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Fixed point approach to the stability of a cubic and quartic mixed type functional equation in non-archimedean spaces

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Abstract

The objective of this article is to establish the generalized Hyers-Ulam stability of the following cubic-quartic(C_3Q_4) functional equation

$$\mathfrak{g}(\upsilon+2\nu)+\mathfrak{g}(\upsilon-2\nu)-4[\mathfrak{g}(\upsilon+\nu)+\mathfrak{g}(\upsilon-\nu)]=3\mathfrak{g}(2\nu)-24\mathfrak{g}(\nu)-6\mathfrak{g}(\upsilon)$$

in non-Archimedean normed spaces by using alternative fixed point theorem.

Keywords: Generalized Hyers-Ulam (HU) stability, cubic-quartic (C_3Q_4) functional equation (FE), non-archimedean (NA) normed spaces, fixed point theorem.

2020 MSC: 39B52, 47H10, 26E30, 46S10, 47S10.

1. Introduction

The concept of stability for a functional equation arises when an FE is replaced by an inequality which is a perturbation of the equation. Ulam [30] proposed the first FE stability problem in 1940. This question has attracted the interest of numerous researchers since then. Hyers [14] responded positively to this question the following year. Aoki [1] generalized Hyers result for additive mapping in 1950. In addition, Rassias [26] generalized the result of Hyers by considering an unbounded Cauchy difference in linear mapping.

Gajda [7] answered Rassias question about $p > 1$, which was raised in 1991. This new concept is known as the HUR stability of FEs. The term HUR stability originates from these historical contexts. Other FEs can also be considered using this terminology. Gavruta [8] was able to further generalize Rassias theorem in 1994 (see also [9]). Ever since, a number of authors have thoroughly researched the stability problem of many FEs, and there are many fascinating findings on the Ulam stability problems in [18–22, 25, 27]. The cubic (C_3) FE

$$\mathfrak{g}(2x+\nu)+\mathfrak{g}(2x-\nu)=2\mathfrak{g}(\upsilon+\nu)+2\mathfrak{g}(\upsilon-\nu)+12\mathfrak{g}(\upsilon) \tag{1.1}$$

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was first proposed by Jun and Kim [16], who also showed its general solution and generalized HUR stability (1.1). The FE (1.1), also referred to as a C_3 FE, is satisfied by the function $\mathfrak{g}(\upsilon) = \upsilon^3$. Every solution to the C_3 FE is referred to as a C_3 function. Moslehian and Rassias [17] proved the generalized HU stability of the Cauchy FE and the quadratic (Q_2) FE in NA spaces. Park and Bae [24] introduced the following quartic (Q_4) FE

$$\mathfrak{g}(\upsilon+2\nu)+\mathfrak{g}(\upsilon-2\nu)=4[\mathfrak{g}(\upsilon+\nu)+\mathfrak{g}(\upsilon-\nu)]-24\mathfrak{g}(\nu)-6\mathfrak{g}(\upsilon) \tag{1.2}$$

and obtained the stability results of (1.2). It is simple to demonstrate that the function $g(\upsilon) = \upsilon^4$ satisfies the FE (1.2), which is known as Q_4 FE, and that each solution of the Q_4 FE is known as a Q_4 function (see also [11]). In 2010, Gordji and Savadkouhi [12] established the generalized HUR stability for the mixed type C_3Q_4 FE

$$\mathfrak{g}(\upsilon+2\nu)+\mathfrak{g}(\upsilon-2\nu)-4[\mathfrak{g}(\upsilon+\nu)+\mathfrak{g}(\upsilon-\nu)]=3\mathfrak{g}(2\nu)-24\mathfrak{g}(\nu)-6\mathfrak{g}(\upsilon) \tag{1.3}$$

in NA normed spaces by Hyers direct method. In the same year, by fixed point approach, Park [23] established the generalized HU stability of the following additive-quadratic-cubic-quartic($A_1Q_2C_3Q_4$) functional equation

$$\mathfrak{g}(\upsilon+2\nu)+\mathfrak{g}(\upsilon-2\nu)=4[\mathfrak{g}(\upsilon+\nu)+\mathfrak{g}(\upsilon-\nu)]-6\mathfrak{g}(\upsilon)+\mathfrak{g}(2\nu)+\mathfrak{g}(-2\nu)-4\mathfrak{g}(\nu)-4\mathfrak{g}(-\nu)$$

in NA Banach spaces. Recently, Gharib et al. [10] investigated the asymptotic stability behaviour of the Pexider-Cauchy FE in NA spaces. Additionally, they addressed a functional inequality and its asymptotic behaviour and demonstrated that, under certain circumstances, if

$$\|f(\upsilon+\nu)-g(\upsilon)-h(\nu)\| \leqslant \epsilon,$$

then f, g and h may be approximated by A_1 mapping in NA normed spaces.

Very recently, Tamilvanan et al. [29] created a new type of generalized mixed-type A_1Q_2 FE and obtained its general solution and proved the Ulam stability of the mixed type A_1Q_2 FE in NA fuzzy φ-2-normed space and NA Banach space using the direct and fixed point approaches by taking into even and odd mapping. The HUR stability for a mixed type C_3Q_4 FE (1.3) for any $\upsilon, \nu \in A$, in NA Banach spaces will be established in this article using the fixed point approach. It is simple to establish that the FE (1.3), sometimes known as a mixed type C_3Q_4 FE, is satisfied by the function $\mathfrak{g}(\upsilon) = \upsilon^3 + \upsilon^4$.

2. Preliminaries

In this section, we will provide some basic notations, definitions and theorems, which will be very useful to prove our main results. By an NA field we mean a field $\mathbb{K}$ equipped with a function (valuation) $|.|$ from $\mathbb{K}$ into $[0, \infty)$ such that $|\varkappa| = 0$ iff $\varkappa = 0$, $|\varkappa\gamma| = |\varkappa|.|\gamma|$, and

$$|\varkappa+\gamma| \leqslant \max\{|\varkappa|, |\gamma|\}$$

for all $\varkappa, \gamma \in \mathbb{K}$. Clearly $|1| = |-1| = 1$ and $|n| \leqslant 1$ for all $n \in \mathbb{N}$. A trivial example of an NA valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Throughout this paper, let us consider A be an NA normed vector space and B as an NA Banach space.

