J. Math. Computer Sci., 33 (2024), 176–188

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

The existence and uniqueness of a fractional q-integro differential equation involving the Caputo-Fabrizio fractional derivative and the fractional q-integral of the Riemann-Liouville type with q-nonlocal condition

Khalid K. Ali[a,*], K. R. Raslan[a], Amira Abd-Elall Ibrahim[b], M. A. Maaty[c]

[a]Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr-City, Cairo, Egypt.

[b]October High Institute for Engineering and Technology, Egypt.

[c]Basic Science Department, Higher Technological Institute, 10th of Ramadan City, Egypt.

Abstract

The fractional integro-differential equations are presented here, together with the novel definitions of the Caputo and Fabrizio differential operators and the q-Riemann-Liouville integral operator. In order to determine whether or not a solution does in fact exist, we employ the Schauder fixed point theorem. We discuss how the solution is unique and how it constantly depends on the constant in the nonlocal condition. In addition to this, a numerical solution to the problem will be found by employing a hybrid approach that combines the forward finite difference and trapezoidal approaches. In conclusion, in order to confirm the primary findings, three examples will be provided as illustrations.

Keywords: Fractional derivative, q-integro-differential equation, existence and uniqueness of solution, applications.

2020 MSC: 26Axx, 34Axx, 03Exx.

1. Introduction

There is a long history of fractional and quantum calculus (q-calculus). Jackson was the first to develop the q-calculus [13]. The properties of q-calculus and its basic definitions can be found in books [5, 15]. The q-calculus is a new branch of mathematics in which the derivative of a real function has no limits. The q-calculus and fractional calculus are generalisations of ordinary calculus. The q-calculus and fractional calculus can be used to describe a wide range of phenomena in real life [19]. Many researchers have recently dedicated themselves to the study of q-fractional differential equations. Ahmad et al. in [1, 8] used the fixed point theory to investigate the existence and uniqueness of some differential equations, including the fractional q derivative of the Caputo type with the nonlcal integral condition and the fractional q integral and fractional q derivative of the Riemann-Liouville type with the q integral condition. On the other hand, there are many definitions for fractional calculus, but the derivatives of Caputo

*Corresponding author

Email addresses: khalidkaram2012@azhar.edu.eg (Khalid K. Ali), kamal_raslan@yahoo.com (K. R. Raslan), amera.1994369@gmail.com (Amira Abd-Elall Ibrahim), Maaty2612@gmail.com (M. A. Maaty)

doi: 10.22436/jmcs.033.02.06

Received: 2023-09-19 *Revised:* 2023-10-28 *Accepted:* 2023-11-16

and Riemann-Liouville are the most commonly used. In [22] Shivaniana and Dinmohammadi studied the existence and uniqueness of a particular class of high-order nonlinear fractional integrodifferential equations involving Rieman-Liouville fractional derivatives. Gameel in [14] investigated the existence of a fractional Caputo-integro differential equation containing the Voltera and Fredholm integrals using the fixed point technique. Recently, Fabrizio and Caputo proposed a new definition for fractional derivation with an exponentially decaying kernel [7]. Several researchers attempted to use it in studying the existence and uniqueness of some fractional differential equations [6, 23]. Many researchers have been interested in the numerical and analytical solutions for various types of integro-differential equations. In [9], the authors examined the possibility of finding a unique solution to the Fredholm integro-differential equation with an initial condition. They used the method of direct computation to find the exact solution and the method of Simpson's finite difference to solve it numerically. The authors of [20] studied the second-order Volterra-Fredholm integro-differential equation with initial and nonlocal conditions, and they used the finite-trapezoidal method to obtain the solution to it numerically. The authors of [10] discussed the solution's existence and uniqueness, as well as the continuous dependence of α_0 for the fractional q-integro differential equation:

$$z''(\sigma) = \kappa\Big(\sigma, z(\sigma), I_q^{\sigma}\mu(\sigma, z'(\sigma))\Big), \quad \sigma \in (0,1], \qquad (1-q)\rho\sum_{i=0}^{\zeta} q^i z(q^i\rho) = \alpha_0, \quad z'(0) = \beta_0, \quad \rho \in (0,1].$$

In addition, they solved it numerically using the cubic B-spline-trapezoidal and finite-trapezoidal methods. The authors of [11] discussed the following equation:

$$z''(\sigma) = \kappa\Big(\sigma, z(\sigma), {}^{CF}D^{\alpha}z(\sigma), I_q^{\sigma}\mu(\sigma, z'(\sigma))\Big), \quad \sigma \in (0,1],$$
$$(1-q)\rho\sum_{i=0}^{\zeta} q^i z(q^i\rho) = \alpha_0, \quad z'(0) = \beta_0, \quad \rho \in (0,1].$$

They discussed its existence and uniqueness, as well as how they obtained its approximate solution using finite-trapezoidal and cubic-trapezoidal methods. In [4] the author studied the following equation, which contains the fractional derivative and integral of Caputo-Fabrizio in addition to the fractional q integral of the Riemann-Liouville type:

$$z''(\sigma) = \kappa\Big(\sigma, z'(\sigma), {}^{CF}I^{\alpha_0}z'(\sigma), z(\sigma), {}^{CF}D^{\beta_0}z(\sigma), I_q^{\gamma_0}\mu(\sigma, z'(\sigma))\Big), \quad \sigma \in (0,1],$$
$$(1-q)\rho\sum_{i=0}^{\zeta} q^i z(q^i\rho) = \alpha_0, \quad z'(0) = \beta_0, \quad \rho \in (0,1].$$

Furthermore, they discussed some different coupled systems of fractional q integro-differential equations numerically and analytically [2, 3, 12].

This paper will discuss the following nonlocal fractional q-integro-differential equation:

$$ {}^{CF}D^{\alpha}z(\sigma) = f\Big(\tau, z(\sigma), {}^{CF}D^{\beta}z(\sigma), I_q^{\gamma}z(\sigma)\Big), \quad \sigma \in (0,1], \tag{1.1}$$
$$(1-q)\rho\sum_{i=0}^{\zeta} q^i z(q^i\rho) = C_0, \quad \rho \in (0,1], \tag{1.2}$$

where ${}^{CF}D^{\alpha}z(\sigma)$, ${}^{CF}D^{\beta}z(\sigma)$ are the Caputo-Fabrizio fractional derivative of the unkwon function $z(\sigma)$, I_q^{γ} is the fractional q-integral of the Riemann Liouville of order $\gamma > 0$, and $q, \alpha, \beta \in (0,1)$.

The following is how this manuscript is organized. In Section 2, we review some important definitions and lemmas. The existence of the solution will be investigated in Section 3. The uniqueness of the solution will be introduced in Section 4. Also, the continuous dependence on the constant C_0 will be studied in Section 5. We give a summary of the finite-trapezoidal method that will be used in our paper in Section 6. In addition, Section 7 introduces three examples. Section 8 provides the conclusion.

2. Preliminaries

First, we go over some key definitions and lemmas that will be useful throughout this paper.

