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Application of Laplace transform to solve fractional integro-differential equations

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Abstract

This paper reveals the solutions to several families of fractional integro-differential equations through the application of a simple fractional calculus method. This approach results in various interesting consequences and also extends the classical Frobenius method. The provided approach is primarily based on established theorems concerning particular solutions of fractional integro-differential equations using the Laplace transform and the extension coefficients of binomial series. Additionally, an illustrative example of such fractional integro-differential equations is presented.

Keywords: Riemann-Liouville fractional integrals, fractional-order differential equation, gamma function, Mittag-Leffler function, Wright function, Laplace transform of the fractional derivative.

2020 MSC: 26A33, 44A10, 34A05, 33D05, 33E12.

1. Introduction

The historical roots of fractional calculus can be attributed to the dilemma concerning the extension of the sense of a derivative to a numerical approximation when n is not an integer. This question was first raised by L'Hopital on September 30th, 1695.

Fractional calculus is used in a wide range of engineering and research fields, including electromagnetics, viscoelasticity, fluid mechanics, electrochemistry, biological population models, optics, and signal processing [23] (we also refer the reader to [1, 3, 4, 7, 13–17, 19, 21, 25] for the classical differential equations and partial differential equations describing these real phenomena). It is employed to simulate

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technical and physical processes that are best represented by fractional differential equations. Fractional derivative models are particularly useful for precise modeling of systems that require accurate damping representation. Numerous analytical and numerical techniques have been proposed in recent years for various disciplines, demonstrating their applicability to new problems. These contributions to science and engineering are firmly rooted in mathematics.

Several well-established mathematical functions are strongly associated with fractional calculus, including the gamma function, beta function, error function, Mittag-Leffler function, and Mellin-Ross function.

In 2011, Tarig and Salih introduced a novel integral transform known as the Tarig transform. They established a relationship between the Laplace and Tarig transforms [5]. Additionally, they applied the Tarig transform to a variable coefficient ordinary differential equation. In 2013, the same authors extended the Tarig transform to a system of integro-differential equations. In 2017, Hassan Taha et al. solved telegraph equations by leveraging the dualities between Kamal and Mahgoub integral transforms and several well-known integral transformations [28].

In 2013, Lin and Lu [18] illuminated the concept of using the extension coefficients of binomial series and the Laplace transform (LT) of fractional derivatives to generate explicit solutions for homogeneous fractional differential equations (see, for example, [2, 6, 8, 18, 20, 22, 24]).

In this article, we develop the concept of a fractional integro-differential equation by employing the Laplace transform (LT) and binomial series extension coefficients. Additionally, we discuss properties related to our focused topic.

2. Preliminaries

In this section, we are listing some preliminaries that are useful throughout the paper.

1. Casual function on a fractional derivative $k(t)$ is defined by

$$\frac{d^\sigma}{dt^\sigma}k(t) = \begin{cases} k^{(m)}(t), & \text{if } \sigma = m \in \mathbb{N}, \\ \frac{1}{\Gamma(m-\sigma)} \int_0^\zeta \frac{k^{(m)}(t)}{(t-x)^{\sigma-m+1}} dt, & \text{if } m-1 < \sigma < m, \end{cases}$$

where the Euler gamma function $\Gamma(\cdot)$ is defined by

$$\Gamma(\nu) = \int_0^\infty t^{\nu-1} e^{-t} dt \quad (\mathbb{R} > 0).$$

2. The LT of a function $k(t)$, $t \in (0, \infty)$, is defined by

$$\mathcal{L}[k(t)](w) = \int_0^\infty e^{-wt} k(t) dt \quad (w \in \mathbb{C}).$$

3. The Mittag-Leffler function [11, 18] is defined by

$$E_{\sigma,\tau}(\nu) = \sum_{\gamma=0}^{\infty} \frac{\nu^\gamma}{\Gamma(\sigma\gamma + \tau)} \quad (\nu, \sigma, \tau \in \mathbb{C}, \mathbb{R}(\sigma) > 0).$$

4. The simplest Wright function [18, 27] is defined by

$$\chi(\sigma, \tau; \nu) = \sum_{\gamma=0}^{\infty} \frac{1}{\Gamma(\sigma\gamma + \tau)} . \frac{\nu^\gamma}{\gamma!} \qquad (\nu, \sigma, \tau \in \mathbb{C}).$$

5. The general Wright function ${}_n\lambda_m(\nu)$ is defined for $\nu \in \mathbb{C}$, $\mathfrak{h}_{1p}, \mathfrak{h}_{2q} \in \mathbb{C}$, [12, 18], and real $\sigma_p, \tau_q \in \mathbb{R}(p = 1, \dots, n; q = 1, \dots, m)$ by the series

$$ {}_n\lambda_m(\nu) = {}_n\lambda_m\left(\begin{matrix} (\mathfrak{h}_{1p}, \sigma_p)_{1,n} \\ (\mathfrak{h}_{2q}, \tau_q)_{1,m} \end{matrix} \middle| \nu \right) = \sum_{\gamma=0}^{\infty} \frac{\prod_{p=1}^{n} \Gamma(\mathfrak{h}_{1p} + \sigma_p \gamma)}{\prod_{q=1}^{m} \Gamma(\mathfrak{h}_{2q} + \tau_q \gamma)} . \frac{\nu^\gamma}{\gamma!}, $$

where $\nu, \mathfrak{h}_{1p}, \mathfrak{h}_{2q} \in \mathbb{C}, \sigma_p, \tau_q \in \mathbb{R}, p = 1, 2, \dots, n$, and $q = 1, 2, \dots, m$.

6. The Riemann-Liouville fractional derivatives [9, 29] $D^{\sigma}_{\mathfrak{h}_1+}y$ and $D^{\tau}_{\mathfrak{h}_2-}y$ of order $\sigma \in \mathbb{C}$ $(\Re(\sigma) \geqslant 0)$ are defined by

$$ (D^{\sigma}_{\mathfrak{h}_1+}y)(x) = \frac{1}{\Gamma(m-\sigma)} (\frac{d}{dx})^m \int_{\mathfrak{h}_1}^{x} \frac{y(t)dt}{(x-t)^{\sigma-m+1}} \ (m = [\mathcal{R}(\sigma)] + 1; x > \mathfrak{h}_1), $$
$$ (D^{\sigma}_{\mathfrak{h}_2-}y)(x) = \frac{1}{\Gamma(m-\sigma)} (-\frac{d}{dx})^m \int_{x}^{\mathfrak{h}_2} \frac{y(t)dt}{(t-x)^{\sigma-m+1}} \ (m = [\mathcal{R}(\sigma)] + 1; x < \mathfrak{h}_2), $$

respectively. Here, the integral part of $\Re[\sigma]$ is $[\Re(\sigma)]$.