Definition 2.1 ([17])**.** Let A be a vector space over a scalar field $\mathbb{K}$ with an NA non-trivial valuation $|\cdot|$. A function $\|\cdot\| : A \to \mathbb{R}$ is an NA norm (valuation) if it satisfies the following conditions:

(i) $\|\varkappa\| = 0$ iff $\varkappa = 0$;
(ii) $\|r\varkappa\| = |r|. \|\varkappa\|$ for all $r \in \mathbb{K}$, $\varkappa \in A$;

(iii) the strong triangle inequality (ultra-metric), namely, $\|\varkappa+\gamma\| \leqslant \max\{\|\varkappa\|, \|\gamma\|\}$ for all $x, y \in A$.

Then $(A, \|.\|)$ is called an NA normed space.

Definition 2.2 ([17]). Let $\{\varkappa_n\}$ be a sequence in an NA normed space A. Then the sequence $\{\varkappa_n\}$ is called Cauchy, if for a given $\epsilon > 0$, there is a positive integer N such that $\|\varkappa_n - \varkappa_m\| \leqslant \epsilon$ for all $m, n \geqslant N$.

Definition 2.3 ([17]). Let $\{\varkappa_n\}$ be a sequence in an NA normed space A. Then the sequence $\{\varkappa_n\}$ is called convergent, if for a given $\epsilon > 0$, there are a positive integer N and an $\varkappa \in A$ such that

$$\|\varkappa_n - \varkappa\| \leqslant \epsilon$$

for all $n \geqslant N$. Then we call $\varkappa \in A$ a limit of the sequence $\{\varkappa_n\}$, and denote it by

$$\lim_{n\to\infty} \varkappa_n = \varkappa.$$

Definition 2.4 ([17]). If every Cauchy sequence converges in A, then the NA normed space A is called an NA Banach space.

Due to the fact that

$$\|\varkappa_n - \varkappa_m\| \leqslant \max\left\{\left\|\varkappa_{j+1} - \varkappa_j\right\| : m \leqslant j \leqslant n-1\right\} \quad (n > m),$$

a sequence $\{\varkappa_n\}$ is Cauchy iff $\{\varkappa_{n+1} - \varkappa_n\}$ converges to zero in an NA space. By a complete NA space we mean one in which every Cauchy sequence is convergent.

Definition 2.5 ([5]). Let A be a set. A function $\rho : A \times A \to [0, \infty]$ is called a generalized metric on A if ρ satisfies the following conditions:

(i) $\rho(\upsilon, \nu) = 0$ iff $\upsilon = \nu$;
(ii) $\rho(\upsilon, \nu) = \rho(\nu, \upsilon)$ for all $\upsilon, \nu \in A$;
(iii) $\rho(\upsilon, \omega) \leqslant \rho(\upsilon, \nu) + \rho(\nu, \omega)$ for all $\upsilon, \nu, \omega \in A$.

We recall some fundamental result in fixed point theory. Fixed theory has a variety of applications, for example, it has an application in the split feasibility problem for modeling inverse problems or in signal processing (see [28]). As usual, we will denote by $\mathbb{N}_0$ the set of all non-negative integers, that is, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Theorem 2.6 ([5, 6]). *Let (A, ρ) be a complete generalized metric space and let $\Omega : A \to A$ be a strictly contractive mapping with Lipschitz constant $Ł < 1$. Then for each given element $\varkappa \in A$, either $\rho(\Omega^n\varkappa, \Omega^{n+1}\varkappa) = \infty$ for all non-negative integers $n \in \mathbb{N}_0$ or there exists a positive integer $n_0 \in \mathbb{N}_0$ such that*

(i) *$\rho(\Omega^n\varkappa, \Omega^{n+1}\varkappa) < \infty$, for all $n \geqslant n_0$;*
(ii) *the sequence $\{\Omega^n\varkappa\}$ converges to a fixed point ν^* of Ω;*
(iii) *γ^* is the unique fixed point of Ω in the set $B = \{\gamma \in A / \rho(\Omega^{n_0}\varkappa, \gamma) < \infty\}$;*
(iv) *for all $\gamma \in B$, we have*

$$\rho(\Omega, \Omega^*) \leqslant \frac{1}{1-Ł}\rho(\gamma, \Omega\gamma).$$

In 1996, Isac and Rassisa [15] were the first provide application of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by number of authors (see [2–4, 13]). Suppose for a function $\mathfrak{g} : A \to B$, we define a function $D_{\mathfrak{g}} : A \to B$ by

$$D_{\mathfrak{g}}(\upsilon, \nu) = \mathfrak{g}(\upsilon + 2\nu) + \mathfrak{g}(\upsilon - 2\nu) - 4\mathfrak{g}(\upsilon + \nu) - 4\mathfrak{g}(\upsilon - \nu) - 3\mathfrak{g}(2\nu) + 24\mathfrak{g}(\nu) + 6\mathfrak{g}(\upsilon)$$

for all $\upsilon, \nu \in A$.

3. HU stability of the mixed type FE (1.3): an odd case

In this part, first we will prove the generalized HU stability of the FE $D\mathfrak{g}(\upsilon, \nu) = 0$ in NA normed spaces.

Theorem 3.1. *Assume that* $\varphi : A \times A \to [0, \infty)$ *is a function such that there is a constant* $Ł < 1$ *with*

$$\varphi(2\upsilon, 2\nu) \leqslant |2|^3 Ł \varphi(\upsilon, \nu) \tag{3.1}$$

for all $\upsilon, \nu \in A$. *Let* $\mathfrak{g} : A \to B$ *be an odd mapping satisfying*

$$\|D_{\mathfrak{g}}(\upsilon, \nu)\| \leqslant \varphi(\upsilon, \nu) \tag{3.2}$$

for all $\upsilon, \nu \in A$. *Then there is a unique* $\mathcal{C}_3$ *mapping* $\mathcal{C} : A \to B$ *such that*

$$\|\mathfrak{g}(\upsilon) - \mathcal{C}(\upsilon)\| \leqslant \frac{1}{|3|.|2|^3(1 - Ł)} \varphi(0, \upsilon) \tag{3.3}$$

for all $\upsilon \in A$.