Definition 2.1 ([15])**.** For each $q \in (0,1)$, we can define the real number σ as

$$[\sigma]_q = \frac{1-q^\sigma}{1-q}.$$

Definition 2.2 ([15])**.** We can define the q-derivative of any function $z(\sigma)$ as:

$$(D_q z)(\sigma) = \frac{z(\sigma)-z(q\sigma)}{\sigma - q\sigma}, \quad \lim_{q\to 1} D_q z(\sigma) = \frac{dz(\sigma)}{d\sigma}.$$

Definition 2.3 ([1])**.** The q-integral of the Riemann-Liouville fractional for any function $z(\sigma)$ is given by

$$(I_q^\gamma z)(\sigma) = \begin{cases} z(\sigma), & \gamma = 0, \\ \frac{1}{\Gamma_q(\gamma)} \int_0^\sigma (\sigma - qs)^{(\gamma-1)} z(s) d_q s, & \gamma > 0, \end{cases} \tag{2.1}$$

where $\Gamma_q(\gamma) = \frac{(1-q)^{(\gamma-1)}}{(1-q)^{\gamma-1}}$ and satisfies $\Gamma_q(\gamma+1) = [\gamma]_q \Gamma_q(\gamma)$,

$$(\sigma-\sigma)^{(0)} = 1, \quad (\sigma-\sigma)^{(\upsilon)} = \prod_{j=0}^{\upsilon-1} (\sigma - q^j\sigma), \upsilon \in \mathbb{N}, \quad (\sigma-\sigma)^{(\rho)} = \sigma^\rho \prod_{j=0}^{\infty} \frac{(\sigma - q^j\sigma)}{(\sigma - q^{j+\rho}\sigma)}, \quad \rho \in \mathbb{R}.$$

Lemma 2.4 ([1])**.** *Using q-integration by parts, for any $\gamma > 0$, we obtain*

$$(I_q^\gamma 1)(\sigma) = \frac{\sigma^{(\gamma)}}{\Gamma_q(\gamma+1)}.$$

Definition 2.5 ([7])**.** Let $0 < \beta < 1$, $z(\sigma) \in H^1(c,d)$. Then, we can define the fractional derivative of Caputo-Fabrizio of order β of a function $z(\sigma)$ as

$${}^{CF}D^\beta z(\sigma) = \frac{N(\beta)}{1-\beta} \int_0^\sigma e^{-\frac{\beta}{1-\beta}(\sigma-s)} z'(s) ds,$$

where $N(\beta)$ is a normalisation function with the property that $N(0) = N(1) = 1$.

Both Losada and Nieto are modified from the above definition later as follows ([17]):

$${}^{CF}D^\beta z(\sigma) = \frac{(2-\beta)N(\beta)}{2(1-\beta)} \int_0^\sigma e^{-\frac{\beta}{1-\beta}(\sigma-s)} z'(s) ds.$$

They showed that $N(\beta) = \frac{2}{2-\beta}, \forall \beta \in (0,1)$. Therefore,

$${}^{CF}D^\beta z(\sigma) = \frac{1}{(1-\beta)} \int_0^\sigma e^{-\frac{\beta}{1-\beta}(\sigma-s)} z'(s) ds. \tag{2.2}$$

Lemma 2.6 ([17])**.** *The unique solution for the following problem ${}^{CF}D^\beta z(\sigma) = \eta(\sigma)$, $\beta \in (0,1)$, is provided by* $z(\sigma) = z(0) + (1-\beta)(\eta(\sigma) - \eta(0)) + \beta \int_0^\sigma \eta(s) ds$.

Theorem 2.7 ([18])**.** *Let $z(\sigma) \in C^1[c,d]$. Then, ${}^{CF}D^\beta z(\sigma) \in C^1[c,d]$.*

Lemma 2.8. *Suppose that* $z(\sigma), \omega(\sigma) \in C[0,1]$. *Therefor,* $|{}^{CF}D^{\beta} z(\sigma) - {}^{CF}D^{\beta}\omega(\sigma)| = \frac{2-\beta}{(1-\beta)^2}\|z-\omega\|$.

Proof.

$$\begin{aligned}
{}^{CF}D^{\beta} z(\sigma) &= \frac{1}{1-\beta}\int_0^{\sigma} e^{-\frac{\beta}{1-\beta}(\sigma-s)} z'(s)ds \\
&= \frac{1}{1-\beta} e^{-\frac{\beta}{1-\beta}(\sigma-s)} z(s)|_0^{\sigma} - \frac{1}{1-\beta}\int_0^{\sigma} \frac{\beta}{1-\beta} e^{-\frac{\beta}{1-\beta}(\sigma-s)} z(s)ds \\
&= \frac{1}{1-\beta} z(\sigma) - \frac{1}{1-\beta} e^{-\frac{\beta\sigma}{1-\beta}} z(0) - \frac{\beta}{(1-\beta)^2}\int_0^{\sigma} e^{-\frac{\beta}{1-\beta}(\sigma-s)} z(s)ds,
\end{aligned}$$

and so,

$$\begin{aligned}
|{}^{CF}D^{\beta} z(\sigma) - {}^{CF}D^{\beta}\omega(\sigma)| &\leqslant \frac{1}{1-\beta}|z(\sigma)-\omega(\sigma)| + \frac{1}{1-\beta} e^{-\frac{\beta\sigma}{1-\beta}} |z(0)-\omega(0)| \\
&\quad + \frac{\beta}{(1-\beta)^2}\int_0^{\sigma} e^{-\frac{\beta}{1-\beta}(\sigma-s)} |z(s)-\omega(s)|ds \\
&\leqslant \frac{2}{1-\beta}\|z-\omega\| + \frac{\beta}{(1-\beta)^2}\|z-\omega\| = \frac{2-\beta}{(1-\beta)^2}\|z-\omega\|.
\end{aligned}$$

□

Lemma 2.9. *The solution of the nonlocal problem* (1.1)-(1.2) *can be represented as*

$$\begin{aligned}
z(\sigma) =& \Omega\left[C_0 - (1-\alpha)(1-q)\rho\sum_{i=0}^{\zeta} q^i \eta(q^i\rho) - \alpha(1-q)\rho\sum_{i=0}^{\zeta} q^i \int_0^{q^i\rho}\eta(s)ds\right] \\
&+ (1-\alpha)\eta(\sigma) + \alpha\int_0^{\sigma}\eta(s)ds,
\end{aligned} \tag{2.3}$$

where $\Omega = \frac{1}{(1-q)\rho\sum_{i=0}^{\zeta} q^i}$ *and* $\eta(\sigma) = f\Big(\sigma, z(\sigma), {}^{CF}D^{\beta} z(\sigma), I_q^{\gamma} z(\sigma)\Big)$.

