J. Math. Computer Sci., 33 (2024), 87–107

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

A compact high resolution semi-variable mesh exponential finite difference method for non-linear boundary value problems of elliptic nature

Geetan Manchanda

Department of Mathematics, Maitreyi College, University of Delhi, Delhi 110021, India.

Abstract

In this research an original exponential approximation of second accuracy in y- and third accuracy in x-axis employing full step discretization has been designed for solving 2D non-linear partial differential equation of elliptic nature in a rectangular domain. We adopted non-constant grid spacing in x-axis and constant grid spacing in y-axis in numerical computation of convection-diffusion equation where convection term dominates. An exhaustive error behaviour of the technique has been analysed. Non-linear elliptic equations are computed using this method. Lastly, proposed idea is scrutinized on simulations of physical repute with emphasis on convection-diffusion equation articulating the efficacy of the technique.

Keywords: Quasilinear elliptic equations, full-step, exponential estimation, error estimate, Burger's equation, convection-diffusion equation, Navier-Stokes equations, multi-harmonic.

2020 MSC: 65M06, 65M12, 65M22, 65N06, 65N12.

1. Introduction

Consider two space nonlinear elliptic boundary value problem (EBVP) as per boundary conditions of Dirichlet type:

$$\phi_{xx} + \phi_{yy} = \rho(x, y, \phi, \phi_x, \phi_y), \quad (x, y) \in \Delta, \tag{1.1}$$

where $\Delta = (0, 1) \times (0, 1)$ and $\partial\Delta$ denotes the boundary of Δ,

$$\phi(x, y) = \phi_0(x, y), (x, y) \in \partial\Delta. \tag{1.2}$$

Suppose for $(x, y) \in (0, 1)$ the following holds:

(i) $\phi(x, y) \in C^6(\Delta)$;

(ii) ρ is continuous;

(iii) $\frac{\partial\rho}{\partial\phi}$, $\left|\frac{\partial\rho}{\partial\phi_x}\right|$, and $\left|\frac{\partial\rho}{\partial\phi_y}\right|$ exist and are continuous;

Email address: gmanchanda@maitreyi.du.ac.in (Geetan Manchanda)

doi: 10.22436/jmcs.033.01.07

Received: 2022-09-23 Revised: 2023-08-02 Accepted: 2023-10-18

(iv) $\frac{\partial \rho}{\partial \phi} \geqslant 0$, $\left|\frac{\partial \rho}{\partial \phi_x}\right| \leqslant R$, $\left|\frac{\partial \rho}{\partial \phi_y}\right| \leqslant S$,

where R and S are > 0 and $C^p(\Delta)$, consists of the family of functions which are continuously differentiable up to p^{th} order.

These conditions profess the solution of (1.1)-(1.2) exists and is unique [5].

The integral of the partial differential equations (PDEs) and associated boundary and initial assumptions are an essential part in modeling various phenomena in the fields of sciences (fluid dynamics, heat flow etc) as well as economics [1, 3, 16, 20, 21]. But very few PDEs possess an analytical solution. Whoever wants to fabricate models based on such equations and their associated conditions, must find out numerical solutions precisely. It is an onerous task to find exact analytical solutions so it becomes imperative to enforce numerical methods to compute approximate solutions of these PDEs, so as to explore the prognosis of the mathematical models. There are several methods based on finite elements or on boundary elements or finite volume method to find numerical solution of the PDEs. However, we found it suitable to use "finite difference methods" due to its candor and the ease with which it is applied [6, 15, 17].

The convection-diffusion mathematical equation is a combination of the diffusion and convection processes and depicts physical phenomena where particles, or other physical units such as mass, momentum, heat and energy are exchanged inside a physical framework. Diffusion and convection always occur simultaneous in nature. The approximate integral of convection-diffusion transport problems emerges in various applications of advanced sciences, for example: physical models of semiconductors, flow of air pollutants, flow of oil in reservoirs etc. Therefore, it becomes quite essential to find a viable, stable, and practical numerical method to compute the convection-diffusion equation [7].

Another significant equation computed in this paper is Poisson's equation which has an enormous use in scientific problems related to heat conduction in steady state, ground water flow, dynamics of stretched loaded membranes, and in the study of the theory of prismatic elastic bodies [23, 24].

