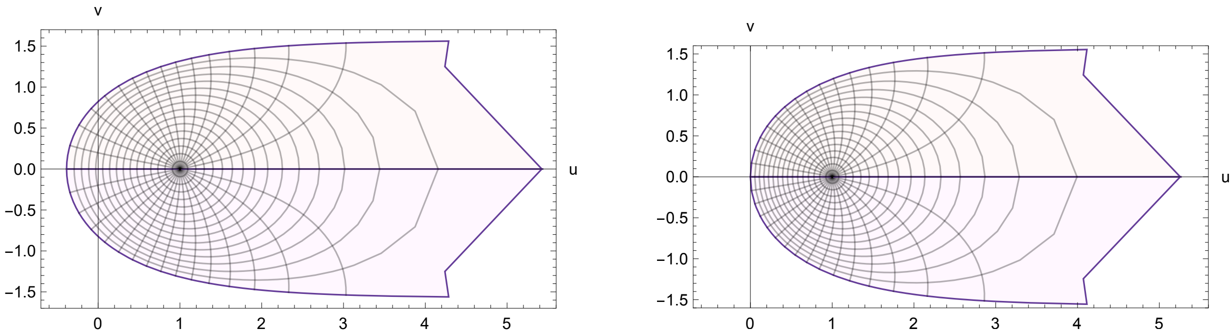
It can be shown by simple calculations that,

$$1+\frac{zg''(z)}{g'(z)} = \frac{1-bz^2}{(1-z)(1-bz)} \quad \text{for} \quad z\in\mathbb{D}.$$

Hence, we get

$$\Re\left\{1+\frac{zg''(z)}{g'(z)}\right\} = \Re\left\{\frac{1-bz^2}{(1-z)(1-bz)}\right\} > \frac{1-b}{2(1+b)} \quad \text{for} \quad z\in\mathbb{D}.$$

This implies that the function g is convex of order $\frac{1-b}{2(1+b)}$. Consequently, the function $\mathscr{L}_{a,b}$ is univalent and convex, but not normalized in the usual sense. A visual representation of this result is illustrated in Figure 2.



$a=1, b=-0.5$ $\qquad$ $a=1, b=\frac{2}{e}-1$

Figure 1: The image of $\mathbb{D}$ under $\mathscr{L}_{a,b}(z)$.

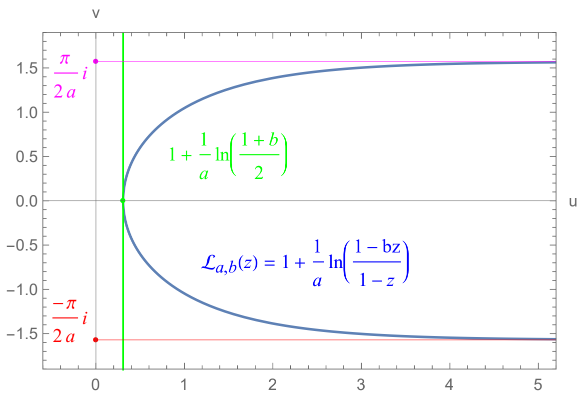


Figure 2: The image of $\partial\mathbb{D}$ under $\mathscr{L}_{a,b}$ for $a=1, b=0$.

Since the bilinear transformation $q(z) := (1-bz)/(1-z)$ maps $|z|=r<1$ onto the disk

$$\left|q(z)-\frac{1-br^2}{1-r^2}\right| \leqslant \frac{(1-b)r}{1-r^2}, \tag{2.4}$$

then by simple calculation, we have

$$\frac{1+br}{1+r} \leqslant \left|\frac{1-bz}{1-z}\right| \leqslant \frac{1-br}{1-r} \quad \text{for} \quad z \in \mathbb{D}.$$

To consider the previous inequality and the increasing property of the function ln on $(0,+\infty)$, we get $\mathscr{L}_{a,b}(-r) \leqslant \Re\{\mathscr{L}_{a,b}(z)\} \leqslant \mathscr{L}_{a,b}(r)$. Furthermore, we have $\Im\{\mathscr{L}_{a,b}(z)\} = \frac{1}{a}\arg\frac{1-bz}{1-z}$. Using relationship (2.4), we can deduce that $|\Im\{\mathscr{L}_{a,b}(z)\}| \leqslant \frac{1}{a}\arcsin\frac{(1-b)r}{1-br^2}$.