7. The shift factorial [10, 26], since $(1)_m = m!$ for $m \in N_0 = \{0, 1, 2, \dots\}$ is given by

$$ (\delta)_m = \begin{cases} 1, & (m = 0), \\ \delta(\delta+1)\cdots(\delta+m-1), & (m \in \mathbb{N}_0/\{0\}). \end{cases} $$

8. The binomial coefficients are defined by

$$ \binom{\delta}{m} = \frac{\delta!}{\delta!(\delta-m)!} = \frac{\delta(\delta-1)(\delta-m+1)}{m!}, $$

where δ and m are integers. Observe that $0! = 1$, then

$$ \binom{\delta}{0} = 1, \binom{\delta}{\delta} = 1 \text{ and } (1-\nu)^{-\delta} = \sum_{\aleph=0}^{\infty} \frac{(\delta)_\aleph}{\aleph!} \nu^n = \sum_{\aleph=0}^{\infty} \binom{\delta+\aleph-1}{\aleph} \nu^\aleph. $$

9. $\mathcal{L}[\chi(\sigma, \tau; t)](w) = \frac{1}{w} E_{\sigma,\tau}(\frac{1}{w})(\sigma > -1, \tau \in \mathbb{C}; \mathbb{R}(w) > 0)$.

10. The generalized Wright function of the LT is given by

$$ \mathcal{L}\left\{ {}_n\lambda_m\left(\begin{matrix} (\mathfrak{h}_{1p}, \sigma_p)_{1,n} \\ (\mathfrak{h}_{2q}, \tau_q)_{1,m} \end{matrix} \middle| -t \right) \right\}(w) = \frac{1}{w} {}_{n+1}\lambda_m\left(\begin{matrix} (1,1), (\mathfrak{h}_{1p}, \sigma_p)_{1,n} \\ (\mathfrak{h}_{2q}, \tau_q)_{1,m} \end{matrix} \middle| -\frac{1}{w} \right) $$

$(\Re > 0), p = 1, 2, \dots, n$ and $q = 1, 2, \dots, m$.

11. $\mathcal{L}[D^\sigma]k(t)(w) = w^\sigma[\mathcal{L}k(t)](w) - \sum_{\gamma=1}^{m} w_{\sigma-\gamma} k^{(\gamma)-1}(0)[10]$, where $\sigma > 0$,

$$ m - 1 < \sigma \leqslant m (m \in \mathbb{N}), k(t) \in \mathcal{C}^m(0, \infty), k^{(m)}(t) \in L_1(0, \mathfrak{h}_2), \ \forall \ \mathfrak{h}_2 > 0. $$

12. The inverse LT is defined by

$$ \mathcal{L}^{-1}\left[\frac{\Gamma[m+1]}{w^{m+1}} \right] = t^m. $$

Remark 2.1. Let us demonstrate Preliminary 11 using the integral transform technique, as shown below; it is not difficult if we take reference Preliminary 2,

$$
\begin{aligned}
\mathcal{L}[D^\sigma k(t)](w) = \int_0^\infty e^{-wt}[D^\sigma k(t)]dt &= \int_0^\infty e^{-wt}.\frac{1}{\Gamma(m-\sigma)}\int_0^t \frac{k^{(m)}(\zeta)}{(t-\zeta)^{\sigma-m+1}}d\zeta dt \\
&= \frac{1}{\Gamma(m-\sigma)}\int_0^\infty\int_\zeta^\infty e^{-wt}\frac{k^{(m)}(\zeta)}{(t-\zeta)^{\sigma-m+1}}dtd\zeta \\
&= \frac{1}{\Gamma(m-\sigma)}\int_0^\infty k^{(m)}(\zeta)\int_0^\infty e^{-w(u+\zeta)}u^{m-\sigma-1}dud\zeta \\
&= \frac{1}{\Gamma(m-\sigma)}\int_0^\infty e^{-w\zeta}k^{(m)}(\zeta)\int_0^\infty e^{-wu}u^{m-\sigma-1}dud\zeta \\
&= \frac{1}{\Gamma(m-\sigma)}\int_0^\infty e^{-w\zeta}k(m)(\zeta)\frac{\Gamma(m-\sigma)}{s^{m-\sigma}}d\zeta \\
&= w^{\sigma-m}\int_0^\infty e^{-w\zeta}k^{(m)}(\zeta)d\zeta = w^{\sigma-m}\mathcal{L}[k^{(m)}(t)](w) \\
&= w^{\sigma-m}(w^m\mathcal{L}[k(t)] - w^{m-1}k(0) - w^{m-2}k'(0) - \cdots - k^{(m-1)}(0)) \\
&= (w^\sigma\mathcal{L}[k(t)] - w^{\sigma-1}k(0) - w^{\sigma-2}k'(0) - \cdots - w^{\sigma-m}k^{(m-1)}(0)) \\
&= w^\sigma\mathcal{L}[k(t)] - \sum_{\gamma=0}^{m} w^{\sigma-\gamma}k^{(\gamma-1)}(0).
\end{aligned}
$$

Note: We use Fubini's theorem to change the order of integration in the preceding derivation.

3. Solutions of the fractional integro-differential equations

In this section, we can strongly suspect that $y(t)$ is sufficient to allow the LT $\mathcal{L}(y)$ to proceed for some value of the parameter w.

Theorem 3.1. *Under assumptions* $1 < \sigma < 2$ *and* $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathbb{R}$ *the fractional integro-differential equation*

$$
y''(t) + \mathfrak{h}_1 y^{(\sigma)}(t) + \mathfrak{h}_2 y(t) = \int_0^s \frac{g(t)}{(s-t)^\tau}dt, \quad 0 < \tau < 1, \tag{3.1}
$$

with initial conditions $y(0) = e_0$ *and* $y'(0) = e_1$ *has the unique solution*