Tian et al. [18] discussed compact computational FDM techniques for computing convection-diffusion models in steady state. Dehghan et al. [2] too contributed in this field. Jain et al. [4, 5] developed techniques for equations, elliptic in nature. Later, Jain et al. applied techniques to family of PDEs with coefficients of variable nature. But these techniques needed modification at nodes of singularity. To resolve this problem, Mohanty and Singh [14] did lot of substantial research. All these schemes [4, 8, 9] had uniform mesh sizes and needed 5 functional valuations. Using classical second order central difference scheme it is not possible to find the exact solution of most of the equations, in order to get the more appropriate numerical solution, we need to increase the number of mesh points and reduce the size, as a consequence it results in covering extra storage space and thereby escalation in computing time. To get more accurate results we need to devise a compact finite difference method of higher accuracy. Mohanty et al. designed an exponential scheme of high accuracy using geometric mesh, employing off and full step discretization [12, 13].

Earlier developed methods by Mohanty et al. lacked proficiency in computing convection diffusion equation for slight values of perturbation factor, therefore, we have made an attempt to overcome that deficiency. Subsequently, the proposed method has an edge over the previous one to deal in a better way, the convection diffusion equation with slighter values of perturbation factor. Noting that, albeit the method developed before was robust it required more algebra and hence more computational cost than the proposed method [10, 11]. The main strength of this article lies in computing convection-diffusion for big values of coefficient of convection term, i.e., slight values of perturbation factor exhibiting boundary layer phenomenon, and that drove us to work with semi-variable mesh.

We construct a method of high resolution to compute nonlinear elliptic PDE employing uniform mesh spacing p in y- direction and variable mesh spacing h_b in x- direction and nine mesh nodes of only one computational cell exploiting full step discretization (see Figure 1).

The article is structured as follows. Section 2 formulates compact scheme of order $O(p^2 + p^2 h_b + h_b^3)$. Section 3 gives descriptive numerical recipe of the scheme.

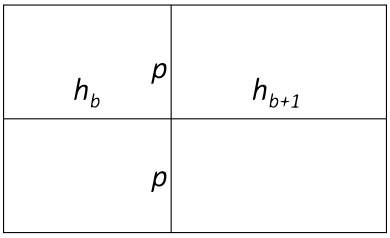

Figure 1: Compact nine point cell.

Furthermore, Section 4 demonstrates convergence of the method for a constant mesh. Thereafter, method to solve system of nonlinear elliptic PDEs has been explained in Section 5. Section 6 contains numerical methods to solve bi-harmonic and tri-harmonic equations. Section 7 illustrates physically relevant numerical simulations to test the effectiveness of the proposed method. Lastly, Section 8 reports the conclusions and discussions.

2. Exponential fitting

To convert (1.1)-(1.2) into a system of equations, superimposing on Δ, a rectangular grid of variable grid spacing $h_b > 0$ in x-axis and constant grid spacing $p > 0$ in y-axis directions. Notating grid node by $(x_b, y_d), 0 = x_0 < x_1 < x_2 < \cdots < x_{M+1} = 1, h_{b+1} = x_{b+1} - x_b$ for $b = 0(1)M$, grid ratio $\zeta = (\frac{h_{b+1}}{h_b}) > 0$ for $b = 1(1)N_0$ and $y_d = dp$, for $d = 0(1)(N_0 + 1)$, where M and N are natural number so that $(N_0 + 1)p = 1$. For $\zeta > 1$ (or $\zeta < 1$) the mesh spacing are rising (or falling) as we proceed in the domain. For $\zeta = 1$ means, $h_{b+1} = h_b = h$, the mesh sizes are same.