For $-1 < t \leqslant 1$, the function $-\log(1-tz)$ is convex in $\mathbb{D}$, and we have the property $\ln(1+|t|r) \leqslant |\log(1-tz)| \leqslant -\ln(1-|t|r)$ for $|z| = r < 1$. Hence, we can conclude that

$$1+\frac{1}{a}\ln\left((1-r)(1+|b|r)\right) \leqslant |\mathscr{L}_{a,b}(z)| \leqslant 1-\frac{1}{a}\ln\left((1-r)(1-|b|r)\right).$$

To conclude this section, we can state the following theorem.

Theorem 2.1. *Let $\mathscr{L}_{a,b}$ be the function defined by (2.1). Then $\mathscr{L}_{a,b}$ is univalent and convex. Also,*

1. *for $z \in \mathbb{D}$,*
$$|\Im\{\mathscr{L}_{a,b}(z)\}| < \frac{\pi}{2a};$$
2. *for $z \in \mathbb{D}$,*
$$\Re\{\mathscr{L}_{a,b}(z)\} > 1+\frac{1}{a}\ln\frac{1+b}{2} =: m(a,b);$$
3. *for $0 < r < 1$, let $\mathbb{D}_r := \{z \in \mathbb{D} \colon |z| \leqslant r\}$, then*
$$\mathscr{L}_{a,b}(\mathbb{D}_r) = \left\{u+iv \colon \frac{1}{a}\ln\frac{1+br}{1+r}\frac{1}{\cos av} \leqslant u-1 \leqslant \frac{1}{a}\ln\frac{1-br}{1-r}\frac{1}{2\cos av},\ |v| < \frac{\pi}{2a}\right\};$$
4. *for $z \in \mathbb{D}$,*
$$\mathscr{L}_{a,b}(\mathbb{D}) = \left\{u+iv \colon u > 1+\frac{1}{a}\ln\frac{1+b}{2\cos av},\ |v| < \frac{\pi}{2a}\right\} =: \mathcal{D}_{a,b};$$
5. *for $z \in \mathbb{D}$,*
$$\mathscr{L}_{a,b}(-|z|) \leqslant \Re\{\mathscr{L}_{a,b}(z)\} \leqslant \mathscr{L}_{a,b}(|z|);$$
6. *for $z \in \mathbb{D}$,*
$$|\Im\{\mathscr{L}_{a,b}(z)\}| \leqslant \frac{1}{a}\arcsin\frac{(1-b)|z|}{1-b|z|^2};$$
7. *for $z \in \mathbb{D}$,*
$$1+\frac{1}{a}\ln\left((1-|z|)(1+|bz|)\right) \leqslant |\mathscr{L}_{a,b}(z)| \leqslant 1-\frac{1}{a}\ln\left((1-|z|)(1-|bz|)\right).$$

Lemma 2.2 ([25])**.** *Let h be convex in $\mathbb{D}$, with $h(0) = 1$. If $p(z) = 1 + p_1 z + \cdots$ satisfies*

$$p(z) + zp'(z) = (zp(z))' \prec h(z) \quad \textit{for} \quad z \in \mathbb{D},$$

then

$$p(z) \prec q(z) \prec h(z) \quad \textit{for} \quad z \in \mathbb{D},$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt \quad \textit{for} \quad z \in \mathbb{D}.$$

Additionally, $q(z)$ is convex and is the best dominant.

By considering Lemma 2.2 and Theorem 2.1, we obtain the following results.

Corollary 2.3. *Suppose that* $f \in \mathcal{A}$ *satisfies the condition* $f'(z) + zf''(z) \prec \mathscr{L}_{a,b}(z)$, *where* $\mathscr{L}_{a,b}(\cdot)$ *given by* (2.1). *Then*