Proof. Using Lemma 2.6, we can get the solution of equation (1.1) as

$$z(\sigma) = z(0) + (1-\alpha)(\eta(\sigma)-\eta(0)) + \alpha\int_0^{\sigma}\eta(s)ds, \quad \sigma \in (0,1]. \tag{2.4}$$

To determine $z(0)$, we use the nonlocal condition (1.2). Then,

$$\begin{aligned}
(1-q)\rho\sum_{i=0}^{\zeta} q^i z(q^i\rho) = z(0)(1-q)\rho\sum_{i=0}^{\zeta} q^i + (1-\alpha)(1-q)\rho\sum_{i=0}^{\zeta} q^i(\eta(q^i\rho)-\eta(0)) \\
+ \alpha(1-q)\rho\sum_{i=0}^{\zeta} q^i\int_0^{q^i\rho}\eta(s)ds.
\end{aligned}$$

Therefore, we obtain

$$z(0) = \Omega\left[C_0 - (1-\alpha)(1-q)\rho\sum_{i=0}^{\zeta} q^i(\eta(q^i\rho)-\eta(0)) - \alpha(1-q)\rho\sum_{i=0}^{\zeta} q^i\int_0^{q^i\rho}\eta(s)ds\right]. \tag{2.5}$$

From (2.4) and (2.5), we can get (2.3). □

3. Existence of solution

It is worth noting that the space $B = C[0,1]=\{z : z(\sigma)$ is continuous on $D = [0,1] : \|z\| = \max_{\sigma\in D} |z(\sigma)|\}$. The following assumptions are required for the following analysis:

1. $f : D \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function, that is, it possesses the two properties listed below:
   (i) for any $\sigma \in D$, $f(\sigma, .)$ is continuous;
   (ii) for any $z \in \mathbb{R}$, $f(., z)$ is measurable;
2. there exist a function $G(\sigma) \in B$ and a positive constants $b > 0$, such that $\forall z, x, y \in \mathbb{R}$, we have
$$|f(\sigma, z, x, y)| \leqslant G(\sigma) + b|z| + b|x| + b|y|.$$
3. $G^* = \sup_{\sigma\in D} G(\sigma)$.

Theorem 3.1. *If the conditions 1-3 are satisfied and in addition,*
$$2b\left(1 + \frac{(2-\beta)}{(1-\beta)^2} + \frac{1}{\Gamma_q(\gamma+1)}(1-\alpha+\frac{\alpha}{(\gamma+1)})\right) < 1,$$
hence, (1.1)-(1.2) *posses a solution at least given by* $z(\sigma) \in B$.

Proof. The H operator associated with (2.3) can be defined as
$$\begin{aligned}Hz(\sigma) = {}& \frac{1}{(1-q)\rho\sum_{i=0}^{\zeta} q^i}\Bigg[C_0 - (1-\alpha)(1-q)\rho\sum_{i=0}^{\zeta} q^i f\Big(q^i\rho, z(q^i\rho), {}^{CF}D^{\beta}z(q^i\rho), I_q^{\gamma}z(q^i\rho)\Big)\\ & - \alpha(1-q)\rho\sum_{i=0}^{\zeta} q^i\int_0^{q^i\rho} f\Big(s, z(s), {}^{CF}D^{\beta}z(s), I_q^{\gamma}z(s)\Big)ds\Bigg]\\ & + (1-\alpha)f\Big(\sigma, z(\sigma), {}^{CF}D^{\beta}z(\sigma), I_q^{\gamma}z(\sigma)\Big) + \alpha\int_0^{\sigma} f\Big(s, z(s), {}^{CF}D^{\beta}z(s), I_q^{\gamma}z(s)\Big)ds.\end{aligned}$$

We will use the Schauder fixed point theorem to investigate that the operator H has a fixed point. Let $D_r = \{z(\sigma) \in \mathbb{R} : \|z\|_C \leqslant r\}$, where
$$r = \frac{\frac{|C_0|}{(1-q)\rho\sum_{i=0}^{\zeta} q^i} + 2G^*}{1 - \left(2b(1 + \frac{(2-\beta)}{(1-\beta)^2} + \frac{1}{\Gamma_q(\gamma+1)}(1-\alpha+\frac{\alpha}{(\gamma+1)})\right)}.$$

We show that $H : D_r \to D_r$. Then, for any arbitrary element $z(\sigma) \in D_r$, using Lemmas 2.4 and 2.8 and assumptions 1-3, we get
$$\begin{aligned}|Hz(\sigma)| = {}& \Bigg|\frac{1}{(1-q)\rho\sum_{i=0}^{\zeta} q^i}\Bigg[C_0 - (1-\alpha)(1-q)\rho\sum_{i=0}^{\zeta} q^i f\Big(q^i\rho, z(q^i\rho), {}^{CF}D^{\beta}z(q^i\rho), I_q^{\gamma}z(q^i\rho)\Big)\\ & - \alpha(1-q)\rho\sum_{i=0}^{\zeta} q^i\int_0^{q^i\rho} f\Big(s, z(s), {}^{CF}D^{\beta}z(s), I_q^{\gamma}z(s)\Big)ds\Bigg]\\ & + (1-\alpha)f\Big(\sigma, z(\sigma), {}^{CF}D^{\beta}z(\sigma), I_q^{\gamma}z(\sigma)\Big) + \alpha\int_0^{\sigma} f\Big(s, z(s), {}^{CF}D^{\beta}z(s), I_q^{\gamma}z(s)\Big)ds\Bigg|\\ \leqslant {}& \frac{1}{(1-q)\rho\sum_{i=0}^{\zeta} q^i}\Bigg[|C_0| + (1-\alpha)(1-q)\rho\sum_{i=0}^{\zeta} q^i\Big|f\Big(q^i\rho, z(q^i\rho), {}^{CF}D^{\beta}z(q^i\rho), I_q^{\gamma}z(q^i\rho)\Big)\Big|\end{aligned}$$

$$
\begin{aligned}
&+\alpha(1-q)\rho\sum_{i=0}^{\zeta} q^i\int_0^{q^i\rho}\left|f\left(s,z(s),{}^{CF}D^{\beta}z(s),I_q^{\gamma}z(s)\right)\right|ds\Bigg]\\
&+(1-\alpha)\left|f\left(\sigma,z(\sigma),{}^{CF}D^{\beta}z(\sigma),I_q^{\gamma}z(\sigma)\right)\right|+\alpha\int_0^{\sigma}\left|f\left(s,z(s),{}^{CF}D^{\beta}z(s),I_q^{\gamma}z(s)\right)\right|ds\\
\leqslant{}& \frac{|C_0|}{(1-q)\rho\sum_{i=0}^{\zeta}q^i}+\frac{1}{(1-q)\rho\sum_{i=0}^{\zeta}q^i}\Bigg[(1-\alpha)(1-q)\rho\sum_{i=0}^{\zeta}q^i\Big(G^*+b|z(q^i\rho)|\\
&+b|{}^{CF}D^{\beta}z(q^i\rho)|+bI_q^{\gamma}|z(q^i\rho)|\Big)+\alpha(1-q)\rho\sum_{i=0}^{\zeta}q^i\int_0^{q^i\rho}\Big(G^*+b|z(s)|+b|{}^{CF}D^{\beta}z(s)|\\
&+bI_q^{\gamma}|z(s)|\Big)ds\Bigg]+(1-\alpha)\Big(G^*+b|z(\sigma)|+b|{}^{CF}D^{\beta}z(\sigma)|+bI_q^{\gamma}|z(\sigma)|\Big)\\
&+\alpha\int_0^{\sigma}\Big(G^*+b|z(s)|+b|{}^{CF}D^{\beta}z(s)|+bI_q^{\gamma}|z(s)|\Big)ds\\
\leqslant{}& \frac{|C_0|}{(1-q)\rho\sum_{i=0}^{\zeta}q^i}+(1-\alpha)G^*+(1-\alpha)b\|z\|+\frac{b(1-\alpha)(2-\beta)}{(1-\beta)^2}\|z\|+\frac{b(1-\alpha)}{\Gamma_q(\gamma+1)}\|z\|\\
&+\alpha G^*+\alpha b\|z\|+\alpha b\frac{2-\beta}{(1-\beta)^2}\|z\|+\frac{\alpha b\|z\|}{(\gamma+1)\Gamma_q(\gamma+1)}+(1-\alpha)G^*+(1-\alpha)b\|z\|\\
&+\frac{b(1-\alpha)(2-\beta)}{(1-\beta)^2}\|z\|+\frac{b(1-\alpha)}{\Gamma_q(\gamma+1)}\|z\|+\alpha G^*+\alpha b\|z\|+\alpha b\frac{2-\beta}{(1-\beta)^2}\|z\|+\frac{\alpha b\|z\|}{(\gamma+1)\Gamma_q(\gamma+1)}\\
\leqslant{}& \frac{|C_0|}{(1-q)\rho\sum_{i=0}^{\zeta}q^i}+2(1-\alpha)G^*+2\alpha G^*+\Bigg(2b(1-\alpha)+\frac{2b(1-\alpha)(2-\beta)}{(1-\beta)^2}+\frac{2b(1-\alpha)}{\Gamma_q(\gamma+1)}\\
&+2\alpha b+2\alpha b\frac{2-\beta}{(1-\beta)^2}+\frac{2\alpha b}{(\gamma+1)\Gamma_q(\gamma+1)}\Bigg)\|z\|\\
\leqslant{}& \frac{|C_0|}{(1-q)\rho\sum_{i=0}^{\zeta}q^i}+2G^*+\Bigg(2b+\frac{2b(2-\beta)}{(1-\beta)^2}+\frac{2b}{\Gamma_q(\gamma+1)}(1-\alpha+\frac{\alpha}{(\gamma+1)})\Bigg)\|z\|=r.
\end{aligned}
$$