Proof. Letting $\upsilon = 0$ in (3.2), we get

$$\left\|\mathfrak{g}(2\nu) - 2^3 \mathfrak{g}(\nu)\right\| \leqslant \frac{1}{|3|} \varphi(0, \nu)$$

for all $\nu \in A$. So

$$\left\|\frac{\mathfrak{g}(2\nu)}{2^3} - \mathfrak{g}(\nu)\right\| \leqslant \frac{1}{|3|} \frac{1}{|2|^3} \varphi(0, \nu) \tag{3.4}$$

for all $\nu \in A$. Consider the set

$$\omega = \{h : A \to B\}$$

and also we define a generalized metric on ω as follows:

$$\rho(f, h) = \inf\left\{\Omega \in \mathbb{R}_+ : \|f(\upsilon) - h(\upsilon)\| \leqslant \Omega\varphi(0, \upsilon), \forall \upsilon \in A\right\},$$

where, as usual, $\inf \varphi = +\infty$. Clearly (ω, ρ) is complete. Now, we take the linear mapping $\Omega : \omega \to \omega$ such that

$$\Omega h(\upsilon) = \frac{1}{2^3} h(2\upsilon)$$

for all $\upsilon \in A$. Let $f, h \in \omega$ be given such that $\rho(f, h) = \varepsilon$. Then

$$\|f(\upsilon) - h(\upsilon)\| \leqslant \varepsilon\varphi(0, \upsilon)$$

for all $\upsilon \in A$. Hence

$$\|\Omega f(\upsilon) - \Omega h(\upsilon)\| = \left\|\frac{1}{2^3} f(2\upsilon) - \frac{1}{2^3} h(2\upsilon)\right\| \leqslant \frac{1}{|2|^3} \varepsilon\varphi(0, 2\upsilon) \leqslant \varepsilon Ł \varphi(0, \upsilon)$$

for all $\upsilon \in A$. So $\rho(f, h) \leqslant \varepsilon$, which implies that $\rho(\Omega f, \Omega h) \leqslant Ł\varepsilon$. This means that

$$\rho(\Omega f, \Omega h) \leqslant Ł\rho(f, h)$$

for all $f, h \in \omega$. It follows from (3.4) that

$$\rho(\mathfrak{g}, \Omega\mathfrak{g}) \leqslant \frac{1}{|3|} \frac{1}{|2|^3} < \infty.$$

By Theorem 2.6, there is a cubic mapping $\mathcal{C} : A \to B$ satisfying the following conditions.

(i) $\mathcal{C}$ is a fixed point of Ω, that is,

$$\mathcal{C}(2\upsilon) = 2^3\mathcal{C}(\upsilon) \tag{3.5}$$

for all $\upsilon \in A$. The mapping $\mathcal{C}$ is a unique fixed point of Ω in the set

$$S = \{f \in \omega : \rho(f, h) < \infty\}.$$

This implies that $\mathcal{C}$ is a unique mapping satisfying (3.5) such that there is a $\Omega \in (0, \infty)$ satisfying

$$\|\mathfrak{g}(\upsilon) - \mathcal{C}(\upsilon)\| \leqslant \Omega.\varphi(0, \upsilon)$$

for all $\upsilon \in A$.

(ii) $\rho(\Omega^n\mathfrak{g}, \mathcal{C}) \to 0$ as $n \to \infty$. This gives that, for all $\upsilon \in A$,

$$\lim_{n\to\infty} (\Omega^n\mathfrak{g})(\upsilon) = \lim_{n\to\infty} \frac{\mathfrak{g}(2^n\upsilon)}{|2|^{3n}} = \mathcal{C}(\upsilon).$$

(iii) $\rho(\mathfrak{g}, \mathcal{C}) \leqslant \frac{1}{1-Ł}\rho(\mathfrak{g}, \Omega^n\mathfrak{g})$, which gives the inequality

$$\rho(\mathfrak{g}, \mathcal{C}) \leqslant \frac{1}{1-Ł}\rho(\mathfrak{g}, \Omega\mathfrak{g}) \leqslant \frac{1}{|3|}\frac{1}{|2|^3(1-Ł)}.$$

This indicates that the inequality (3.3) holds.

As a result of (3.1) and (3.2),

$$\|D\mathcal{C}(\upsilon, \nu)\| = \lim_{n\to\infty} \|2^{-3n}D\mathfrak{g}(2^n\upsilon, 2^n\nu)\| \leqslant \lim_{n\to\infty} \frac{1}{|2|^{3n}}\varphi(2^n\upsilon, 2^n\nu) \leqslant \lim_{n\to\infty} Ł^n\varphi(\upsilon, \nu) = 0$$

for all $\upsilon, \nu \in A$ and $n \in \mathbb{N}$. So, $\|D\mathcal{C}(\upsilon, \nu)\| = 0$. Since $\mathfrak{g}$ is odd, $\mathcal{C}_3$ is odd. Thus the mapping $\mathcal{C} : A \to B$ is $\mathcal{C}_3$ mapping, as desired. □

Corollary 3.2. *Let $\theta \geqslant 0$ and σ be a positive real number with $\sigma < 3$. Let $\mathfrak{g} : A \to B$ be an odd mapping satisfying $\mathfrak{g}(0) = 0$ and*

$$\|D_\mathfrak{g}(\upsilon, \nu)\| \leqslant \theta(\|\upsilon\|^\sigma + \|\nu\|^\sigma).$$

Then the limit

$$\mathcal{C}(\upsilon) = \lim_{n\to\infty} \frac{\mathfrak{g}(2^n\upsilon)}{8^n}$$

exists for all $\upsilon \in A$ and $\mathcal{C} : A \to B$ is a cubic mapping such that

$$\|\mathfrak{g}(\upsilon) - \mathcal{C}(\upsilon)\| \leqslant \frac{|2|^\sigma\theta}{|3||8|(|2|^\sigma - |2|^3)}\|\upsilon\|^\sigma$$

for all $\upsilon \in A$.