Denote by $\Phi_{b,d}$ to be the analytical and $\phi_{b,d}$ to be the numerical value of $\phi(x_b, y_d)$, respectively. Further, at the mesh point (x_b, y_d), denote

$$P = \zeta^2 + \zeta - 1, Q = (1 + \zeta)(1 + 3\zeta + \zeta^2), R = \zeta(1 + \zeta - \zeta^2).$$

Suppose

(i) $\rho_{b,d} = \rho(x_b, y_d, \Phi_{b,d}, \Phi_{x_{b,d}}, \Phi_{y_{b,d}})$;
(ii) $\rho_{b,d} = \rho(x_b, y_d, \phi_{b,d}, \phi_{x_{b,d}}, \phi_{y_{b,d}})$,

At each mesh point (x_b, y_d), (1.1) is written as

$$\Phi_{xx_{b,d}} + \Phi_{yy_{b,d}} = \rho_{b,d}.$$

Let the central operator be $\delta_x \Phi_l = \left(\Phi_{l+\frac{1}{2}} - \Phi_{l-\frac{1}{2}}\right)$ and the average operator be $\mu_x \Phi_l = \frac{1}{2}\left(\Phi_{l+\frac{1}{2}} + \Phi_{l-\frac{1}{2}}\right)$ in x-direction etc. The approximations after discretizing the PDE (1.1) are defined as below:

$$\overline{\Phi}_{x_{b,d}} = \frac{\Phi_{b+1,d} - (1-\zeta^2)\Phi_{b,d} - \zeta^2\Phi_{b-1,d}}{h_b\zeta(1+\zeta)}, \tag{2.1a}$$

$$\overline{\Phi}_{x_{b+1},d} = \frac{1}{\zeta(1+\zeta)h_b}((1+2\zeta)\Phi_{b+1,d} - (1+\zeta)^2\Phi_{b,d} + \zeta^2\Phi_{b-1,d}), \tag{2.1b}$$

$$\overline{\Phi}_{x_{b-1},d} = \frac{1}{\zeta(1+\zeta)h_b}(-\Phi_{b+1,d} + (1+\zeta)^2\Phi_{b,d} - \zeta(2+\zeta)\Phi_{b-1,d}), \tag{2.1c}$$

$$\overline{\Phi}_{y_{b,d}} = \frac{1}{2p}(\Phi_{b,d+1} - \Phi_{b,d-1}), \tag{2.2a}$$

$$\overline{\Phi}_{y_{b+1,d}} = \frac{1}{2p}(\Phi_{b+1,d+1} - \Phi_{b+1,d-1}), \tag{2.2b}$$

$$\overline{\Phi}_{y_{b-1,d}} = \frac{1}{2p}(\Phi_{b-1,d+1} - \Phi_{b-1,d-1}), \tag{2.2c}$$

$$\overline{\Phi}_{yy_{b+1,d}} = \frac{1}{p^2}(\Phi_{b+1,d+1} - 2\Phi_{b+1,d} + \Phi_{b+1,d-1}), \tag{2.3a}$$
$$\overline{\Phi}_{yy_{b-1,d}} = \frac{1}{p^2}(\Phi_{b-1,d+1} - 2\Phi_{b-1,d} + \Phi_{b-1,d-1}). \tag{2.3b}$$

Define

$$\overline{\rho}_{b+1,d} = \rho(x_{b+1}, y_d, \Phi_{b+1,d}, \overline{\Phi}_{x_{b+1,d}}, \overline{\Phi}_{y_{b+1,d}}), \quad \overline{\rho}_{b-1,d} = \rho(x_{b-1}, y_d, \Phi_{b-1,d}, \overline{\Phi}_{x_{b-1,d}}, \overline{\Phi}_{y_{b-1,d}}).$$

Next, we employ following approximations

$$\check{\Phi}_{x_{b,d}} = \overline{\Phi}_{x_{b,d}} - \frac{h}{12}(\overline{\rho}_{b+1,d} - \overline{\rho}_{b-1,d}) + \frac{h}{12}(\overline{\Phi}_{yy_{b+1,d}} - \overline{\Phi}_{yy_{b-1,d}}),$$
$$\check{\check{\Phi}}_{x_{b,d}} = \overline{\Phi}_{x_{b,d}} - \frac{h}{4}(\overline{\rho}_{b+1,d} - \overline{\rho}_{b-1,d}) + \frac{h}{4}(\overline{\Phi}_{yy_{b+1,d}} - \overline{\Phi}_{yy_{b-1,d}}),$$

and let

$$\check{\rho}_{b,d} = \rho(x_b, y_d, \Phi_{b,d}, \check{\Phi}_{x_{b,d}}, \overline{\Phi}_{y_{b,d}}), \quad \check{\check{\rho}}_{b,d} = \rho(x_b, y_d, \Phi_{b,d}, \check{\check{\Phi}}_{x_{b,d}}, \overline{\Phi}_{y_{b,d}}).$$