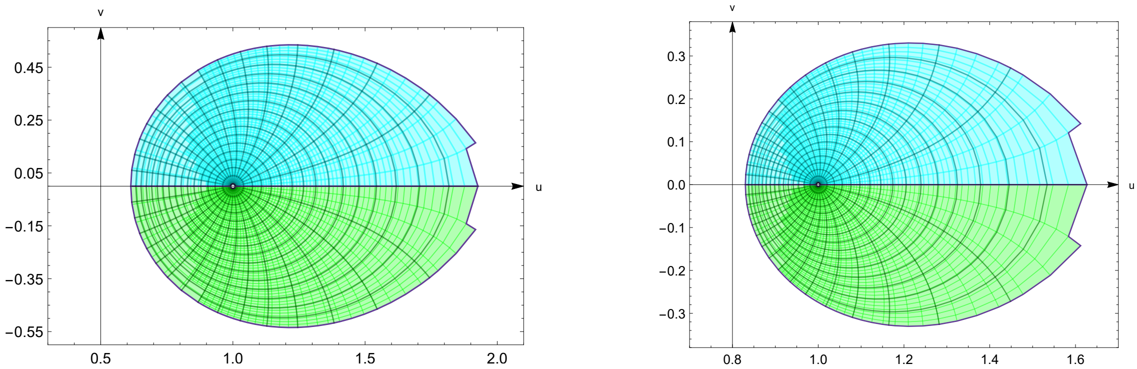
$$f'(z) \prec q_{a,b}(z) \quad \textit{for} \quad z \in \mathbb{D},$$

where $q_{a,b}$ *given by* (2.5) *and the function* $q_{a,b}$ *is convex and the best dominant. Suppose that* $f \in \mathcal{A}$ *satisfies the condition* $f'(z) \prec \mathscr{L}_{a,b}(z)$, *where* $\mathscr{L}_{a,b}(\cdot)$ *given by* (2.1). *Then*

$$\frac{f(z)}{z} \prec q_{a,b}(z) \quad \textit{for} \quad z \in \mathbb{D},$$

where $q_{a,b}$ *(see Fig. 3) is given by* (2.5) *and the function* $q_{a,b}$ *is convex and the best dominant.*

$$q_{a,b}(z) = \begin{cases} 1 + \frac{1}{az}\left[(1-z)\log(1-z) - \frac{1-bz}{b}\log(1-bz)\right], & \textit{for} \quad b \neq 0, \\ 1 + \frac{1}{az}\left[z + (1-z)\log(1-z)\right], & \textit{for} \quad b = 0. \end{cases} \tag{2.5}$$



$a = 1, b = 0$ $\qquad$ $a = 1, b = 0.5$

Figure 3: The image of $\mathbb{D}$ under $q_{a,b}(z)$.

3. The class $\mathcal{ST}_{sis}(\alpha)$ and its properties

Definition 3.1. For $-1 < b < 1$ and $a > 0$, the subclass $\mathcal{ST}_{sis}(a, b)$ of $\mathcal{A}$ consists of functions f that satisfy the condition

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{1}{a}\log\frac{1-bz}{1-z} \quad \text{for} \quad z \in \mathbb{D}.$$

Geometrically, this condition means that the expression $zf'(z)/f(z)$ lies within a semi-infinite strip defined by $u > 1 + (1/a)\ln[(1+b)/2]$ and $|v| < \pi/(2a)$. By the properties of $m(a, b)$ given in Theorem 2.1, it follows that for $f \in \mathcal{ST}_{sis}(a, b)$, the real part of $zf'(z)/f(z)$ is greater than $m(a, b)$ for all $z \in \mathbb{D}$.

According to the theorem, the functions $\mathscr{L}_{a,b}$ have a symmetric domain with respect to the real axis and are starlike with respect to $\mathscr{L}_{a,b}(0) = 1$, $\mathscr{L}'_{a,b}(0) = (1-b)/a > 0$, and for $z \in \mathbb{D}$, $\mathfrak{R}\{\mathscr{L}_{a,b}(z)\} > m(a, b)$. It is assumed that $m(a, b) \geqslant 0$ or $a \geqslant \ln(2/(1+b))$, unless stated otherwise. Thus,

$$\mathcal{ST}_{sis}\left(a, \frac{2}{e^a} - 1\right) \subset \mathcal{ST}, \quad \mathcal{ST}_{sis}(a, b) \subset \mathcal{ST}(\beta) \quad \text{for} \quad 0 \leqslant \beta \leqslant 1 + \frac{1}{a}\ln\frac{1+b}{2}.$$

Also,

$$\mathcal{ST}_{sis}(a_1, b_1) \subset \mathcal{ST}_{sis}(a_2, b_2) \Longleftrightarrow b_2 \leqslant b_1, \; a_2 \geqslant a_1.$$