Proof. The proof follows from Theorem 3.1, if we get

$$\varphi(\upsilon, \nu) = \theta(\|x\|^\sigma + \|y\|^\sigma)$$

for all $\upsilon, \nu \in A$. In fact, if we choose $Ł = |2|^{3-\sigma}$, then we get the desired result. □

Theorem 3.3. *Assume that $\varphi : A \times A \to [0, \infty)$ is a function such that there is a constant $Ł < 1$ with*

$$\varphi\left(\frac{\upsilon}{2}, \frac{\nu}{2}\right) \leqslant \frac{Ł}{|2|^3}\varphi(\upsilon, \nu)$$

for all $\upsilon, \nu \in A$. Let $\mathfrak{g} : A \to B$ be an odd mapping satisfying (3.2). Then there is a unique $\mathcal{C}_3$ mapping $\mathcal{C} : A \to B$

such that

$$\|\mathfrak{g}(\upsilon) - \mathcal{C}(\upsilon)\| \leqslant \frac{1}{|3|} \frac{Ł}{|2|^3(1-Ł)} \varphi(0, \upsilon) \tag{3.6}$$

for all $\upsilon \in A$.

Proof. Letting $\upsilon = 0$ in (3.2), we get

$$\left\| \mathfrak{g}(\nu) - 2^3 \mathfrak{g}(\frac{\nu}{2}) \right\| \leqslant \frac{1}{|3|} \varphi\left(0, \frac{\nu}{2}\right) \tag{3.7}$$

for all $\nu \in A$. Define $\rho(f, g)$ as in Theorem 3.1. Consider the linear mapping $\Omega : \omega \to \omega$ such that

$$\Omega h(\upsilon) = 2^3 h\left(\frac{\upsilon}{2}\right)$$

for all $\upsilon \in A$. Let $f, h \in \omega$ be such that $\rho(f, h) = \varepsilon$. Then

$$\|f(\upsilon) - h(\upsilon)\| \leqslant \varepsilon\varphi(0, \upsilon)$$

for all $\upsilon \in A$ and so

$$\|\Omega f(\upsilon) - \Omega h(\upsilon)\| = \left\| 2^3 f\left(\frac{\upsilon}{2}\right) - 2^3 h\left(\frac{\upsilon}{2}\right) \right\| \leqslant |2|^3 \varepsilon\varphi\left(0, \frac{\upsilon}{2}\right) \leqslant |2|^3 \varepsilon \frac{Ł}{|2|^3} \varphi(0, \upsilon)$$

for all $\upsilon \in A$. Thus $\rho(\upsilon, \nu) = \varepsilon$ implies that $\rho(\Omega f, \Omega h) \leqslant \varepsilon$. This means that

$$\rho(\Omega f, \Omega h) \leqslant Ł\rho(f, h)$$

for all $f, h \in \omega$. If follows from (3.7) that

$$\rho(\mathfrak{g}, \Omega\mathfrak{g}) \leqslant \frac{1}{|3|} \frac{Ł}{|2|^3} < \infty.$$

So

$$\rho(\mathfrak{g}, \mathcal{C}) \leqslant \frac{1}{|3|} \frac{Ł}{|2|^3(1-Ł)}.$$

Therefore we obtain (3.6). This proof follows from the same pattern as that of Theorem 3.1. □

Corollary 3.4. *Let* $\theta \geqslant 0$ *and* σ *be a positive real number with* $\sigma > 4$. *Let* $\mathfrak{g} : A \to B$ *be an odd mapping satisfying* $\mathfrak{g}(0) = 0$ *and*

$$\|D_{\mathfrak{g}}(\upsilon, \nu)\| \leqslant \theta(\|\upsilon\|^\sigma + \|\nu\|^\sigma).$$

Then the limit

$$\mathcal{C}(\upsilon) = \lim_{n\to\infty} |8|^n \frac{\mathfrak{g}(\upsilon)}{2^n}$$

exists for all $\upsilon \in A$ *and* $\mathcal{C} : A \to B$ *is a* $\mathcal{C}_3$ *mapping such that*

$$\|\mathfrak{g}(\upsilon) - \mathcal{C}(\upsilon)\| \leqslant \frac{|2|^\sigma\theta}{|3||8|(|2|^3 - |2|^\sigma)} \|\upsilon\|^\sigma$$

for all $\upsilon \in A$.

Proof. The proof follows from Theorem 3.3, if we get

$$\varphi(\upsilon, \nu) = \theta(\|\upsilon\|^\sigma + \|\nu\|^\sigma)$$

for all $\upsilon, \nu \in A$. In fact, if we choose $Ł = |2|^{\sigma-3}$, then we get the desired result. □

Example 3.5. Let $p > 2$ be a prime number and $A = B = \mathbb{Q}_p$. Define $\mathfrak{g} : A \to B$ by $f(\upsilon) = \upsilon^3 + 1$ for all $\upsilon \in A$. Since $|2| = 1$,

$$\|D_{\mathfrak{g}}(\upsilon, \nu)\| = |27| \leqslant 1 \quad (\upsilon \in A).$$

Therefore, $\varphi(\upsilon, \nu) = |27|$, by equation (3.2) and the conditions of Theorem 3.3 are satisfied. In fact, for $2 = p$, we have

$$p^{3n}\mathfrak{g}(p^{-n}\upsilon) = \upsilon^3 + p^{3n}, \quad \forall \upsilon \in A,$$

and

$$\left\|p^{3n}\mathfrak{g}(p^{-n}\upsilon) - \upsilon^3\right\| = \left|p^{3n}\right| = \frac{1}{p^{3n}}, \quad \forall \upsilon \in A.$$

Therefore, $\lim_{n\to\infty} p^{3n}\mathfrak{g}(p^{-n}\upsilon) = \upsilon^3$, which implies that there exists a cubic ($\mathcal{C}_3$) mapping on A such that

$$0 = \|\mathfrak{g}(\upsilon) - \mathcal{C}(\upsilon)\| < \varphi(\upsilon), \quad \forall \upsilon \in A.$$

4. HU stability of the mixed type FE (1.3): an even case

In this part, we will prove the generalized HU stability of the mixed type $\mathcal{C}_3\mathcal{Q}_4$ FE $D\mathfrak{g}(\upsilon, \nu) = 0$ in NA normed spaces.