$$
\begin{aligned}
y(t) = e_0 &\sum_{\gamma=0}^\infty \frac{(-\mathfrak{h}_2)^\gamma t^{2\gamma}}{\gamma!}\sum_{\aleph=0}^\infty \frac{\Gamma(\aleph+\gamma+1)(-\mathfrak{h}_1 t^{2-\sigma})^\aleph}{\Gamma[(2-\sigma)\aleph+2\gamma+1]\aleph!} \\
&+ e_1\sum_{\gamma=0}^\infty \frac{(-\mathfrak{h}_2)^\gamma t^{2\gamma+1}}{\gamma!}\sum_{\aleph=0}^\infty \frac{\Gamma(\aleph+\gamma+1)(-\mathfrak{h}_1 t^{2-\sigma})^\aleph}{\Gamma[(2-\sigma)\aleph+2\gamma+2]\aleph!} \\
&+ ae_0\sum_{\gamma=0}^\infty \frac{(-\mathfrak{h}_2)^\gamma t^{2\gamma-\sigma+2}}{\gamma!}\sum_{\aleph=0}^\infty \frac{\Gamma(\aleph+\gamma+1)(-\mathfrak{h}_1 t^{2-\sigma})^\aleph}{\Gamma[(2-\sigma)\aleph+2\gamma-\sigma+3]\aleph!} \\
&+ ae_1\sum_{\gamma=0}^\infty \frac{(-\mathfrak{h}_2)^\gamma t^{2\gamma-\sigma+3}}{\gamma!}\sum_{\aleph=0}^\infty \frac{\Gamma(\aleph+\gamma+1)(-\mathfrak{h}_1 t^{2-\sigma})^\aleph}{\Gamma[(2-\sigma)\aleph+2\gamma-\sigma+4]\aleph!} \\
&+ \frac{\sin(\tau\pi)}{\pi}\frac{d}{ds}\int_0^s (s-t)^{\tau-1}k(t)dt\sum_{\gamma=0}^\infty \frac{(-\mathfrak{h}_2)^\gamma}{\gamma!}t^{2\gamma+1}\sum_{\aleph=0}^\infty \frac{\Gamma(\gamma+\aleph+1)}{\aleph!}\frac{(-\mathfrak{h}_1 t^{(2-\sigma)})^n}{\Gamma[(2-\sigma)n+2\gamma+2]}.
\end{aligned} \tag{3.2}
$$

Proof. Utilizing the LT in (3.1) and taking into consideration, we have

$$w^2\mathcal{L}[y]-e_0w-e_1+\mathfrak{h}_1w^{\sigma}\mathcal{L}[y]-ae_0w^{\sigma-1}-\mathfrak{h}_1e_1w^{\sigma-2}+\mathfrak{h}_2\mathcal{L}[y]=\mathcal{L}[k(t)], \tag{3.3}$$

where $k(t)=\int_0^s \frac{g(t)}{(s-t)^{\tau}}dt$. Equation (3.3) generates

$$\begin{aligned}\mathcal{L}[y]=&\frac{e_0w+\mathfrak{h}_1e_0w^{\sigma-1}+\mathfrak{h}_1e_1w^{\sigma-2}+\mathcal{L}[k(t)]}{w^2+aw^{\sigma}+\mathfrak{h}_2}\\&+e_0\sum_{\gamma=0}^{\infty}(-\mathfrak{h}_2)^{\gamma}\sum_{\aleph=0}^{\infty}\binom{\gamma+\aleph}{\aleph}(-\mathfrak{h}_1)^{\aleph}w^{(\sigma-2)\aleph-2\gamma-1}\\&+e_1\sum_{\gamma=0}^{\infty}(-\mathfrak{h}_2)^{\gamma}\sum_{\aleph=0}^{\infty}\binom{\gamma+\aleph}{\aleph}(-\mathfrak{h}_1)^{\aleph}w^{(\sigma-2)\aleph-2\gamma-2}\\&+\mathfrak{h}_1e_0\sum_{\gamma=0}^{\infty}(-\mathfrak{h}_2)^{\gamma}\sum_{\aleph=0}^{\infty}\binom{\gamma+\aleph}{\aleph}(-\mathfrak{h}_1)^{\aleph}w^{(\sigma-2)\aleph-2\gamma+\sigma-3}\\&+\mathfrak{h}_1e_1\sum_{\gamma=0}^{\infty}(-\mathfrak{h}_2)^{\gamma}\sum_{\aleph=0}^{\infty}\binom{\gamma+\aleph}{\aleph}(-\mathfrak{h}_1)^{\aleph}w^{(\sigma-2)\aleph-2\gamma+\sigma-4}\\&+\mathfrak{p}\mathcal{L}\left[\int_0^s(s-t)^{\tau-1}k(t)dt\right]\frac{\sin(\tau\pi)}{\pi}\sum_{\gamma=0}^{\infty}(-\mathfrak{h}_2)^{\gamma}\sum_{\aleph=0}^{\infty}(-\mathfrak{h}_1)^{\aleph}\binom{\gamma+\aleph}{\aleph}w^{(\sigma-2)\aleph-2\gamma-2},\end{aligned} \tag{3.4}$$

since

$$\begin{aligned}\frac{1}{w^2+\mathfrak{h}_1w^{\sigma}+\mathfrak{h}_2}&=\frac{w^{-\sigma}}{w^{2-\sigma}+\mathfrak{h}_1+\mathfrak{h}_2w^{-\sigma}}\\&=\frac{w^{-\sigma}}{(w^{2-\sigma}+\mathfrak{h}_1)(1+\frac{\mathfrak{h}_2w^{-\sigma}}{w^{2-\sigma}+\mathfrak{h}_1})}\\&=\frac{w^{-\sigma}}{w^{2-\sigma}+\mathfrak{h}_1}\sum_{\gamma=0}^{\infty}(\frac{-\mathfrak{h}_2w^{-\sigma}}{w^{2-\sigma}+\mathfrak{h}_1})^{\gamma}\\&=\sum_{\gamma=0}^{\infty}\frac{(-\mathfrak{h}_2)^{\gamma}w^{-\sigma\gamma-\sigma}}{(w^{2-\sigma}+\mathfrak{h}_1)^{\gamma+1}}\\&=\sum_{\gamma=0}^{\infty}\frac{(-\mathfrak{h}_2)^{\gamma}w^{-2\gamma-2}}{(1+\mathfrak{h}_1w^{\sigma-2})^{\gamma+1}}\\&=\sum_{\gamma=0}^{\infty}(-\mathfrak{h}_2)^{\gamma}w^{-2\gamma-2}\sum_{\aleph=0}^{\infty}(-\mathfrak{h}_1w^{\sigma-2})^{\aleph}\binom{\gamma+\aleph}{\aleph}\\&=\sum_{\gamma=0}^{\infty}(-\mathfrak{h}_2)^{\gamma}\sum_{\aleph=0}^{\infty}\binom{\gamma+\aleph}{\aleph}(-\mathfrak{h}_1)^{\aleph}w^{(\sigma-2)\mathfrak{h}_2-2\gamma-2}\end{aligned}$$

and

$$\mathcal{L}[k(t)]=\mathcal{L}\left[\int_0^s\frac{g(t)}{(s-t)^{\tau}}dt\right].$$