Especially, let $b \geqslant 0$ and $a \geqslant \ln 2$, we have

$$\mathcal{ST}_{sis}(a, b) \subset \mathcal{ST}_{sis}(a, 0).$$

Additionally, we have $\mathcal{ST}_{sis}(a, b) \subset \mathcal{N}$ for $e^{-a} \leqslant (1+b)/2$ (Fig. 4), where $\mathcal{N}$ is the family of functions that are univalent, starlike in one direction (see [26]), and satisfying,

$$-\frac{1}{2} < \Re\left(\frac{zf'(z)}{f(z)}\right).$$

By Theorem 2.1, it can be clearly seen that the largest disk with center $(1, 0)$ that contains $\mathcal{L}_{a,b}(z)$ (see Fig. 5) is $\{w \in \mathbb{C} \colon |w-1| < R\}$, where $R := \frac{1}{a} \ln \frac{2}{1+b}$. Thus

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{1}{a} \ln \frac{2}{1+b} = 1 - m(a, b) \Longrightarrow f \in \mathcal{ST}_{sis}(a, b) \quad \text{for} \quad z \in \mathbb{D}. \tag{3.1}$$

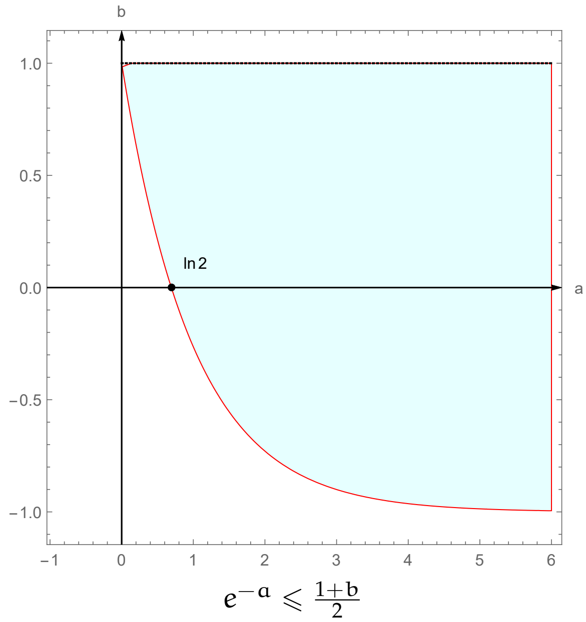


Figure 4: The range of the parameters a and b.