Therefore, $\|Hz\|\leqslant r$ for every $z(\sigma)\in D_r$, which implies that $H: D_r\to D_r$ and $\{Hz(\sigma)\}$ is uniformly bounded in D_r. Now, we prove that H operator is equicontinuous. Let $\sigma_1,\sigma_2\in(0,1]$ such that $\sigma_1<\sigma_2$; thus,

$$
\begin{aligned}
|Hz(\sigma_2)-Hz(\sigma_1)|\leqslant{}&(1-\alpha)\left|f\left(\sigma_2,z(\sigma_2),{}^{CF}D^{\beta}z(\sigma_2),I_q^{\gamma}z(\sigma_2)\right)-f\left(\sigma_1,z(\sigma_1),{}^{CF}D^{\beta}z(\sigma_1),I_q^{\gamma}z(\sigma_1)\right)\right|\\
&+\alpha\left|\int_0^{\sigma_2}f\left(s,z(s),{}^{CF}D^{\beta}z(s),I_q^{\gamma}z(s)\right)ds-\int_0^{\sigma_1}f\left(s,z(s),{}^{CF}D^{\beta}z(s),I_q^{\gamma}z(s)\right)ds\right|\\
\leqslant{}&(1-\alpha)\left|f\left(\sigma_2,z(\sigma_2),{}^{CF}D^{\beta}z(\sigma_2),I_q^{\gamma}z(\sigma_2)\right)-f\left(\sigma_1,z(\sigma_1),{}^{CF}D^{\beta}z(\sigma_1),I_q^{\gamma}z(\sigma_1)\right)\right|\\
&+\alpha\int_{\sigma_1}^{\sigma_2}\left|f\left(s,z(s),{}^{CF}D^{\beta}z(s),I_q^{\gamma}z(s)\right)\right|ds.
\end{aligned}
$$

Clearly, $|Hz(\sigma_2)-Hz(\sigma_1)|\to 0$, as $\sigma_1\to\sigma_2$. Hence, $\{Hz(\sigma)\}$ is equi-continuous in the set D_r. Let $z_n(\sigma)\in D_r$, $z_n(\sigma)\to z(\sigma)(n\to\infty)$, therefore the continuity of function f implies that $f(\sigma,z_n,x_n,y_n)\to f(\sigma,z,x,y)$ as $n\to\infty$. Also,

$$
\begin{aligned}
&\lim_{n\to\infty}Hz_n(\sigma)\\
&\quad=\lim_{n\to\infty}\Bigg[\frac{1}{(1-q)\rho\sum_{i=0}^{\zeta}q^i}\Bigg[C_0-(1-\alpha)(1-q)\rho\sum_{i=0}^{\zeta}q^if\left(q^i\rho,z_n(q^i\rho),{}^{CF}D^{\beta}z_n(q^i\rho),I_q^{\gamma}z_n(q^i\rho)\right)
\end{aligned}
$$

$$- \alpha(1-q)\rho \sum_{i=0}^{\zeta} q^i \int_0^{q^i\rho} f\Big(s, z_n(s), {}^{CF}D^{\beta} z_n(s), I_q^{\gamma} z_n(s)\Big) ds\Big]$$
$$+ (1-\alpha) \lim_{n\to\infty} f\Big(\tau, z_n(\sigma), {}^{CF}D^{\beta} z_n(\sigma), I_q^{\gamma} z_n(\sigma)\Big) + \alpha \lim_{n\to\infty} \int_0^{\sigma} f\Big(s, z_n(s), {}^{CF}D^{\beta} z_n(s), I_q^{\gamma} z_n(s)\Big) ds\Big].$$

Using the assumption 1 and the theorem of Lebesgue-dominated convergence [16], then

$$\begin{aligned}
\lim_{n\to\infty} Hz_n(\sigma) = \Big[&\frac{1}{(1-q)\rho\sum_{i=0}^{\zeta} q^i}\Big[C_0 - (1-\alpha)(1-q)\rho \sum_{i=0}^{\zeta} q^i \lim_{n\to\infty} f\Big(q^i\rho, z_n(q^i\rho), {}^{CF}D^{\beta} z_n(q^i\rho), \\
& I_q^{\gamma} z_n(q^i\rho)\Big) - \alpha(1-q)\rho \sum_{i=0}^{\zeta} q^i \int_0^{q^i\rho} \lim_{n\to\infty} f\Big(s, z_n(s), {}^{CF}D^{\beta} z_n(s), I_q^{\gamma} z_n(s)\Big) ds\Big] \\
& + (1-\alpha) \lim_{n\to\infty} f\Big(\sigma, z_n(\sigma), {}^{CF}D^{\beta} z_n(\sigma), I_q^{\gamma} z_n(\sigma)\Big) \\
& + \alpha \int_0^{\sigma} \lim_{n\to\infty} f\Big(s, z_n(s), {}^{CF}D^{\beta} z_n(s), I_q^{\gamma} z_n(s)\Big) ds\Big] = Hz(\sigma).
\end{aligned}$$

Then, $Hz_n(\sigma) \to Hz(\sigma)$ as $n \to \infty$. Therefore, the operator H is continuous in D_r. As a result, using the Arzela-Ascoli and Schauder fixed theorems, we conclude that (1.1)-(1.2) has a solution at least given by $z(\sigma) \in B$. □

4. Uniqueness of the solution

(a) Let $f : D \times \mathbb{R}^3 \to \mathbb{R}$ is continuouse and measurable $\forall z, x, y \in \mathbb{R}$ for almost all $\sigma \in D$ and satisfies the following Lipschitz condition:

$$|f(\sigma, z, x, y) - f(\sigma, z_1, x_1, y_1)| \leqslant b|z - z_1| + b|x - x_1| + b|y - y_1|.$$