Theorem 4.1. *Assume that $\varphi : A \times A \to [0, \infty)$ is a function such that there is a constant $Ł < 1$ with*

$$\varphi(2\upsilon, 2\nu) \leqslant |2|^4 Ł \varphi(\upsilon, \nu) \tag{4.1}$$

for all $\upsilon, \nu \in A$. Let $\mathfrak{g} : A \to B$ be an even mapping satisfying

$$\|D_{\mathfrak{g}}(\upsilon, \nu)\| \leqslant \varphi(\upsilon, \nu) \tag{4.2}$$

for all $\upsilon, \nu \in A$. Then there is a unique $\mathcal{Q}_4$ mapping $\mathcal{Q} : A \to B$ such that

$$\|\mathfrak{g}(\upsilon) - \mathcal{Q}(\upsilon)\| \leqslant \frac{1}{|2|^4(1 - Ł)}\varphi(0, \upsilon) \tag{4.3}$$

for all $\upsilon \in A$.

Proof. Letting $\upsilon = 0$ in (4.2), we get

$$\left\|\mathfrak{g}(2\nu) - 2^4\mathfrak{g}(\nu)\right\| \leqslant \varphi(0, \nu)$$

for all $\nu \in A$. So

$$\left\|\frac{\mathfrak{g}(2\nu)}{2^4} - \mathfrak{g}(\nu)\right\| \leqslant \frac{1}{|2|^4}\varphi(0, \nu) \tag{4.4}$$

for all $\nu \in A$. Consider the set

$$\omega = \{h : A \to B\}$$

and also we define a generalized metric on ω as follows:

$$\rho(f, h) = \inf\left\{\Omega \in \mathbb{R}_+ : \|f(\upsilon) - h(\upsilon)\| \leqslant \Omega\varphi(0, \upsilon), \forall \upsilon \in A\right\},$$

where, as usual, $\inf \varphi = +\infty$. It is simple to see that (ω, ρ) is complete. Now, we take the linear mapping $\Omega : \omega \to \omega$ such that

$$\Omega h(\upsilon) := \frac{1}{2^4}h(2\upsilon)$$

for all $\upsilon \in A$. Let $f, h \in \omega$ be given such that $\rho(f, h) = \varepsilon$. Then

$$\|g(\upsilon) - h(\upsilon)\| \leqslant \varepsilon\varphi(0, \upsilon)$$

for all $\upsilon \in A$. Hence

$$\|\Omega f(\upsilon) - \Omega h(\upsilon)\| = \left\|\frac{1}{2^4} f(2\upsilon) - \frac{1}{2^4} h(2\upsilon)\right\| \leqslant \frac{1}{|2|^4}\varepsilon\varphi(0, 2\upsilon) \leqslant \varepsilon Ł\varphi(0, \upsilon)$$

for all $\upsilon \in A$. So $\rho(f, h) \leqslant \varepsilon$, which implies that $\rho(\Omega f, \Omega h) \leqslant Ł\varepsilon$. This means that

$$\rho(\Omega f, \Omega h) \leqslant Ł\rho(f, h)$$

for all $f, h \in \omega$. It follows from (4.4) that

$$\rho(\mathfrak{g}, \Omega\mathfrak{g}) \leqslant \frac{1}{|2|^4} < +\infty.$$

By Theorem 2.6, there exists a mapping $\mathcal{Q} : A \to B$ satisfying the following conditions.

(i) $\mathcal{Q}$ is a fixed point of Ω, that is,

$$\mathcal{Q}(2\upsilon) = 2^4\mathcal{Q}(\upsilon) \tag{4.5}$$

for all $\upsilon \in A$. The mapping $\mathcal{Q}$ is a unique fixed point of Ω in the set

$$S = \{g \in \omega : \rho(g, h) < \infty\}.$$

This implies that $\mathcal{Q}$ is a unique mapping satisfying (4.5) such that there is an $\Omega \in (0, \infty)$ satisfying

$$\|\mathfrak{g}(\upsilon) - \mathcal{Q}(\upsilon)\| \leqslant \Omega\varphi(0, \upsilon)$$

for all $\upsilon \in A$.

(ii) $\rho(\Omega^n\mathfrak{g}, \mathcal{Q}) \to 0$ as $n \to \infty$. This gives that, for all $\upsilon \in A$,

$$\lim_{n\to\infty} (\Omega^n\mathfrak{g})(\upsilon) = \lim_{n\to\infty} \frac{\mathfrak{g}(2^n\upsilon)}{|2|^{4n}} = \mathcal{Q}(\upsilon).$$

(iii) $\rho(\mathfrak{g}, \mathcal{Q}) \leqslant \frac{1}{1 - Ł}\rho(\mathfrak{g}, \Omega^n\mathfrak{g})$, which gives the inequality

$$\rho(\mathfrak{g}, \mathcal{Q}) \leqslant \frac{1}{1 - Ł}\rho(\mathfrak{g}, \Omega\mathfrak{g}) \leqslant \frac{1}{|2|^4(1 - Ł)}.$$

This indicates that the inequality (4.3) holds.

As a result of (4.1) and (4.2),

$$\|D\mathcal{Q}(\upsilon, \nu)\| = \lim_{n\to\infty} \|2^{-4n} D\mathfrak{g}(2^n\upsilon, 2^n\nu)\| \leqslant \lim_{n\to\infty} \frac{1}{|2|^{4n}}\varphi(2^n\upsilon, 2^n\nu) \leqslant \lim_{n\to\infty} Ł^n\varphi(\upsilon, \nu) = 0$$

for all $\upsilon, \nu \in A$ and $n \in \mathbb{N}$. So, $\|D\mathcal{Q}(\upsilon, \nu)\| = 0$. Since $\mathfrak{g}$ is even, $\mathcal{Q}$ is even. Thus the mapping $\mathcal{Q} : A \to B$ is $\mathcal{Q}_4$ mapping as desired. □

Corollary 4.2. *Let $\theta \geqslant 0$ and σ be a positive real number with $\sigma < 4$. Let $\mathfrak{g} : A \to B$ be an even mapping satisfying $\mathfrak{g}(0) = 0$ and*

$$\|D_{\mathfrak{g}}(\upsilon, \nu)\| \leqslant \theta(\|\upsilon\|^{\sigma} + \|\nu\|^{\sigma}),$$

then the limit

$$\mathcal{Q}(\upsilon) = \lim_{n\to\infty} \frac{\mathfrak{g}(2^n\upsilon)}{16^n}$$

exists for all $\upsilon \in A$ and $\mathcal{Q} : A \to B$ is a $\mathcal{Q}_4$ mapping such that

$$\|\mathfrak{g}(\upsilon) - \mathcal{Q}(\upsilon)\| \leqslant \frac{|2|^{\sigma}\theta}{|16|(|2|^{\sigma} - |2|^{4})}\|\upsilon\|^{\sigma}$$

for all $\upsilon \in A$.