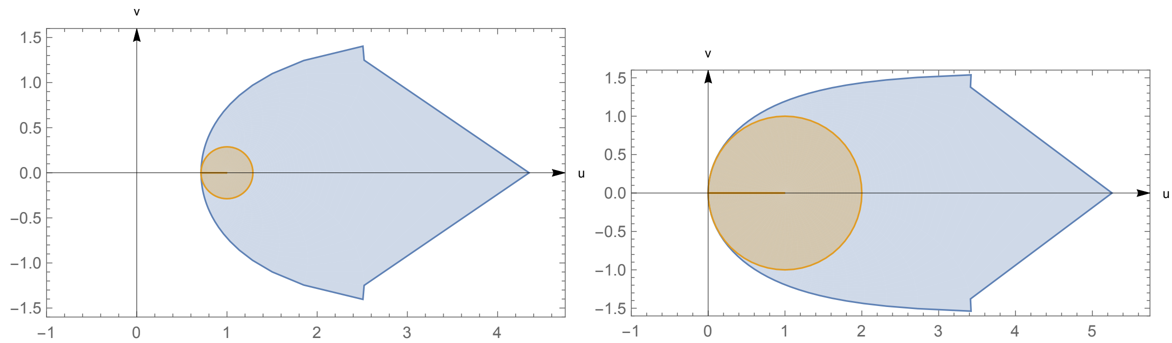


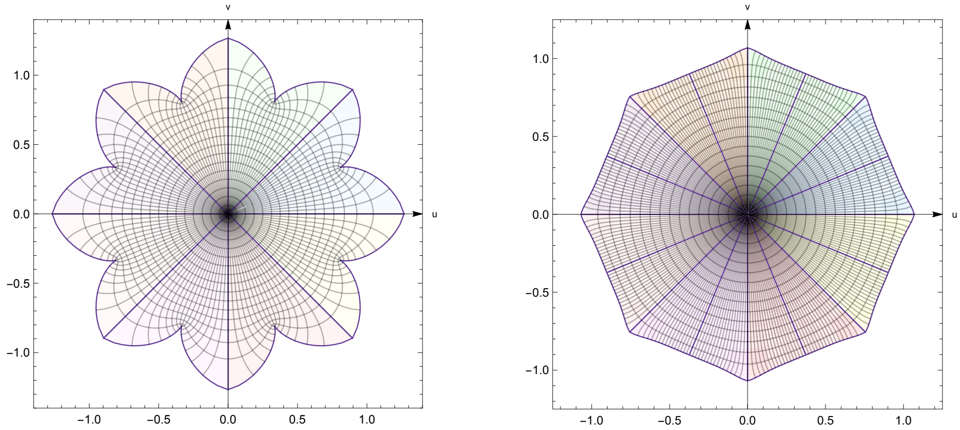
Figure 5: The image of $\mathbb{D}$ under $\mathcal{L}_{a,b}(z)$, $\left(\frac{1}{a} \ln \frac{2}{1+b}\right) z + 1$ for $a = 1, b = 0.5, \frac{2}{e} - 1$.

A function g is in the class $\mathcal{ST}_{sis}(a, b)$ if and only if there exist an analytic function q; $q \prec \mathcal{L}_{a,b}$, such that

$$g(z) = z \exp\left(\int_0^z \frac{q(t)-1}{t} dt\right) \quad \text{for} \quad z \in \mathbb{D}.$$

For $q(t) = \mathscr{L}_{a,b}(t^n)$ with $n = 1, 2, \ldots$ and $t \in \mathbb{D}$, we have (see Fig. 6)

$$h_{a,b,n}(z) = z \exp\left(\frac{1}{a}\int_0^z \frac{\log\frac{1-bt^n}{1-t^n}}{t}\, dt\right).$$



$a = 1, b = 0.5$ $\qquad$ $a = 2, b = 0.5$

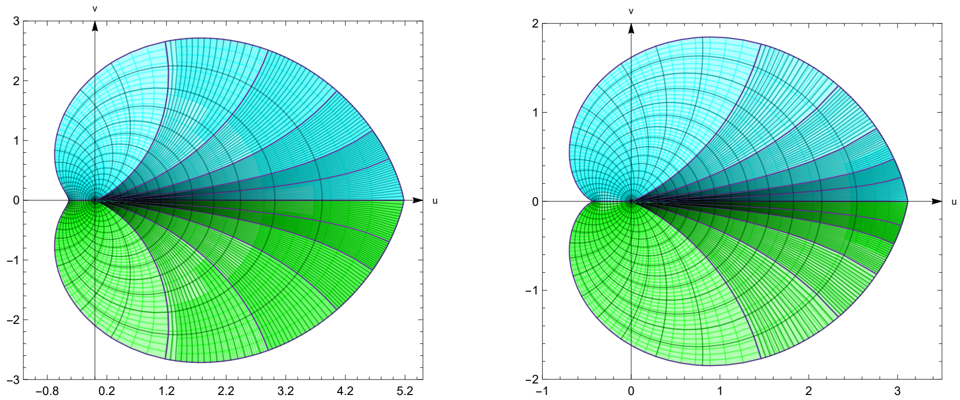
Figure 6: The image of $\mathbb{D}$ under $h_{a,b,n}(z)$ for $n = 8$.

Since $\Re\{1 - z\}$ and $\Re\{1 - bz\}$ are positive for all $z \in \mathbb{D}$ and $a \geqslant \ln \frac{2}{1+b}$, then we can write the previous relationship as follows,

$$\begin{aligned} h_{a,b,n}(z) &= z \exp\left(\frac{1}{a}\int_0^z \frac{\log(1-bt^n)}{t}\, dt - \frac{1}{a}\int_0^z \frac{\log(1-t^n)}{t}\, dt\right) \\ &= z \exp\left(\frac{\mathrm{Li}_2(z^n) - \mathrm{Li}_2(bz^n)}{an}\right) \qquad (3.2) \\ &= z + \frac{1-b}{an} z^{n+1} + \frac{(1-b)\,[an + 2 + (an-2)b]}{4a^2n^2} z^{2n+1} + \cdots . \end{aligned}$$

These are extremal functions for several problems in the class $\mathcal{ST}_{sis}(a, b)$. Especially for $n = 1$ we obtain (see Fig. 7)

$$h_{a,b}(z) := h_{a,b,1}(z) = z \exp\left(\frac{\mathrm{Li}_2(z) - \mathrm{Li}_2(bz)}{a}\right). \qquad (3.3)$$



$a = 2, b = 2/e - 1$ $\qquad$ $a = 1, b = 0$

Figure 7: The image of $\mathbb{D}$ under $h_{a,b}(z)$.