Theorem 4.1. *Let the assumption* (a) *be satisfied, then the problem* (1.1)-(1.2)*, posses one solution.*

Proof. Assume that $z(\sigma), \omega(\sigma)$ be two solutions of (1.1)-(1.2), therefore

$$\begin{aligned}
|z(\sigma) - \omega(\sigma)| \leqslant & \frac{1}{(1-q)\rho\sum_{i=0}^{\zeta} q^i}\Big[(1-\alpha)(1-q)\rho \sum_{i=0}^{\zeta} q^i \Big| f\Big(q^i\rho, \omega(q^i\rho), {}^{CF}D^{\beta}\omega(q^i\rho), I_q^{\gamma}\omega(q^i\rho)\Big) \\
& - f\Big(q^i\rho, z(q^i\rho), {}^{CF}D^{\beta} z(q^i\rho), I_q^{\gamma} z(q^i\rho)\Big)\Big| \\
& + \alpha(1-q)\rho \sum_{i=0}^{\zeta} q^i \int_0^{q^i\rho} \Big| f\Big(s, \omega(s), {}^{CF}D^{\beta}\omega(s), I_q^{\gamma}\omega(s)\Big) - f\Big(s, z(s), {}^{CF}D^{\beta} z(s), I_q^{\gamma} z(s)\Big)\Big| ds\Big] \\
& + (1-\alpha)\Big| f\Big(\sigma, z(\sigma), {}^{CF}D^{\beta} z(\sigma), I_q^{\gamma} z(\sigma)\Big) - f\Big(\sigma, \omega(\sigma), {}^{CF}D^{\beta}\omega(\sigma), I_q^{\gamma}\omega(\sigma)\Big)\Big| \\
& + \alpha \int_0^{\sigma} \Big| f\Big(s, z(s), {}^{CF}D^{\beta} z(s), I_q^{\gamma} z(s)\Big) - f\Big(s, \omega(s), {}^{CF}D^{\beta}\omega(s), I_q^{\gamma}\omega(s)\Big)\Big| ds \\
\leqslant & \frac{1}{(1-q)\rho\sum_{i=0}^{\zeta} q^i}\Big[(1-\alpha)(1-q)\rho \sum_{i=0}^{\zeta} q^i \Big(b|\omega(q^i\rho) - z(q^i\rho)| + b|{}^{CF}D^{\beta}\omega(q^i\rho) \\
& - {}^{CF}D^{\beta} z(q^i\rho)| + b I_q^{\gamma}|\omega(q^i\rho) - z(q^i\rho)|\Big) + \alpha(1-q)\rho \sum_{i=0}^{\zeta} q^i \int_0^{q^i\rho} \Big(b|\omega(s) - z(s)|
\end{aligned}$$

$$
\begin{aligned}
&+ b|{}^{CF}D^{\beta}\omega(s) - {}^{CF}D^{\beta}z(s)| + bI_q^{\gamma}|\omega(s) - z(s)|\Big)ds\Big] \\
&+ (1-\alpha)\Big(b|z(\sigma) - \omega(\sigma)| + b|{}^{CF}D^{\beta}z(\sigma) - {}^{CF}D^{\beta}\omega(\sigma)| + bI_q^{\gamma}|z(\sigma) - \omega(\sigma)|\Big) \\
&+ \alpha\int_0^{\sigma}\Big(b|z(s) - \omega(s)| + b|{}^{CF}D^{\beta}z(s) - {}^{CF}D^{\beta}\omega(s)| + bI_q^{\gamma}|z(s) - \omega(s)|\Big)ds \\
\leqslant &\left(2b + 2b\frac{(2-\beta)}{(1-\beta)^2} + \frac{2b(1-\alpha)}{\Gamma_q(\gamma+1)} + \frac{2b\alpha}{(\gamma+1)\Gamma_q(\gamma+1)}\right)\|\omega - z\|.
\end{aligned}
$$

Hence,

$$
\left[1 - 2b\left(1 + \frac{(2-\beta)}{(1-\beta)^2} + \frac{1}{\Gamma_q(\gamma+1)}((1-\alpha) + \frac{\alpha}{(\gamma+1)})\right)\right]\|\omega - z\| \leqslant 0.
$$

Since $2b\left(1 + \frac{(2-\beta)}{(1-\beta)^2} + \frac{1}{\Gamma_q(\gamma+1)}((1-\alpha) + \frac{\alpha}{(\gamma+1)})\right) < 1$, then $\omega(\sigma) = z(\sigma)$. Therefore, the problem (1.1)-(1.2) possess one solution $z(\sigma)$ that belongs to B. □

5. Continuous reliance

In the following part, we will investigate the continuous dependency of a solution on a constant that will be designated by C_0.

5.1. Continuous reliance on C_0.

Definition 5.1. $z(\sigma) \in B$ is a solution of (1.1)-(1.2) and is a continuous dependence on C_0, if for all $\epsilon > 0$, there exist $\delta_0(\epsilon)$ such that $|C_0 - C_0^*| < \delta_0 \Rightarrow \|z - z^*\| < \epsilon$, where $z^*(\sigma)$ is the solution for the following:

$$
D^{\alpha}z^*(\sigma) = f\Big(\sigma, z^*(\sigma), {}^{CF}D^{\beta}z^*(\sigma), I_q^{\gamma}z^*(\sigma)\Big), \quad \sigma \in (0,1], \tag{5.1}
$$

$$
(1-q)\rho\sum_{i=0}^{\zeta} q^i z^*(q^i\rho) = C_0^*. \tag{5.2}
$$

Theorem 5.2. *If all three of the requirements of the Theorem (4.1) are met, then the solution of (1.1)-(1.2) is inextricably linked to the variable* C_0.

Proof. Assume that $z(\sigma)$ and $z^*(\sigma)$ are two solutions of (1.1)-(1.2) and (5.1)-(5.2), respectively. Then,