Proof. The proof follows from Theorem 4.1, if we get

$$\varphi(\upsilon, \nu) = \theta(\|\upsilon\|^{\sigma} + \|\nu\|^{\sigma})$$

for all $\upsilon, \nu \in A$. In fact, if we choose $Ł = |2|^{4-\sigma}$, then we get the desired result. □

Theorem 4.3. *Assume that $\varphi : A \times A \to [0, \infty)$ is a function such that there is a constant $Ł < 1$ with*

$$\varphi\left(\frac{\upsilon}{2}, \frac{\nu}{2}\right) \leqslant \frac{Ł}{|2|^4}\varphi(\upsilon, \nu)$$

for all $\upsilon, \nu \in A$. Let $\mathfrak{g} : A \to B$ be an even mapping satisfying (3.2). Then there is a unique $\mathcal{Q}_4$ mapping $\mathcal{Q} : A \to B$ such that

$$\|\mathfrak{g}(\upsilon) - \mathcal{Q}(\upsilon)\| \leqslant \frac{Ł}{|2|^4(1 - Ł)}\varphi(0, \upsilon) \tag{4.6}$$

for all $\upsilon \in A$.

Proof. Letting $\upsilon = 0$ in (3.2), we get

$$\left\|\mathfrak{g}(\nu) - 2^4\mathfrak{g}\left(\frac{y}{2}\right)\right\| \leqslant \varphi\left(0, \frac{\nu}{2}\right) \tag{4.7}$$

for all $\nu \in A$. Define $\rho(f, g)$ as in Theorem 3.1. Consider the linear mapping $\Omega : \omega \to \omega$ such that

$$\Omega h(\upsilon) = 2^4 h\left(\frac{\upsilon}{2}\right)$$

for all $\upsilon \in A$. Let $f, h \in \omega$ be such that $\rho(f, h) = \varepsilon$. Then

$$\|f(\upsilon) - h(\upsilon)\| \leqslant \varepsilon\varphi(0, \upsilon)$$

for all $\upsilon \in A$ and so

$$\|\Omega f(\upsilon) - \Omega h(\upsilon)\| = \left\|2^4 f\left(\frac{\upsilon}{2}\right) - 2^4 h\left(\frac{\upsilon}{2}\right)\right\| \leqslant |2|^4\varepsilon\varphi\left(0, \frac{\upsilon}{2}\right) \leqslant |2|^4\varepsilon\frac{Ł}{|2|^4}\varphi(0, \upsilon)$$

for all $\upsilon \in A$. Thus $\rho(\upsilon, \nu) = \varepsilon$ implies that $\rho(\Omega f, \Omega h) \leqslant \varepsilon$. This means that

$$\rho(\Omega f, \Omega h) \leqslant Ł\rho(f, h)$$

for all $f, h \in \omega$. If follows from (4.7) that

$$\rho(\mathfrak{g}, \Omega\mathfrak{g}) \leqslant \frac{Ł}{|2|^4} < \infty.$$

So

$$\rho(\mathfrak{g}, \mathcal{Q}) \leqslant \frac{Ł}{|2|^4(1 - Ł)}.$$

Therefore we obtain (4.6). This proof follows same pattern as that of Theorem 4.1. □

Corollary 4.4. *Let $\theta \geqslant 0$ and r be a positive real number with $\sigma > 4$. Let $\mathfrak{g} : A \to B$ be an even mapping satisfying $\mathfrak{g}(0) = 0$ and*

$$\|D_{\mathfrak{g}}(\upsilon, \nu)\| \leqslant \theta(\|\upsilon\|^{\sigma} + \|\nu\|^{\sigma}),$$

then the limit

$$\mathcal{Q}(\upsilon) = \lim_{n\to\infty} |16|^n \frac{\mathfrak{g}(\upsilon)}{2^n}$$

exists for all $\upsilon \in A$ and $\mathcal{Q} : A \to B$ is a $\mathcal{Q}_4$ mapping such that

$$\|\mathfrak{g}(\upsilon) - \mathcal{Q}(\upsilon)\| \leqslant \frac{|2|^{\sigma}\theta}{|16|(|2|^4 - |2|^{\sigma})}\|\upsilon\|^{\sigma}$$

for all $\upsilon \in A$.

Proof. The proof follows from Theorem 4.3, if we get

$$\varphi(\upsilon, \nu) = \theta(\|\upsilon\|^{\sigma} + \|\nu\|^{\sigma})$$

for all $\upsilon, \nu \in A$. In fact, if we choose $Ł = |2|^{\sigma-4}$, then we get the desired result. □

Example 4.5. Let $p > 2$ be a prime number and $A = B = \mathbb{Q}_p$. Define $\mathfrak{g} : A \to B$ by $\mathfrak{g}(\upsilon) = \upsilon^4 + 1$ for all $\upsilon \in A$. Since $|2| = 1$,

$$\|D_{\mathfrak{g}}(\upsilon, \nu)\| = |27| \leqslant 1 \quad (\upsilon \in A).$$

Therefore, $\varphi(\upsilon, \nu) = |27|$, by equation (3.2) and the conditions of Theorem 4.3 are satisfied. In fact, for $2 = p$, we have

$$p^{4n}\mathfrak{g}(p^{-n}\upsilon) = \upsilon^4 + p^{4n}, \quad \forall \upsilon \in A,$$

and

$$\left\|p^{4n}\mathfrak{g}(p^{-n}\upsilon) - \upsilon^4\right\| = \left|p^{4n}\right| = \frac{1}{p^{4n}}, \quad \forall \upsilon \in A.$$

Therefore, $\lim_{n\to\infty} p^{4n}\mathfrak{g}(p^{-n}\upsilon) = \upsilon^4$, which implies that there exists a quartic ($\mathcal{Q}_4$) mapping on A such that

$$0 = \|\mathfrak{g}(\upsilon) - \mathcal{Q}(\upsilon)\| < \varphi(\upsilon), \quad \forall \upsilon \in A.$$

5. HU stability of the mixed type FE (1.3)

In this part, we will establish the generalized HU stability of the mixed type $\mathcal{C}_3\mathcal{Q}_4$ FE (1.3) in NA normed spaces. For a given function $\mathfrak{g} : A \to B$, let

$$\mathfrak{g}_o(\upsilon) = \frac{\mathfrak{g}(\upsilon) - \mathfrak{g}(-\upsilon)}{2} \quad \text{and} \quad \mathfrak{g}_e(\upsilon) = \frac{\mathfrak{g}(\upsilon) + \mathfrak{g}(-\upsilon)}{2}.$$

Then $\mathfrak{g}_o$ is odd and $\mathfrak{g}_e$ is even.