Corollary 3.2. *The class $\mathcal{ST}_{sis}(a,b)$ is nonempty. The following functions are the examples of their members.*

1. *Let $a_n \in \mathbb{C}$ with $n = 2, 3, \ldots$. Then $h(z) = z + a_n z^n \in \mathcal{ST}_{sis}(a,b) \Longleftrightarrow |a_n| \leqslant \frac{1-m(a,b)}{n-m(a,b)}$.*
2. *Let $A \in \mathbb{C}$. Then $h(z) = z/(1-Az)^2 \in \mathcal{ST}_{sis}(a,b) \Longleftrightarrow |A| \leqslant \frac{1-m(a,b)}{3-m(a,b)}$.*

Proof.

1. The function $h(z) = z + a_n z^n$ is univalent, if and only if $|a_n| \leqslant 1/n$. Logarithmic differentiation of non-zero univalent function $h(z)/z$ in $\mathbb{D}$ yields:

$$\frac{zh'(z)}{h(z)} - 1 = \frac{(n-1)a_n z^{n-1}}{1 + a_n z^{n-1}} \quad \text{for} \quad z \in \mathbb{D}.$$

Thus

$$\left|\frac{zh'(z)}{h(z)} - 1\right| = \left|\frac{(n-1)a_n z^{n-1}}{1 + a_n z^{n-1}}\right| < \frac{(n-1)|a_n|}{1-|a_n|} \quad \text{for} \quad z \in \mathbb{D}.$$

From (3.1), the function $z + a_n z^n$ is in $\mathcal{ST}_{sis}(a,b)$ if and only if

$$\frac{(n-1)|a_n|}{1-|a_n|} \leqslant 1 - m(a,b).$$

Thus the Case 1 is obtained.

2. The proof is very similar to that of part 1. Therefore, we only sketch the proof. Obviously, the rotated Koebe function $z/(1-Az)^2$ with A, a complex number with absolute value 1, is not the element of $\mathcal{ST}_{sis}(a,b)$. Since $\mathcal{ST}_{sis}(a,b) \subset \mathcal{ST}$, we conclude that $|A| < 1$. Therefore

$$\left|\frac{zh'(z)}{h(z)} - 1\right| = \left|\frac{2Az}{1-Az}\right| < \frac{2|A|}{1-|A|} \quad \text{for} \quad z \in \mathbb{D}.$$

From (3.1), the function $z/(1-Az)^2$ is in $\mathcal{ST}_{sis}(a,b)$ if and only if

$$\frac{2|A|}{1-|A|} \leqslant 1 - m(a,b).$$

Thus the Case 2 is obtained. □

The following corollary is the consequence of Theorems in [20].

Corollary 3.3. *Let $f \in \mathcal{ST}_{sis}(a,b)$ and $|z| = r < 1$. Then*

1. $-h_{a,b}(-r) \leqslant |f(z)| \leqslant h_{a,b}(r)$;
2. $\left|\operatorname{Arg}\left(\frac{f(z)}{z}\right)\right| \leqslant \max_{|z|=r} \operatorname{Arg}\left(\frac{h_{a,b}(z)}{z}\right)$*, equality holds at a given point other than 0 for functions $\overline{\mu} h_{a,b}(\mu z)$ with $|\mu| = 1$;*
3. $\frac{f(z)}{z} \prec \frac{h_{a,b}(z)}{z}$ $(z \in \mathbb{D})$;
4. *if $f \in \mathcal{ST}_{sis}(a,b)$, then either f is a rotation of $h_{a,b}$ given by (3.3) or*

$$\{w \in \mathbb{C} \colon |w| \leqslant -h_{a,b}(-1)\} \subset f(\mathbb{D}),$$

where $h_{a,b}(-1) = \lim_{r \to 1^+} h_{a,b}(-r)$.