$$
\begin{aligned}
|z(\sigma) - z^*(\sigma)| \leqslant &\frac{1}{(1-q)\rho\sum_{i=0}^{\zeta}q^i}|C_0 - C_0^*| + \frac{1}{(1-q)\rho\sum_{i=0}^{\zeta}q^i}\Big[(1-\alpha)(1-q)\rho \\
&\times\sum_{i=0}^{\zeta}q^i\Big|f\Big(q^i\rho, z^*(q^i\rho), {}^{CF}D^{\beta}z^*(q^i\rho), I_q^{\gamma}z^*(q^i\rho)\Big) - f\Big(q^i\rho, z(q^i\rho), {}^{CF}D^{\beta}z(q^i\rho), I_q^{\gamma}z(q^i\rho)\Big)\Big| \\
&+ \alpha(1-q)\rho\sum_{i=0}^{\zeta}q^i\int_0^{q^i\rho}\Big|f\Big(s, z^*(s), {}^{CF}D^{\beta}z^*(s), I_q^{\gamma}z^*(s)\Big) \\
&- f\Big(s, z(s), {}^{CF}D^{\beta}z(s), I_q^{\gamma}z(s)\Big)\Big|ds\Big] + (1-\alpha)\Big|f\Big(\sigma, z(\sigma), {}^{CF}D^{\beta}z(\sigma), I_q^{\gamma}z(\sigma)\Big) \\
&- f\Big(\sigma, z^*(\sigma), {}^{CF}D^{\beta}z^*(\sigma), I_q^{\gamma}z^*(\sigma)\Big)\Big| + \alpha\int_0^{\sigma}\Big|f\Big(s, z(s), {}^{CF}D^{\beta}z(s), I_q^{\gamma}z(s)\Big) \\
&- f\Big(s, z^*(s), {}^{CF}D^{\beta}z^*(s), I_q^{\gamma}z^*(s)\Big)\Big|ds
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \frac{|C_0 - C_0^*|}{(1-q)\rho\sum_{i=0}^{\zeta} q^i} + \frac{1}{(1-q)\rho\sum_{i=0}^{\zeta} q^i}\Bigg[(1-\alpha)(1-q)\rho\sum_{i=0}^{\zeta} q^i\Big(b|z^*(q^i\rho) - z(q^i\rho)| \\
&\quad + b|{}^{CF}D^{\beta} z^*(q^i\rho) - {}^{CF}D^{\beta} z(q^i\rho)| + bI_q^{\gamma}|z^*(q^i\rho) - z(q^i\rho)|\Big) \\
&\quad + \alpha(1-q)\rho\sum_{i=0}^{\zeta} q^i \int_0^{q^i\rho}\Big(b|z^*(s) - z(s)| + b|{}^{CF}D^{\beta} z^*(s) - {}^{CF}D^{\beta} z(s)| + bI_q^{\gamma}|z^*(s) - z(s)|\Big)ds\Bigg] \\
&\quad + (1-\alpha)\Big(b|z(\sigma) - z^*(\sigma)| + b|{}^{CF}D^{\beta} z(\sigma) - {}^{CF}D^{\beta} z^*(\sigma)| + bI_q^{\gamma}|z(\sigma) - z^*(\sigma)|\Big) \\
&\quad + \alpha\int_0^{\sigma}\Big(b|z(s) - z^*(s)| + b|{}^{CF}D^{\beta} z(s) - {}^{CF}D^{\beta} z^*(s)| + bI_q^{\gamma}|z(s) - z^*(s)|\Big)ds \\
&\leqslant \frac{\delta_0}{(1-q)\rho\sum_{i=0}^{\zeta} q^i} + \left(2b + 2b\frac{(2-\beta)}{(1-\beta)^2} + \frac{2b(1-\alpha)}{\Gamma_q(\gamma+1)} + \frac{2b\alpha}{(\gamma+1)\Gamma_q(\gamma+1)}\right)\|z - z^*\|.
\end{aligned}
$$

Hence,

$$
\|z - z^*\| \leqslant \frac{\frac{\delta_0}{(1-q)\rho\sum_{i=0}^{\zeta} q^i}}{1 - 2b\left(1 + \frac{(2-\beta)}{(1-\beta)^2} + \frac{1}{\Gamma_q(\gamma+1)}(1 - \alpha + \frac{\alpha}{(\gamma+1)})\right)}.
$$

Therefore, the solution of (1.1)-(1.2) is continuously dependent on C_0. □

6. Methodology of numerical technique

Now, we use the merge of the forward finite difference with the trapezoidal methods to solve the problem (1.1)-(1.2) numerically. To begin, we can write the problem (1.1)-(1.2) as follows:

$$
{}^{CF}D^{\alpha} z(\sigma) - b\varphi(z(\sigma)) - b\,{}^{CF}D^{\beta} z(\sigma) - bI_q^{\gamma} z(\sigma) = G(\sigma), \quad (1-q)\rho\sum_{i=0}^{\zeta} q^i z(q^i\rho) = C_0. \tag{6.1}
$$

Using (2.1) and (2.2), then (6.1) may be written as follows:

$$
\frac{1}{1-\alpha}\int_0^{\sigma} e^{\frac{-\alpha(\sigma-s)}{1-\alpha}} z'(s)ds - b\varphi(z(\sigma)) - \frac{b}{1-\beta}\int_0^{\sigma} e^{\frac{-\beta(\sigma-s)}{1-\beta}} z'(s)ds - \frac{b}{\Gamma_q(\gamma)}\int_0^{\sigma}(\sigma - qs)^{(\gamma-1)} z(s)d_q s = G(\sigma), \tag{6.2}
$$

where $\varphi(z(\sigma))$ is the nonlinear term of the unknown function $z(\sigma)$. Now, we divide the interval of integration $[0,\sigma]$ of equation (6.2) into μ equal subintervals. The width of it is given by $h = \sigma_i/i$, [21]. To simplify, we take $z_i = z(\sigma_i)$, $z_j' = z'(s_j)$, $G_i = G(\sigma_i)$, $K_{ij} = \frac{1}{1-\alpha}e^{\frac{-\alpha(\sigma_i - s_j)}{1-\alpha}}$, $Y_{ij} = \frac{b}{1-\beta}e^{\frac{-\beta(\sigma_i - s_j)}{1-\beta}}$, and $X_{ij} = \frac{b}{\Gamma_q(\gamma)}(\sigma_i - qs_j)^{(\gamma-1)}$. Then, we can write (6.2) as follows:

$$
\int_0^{\sigma_i} K_{ij} z_j' ds - b\varphi(z_i) - \int_0^{\sigma_i} Y_{ij} z_j' ds - \int_0^{\sigma_i} X_{ij} z_j d_q s = G_i. \tag{6.3}
$$

To get the numerical solution of (6.3), we approximate the integral part of it by using the trapezoidal method and the derivative part by using the forward finite difference method as follows:

$$
\int_0^{\sigma_i} K_{ij} z_j' ds \approx \frac{h}{2}\left[K_{i0} z_0' + 2\sum_{j=1}^{i-1} K_{ij} z_j' + K_{ii} z_i'\right], \qquad \int_0^{\sigma_i} Y_{ij} z_j' ds \approx \frac{h}{2}\left[Y_{i0} z_0' + 2\sum_{j=1}^{i-1} Y_{ij} z_j' + Y_{ii} z_i'\right],
$$

$$
\int_0^{\sigma_i} X_{ij} z_j ds \approx \frac{h}{2}\left[X_{i0} z_0 + 2\sum_{j=1}^{i-1} X_{ij} z_j + X_{ii} z_i\right], \qquad v_j' \approx \frac{v_{j+1} - v_j}{h}, \quad i = 0, 1, 2, 3, \ldots, \mu,
$$

Then, (6.3) can be written as follows:

$$\frac{h}{2}\Big[K_{i0}(\frac{z_1-z_0}{h})+2\sum_{j=1}^{i-1}K_{ij}(\frac{z_{j+1}-z_j}{h})+K_{ii}(\frac{z_{i+1}-z_i}{h})\Big]-b\varphi(z_i)-\frac{h}{2}\Big[Y_{i0}(\frac{z_1-z_0}{h}) \\ +2\sum_{j=1}^{i-1}Y_{ij}(\frac{z_{j+1}-z_j}{h})+Y_{ii}(\frac{z_{i+1}-z_i}{h})\Big]-\frac{h}{2}\Big[X_{i0}z_0+2\sum_{j=1}^{i-1}X_{ij}z_j+X_{ii}z_i\Big]=G_i,\quad i=0,1,\dots,\mu. \tag{6.4}$$

7. Test problems

Now that we have several instances to work with, we will apply the assumptions of the Theorem (3.1) to them, and we will solve them numerically by combining the forward finite difference method with the trapezoidal method. In order to illustrate that the technique being applied yields accurate results, the outcomes of these cases will be contrasted with the precise answers.