Then at (x_b, y_d), (1.1) is discretized by technique written below:

$$\begin{aligned} &L_1(\Phi_{b+1,d+1} + \Phi_{b+1,d-1}) + L_2(\Phi_{b-1,d+1} + \Phi_{b-1,d-1}) + L_3(\Phi_{b,d+1} + \Phi_{b,d-1}) \\ &\quad + L_4(\Phi_{b+1,d}) + L_5(\Phi_{b-1,d}) + L_6\Phi_{b,d} \\ &= \zeta(1+\zeta)\frac{h^2}{2}\check{\rho}_{b,d} \exp\left(\frac{P\overline{\rho}_{b+1,d} + R\overline{\rho}_{b-1,d} - (P+R)\check{\check{\rho}}_{b,d}}{6\zeta(1+\zeta)\check{\rho}_{b,d}}\right) + I_{b,d}, \end{aligned} \tag{2.4}$$

where $I_{b,d} = O(p^2h_b^2 + p^2h_b^3 + h_b^5)$, $[b = 1(1)M, d = 1(1)N]$,

$$L_1 = P\frac{h_b^2}{12p^2}, \qquad L_2 = R\frac{h_b^2}{12p^2}, \qquad L_3 = Q\frac{h_b^2}{12p^2},$$
$$L_4 = 1 - 2P\frac{h_b^2}{12p^2}, \qquad L_5 = \zeta - 2R\frac{h_b^2}{12p^2}, \qquad L_6 = -(1+\zeta) - 2Q\frac{h_b^2}{12p^2}.$$

3. Formulation of technique

To deduce the technique (2.4), we use the notations where (b, d) symbolizes at node (x_b, y_d),

$$\kappa_{b,d} = \left(\frac{\partial \rho}{\partial \Phi_x}\right)_{b,d}, \qquad \delta_{b,d} = \left(\frac{\partial \rho}{\partial \Phi_y}\right)_{b,d}.$$

We simplify the approximations and get

$$\overline{\Phi}_{x_{b,d}} = \Phi_{x_{b,d}} + \frac{\zeta h_b^2}{6}\Phi_{xxx_{b,d}} + O(h_b^3), \tag{3.1a}$$
$$\overline{\Phi}_{x_{b+1,d}} = \Phi_{x_{b+1,d}} - \frac{\zeta(1+\zeta)}{6}h_b^2\Phi_{xxx_{b,d}} + O(h_b^3), \tag{3.1b}$$
$$\overline{\Phi}_{x_{b-1,d}} = \Phi_{x_{b-1,d}} - \frac{(1+\zeta)}{6}h_b^2\Phi_{xxx_{b,d}} + O(h_b^3), \tag{3.1c}$$

$$\overline{\Phi}_{y_{b,d}} = \Phi_{y_{b,d}} + \frac{p^2}{6}\Phi_{yyy_{b,d}} + O(p^4), \tag{3.1d}$$
$$\overline{\Phi}_{y_{b+1,d}} = \Phi_{y_{b+1,d}} + \frac{p^2}{6}\Phi_{yyy_{b,d}} \pm O(h_b p^2), \tag{3.1e}$$
$$\overline{\Phi}_{y_{b-1,d}} = \Phi_{y_{b-1,d}} + \frac{p^2}{6}\Phi_{yyy_{b,d}} \pm O(h_b p^2), \tag{3.1f}$$
$$\overline{\Phi}_{yy_{b+1,d}} = \Phi_{yy_{b+1,d}} + O(p^2 + p^2 h_b), \tag{3.1g}$$
$$\overline{\Phi}_{yy_{b-1,d}} = \Phi_{yy_{b-1,d}} + O(p^2 + p^2 h_b). \tag{3.1h}$$