Now, the convolution integral

$$K(\mathfrak{P})=F(\mathfrak{P})G(\mathfrak{P}),$$

where

$$K(\mathfrak{P})\text{ is the LT of }K(\mathfrak{S})=w^{-\tau},\quad \mathcal{L}[K(\mathfrak{S})]=w^{-\tau}F(\mathfrak{P})=\frac{\Gamma(-\tau+1)}{\mathfrak{p}^{-\tau+1}}F(\mathfrak{P})=\mathfrak{p}^{-1+\tau}\Gamma(1-\tau),$$

and

$$G(\mathfrak{P}) = \frac{K(\mathfrak{P})}{G(\mathfrak{P})}, \qquad G(\mathfrak{P}) = \frac{K(\mathfrak{P})}{\mathfrak{p}^{-1+\tau}\Gamma(1-\tau)},$$
$$G(\mathfrak{P}) = \frac{\mathfrak{p}[\Gamma(\tau)\mathfrak{p}^{(-\tau)}K(\mathfrak{P})]}{\pi\operatorname{cosec}\pi\tau}, \quad G(\mathfrak{P}) = \frac{\sin\tau\pi}{\pi}\mathfrak{p}\mathcal{L}\left[\int_0^s (s-t)^{\tau-1}k'(t)dt\right].$$

Thus, the inverse LT to equation (3.4) yields the solution (3.2),

$$\begin{aligned} y(t) = {} & e_0 \sum_{\gamma=0}^{\infty} \frac{(-\mathfrak{h}_2)^{\gamma}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{(\gamma+\aleph)!(-\mathfrak{h}_1)^{\aleph}}{\Gamma[(2-\sigma)\aleph+2\gamma+1]} \frac{t^{(2-\sigma)\aleph+2\gamma}}{\aleph!} \\ & + e_1 \sum_{\gamma=0}^{\infty} \frac{(-\mathfrak{h}_2)^{\gamma}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{(\gamma+\aleph)!(-\mathfrak{h}_1)^{\aleph}}{\Gamma[(2-\sigma)\aleph+2\gamma+2]} \frac{t^{(2-\sigma)\aleph+2\gamma+1}}{\aleph!} \\ & + \mathfrak{h}_1 e_0 \sum_{\gamma=0}^{\infty} \frac{(-\mathfrak{h}_2)^{\gamma}\gamma!}{\sum_{\aleph=0}^{\infty}} \frac{(\gamma+\aleph)!(-\mathfrak{h}_1)^{\aleph}}{\Gamma[(2-\sigma)\aleph+2\gamma-\sigma+3]} \frac{t^{(2-\sigma)\aleph+2\gamma-\sigma+2}}{\aleph!} \\ & + \mathfrak{h}_1 e_1 \sum_{\gamma=0}^{\infty} \frac{(-\mathfrak{h}_2)^{\gamma}\gamma!}{\sum_{\aleph=0}^{\infty}} \frac{(\gamma+\aleph)!(-\mathfrak{h}_1)^{\aleph}}{\Gamma[(2-\sigma)\aleph+2\gamma-\sigma+4]} \frac{t^{(2-\sigma)\aleph+2\gamma-\sigma+3}}{\aleph!} \\ & + \frac{\sin(\tau\pi)}{\pi}\frac{d}{ds}\int_0^s (s-t)^{\tau-1}k(t)dt \sum_{\gamma=0}^{\infty} \frac{(-\mathfrak{h}_2)^{\gamma}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{(\gamma+\aleph)!}{\aleph!}(-\mathfrak{h}_1)^{\aleph} \frac{(t^{(2-\sigma)})^{\aleph}+2\gamma+1}{\Gamma[(2-\sigma)\aleph+2\gamma+2]}. \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) = {} & e_0 \sum_{\gamma=0}^{\infty} \frac{(-\mathfrak{h}_2)^{\gamma} t^{2\gamma}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\aleph+\gamma+1)(-\mathfrak{h}_1 t^{2-\sigma})^{\aleph}}{\Gamma[(2-\sigma)\aleph+2\gamma+1]\aleph!} \\ & + e_1 \sum_{\gamma=0}^{\infty} \frac{(-\mathfrak{h}_2)^{\gamma} t^{2\gamma+1}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\aleph+\gamma+1)(-\mathfrak{h}_1 t^{2-\sigma})^{\aleph}}{\Gamma[(2-\sigma)\aleph+2\gamma+2]\aleph!} \\ & + \mathfrak{h}_1 e_0 \sum_{\gamma=0}^{\infty} \frac{(-\mathfrak{h}_2)^{\gamma} t^{2\gamma-\sigma+2}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\aleph+\gamma+1)(-\mathfrak{h}_1 t^{2-\sigma})^{\aleph}}{\Gamma[(2-\sigma)\aleph+2\gamma-\sigma+3]\aleph!} \\ & + \mathfrak{h}_1 e_1 \sum_{\gamma=0}^{\infty} \frac{(-b)^{\gamma} t^{2\gamma-\sigma+3}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\aleph+\gamma+1)(-a t^{2-\sigma})^{\aleph}}{\Gamma[(2-\sigma)\aleph+2\gamma-\sigma+4]\aleph!} \\ & + \frac{\sin(\tau\pi)}{\pi}\frac{d}{ds}\int_0^s (s-t)^{\tau-1}k(t)dt \sum_{\gamma=0}^{\infty} \frac{(-\mathfrak{h}_2)^{\gamma}}{\gamma!} t^{2\gamma+1} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\gamma+\aleph+1)}{\aleph!} \frac{(-\mathfrak{h}_1 t^{(2-\sigma)})^{\aleph}}{\Gamma[(2-\sigma)\aleph+2\gamma+2]}, \end{aligned}$$

which is (3.2). This completes the proof of the theorem. □

Example 3.2. A generalized viscoelastic free damping oscillation's fractional integro-differential equation [2, 18]

$$y''(t) + \mathfrak{h}_1 y^{(\frac{3}{2})}(t) + \mathfrak{h}_2 y(t) = \int_0^s \frac{g(t)}{(s-t)^{\frac{1}{2}}} dt \tag{3.5}$$

with initial conditions $y(0) = e_0$ and $y'(0) = e_1$ has the unique solution

$$\begin{aligned} y(t) = {} & e_0 \sum_{\gamma=0}^{\infty} \frac{(-\mathfrak{h}_2)^{\gamma} t^{2\gamma}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\aleph+\gamma+1)(-\mathfrak{h}_1 t^{\frac{1}{2}})^{\aleph}}{\Gamma[(\frac{1}{2})\aleph+2\gamma+1]\aleph!} \\ & + e_1 \sum_{\gamma=0}^{\infty} \frac{(-\mathfrak{h}_2)^{\gamma} t^{2\gamma+1}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\aleph+\gamma+1)(-\mathfrak{h}_1 t^{\frac{1}{2}})^{\aleph}}{\Gamma[(\frac{1}{2})\aleph+2\gamma+2]\aleph!} \end{aligned}$$