Problem 7.1. In (6.2), we decided to go with $G(\sigma) = (5.0546\times10^{-7}\sigma^6 - 2.88749\times10^{-5}\sigma^4 + 8.9632\times10^{-4}\sigma^2 - 0.0115674)\sigma^{8/3} - e^{-\sigma} + 0.0114286e^{-0.3333\sigma} + 0.937143\sin(\sigma) + 0.988571\cos(\sigma)$, $\varphi(z(\sigma)) = z(\sigma)$, $b = \frac{1}{35}$, $\gamma = \frac{5}{3}$, $q = 0.5$, $\rho = 0.2$, $\alpha = 0.5$, $\beta = 0.25$, $C_0 = 0.026108$, $\zeta = 2$. The assumptions 1-3 of Theorem 3.1 are fulfilled and it is clear that $2b\left(1+\frac{(2-\beta)}{(1-\beta)^2}+\frac{1}{\Gamma_q(\gamma+1)}(1-\alpha+\frac{\alpha}{(\gamma+1)})\right) < 1$. Therefore, the test Problem 7.1 has one solution given by $z(\sigma) = \sin(\sigma)$. If we take $\mu = 80$, the result of this problem can be obtained as tabulated in Table 1. Moreover, we discuss the continuous dependence on C_0. Taking $|C_0 - C_0^*| = 10^{-5} \Rightarrow |z(0.1) - z^*(0.1)| = 5.71244\times10^{-5}$. As a result, $z(\sigma)$ is continuous dependence on C_0.

Table 1: The precise and numerical answers to Problem 7.1 of the examination.

σ_i	Num. sol. of z	Exact solution of z	Absolute error of z
0.1	0.100417	0.099833	5.83177×10^{-4}
0.2	0.198026	0.198669	6.43236×10^{-4}
0.3	0.294089	0.295520	1.43096×10^{-3}
0.4	0.387528	0.389418	1.88991×10^{-3}
0.5	0.477326	0.479426	2.09938×10^{-3}
0.6	0.562525	0.564642	2.11701×10^{-3}
0.7	0.642233	0.644218	1.98513×10^{-3}
0.8	0.715621	0.717356	1.73531×10^{-3}
0.9	0.781935	0.783327	1.39155×10^{-3}

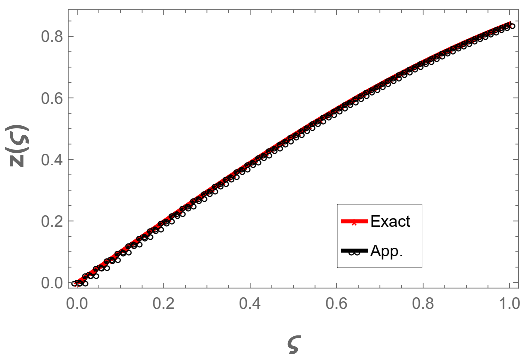

Figure 1: Examining the differences between the numerical and the precise answers to test Problem 7.1.

Together, Figure 1 and Table 1 provide a comprehensive and robust demonstration of the efficiency of the approach. They visually depict the positive outcomes and offer quantitative evidence to support the conclusions. These visuals are valuable tools for effectively communicating the approach's efficacy to a wider audience, facilitating a better understanding and appreciation of the research or project.

Problem 7.2. In (6.2), we decided to go with $G(\sigma) = -0.0208668(0.00496301\sigma^4 + 0.0254659\sigma^3 + 0.105795\sigma^2 + 0.337215\sigma + 0.754727)\sigma^{8/7} + 1.42857e^{-0.428571\sigma}\left(0.7e^{1.42857\sigma} - 0.7\right) - 0.0666667e^{-2.33333\sigma}\left(0.3e^{3.33333\sigma} - 0.3\right) - \frac{e^{\sigma}}{50}$, $b = \frac{1}{50}$, $\gamma = \frac{8}{7}$, $q = 0.4$, $\rho = 0.5$, $\alpha = 0.3$, $\beta = 0.7$, $C_0 = 0.641185$, $\zeta = 1$. The assumptions 1-3 of Theorem 3.1 are fulfilled and it is clear that $2b\left(1 + \frac{(2-\beta)}{(1-\beta)^2} + \frac{1}{\Gamma_q(\gamma+1)}(1 - \alpha + \frac{\alpha}{(\gamma+1)})\right) < 1$. Therefore, the test Problem 7.2 has one solution given by $z(\sigma) = \exp(\sigma)$. If we take $\mu = 80$, the result of this problem can be obtained as tabulated in Table 2.

Table 2: The precise and numerical answers to Problem 7.2 of the examination

σ_i	Num. sol. of z	Exact solution of z	Absolute error of z
0.1	1.11050	1.10517	5.33121×10^{-3}
0.2	1.22483	1.22140	3.42772×10^{-3}
0.3	1.35158	1.34986	1.72346×10^{-3}
0.4	1.49197	1.49182	1.43744×10^{-4}
0.5	1.64735	1.64872	1.37109×10^{-3}
0.6	1.81925	1.82212	2.86955×10^{-3}
0.7	2.00936	2.01375	4.39189×10^{-3}
0.8	2.21957	2.22554	5.97236×10^{-3}
0.9	2.45196	2.45960	7.64084×10^{-3}

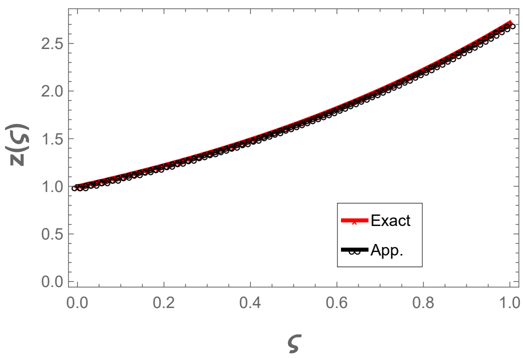

Figure 2: Examining the differences between the numerical and the precise answers to test Problem 7.2.

The efficiency of the approach that was used is demonstrated above in both Figure 2 and Table 2.

Problem 7.3. In (6.2), we decided to go with $G(\sigma) = -0.00722342\sigma^{11/5} - 2.5e^{-0.666667\sigma} + 0.1e^{-0.2\sigma} - 0.0166667\cos(\sigma) + 2.4$, $\varphi(z(\sigma)) = \cos(z(\sigma))$, $b = \frac{1}{60}$, $\gamma = \frac{6}{5}$, $q = 0.2$, $\rho = 0.5$, $\alpha = 0.4$, $\beta = \frac{1}{6}$, $C_0 = 0.208$, $\zeta = 1$. The assumptions 1-3 of Theorem 3.1 are fulfilled and it is clear that $2b\left(1 + \frac{(2-\beta)}{(1-\beta)^2} + \frac{1}{\Gamma_q(\gamma+1)}(1 - \alpha + \frac{\alpha}{(\gamma+1)})\right) < 1$. Therefore, the test Problem 7.3 has one solution given by $z(\sigma) = \sigma$. If we take $\mu = 80$, the result of this problem can be obtained as tabulated in Table 3. The efficiency of the approach that was used is demonstrated above in both Figure 3 and Table 3.