Further, by Taylor series expansion we obtain

$$\begin{aligned} &L_1(\Phi_{b+1,d+1} + \Phi_{b+1,d-1}) + L_2(\Phi_{b-1,d+1} + \Phi_{b-1,d-1}) + L_3(\Phi_{b,d+1} + \Phi_{b,d-1}) \\ &\quad + L_4(\Phi_{b+1,d}) + L_5(\Phi_{b-1,d}) + L_6\Phi_{b,d} \\ &= \zeta(1+\zeta)\frac{h^2}{2}\rho_{b,d}\exp\left(\frac{P\rho_{b+1,d} + R\rho_{b-1,d} - (P+R)\rho_{b,d}}{6\zeta(1+\zeta)\rho_{b,d}}\right) + I_{b,d}, \end{aligned} \tag{3.2}$$

where $I_{b,d} = O(p^2 h_b^2 + p^2 h_b^3 + h_b^5)$, $[b = 1(1)M, d = 1(1)N]$. Using approximations (3.1b), (3.1c), (3.1e), and (3.1f) we obtain

$$\overline{\rho}_{b+1,d} = \rho_{b+1,d} - \frac{\zeta(1+\zeta)}{6}h_b^2\Phi_{xxx}\kappa + O(p^2 + p^2 h_b + h_b^3), \tag{3.3a}$$
$$\overline{\rho}_{b-1,d} = \rho_{b-1,d} - \frac{(1+\zeta)}{6}h_b^2\Phi_{xxx}\kappa + O(p^2 + p^2 h_b + h_b^3). \tag{3.3b}$$

Now let,

$$\check{\Phi}_{x_{b,d}} = \overline{\Phi}_{x_{b,d}} + a_1 h_b(\overline{\rho}_{b+1,d} - \overline{\rho}_{b-1,d}) + a_2 h_b(\overline{\Phi}_{yy_{b+1,d}} - \overline{\Phi}_{yy_{b-1,d}}),$$

where a_1, a_2 are to be determined. Then using Taylor's expansion and (3.1a), (3.1g), (3.1h), (3.3a), and (3.3b) we have

$$\begin{aligned} \check{\Phi}_{x_{b,d}} &= \overline{\Phi}_{x_{b,d}} + a_1 h_b(\overline{\rho}_{b+1,d} - \overline{\rho}_{b-1,d}) + a_2 h_b(\overline{\Phi}_{yy_{b+1,d}} - \overline{\Phi}_{yy_{b-1,d}}) \\ &= \Phi_{b,d} + \frac{h_b^2}{6}(\zeta + 6a_1(1+\zeta))\Phi_{xxx} + h_b^2(a_1 + a_2)(1+\zeta)\Phi_{xyy} + O(p^2 + p^2 h_b + h_b^3). \end{aligned}$$

Now,

$$\check{\Phi}_{x_{b,d}} = \Phi_{x_{b,d}} + O(p^2 + p^2 h_b + h_b^3), \quad \text{if} \quad a_1 = -a_2 = -\frac{\zeta}{6(1+\zeta)},$$

and consequently,

$$\check{\rho}_{b,d} = \rho(x_b, y_d, \Phi_{b,d}, \check{\Phi}_{x_{b,d}}, \overline{\Phi}_{y_{b,d}}) = \rho_{b,d} + O(p^2 + p^2 h_b + h_b^3).$$

Again, let

$$\check{\check{\Phi}}_{x_{b,d}} = \overline{\Phi}_{x_{b,d}} + b_1 h_b(\overline{\rho}_{b+1,d} - \overline{\rho}_{b-1,d}) + b_2 h_b(\overline{\Phi}_{yy_{b+1,d}} - \overline{\Phi}_{yy_{b-1,d}}),$$

where b_1, b_2 are to be determined. Then using Taylor's expansion and (3.1a), (3.1g), (3.1h), (3.3a), and (3.3b) we have

$$\check{\check{\Phi}}_{x_{b,d}} = \Phi_{b,d} + \frac{h_b^2}{6}(\zeta + 6b_1(1+\zeta))\Phi_{xxx} + h_b^2(b_1 + b_2)(1+\zeta)\phi_{xyy} + O(p^2 + p^2 h_b + h_b^3). \tag{3.4}$$