$$+ \mathfrak{h}_1 e_0 \sum_{\gamma=0}^{\infty} \frac{(-\mathfrak{h}_2)^{\gamma} t^{2\gamma+\frac{1}{2}}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\aleph+\gamma+1)(-\mathfrak{h}_1 t^{\frac{1}{2}})^{\aleph}}{\Gamma[(\frac{1}{2})\aleph+2\gamma+\frac{3}{2}]\aleph!}$$
$$+ \mathfrak{h}_1 e_1 \sum_{\gamma=0}^{\infty} \frac{(-\mathfrak{h}_2)^{\gamma} t^{2\gamma+\frac{3}{2}}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\aleph+\gamma+1)(-\mathfrak{h}_1 t^{\frac{1}{2}})^{\aleph}}{\Gamma[(\frac{1}{2})\aleph+2\gamma+\frac{5}{2}]\aleph!}$$
$$+ \frac{1}{\pi}\frac{d}{ds}\int_0^s (s-t)^{-\frac{1}{2}} k(t)dt \sum_{\gamma=0}^{\infty} \frac{(-\mathfrak{h}_2)^{\gamma}}{\gamma!} t^{2\gamma+1} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\gamma+\aleph+1)}{\aleph!} \frac{(-\mathfrak{h}_1 t^{(\frac{1}{2})})^{\aleph}}{\Gamma[(\frac{1}{2})\aleph+2\gamma+2]}$$

with specific, if $\mathfrak{h}_1 = \sqrt{5}$ and $\mathfrak{h}_2 = 10$, then (3.5) is

$$y''(t) + \sqrt{5} y^{(\frac{3}{2})}(t) + 10 y(t) = \int_0^s \frac{g(t)}{(s-t)^{\frac{1}{2}}} dt$$

with initial conditions $y(0) = e_0$ and $y'(0) = e_1$, then the solution is given by

$$y(t) = e_0 \sum_{\gamma=0}^{\infty} \frac{(-10)^{l} t^{2\gamma}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\aleph+\gamma+1)(-\sqrt{5} t^{\frac{1}{2}})^{\aleph}}{\Gamma[(\frac{1}{2})\aleph+2\gamma+1]\aleph!}$$
$$+ e_1 \sum_{\gamma=0}^{\infty} \frac{(-10)^{\gamma} t^{2\gamma+1}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\aleph+\gamma+1)(-\sqrt{5} t^{\frac{1}{2}})^{\aleph}}{\Gamma[(\frac{1}{2})\aleph+2\gamma+2]\aleph!}$$
$$+ \sqrt{5} e_0 \sum_{\gamma=0}^{\infty} \frac{(-10)^{\gamma} t^{2\gamma+\frac{1}{2}}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\aleph+\gamma+1)(-\sqrt{5} t^{\frac{1}{2}})^{\aleph}}{\Gamma[(\frac{1}{2})\aleph+2\gamma+\frac{3}{2}]\aleph!}$$
$$+ \sqrt{5} e_1 \sum_{\gamma=0}^{\infty} \frac{(-10)^{\gamma} t^{2\gamma+\frac{3}{2}}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\aleph+l+1)(-\sqrt{5} t^{\frac{1}{2}})^{\aleph}}{\Gamma[(\frac{1}{2})\aleph+2\gamma+\frac{5}{2}]\aleph!}$$
$$+ \frac{1}{\pi}\frac{d}{ds}\int_0^s (s-t)^{-\frac{1}{2}} k(t)dt \sum_{\gamma=0}^{\infty} \frac{(-10)^{\gamma}}{\gamma!} t^{2\gamma+1} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\gamma+\aleph+1)}{\aleph!} \frac{(-\sqrt{5} t^{(\frac{1}{2})})^{\aleph}}{\Gamma[(\frac{1}{2})\aleph+2\gamma+2]}.$$

Figure 1 illustrates the solution behavior of the fractional integro-differential equation of Example 3.2 at various values of σ with the initial conditions $e_0 = 1$ and $e_1 = 1$.

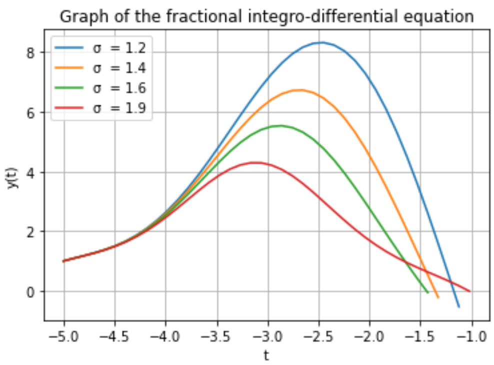

Figure 1: The solution behavior of Example 3.2.

Theorem 3.3. *Under assumptions* $1 < \sigma < 2$ *and* $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathbb{R}$ *the fractional integro-differential equation*

$$y^{\sigma} + \mathfrak{h}_1 y'(t) + \mathfrak{h}_2 y(t) = \int_0^s \frac{g(t)}{(s-t)^{\tau}} dt, \quad 0 < \tau < 1, \tag{3.6}$$

with initial conditions $y(0) = e_0$ *and* $y'(0) = e_1$ *has the unique solution*

$$\begin{aligned} y(t) = e_0 \sum_{\gamma=0}^{\infty} \frac{(-\mathfrak{h}_2)^{\gamma}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\aleph+\gamma+1)(-\mathfrak{h}_1)^{\aleph} t^{(\sigma-1)\aleph+\sigma\gamma}}{\Gamma[(\sigma-\gamma)\aleph+\sigma\gamma+1]\aleph!} \\ + e_1 \sum_{\gamma=0}^{\infty} \frac{(-\mathfrak{h}_2)^{\gamma}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\aleph+\gamma+1)(-\mathfrak{h}_1)^{\aleph} t^{(\sigma-1)\aleph+\sigma\gamma+1}}{\Gamma[(\sigma-1)\aleph+\sigma\gamma+2]\aleph!} \\ + \mathfrak{h}_1 e_0 \sum_{\gamma=0}^{\infty} \frac{(-\mathfrak{h}_2)^{\gamma}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\aleph+\gamma+1)(-\mathfrak{h}_1)_1^{\mathfrak{h}} t^{(\sigma-1)\aleph+\sigma\gamma+\sigma-1)\aleph}}{\Gamma[(\sigma-1)\aleph+\sigma\gamma-\sigma]\aleph!} \\ + \frac{\sin(\tau\pi)}{\pi} \frac{d}{ds} \int_0^s (s-t)^{\tau-1} k(t) dt \sum_{\gamma=0}^{\infty} \frac{(-\mathfrak{h}_2)^{\gamma}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{(-\mathfrak{h}_1)^{\aleph}\Gamma(\gamma+\aleph+1)}{\aleph!} \frac{t^{(\sigma-1)\aleph+\sigma\gamma+\sigma-1}}{\Gamma[(\sigma-1)\aleph+\sigma\gamma+\sigma]}. \end{aligned} \tag{3.7}$$