Theorem 5.1. *Let $\varphi : A \times A \to [0, \infty)$ is a function such that there is an $Ł < 1$ with*

$$\varphi(2\upsilon, 2\nu) \leqslant |2|^4 Ł\varphi(\upsilon.\nu) \tag{5.1}$$

for all $\upsilon, \nu \in A$. Suppose $\mathfrak{g} : A \to B$ is a mapping satisfying

$$\|D_{\mathfrak{g}}(\upsilon, \nu)\| \leqslant \varphi(\upsilon, \nu) \tag{5.2}$$

for all $\upsilon, \nu \in A$. Then there are a unique $\mathcal{C}_3$ mapping $\mathcal{C} : A \to B$ and a unique $\mathcal{Q}_4$ mapping $\mathcal{Q} : A \to B$ such that

$$\|\mathfrak{g}(\upsilon) - \mathcal{C}(\upsilon) - \mathcal{Q}(\upsilon)\| \leqslant \frac{1}{|2|^5(1 - Ł)} \max\{\varphi(0, \upsilon), \varphi(0, -\upsilon)\}$$

for all $\upsilon \in A$.

Proof. Assume that $\mathfrak{g}(\upsilon) = \mathfrak{g}_e(\upsilon) + \mathfrak{g}_o(\upsilon)$. Let

$$\varphi(\upsilon,\nu) = \frac{1}{|2|}\max\{\varphi(\upsilon,\nu), \varphi(-\upsilon,-\nu)\}.$$

Then by (5.1) and (5.2), we have

$$\varphi(2\upsilon,2\nu) \leqslant |2|^4 Ł\varphi(\upsilon,\nu) \leqslant |2|^3 Ł\varphi(\upsilon,\nu), \qquad \|D\mathfrak{g}_o(\upsilon,\nu)\| \leqslant \varphi(\upsilon,\nu), \qquad \|D\mathfrak{g}_e(\upsilon,\nu)\| \leqslant \varphi(\upsilon,\nu).$$

Hence by Theorems 3.1 and 4.1, there are a unique $\mathcal{C}_3$ mapping $\mathcal{C} : A \to B$ and a unique $\mathcal{Q}_4$ mapping $\mathcal{Q} : A \to B$ such that

$$\|\mathfrak{g}_o(\upsilon) - \mathcal{C}(\upsilon)\| \leqslant \frac{1}{|3|}\frac{1}{|2|^3(1-Ł)} \quad \text{and} \quad \|\mathfrak{g}_e(\upsilon) - \mathcal{Q}(\upsilon)\| \leqslant \frac{1}{|2|^4(1-Ł)}.$$

Therefore

$$\begin{aligned}\|\mathfrak{g}(\upsilon) - \mathcal{C}(\upsilon) - \mathcal{Q}(\upsilon)\| &= \|\mathfrak{g}_o(\upsilon) + \mathfrak{g}_e(\upsilon) - \mathcal{C}(\upsilon) - \mathcal{Q}(\upsilon)\| \\ &\leqslant \max\{\|\mathfrak{g}_o(\upsilon) - \mathcal{C}(\upsilon)\|, \|\mathfrak{g}_e(\upsilon) - \mathcal{Q}(\upsilon)\|\} \\ &\leqslant \max\left\{\frac{\varphi(0,\upsilon)}{|3|\,|2|^3(1-Ł)}, \frac{\varphi(0,\upsilon)}{|2|^4(1-Ł)}\right\} \leqslant \frac{1}{|2|^5(1-Ł)}\max\{\varphi(0,\upsilon), \varphi(0,-\upsilon)\}\end{aligned}$$

for all $\upsilon \in A$. □

Corollary 5.2. *Let $\theta \geqslant 0$ and σ be a positive real number with $\sigma < 4$. Let $\mathfrak{g} : A \to B$ be a mapping satisfying $\mathfrak{g}(0) = 0$ and*

$$\|D_{\mathfrak{g}}(\upsilon,\nu)\| \leqslant \theta\left(\|\upsilon\|^\sigma + \|\nu\|^\sigma\right).$$

Then there are a unique $\mathcal{C}_3$ mapping $\mathcal{C} : A \to B$ and a unique Q_4 mapping $Q : A \to B$ such that

$$\|\mathfrak{g}(\upsilon) - \mathcal{C}(\upsilon) - Q(\upsilon)\| \leqslant \frac{\theta}{|32|(|2|^\sigma - |2|^4)}\|\upsilon\|^\sigma$$

for all $\upsilon \in A$.

Proof. The proof follows from Theorem 5.1, if we get

$$\varphi(\upsilon,\nu) = \theta(\|\upsilon\|^\sigma + \|\nu\|^\sigma)$$

for all $\upsilon,\nu \in A$. In fact, if we choose $Ł = |2|^{4-\sigma}$, then we get the desired result. □

Theorem 5.3. *Let $\varphi : A \times A \to [0,\infty)$ be a function such that there is an $Ł < 1$ with*

$$\varphi\left(\frac{\upsilon}{2}, \frac{\nu}{2}\right) \leqslant \frac{Ł}{|2|^4}\varphi(\upsilon.\nu)$$

for all $\upsilon,\nu \in A$. Suppose $\mathfrak{g} : A \to B$ is a mapping satisfying

$$\|D_{\mathfrak{g}}(\upsilon,\nu)\| \leqslant \varphi(\upsilon,\nu) \tag{5.3}$$

for all $\upsilon,\nu \in A$. Then there are a unique $\mathcal{C}_3$ mapping $\mathcal{C} : A \to B$ and a unique $\mathcal{Q}_4$ mapping $\mathcal{Q} : A \to B$ such that

$$\|\mathfrak{g}(\upsilon) - \mathcal{C}(\upsilon) - \mathcal{Q}(\upsilon)\| \leqslant \frac{Ł}{|2|^5(1-Ł)}.\max\{\varphi(0,\upsilon), \varphi(0,-\upsilon)\}$$

for all $\upsilon \in A$.