Using (3.1d) and (3.4) we get

$$\check{\check{\rho}}_{b,d} = \rho(x_b, y_d, \Phi_{b,d}, \check{\check{\Phi}}_{x_{b,d}}, \overline{\Phi}_{y_{b,d}})$$
$$= \rho_{b,d} + \frac{h_b^2}{6}(\zeta + 6b_1(1+\zeta))\Phi_{xxx}\kappa + h_b^2(b_1+b_2)(1+\zeta)\Phi_{xyy} + O(p^2 + p^2 h_b + h_b^3).$$

Now,

$$\begin{aligned}
&P\overline{\rho}_{b+1,d} + R\overline{\rho}_{b-1,d} - (P+R)\check{\check{\rho}}_{b,d} \\
&= P\overline{\rho}_{b+1,d} + R\overline{\rho}_{b-1,d} - (P+R)\rho_{b,d} - P\frac{\eta(1+\eta)}{6}\frac{h_b^2}{6}\Phi_{xxx}\kappa - R\frac{(1+\eta)}{6}h_b^2\Phi_{xxx}\kappa \\
&\quad - (P+R)\frac{h_b^2}{6}(\zeta + 6b_1(1+\zeta))\Phi_{xxx}\kappa - (P+R)h_b^2(b_1+b_2)(1+\zeta)\Phi_{xyy}\kappa + O(p^2 + p^2 h_b + h_b^3) \\
&= P\rho_{b+1,d} + R\rho_{b-1,d} - (P+R)\rho_{b,d} - (1+\zeta)(2\zeta^2 - (1 - 3\zeta + \zeta^2)(\zeta + 6b_1(1+\zeta)))\frac{h_b^2}{6}\Phi_{xxx}\kappa \\
&\quad + (1+\zeta)^2(1 - 3\zeta + \zeta^2)h_b^2(b_1+b_2)U_{xyy}\kappa + O(k^2 + k^2 h_b + h_b^3) \\
&= P\rho_{b+1,d} + R\rho_{b-1,d} - (P+R)\rho_{b,d} + O(p^2 + p^2 h_b + h_b^3),
\end{aligned}$$

if

$$2\zeta^2 - (1 - 3\zeta + \zeta^2)(\zeta + 6b_1(1+\zeta)) = 0 \quad \text{and} \quad b_1 + b_2 = 0.$$

This implies

$$b_1 = -b_2 = -\frac{\zeta(1 - 5\zeta + \zeta^2)}{6(1+\zeta)(1 - 3\zeta + \zeta^2)}.$$

4. Error analysis

We take $\zeta = 1$ for our convenience. Then the method (2.4) for the equation (1.1) becomes

$$(\delta x^2 + \frac{h^2}{p^2}\delta y^2 + \frac{h^2}{12p^2}\delta x^2 \delta y^2)\Phi_{b,d} = h^2 \check{\rho}_{b,d} \exp\left(\frac{\overline{\rho}_{b+1,d} + \overline{\rho}_{b-1,d} - 2\check{\check{\rho}}_{b,d}}{12\check{\rho}_{b,d}}\right)$$

or

$$\begin{aligned}
&\xi_1(\Phi_{b+1,d} + \Phi_{b-1,d}) + \xi_2(\Phi_{b,d+1} + \Phi_{b,d-1}) + \xi_3(\Phi_{b+1,d+1} \\
&\quad + \Phi_{b+1,d-1} + \Phi_{b-1,d+1} + \Phi_{b-1,d-1} - (20 + 24\frac{p^2}{h_b^2})\Phi_{b,d}) \\
&\quad = h^2 \check{\rho}_{b,d} \exp\left(\frac{\overline{\rho}_{b+1,d} + \overline{\rho}_{b-1,d} - 2\check{\check{\rho}}_{b,d}}{12\check{\rho}_{b,d}}\right) + I_{b,d}, \quad [b = 1(1)M, d = 1(1)N_0],
\end{aligned} \tag{4.1}$$

where $I_{b,d} = O(p^2 h_b^2 + h_b^6)$, $\xi_1 = 1 - \frac{h_b^2}{6p^2}$, $\xi_2 = \frac{5h_b^2}{6p^2}$, and $\xi_3 = \frac{h_b^2}{12p^2}$. We shall assume in (4.1) that $\xi_1 \geqslant 0$ and $\xi_2 \geqslant 0$. Let

$$\tau_{b,d} = h^2 \check{\rho}_{b,d} \exp\left(\frac{\overline{\rho}_{b+1,d} + \overline{\rho}_{b-1,d} - 2\check{\check{\rho}}_{b,d}}{12\check{\rho}_{b,d}}\right).$$