Proof. Utilizing the LT (Preliminary 11) in (3.6), we have

$$w^2\mathcal{L}[y] - w^{\sigma-1}y(0) - w^{\sigma-2}y'(0) + \mathfrak{h}_1 w\mathcal{L}[y] - \mathfrak{h}_1 y'(0) + \mathfrak{h}_2\mathcal{L}[y] = \mathcal{L}[k(t)], \tag{3.8}$$

where $k(t) = \int_0^s \frac{g(t)}{(s-t)^{\tau}} dt$, that is,

$$(w^{\sigma} + \mathfrak{h}_1 w + \mathfrak{h}_2)\mathcal{L}[y] = e_0 w^{\sigma-1} + e_1 w^{\sigma-2} + \mathfrak{h}_1 e_0 + \mathcal{L}[k(t)].$$

Hence, the equation (3.8) yields

$$\begin{aligned} \mathcal{L}[y] = \frac{e_0 w^{\sigma-1} + e_1 w^{\sigma-2} + \mathfrak{h}_1 e_0 + \mathcal{L}[k(t)]}{w^{\sigma} + \mathfrak{h}_1 w + \mathfrak{h}_2} \\ + e_0 \sum_{\gamma=0}^{\infty} (-\mathfrak{h}_2)^{\gamma} \sum_{\aleph=0}^{\infty} \binom{\gamma+\aleph}{\aleph} (-\mathfrak{h}_1)^{\aleph} w^{\aleph-\sigma\aleph-\sigma\gamma-1} \\ + e_1 \sum_{\gamma=0}^{\infty} (-\mathfrak{h}_2)^{\gamma} \sum_{\aleph=0}^{\infty} \binom{\gamma+\aleph}{\aleph} (-\mathfrak{h}_1)^{\aleph} w^{\aleph-\sigma\aleph-\sigma\gamma-2} \\ + \mathfrak{h}_1 e_0 \sum_{\gamma=0}^{\infty} (-\mathfrak{h}_2)^{\gamma} \sum_{\aleph=0}^{\infty} \binom{\gamma+\aleph}{\aleph} (-\mathfrak{h}_1)^{\aleph} w^{\aleph-\sigma\aleph-\sigma\gamma-\sigma} \\ + \frac{\sin(\tau\pi)}{\pi} p\mathcal{L}\left[\int_0^s (s-t)^{\tau-1} k'(t) dt\right] \sum_{\gamma=0}^{\infty} (-\mathfrak{h}_2)^{l} \sum_{\aleph=0}^{\infty} (-\mathfrak{h}_1)^{\aleph} \binom{\gamma+\aleph}{\aleph} w^{\aleph-\sigma\aleph-\sigma\gamma-\sigma}. \end{aligned} \tag{3.9}$$

Since

$$\begin{aligned} \frac{1}{w^{\sigma} + \mathfrak{h}_1 w + \mathfrak{h}_2} &= \frac{w^{-1}}{w^{\sigma-1} + \mathfrak{h}_1 + \mathfrak{h}_2 w^{-1}} \\ &= \frac{w^{-1}}{(w^{\sigma-1} + \mathfrak{h}_1)(1 + \frac{\mathfrak{h}_2 w^{-1}}{w^{\sigma-1}+\mathfrak{h}_1})} \\ &= \frac{w^{-1}}{w^{\sigma-1} + \mathfrak{h}_1} \sum_{\gamma=0}^{\infty} (-1)^{\gamma} (\frac{-\mathfrak{h}_2 w^{-1}}{w^{\sigma-1} + \mathfrak{h}_1})^{\gamma} \\ &= \sum_{\gamma=0}^{\infty} \frac{(-\mathfrak{h}_2)^{\gamma} w^{-\gamma-1}}{(w^{\sigma-1} + \mathfrak{h}_1)^{\gamma+1}} \\ &= \sum_{\gamma=0}^{\infty} \frac{(-\mathfrak{h}_2)^{\gamma} w^{-\sigma\gamma-\sigma}}{(1 + \mathfrak{h}_1 w^{1-\sigma})^{\gamma+1}} \end{aligned}$$

$$= \sum_{\gamma=0}^{\infty}(-\mathfrak{h}_2)^{\gamma} w^{-\sigma\gamma-\sigma} \sum_{\aleph=0}^{\infty}(-\mathfrak{h}_1 w^{1-\sigma})^{\aleph}\binom{\gamma+\aleph}{\aleph}$$

$$= \sum_{\gamma=0}^{\infty}(-\mathfrak{h}_2)^{\gamma} \sum_{\mathfrak{h}_2=0}^{\infty}\binom{\gamma+\mathfrak{h}_2}{\aleph}(-\mathfrak{h}_1)^{\aleph} w^{\aleph-(\sigma)\aleph-\sigma\gamma-\sigma}$$

and

$$\mathcal{L}[k(t)] = \mathcal{L}\left[\int_0^s \frac{g(t)}{(s-t)^{\tau}}dt\right],$$

so we have

$$G(\mathfrak{P}) = \frac{\sin(\tau\pi)}{\pi} p\mathcal{L}\left[\int_0^s (s-t)^{\tau-1}k'(t)dt\right].$$

Thus, the inverse LT to equation (3.9) yields

$$\begin{aligned}
y(t) &= e_0\sum_{\gamma=0}^{\infty}\frac{(-\mathfrak{h}_2)^{\gamma}}{\gamma!}\sum_{\aleph=0}^{\infty}\frac{\Gamma(\aleph+\gamma+1)(-\mathfrak{h}_1)^{\aleph}t^{(\sigma-1)\aleph+\sigma\gamma}}{\Gamma[(\sigma-1)\aleph+\sigma\gamma+1]\aleph!}\\
&+ e_1\sum_{\gamma=0}^{\infty}\frac{(-\mathfrak{h}_2)^{\gamma}}{\gamma!}\sum_{\aleph=0}^{\infty}\frac{\Gamma(\aleph+\gamma+1)(-\mathbf{a})^{\aleph}t^{(\sigma-1)\aleph+\sigma\gamma+1}}{\Gamma[(\sigma-1)\aleph+\sigma\gamma+2]\aleph!}\\
&+ \mathfrak{h}_1 e_0\sum_{\gamma=0}^{\infty}\frac{(-\mathfrak{h}_2)^{\gamma}}{\gamma!}\sum_{\aleph=0}^{\infty}\frac{\Gamma(\aleph+\gamma+1)(-\mathfrak{h}_1)^{\aleph}t^{(\sigma-1)\aleph+\sigma\gamma+\sigma-1)\aleph}}{\Gamma[(\sigma-1)\aleph+\sigma\gamma+\sigma]\aleph!}\\
&+ \frac{\sin(\tau\pi)}{\pi}\frac{d}{ds}\int_0^s (s-t)^{\tau-1}k(t)dt\sum_{\gamma=0}^{\infty}\frac{(-\mathfrak{h}_2)^{\gamma}}{\gamma!}\sum_{\aleph=0}^{\infty}\frac{(-\mathfrak{h}_1)^{\aleph}\Gamma(\gamma+\aleph+1)}{\aleph!}\frac{t^{(\sigma-1)\aleph+\sigma\gamma+\sigma-1}}{\Gamma[(\sigma-1)\aleph+\sigma\gamma+\sigma]},
\end{aligned}$$