The bound for the Fekete-Szegö inequality for the class $\mathcal{ST}_{sis}(a,b)$ can be estimated as in [20, Theorem 1, p.38]. If $f \in \mathcal{ST}_{sis}(a,b)$ be given by (1.2), then for complex number $\lambda \in \mathbb{C}$,

$$|a_3 - \lambda a_2^2| \leqslant \frac{1-b}{2a} \max\left\{1, \left|\lambda \frac{2(1-b)}{a} - \frac{1-b}{a} - \frac{1+b}{2}\right|\right\}.$$

The inequalities are sharp for the functions $h_{a,b,2}$ and $h_{a,b}$ where are given by (3.2) and (3.3). Let $f \in \mathcal{ST}_{sis}(a,b)$ be given by (1.2). Then

$$|a_3 - a_2^2| \leqslant \frac{1-b}{2a}, \quad |a_2| \leqslant \frac{1-b}{a},$$

and

$$|a_3| \leqslant \frac{1-b}{2a} \begin{cases} 1, & \text{for } a \geqslant 2, \\ \frac{a(1+b)+2(1-b)}{2a}, & \text{for } a \leqslant 2. \end{cases}$$

These inequalities are sharp.

If $f \in \mathcal{S}$, then the function $f(z)/z$ is analytic and does not vanish in $\mathbb{D}$. Furthermore, for such $f \in \mathcal{S}$, the logarithmic function as given by equation

$$\log \frac{f(z)}{z} = \sum_{n=1}^{\infty} 2\gamma_n(f) z^n \tag{3.4}$$

is well-defined for $z \in \mathbb{D}$, where $\gamma_n(f)$ denotes the n^{th} coefficient in the Laurent expansion of f around $z = 0$.

Theorem 3.4 ([29])**.** *Let $g(z) = \sum_{n=1}^{\infty} g_n z^n$ be analytic and convex univalent in $\mathbb{D}$. If $h(z) = \sum_{n=1}^{\infty} h_n z^n$ is analytic in $\mathbb{D}$ and satisfies the subordination $h(z) \prec g(z)$ in $\mathbb{D}$, then*

$$|h_n| \leqslant |g_1| \quad \textit{for} \quad n = 1, 2, \ldots.$$

Theorem 3.5. *Let $f \in \mathcal{A}$ belongs to the class $\mathcal{ST}_{sis}(a,b)$. Then the logarithmic coefficients of f satisfy the inequality*

$$|\gamma_n(f)| \leqslant \frac{1-b}{2a} \cdot \frac{1}{n} \quad \textit{for} \quad n = 1, 2, \ldots. \tag{3.5}$$

The inequality is sharp. Equality holds for functions $h_{a,b,n}(z)$ given by (3.2).

Proof. Let $f \in \mathcal{ST}_{sis}(a,b)$. Then we have

$$z\left(\log \frac{f(z)}{z}\right)' \prec \frac{1}{a} \log \frac{1-bz}{1-z} \quad \text{for} \quad z \in \mathbb{D}.$$

The subordination relation (3.4) implies that

$$\sum_{n=1}^{\infty} 2n\gamma_n(f) z^n \prec \sum_{n=1}^{\infty} B_n z^n,$$

where $B_n = (1-b^n)/(an)$ is defined in (2.1). Applying Theorem 3.4, we obtain the inequality $2n|\gamma_n(f)| \leqslant |B_1| = (1-b)/a$, and this completes inequality (3.5). The inequality is sharp when $f(z) = h_{a,b,n}(z)$, which is given by (3.2). □

4. Conclusions

The paper presented a thorough examination of the subset horizontal semi-infinite strip domain curve, exploring various parameters and their effects. The authors investigated families of starlike and convex functions that were situated within the regions bounded by the curve, providing examples and highlighting the properties of extremal functions within these families. The study also established both upper and lower bounds for the real and imaginary components of these functions, identifying key features of these bounds. Furthermore, the paper investigated logarithmic coefficients. Overall, this research provided valuable insights into the nature of the subset horizontal semi-infinite strip domain curve and its associated function. In the future we intend to study more inequalities [9, 10, 27] and new results related to improper integrals as in [1, 2, 31].

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