Then expressing method (4.1) in matrix form is

$$\mathbf{C\Phi} + \boldsymbol{\tau}\mathbf{\Phi} + \mathbf{I} = \mathbf{0}, \tag{4.2}$$

where $\mathbf{C} = [\mathbf{C_1}, \mathbf{C_2}, \mathbf{C_1}]_{\mathbf{N_0^2 \times N_0^2}}$ is a three-block-diagonal matrix and

$$\mathbf{C_1} = [-\boldsymbol{\xi_3}, \boldsymbol{\xi_2}, -\boldsymbol{\xi_3}]_{\mathbf{N_0 \times N_0}} \quad \text{and} \quad \mathbf{C_2} = [-\boldsymbol{\xi_1}, \mathbf{20}\boldsymbol{\xi_3} + \mathbf{2}, -\boldsymbol{\xi_1}]_{\mathbf{N_0 \times N_0}}$$

are tri-diagonal matrices. We solve $(N_0^2 \times N_0^2)$ system as given below to find an appropriate approximation $\boldsymbol{\phi}$ for $\boldsymbol{\Phi}$.

$$\mathbf{C}\boldsymbol{\phi} + \boldsymbol{\tau\phi} = \mathbf{0}. \tag{4.3}$$

Let

$$\varepsilon_{b,d} = \phi_{b,d} - \Phi_{b,d}, \quad (b = 1(1)N_0, \ d = 1(1)N_0),$$

and

$$\boldsymbol{\upsilon} = \boldsymbol{\phi} - \boldsymbol{\Phi}.$$

Let

$$\begin{aligned}
\overline{\rho}_{b\pm1,d} &= \rho(x_{b\pm1}, y_d, \bar{\phi}_{b\pm1,d}, \bar{\phi}_{x_{b\pm1,d}}, \bar{\phi}_{y_{b\pm1,d}}) \approx \overline{\rho}_{b\pm1,d}, \\
\check{\rho}_{b,d} &= \rho(x_p, y_q, \phi_{b,d}, \check{\phi}_{x_{b,d}}, \overline{\phi}_{y_{b,d}}) \approx \check{\rho}_{b,d}, \\
\check{\check{\rho}}_{b,d} &= \rho(x_b, y_d, \phi_{b,d}, \check{\check{\phi}}_{x_{b,d}}, \overline{\phi}_{y_{b,d}}) \approx \check{\check{\rho}}_{b,d}.
\end{aligned}$$

We may write

$$\begin{aligned}
\overline{\rho}_{b\pm1,d} - \overline{\rho}_{b\pm1,d} &= (\overline{\phi}_{b\pm1,d} - \overline{\Phi}_{b\pm1,d})G^{(1)}_{b\pm1,d} + (\overline{\phi}_{x_{b\pm1,d}} - \overline{\Phi}_{x_{b\pm1,d}})H^{(1)}_{b\pm1,d} + (\overline{\phi}_{y_{b\pm1,d}} - \overline{\Phi}_{y_{b\pm1,d}})V^{(1)}_{b\pm1,d}, \\
\check{\rho}_{b,d} - \check{\rho}_{b,d} &= \varepsilon_{b,d}G^{(2)}_{b,d} + (\check{\phi}_{x_{b,d}} - \check{\Phi}_{x_{b,d}})H^{(2)}_{b,d} + (\overline{\phi}_{y_{b,d}} - \overline{\Phi}_{y_{b,d}})V^{(2)}_{b,d}, \\
\check{\check{\rho}}_{b,d} - \check{\check{\rho}}_{b,d} &= \varepsilon_{b,d}G^{(3)}_{b,d} + (\check{\check{\phi}}_{x_{b,d}} - \check{\check{\Phi}}_{x_{b,d}})H^{(3)}_{b,d} + (\overline{\phi}_{y_{b,d}} - \overline{\Phi}_{y_{b,d}})V^{(3)}_{b,d},
\end{aligned} \tag{4.4}$$