which is (3.7) and this completes the proof of the theorem. Aslo, the Wright function can express this solution as

$$\begin{aligned}
y(t) &= e_0\sum_{\gamma=0}^{\infty}\frac{(-\mathfrak{h}_2)^{\gamma}t^{\sigma\gamma}}{\gamma!}\ {}_1\lambda_1\left(\begin{matrix}(\gamma+1,1)\\(\sigma\gamma+1,\sigma-1)\end{matrix}\,\middle|\,-\mathfrak{h}_1 t^{\sigma-1}\right)\\
&+ e_1\sum_{\gamma=0}^{\infty}\frac{(-\mathfrak{h}_2)^{\gamma}t^{\sigma\gamma+1}}{\gamma!}\ {}_1\lambda_1\left(\begin{matrix}(\gamma+1,1)\\(\sigma\gamma+2,\sigma-1)\end{matrix}\,\middle|\,-\mathfrak{h}_1 t^{\sigma-1}\right)\\
&+ a e_0\sum_{\gamma=0}^{\infty}\frac{(-\mathfrak{h}_2)^{\gamma}t^{\sigma\gamma+\sigma-1}}{\gamma!}\ {}_1\lambda_1\left(\begin{matrix}(\gamma+1,1)\\(\sigma\gamma+1\sigma,\sigma-1)\end{matrix}\,\middle|\,-\mathfrak{h}_1 t^{\sigma-1}\right)\\
&+ \frac{\sin(\tau\pi)}{\pi}\frac{d}{ds}\int_0^s (s-t)^{\tau-1}k(t)dt\sum_{\gamma=0}^{\infty}\frac{(-\mathfrak{h}_2)^{\gamma}t^{\sigma\gamma+\sigma-1}}{\gamma!}\ {}_1\lambda_1\left(\begin{matrix}(\gamma+1,1)\\(\sigma\gamma+1\sigma,\sigma-1)\end{matrix}\,\middle|\,-\mathfrak{h}_1 t^{\sigma-1}\right).
\end{aligned}$$

□

Example 3.4. If we consider $\sigma = \frac{3}{2}, \tau = \frac{1}{2}, \mathfrak{h}_1 = -3$, and $\mathfrak{h}_2 = -5$ in Theorem 3.3, then the fractional integro-differential equation

$$y^{\frac{3}{2}} - 3y'(t) - 5y(t) = \int_0^s \frac{g(t)}{(s-t)^{\frac{1}{2}}}dt$$

with initial conditions $y(0) = e_0$ and $y'(0) = e_1$ has the unique solution

$$\begin{aligned} y(t) = {} & e_0 \sum_{\gamma=0}^{\infty} \frac{(5)^{\gamma}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\aleph+\gamma+1)(3)^{\aleph} t^{(\frac{1}{2})\aleph+\frac{3}{2}\gamma}}{\Gamma[(\frac{1}{2})\aleph+\frac{3}{2}\gamma+1]\aleph!} \\ & + e_1 \sum_{\gamma=0}^{\infty} \frac{(5)^{\gamma}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\aleph+\gamma+1)(3)^{\aleph} t^{(\frac{1}{2})\aleph+(\frac{3}{2})\gamma+1}}{\Gamma[(\frac{1}{2})\aleph+\frac{3}{2}\gamma+2]\aleph!} \\ & - 3e_0 \sum_{\gamma=0}^{\infty} \frac{(5)^{\gamma}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{\Gamma(\aleph+\gamma+1)(3)^{\aleph} t^{(\frac{1}{2})\aleph+\frac{3}{2}\gamma+\frac{1}{2}}}{\Gamma[(\frac{1}{2})\aleph+\frac{3}{2}\gamma-\frac{3}{2}]\aleph!} \\ & + \frac{1}{\pi}\frac{d}{ds}\int_0^s (s-t)^{-\frac{1}{2}} k(t)dt \sum_{\gamma=0}^{\infty} \frac{(5)^{\gamma}}{\gamma!} \sum_{\aleph=0}^{\infty} \frac{(3)^{\aleph}\Gamma(\gamma+\aleph+1)}{\aleph!} \frac{t^{(\frac{1}{2})\aleph+\frac{3}{2}\gamma+\frac{1}{2}}}{\Gamma((\frac{1}{2})\aleph+\frac{3}{2}\gamma+\frac{3}{2})}. \end{aligned}$$

Figure 2 illustrates the solution behavior of the fractional integro-differential equation of Example 3.4 at various values of σ and initial conditions as $e_0 = 1$ and $e_1 = 1$.

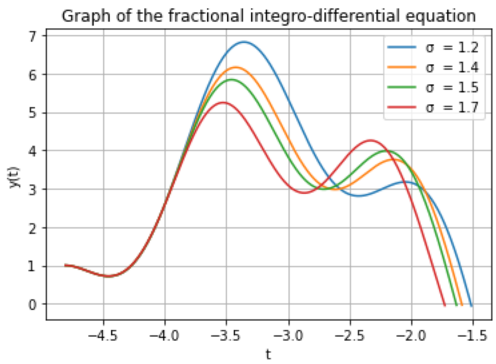

Figure 2: The solution behavior of Example 3.4.

Proposition 3.5. *Under assumptions* $0 < \sigma, \tau < 1$, *and* $\mathfrak{h}_2 \in \mathbb{R}$, *the fractional integro-differential equation*

$$y^{(\sigma)}(t) - \mathfrak{h}_2 y(t) = \int_0^s \frac{g(t)}{(s-t)^{\tau}} dt \tag{3.10}$$

with the initial condition $y(0) = e_0$ *has the unique solution*

$$\begin{aligned} y(t) &= e_0 \sum_{\gamma=0}^{\infty} \mathfrak{h}_2^{\gamma} \frac{t^{\sigma\gamma}}{\Gamma(\sigma\gamma+1)} + \frac{\sin(\tau\pi)}{\pi} \frac{d}{ds} \int_0^s (s-t)^{\tau-1} k(t) dt \sum_{\gamma=0}^{\infty} \frac{\mathfrak{h}_2^{\gamma} t^{\sigma+\sigma\gamma-1}}{\Gamma(\sigma+\sigma\gamma)} \\ &= e_0 E_{\sigma}(\mathfrak{h}_2 t^{\sigma}) + \frac{\sin(\tau\pi)}{\pi} \frac{d}{ds} \int_0^s (s-t)^{\tau-1} k(t) dt \, t^{\sigma-1} E_{\sigma,\sigma}(\mathfrak{h}_2 t^{\sigma}). \end{aligned} \tag{3.11}$$