Table 3: The precise and numerical answers to Problem 7.3 of the examination.

σ_i	Num. sol. of z	Exact solution of z	Absolute error of z
0.2	0.200146	0.2	1.45719×10^{-4}
0.3	0.299985	0.3	1.49185×10^{-5}
0.4	0.399912	0.4	8.82343×10^{-5}
0.5	0.499919	0.5	8.07471×10^{-5}
0.6	0.600004	0.6	3.95371×10^{-6}
0.7	0.700165	0.7	1.64711×10^{-4}
0.8	0.800402	0.8	4.02383×10^{-4}
0.9	0.900719	0.9	7.19467×10^{-4}

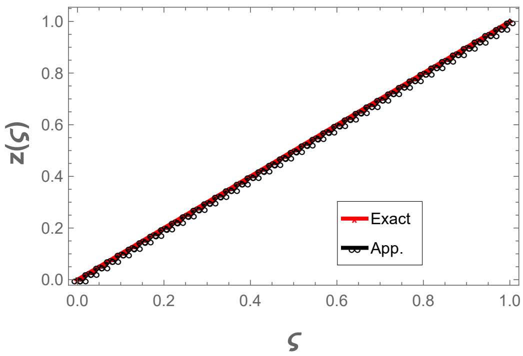

Figure 3: Examining the differences between the numerical and the precise answers to test Problem 7.3.

8. Conclusion

This article included a study of a new equation that had not been studied before, which contains the fractional q integral of the Riemann-Liouville type and the fractional derivative of Caputo-Fabrizio. The topic of whether a solution can be found or not is covered in this article. Furthermore, the solution's continuous dependence on C_0, as well as its uniqueness are investigated. In addition, the numerical solution to the issue is obtained by employing both the finite-forward approach and the trapezoidal method. In conclusion, three cases are solved numerically, and then those solutions are compared with the precise solutions in order to demonstrate how successful the method that was used is. The results recorded in the previous tables and represented graphically in the previous figures have shown that the method used is extremely effective.

References

[1] B. Ahmad, J. J. Nieto, A. Alsaedi, H. Al-Hutami, *Existence of solutions for nonlinear fractional q-difference integral equations with two fractional orders and nonlocal four-point boundary conditions*, J. Franklin Inst., **351** (2014), 2890–2909. 1, 2.3, 2.4

[2] K. K. Ali, K. R. Raslan, A. A.-E. Ibrahim, D. Baleanu, *The nonlocal coupled system of Caputo–Fabrizio fractional q-integro differential equation*, Math. Methods Appl. Sci., **2023** (2023), 17 pages. 1

[3] K. K. Ali, K. R. Raslan, A. A.-E. Ibrahim, M. S. Mohamed, *On study the existence and uniqueness of the solution of the Caputo-Fabrizio coupled system of nonlocal fractional q-integro differential equations*, Math. Methods Appl. Sci., **46** (2023), 13226–13242. 1

[4] K. K. Ali, K. R. Raslan, A. A.-E. Ibrahim, M. S. Mohamed, *On study the fractional Caputo-Fabrizio integro differential equation including the fractional q-integral of the Riemann-Liouville type*, AIMS Math., **8** (2023), 18206–18222. 1

[5] M. H. Annaby, Z. S. Mansour, *q-fractional calculus and equations*, Springer, Heidelberg, (2012). 1

[6] D. Baleanu, A. Mousalou, S. Rezapour, *A new method for investigating approximate solutions of some fractional integro-differential equations involving the Caputo-Fabrizio derivative*, Adv. Difference Equ., **2017** (2017), 12 pages. 1

[7] M. Caputo, M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Progr. Fract. Differ. Appl., **1** (2015), 73–85. 1, 2.5

[8] S. Etemad, S. K. Ntouyas, B. Ahmad, *Existence theory for a fractional q-integro-difference equation with q-integral boundary conditions of different orders*, Mathematics, **7** (2019), 15 pages. 1

[9] A. A.-E. Ibrahim, A. A. S. Zaghrout, K. R. Raslan, K. K. Ali, *On the analytical and numerical study for nonlinear Fredholm integro-differential equations*, Appl. Math. Inf. Sci., **14** (2020), 921–929. 1

[10] A. A.-E. Ibrahim, A. A. S. Zaghrout, K. R. Raslan, K. K. Ali, *On the analytical and numerical study for fractional q-integrodifferential equations*, Bound. Value Probl., **2022** (2022), 15 pages. 1

[11] A. A. Ibrahim, A. A. S. Zaghrout, K. R. Raslan, K. K. Ali, *On study nonlocal integro differetial equation involving the Caputo-Fabrizio Fractional derivative and q-integral of the Riemann Liouville Type*, Appl. Math. Inf. Sci., **16** (2022), 983–993. 1

[12] A. A.-E. Ibrahim, A. A. S. Zaghrout, K. R. Raslan, K. K. Ali, *On study of the coupled system of nonlocal fractional q-integro-differential equations*, Int. J. Model. Simul. Sci. Comput., **14** (2023), 26 pages. 1

[13] F. H. Jackson, *XI.—On q-functions and a certain difference operator*, Trans. Roy. Soc. Edinb., **46** (1908), 253–281. 1

[14] S. A. M. Jameel, *On the existence and stability of Caputo Volterra-Fredholm systems*, J. Math. Comput. Sci., **30** (2023), 322–331. 1

[15] V. Kac, P. Cheung, *Quantum Calculus*, Springer-Verlag, New York, (2002). 1, 2.1, 2.2

[16] A. N. Kolomogorov, S. V. Fomīn, *Introductory real analysis*, Dover Publications, New York, (1975). 3

[17] J. Losada, J. J. Nieto, *Properties of a New Fractional Derivative without Singular Kernel*, Progr. Fract. Differ. Appl., **1** (2015), 87–92. 2.5, 2.6

[18] G. A. Mboro Nchama, L. D. Lau-Alfonso, A. M. León Mecías, M. Rodríguez Ricard, *Properties of the Caputo-Fabrizio fractional derivative*, Appl. Math. Inf. Sci., **14** (2020), 761–769. 2.7

[19] R. Prasad, K. Kumar, R. Dohare, *Caputo fractional order derivative model of Zika virus transmission dynamics*, J. Math. Comput. Sci., **2023** (2023), 145–175. 1

[20] K. R. Raslan, K. K. Ali, R. G. Ahmed, H. K. Al-Jeaid, A. A.-E. Ibrahim, *Study of nonlocal boundary value problem for the Fredholm-Volterra integro-differential equation*, J. Funct. Spaces, **2022** (2022), 16 pages. 1

[21] R. Saadati, B. Raftari, H. Adibi, S. M. Vaezpour, S. Shakeri, *A comparison between the variational iteration method and trapezoidal rule for solving linear integro-differential equations*, World Appl. Sci. J., **4** (2008), 321–325. 6

[22] E. Shivanian, A. Dinmohammadi, *On the solution of a nonlinear fractional integro-differential equation with non-local boundary condition*, Int. J. Nonlinear Anal. Appl., **2023** (2023), 12 pages. 1

[23] S. Toprakseven, *The existence and uniqueness of initial-boundary value problems of the fractional Caputo-Fabrizio differential equations*, Univ. J. Math. Appl., **2** (2019), 100–106. 1