Proof. Consider $\mathfrak{g}(\upsilon) = \mathfrak{g}_e(\upsilon) + \mathfrak{g}_o(\upsilon)$. Let

$$\varphi(\upsilon, \nu) = \frac{1}{|2|} \max\{\varphi(\upsilon, \nu), \varphi(-\upsilon, -\nu)\}.$$

Then by (5.1) and (5.2), we have

$$\varphi\left(\frac{\upsilon}{2}, \frac{\nu}{2}\right) \leqslant \frac{Ł}{|2|^4}\varphi(\upsilon, \nu) \leqslant \frac{Ł}{|2|^3}\varphi(\upsilon, \nu), \quad \|D\mathfrak{g}_o(\upsilon, \nu)\| \leqslant \varphi(\upsilon, \nu), \quad \|D\mathfrak{g}_e(\upsilon, \nu)\| \leqslant \varphi(\upsilon, \nu).$$

Hence by Theorems 3.1 and 4.1, there are a unique $\mathcal{C}_3$ mappping $\mathcal{C} : A \to B$ and a unique $\mathcal{Q}_4$ mapping $\mathcal{Q} : A \to B$ such that

$$\|\mathfrak{g}_o(\upsilon) - \mathcal{C}(\upsilon)\| \leqslant \frac{1}{|3|}\frac{Ł}{|2|^3(1-Ł)} \quad \text{and} \quad \|\mathfrak{g}_e(\upsilon) - \mathcal{Q}(\upsilon)\| \leqslant \frac{Ł}{|2|^4(1-Ł)}.$$

Therefore

$$\begin{aligned}
\|\mathfrak{g}(\upsilon) - \mathcal{C}(\upsilon) - \mathcal{Q}(\upsilon)\| &= \|\mathfrak{g}_o(\upsilon) + \mathfrak{g}_e(\upsilon) - \mathcal{C}(\upsilon) - \mathcal{Q}(\upsilon)\| \\
&\leqslant \max\{\|\mathfrak{g}_o(\upsilon) - \mathcal{C}(\upsilon)\|, \|\mathfrak{g}_e(\upsilon) - \mathcal{Q}(\upsilon)\|\} \\
&\leqslant \max\left\{\frac{Ł\varphi(0,\upsilon)}{|3||2|^3(1-Ł)}, \frac{Ł\varphi(0,\upsilon)}{|2|^4(1-Ł)}\right\} \leqslant \frac{Ł}{|2|^5(1-Ł)} \max\{\varphi(0,\upsilon), \varphi(0,-\upsilon)\}
\end{aligned}$$

for ever $\upsilon \in A$. □

Corollary 5.4. *Let $\theta \geqslant 0$ and σ be a positive real number with $\sigma > 4$. Let $\mathfrak{g} : A \to B$ be a mapping satisfying $\mathfrak{g}(0) = 0$ and*

$$\|D_{\mathfrak{g}}(\upsilon, \nu)\| \leqslant \theta(\|\upsilon\|^\sigma + \|\nu\|^\sigma).$$

Then there are a unique $\mathcal{C}_3$ mapping $\mathcal{C} : A \to B$ and a unique $\mathcal{Q}_4$ mapping $\mathcal{Q} : A \to B$ such that

$$\|\mathfrak{g}(\upsilon) - \mathcal{C}(\upsilon) - \mathcal{Q}(\upsilon)\| \leqslant \frac{|2|^\sigma.\theta}{|32|(|2|^4 - |2|^\sigma)}\|\upsilon\|^\sigma,$$

for all $\upsilon \in A$.

Proof. The proof follows from Theorem 5.3, if we get

$$\varphi(\upsilon, \nu) = \theta(\|\upsilon\|^\sigma + \|\nu\|^\sigma)$$

for all $\upsilon, \nu \in A$. In fact, if we choose $Ł = |2|^{\sigma-4}$, then we get the desired result. □

Example 5.5. Let $p > 2$ be a prime number and $A = B = \mathbb{Q}_p$. Define $\mathfrak{g} : A \to B$ by $\mathfrak{g}(\upsilon) = \upsilon^3 + \upsilon^4 + 1$ for all $\upsilon \in A$. Since $|2| = 1$,

$$\|D_{\mathfrak{g}}(\upsilon, \nu)\| = |29| \leqslant 1 \quad (\upsilon \in A).$$

Therefore, $\varphi(\upsilon, \nu) = |29|$, by equation (5.3) and the conditions of Theorem 5.3 are satisfied. In fact, for $2 = p$, we have

$$p^{4n}\mathfrak{g}(p^{-n}\upsilon) = \upsilon^3 + \upsilon^4 + p^{4n}, \quad \forall \upsilon \in A,$$

and

$$\left\|p^{4n}\mathfrak{g}(p^{-n}\upsilon) - \upsilon^3 + \upsilon^4\right\| = \left|p^{4n}\right| = \frac{1}{p^{4n}}, \quad \forall \upsilon \in A.$$

Therefore, $\lim_{n\to\infty} p^{4n}\mathfrak{g}(p^{-n}\upsilon) = \upsilon^3 + \upsilon^4$, which implies that there exist a cubic ($\mathcal{C}_3$) mapping and a quartic ($\mathcal{Q}_4$) mapping on A such that

$$0 = \|\mathfrak{g}(\upsilon) - \mathcal{C}(\upsilon) - \mathcal{Q}(\upsilon)\| < \varphi(\upsilon), \quad \forall \upsilon \in A.$$

6. Conclusion

In this article, we established the sufficient criteria for HUR stability of a mixed type $\mathcal{C}_3\mathcal{Q}_4$ FE in NA normed spaces with the help of Diaz and Margolis fixed point theorem.

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