Proof. Utilizing the LT (Preliminary 11) in (3.10), we have

$$w^{\sigma}\mathcal{L}[y] - w^{\sigma-1} y(0) - \mathfrak{h}_2 \mathcal{L}[y] = \mathcal{L}[k(t)],$$

where $k(t) = \int_0^s \frac{g(t)}{(s-t)^{\tau}} dt$,

$$w^{\sigma}\mathcal{L}[y] - e_0 w^{\sigma-1} - \mathfrak{h}_2\mathcal{L}[y] = \mathcal{L}[k(t)], \quad \mathcal{L}[y](w^{\sigma} - \mathfrak{h}_2) = e_0 w^{\sigma-1} + \mathcal{L}[k(t)], \quad \mathcal{L}[y] = \frac{e_0 w^{\sigma-1} + \mathcal{L}[k(t)]}{w^{\sigma} - \mathfrak{h}_2}.$$

Considering

$$\frac{1}{w^{\sigma}-\mathfrak{h}_2}=\frac{1}{w^{\sigma}(1-\frac{\mathfrak{h}_2}{w^{\sigma}})}=\frac{w^{-\sigma}}{1-\mathfrak{h}_2 w^{-\sigma}}$$
$$=w^{-\sigma}(1-\mathfrak{h}_2 w^{-\sigma})^{-1}$$
$$=w^{-\sigma}[1+\mathfrak{h}_2 w^{-\sigma}+(\mathfrak{h}_2 w^{-\sigma})^2+\cdots]=w^{-\sigma}\sum_{\gamma=0}^{\infty}(\mathfrak{h}_2 w^{-\sigma})^{\gamma},$$

we have

$$\begin{aligned}\mathcal{L}[y]&=e_0 w^{\sigma-1}\left[w^{-\sigma}\sum_{\gamma=0}^{\infty}(\mathfrak{h}_2 w^{-\sigma})^{\gamma}\right]+w^{-\sigma}\sum_{\gamma=0}^{\infty}(\mathfrak{h}_2 w^{-\sigma})^{\gamma}\mathcal{L}[k(t)]\\&=e_0 w^{-1}\sum_{\gamma=0}^{\infty}\mathfrak{h}_2^{\gamma}w^{-\sigma\gamma}+\frac{\sin(\tau\pi)}{\pi}p\mathcal{L}\left[\int_0^s(s-t)^{\tau-1}k'(t)dt\right]w^{-\sigma}(\mathfrak{h}_2 w^{-\sigma})^{\gamma}.\end{aligned}\tag{3.12}$$

Thus, the inverse LT to (3.12) yields

$$\begin{aligned}y(t)&=e_0\sum_{\gamma=0}^{\infty}\mathfrak{h}_2^{\gamma}\mathcal{L}^{-1}(w^{\sigma\gamma+1})+\frac{\sin(\tau\pi)}{\pi}\frac{d}{ds}\int_0^s(s-t)^{\tau-1}k(t)dt\sum_{\gamma=0}^{\infty}\mathfrak{h}_2^{\gamma}\mathcal{L}^{-1}(w^{(\sigma+\sigma\gamma)})\\&=e_0\sum_{\gamma=0}^{\infty}\mathfrak{h}_2^{\gamma}\frac{t^{\sigma\gamma}}{\Gamma(\sigma\gamma+1)}+\frac{\sin(\tau\pi)}{\pi}\frac{d}{ds}\int_0^s(s-t)^{\tau-1}k(t)dt\sum_{\gamma=0}^{\infty}\frac{\mathfrak{h}_2^{\gamma}t^{\sigma+\sigma\gamma-1}}{\Gamma(\sigma+\sigma\gamma)}\\&=e_0E_{\sigma}(\mathfrak{h}_2 t^{\sigma})+\frac{\sin(\tau\pi)}{\pi}\frac{d}{ds}\int_0^s(s-t)^{\tau-t}k(t)dt\ t^{\sigma-1}\sum_{\gamma=0}^{\infty}\frac{(\mathfrak{h}_2 t^{\sigma})^{\gamma}}{\Gamma(\sigma\gamma+\sigma)}\\&=e_0E_{\sigma}(\mathfrak{h}_2 t^{\sigma})+\frac{\sin(\tau\pi)}{\pi}\frac{d}{ds}\int_0^s(s-t)^{\tau-1}k(t)dt\ t^{\sigma-1}E_{\sigma,\sigma}(\mathfrak{h}_2 t^{\sigma}),\end{aligned}$$

which is (3.11). This completes the proof of the theorem. □

Remark 3.6. If $\mathfrak{h}_1=0$ in equation (3.6), then the fractional integro-differential equation

$$y^{\sigma}(t)+\mathfrak{h}_2 y(t)=\int_0^s\frac{g(t)}{(s-t)^{\tau-1}}dt,\quad 1<\sigma\leqslant 2;\ 0<\tau<1,\tag{3.13}$$

with initial conditions $y(0)=e_0$ and $y'(0)=e_1$ has the unique solution

$$y(t)=e_0E_{\sigma,1}(-\mathfrak{h}_2 t^{\sigma})+e_1tE_{\sigma,2}(-\mathfrak{h}_2 t^{\sigma})+\frac{\sin(\tau\pi)}{\pi}\frac{d}{ds}\int_0^s(s-t)^{\tau-1}k(t)dt\ t^{\sigma-1}E_{\sigma,\sigma}(-\mathfrak{h}_2 t^{\sigma}).$$

Proposition 3.7. *A nearly simple harmonic vibration integro-differential equation ([27])*

$$y^{\sigma}+\mathfrak{r}^2y(t)=\int_0^s\frac{g(t)}{(s-t)^{\tau-1}}dt,\quad 1<\sigma\leqslant 2;\ 0<\tau<1,$$

with initial conditions $y(0)=e_0$ *and* $y'(0)=e_1$ *has the unique solution*

$$y(t)=e_0E_{\sigma,1}(-\mathfrak{r}^2t^{\sigma})+e_1tE_{\sigma,2}(-\mathfrak{r}^2t^{\sigma})+\frac{\sin(\tau\pi)}{\pi}\frac{d}{ds}\int_0^s(s-t)^{\tau-1}k(t)dt\ t^{\sigma-1}E_{\sigma,\sigma}(-\mathfrak{r}^2t^{\sigma}).$$

Proof. The above proof is accomplished by implanting $\mathfrak{h}_2=\mathfrak{r}^2$ in equation (3.13). □

4. Conclusion

In this article, the utilization of the LT for solving fractional integro-differential equations is demonstrated. The relationships between the LT and other transforms reveal deeper connections, enabling the discovery of additional interactions unique to the LT. A distinct methodology for solving fractional integro-differential equations is introduced, involving the application of the LT alongside binomial series extension coefficients. Furthermore, the study focuses on examining various properties and presenting illustrative examples.

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