for suitable $E^{(1)}_{b\pm1,d}$, $E^{(2)}_{b,d}$, and $E^{(3)}_{b,d}$, where $E = G, H$ and V. For $E = H$ and V, we write

$$E^{(1)}_{b\pm1,d} = E^{(1)}_{b,d} \pm hE^{(1)}_{x_{b,d}} + O(h^2), \qquad G^{(1)}_{b\pm1,d} = G^{(1)}_{b,d} \pm O(h). \tag{4.5}$$

Using (4.4) and (4.5) we yield

$$\boldsymbol{\tau\phi} - \boldsymbol{\tau\Phi} = \boldsymbol{\vartheta\upsilon}, \tag{4.6}$$

where $\boldsymbol{\vartheta} = (\vartheta_{i,j}), ((i = 1(1)N_0^2), j = 1(1)N_0^2)$ is the three-block-diagonal matrix with

$$\begin{aligned}
\vartheta_{(d-1)N_0+b,(d-1)N_0+b} &= h^2[G^{(2)}_{b,d} - \frac{1}{6}G^{(3)}_{b,d} - \frac{1}{3}H^{(1)}_{x_{b,d}} + \frac{1}{3}H^{(1)}_{b,d}H^{(2)}_{b,d} - \frac{1}{6}H^{(1)}_{b,d}H^{(3)}_{b,d}] \\
&\quad + O(p^2 + h^4), \quad ((b = 1(1)N_0), d = 1(1)N_0), \\
\vartheta_{(d-1)N_0+b,(d-1)N_0+b\pm1} &= \frac{h}{12}[\pm 6H^{(2)}_{b,d} \pm H^{(1)}_{b,d} \mp H^{(3)}_{b,d}] \\
&\quad + \frac{h^2}{2}[-2H^{(1)}_{b,d}H^{(2)}_{b,d} + G^{(1)}_{b,d} + 2H^{(1)}_{x_{b,d}} + H^{(1)}_{b,d}H^{(3)}_{b,d}] \\
&\quad + O(p^2 + h^3), \quad ((b = 1(1)N_0 - 1, 2(1)N_0), d = 1(1)N_0), \\
\vartheta_{(d-1)N_0+b,(d-1\pm1)N_0+b} &= \frac{h^2}{12p}[\pm 6V^{(2)}_{b,d} \mp V^{(3)}_{b,d}] + o(p^2 + h^3), \quad ((b = 1(1)N_0), d = 1(1)N_0 - 1, 2(1)N_0), \\
\vartheta_{(d-1)N_0+b,dN_0+b\pm1} &= \frac{h^2}{24p}[\pm V^{(1)}_{b,d}] + O(p^2 + h^3), \quad ((b = 1(1)N_0 - 1, 2(1)N_0), d = 1(1)N_0 - 1),
\end{aligned}$$

$$\vartheta_{(d-1)N_0+b,(d-2)N_0+b\pm1} = \frac{h^2}{24p}[\pm V^{(1)}_{b,d}] + O(p^2+h^3), \quad ((b = 1(1)N_0-1, 2(1)N_0), d = 1(1)N_0-1).$$

Equation of error, using (4.6) in (4.2) and (4.3) is:

$$(\mathbf{C}+\boldsymbol{\vartheta})\boldsymbol{\upsilon} = \mathbf{I}. \tag{4.7}$$

Let $G^{(2)}_{*} = \min\limits_{(x,y)\in\overline{\Delta}} \frac{\partial\rho}{\partial\Phi}$ and $G^{*}_{(2)} = \max\limits_{(x,y)\in\overline{\Delta}} \frac{\partial\rho}{\partial\Phi}$, where $\overline{\Delta} = \Delta\cup\partial\Delta$. Then $0 < G^{(2)}_{*} \leqslant G^{(1)}_{b\pm1,d}, G^{(2)}_{b,d}, G^{(3)}_{b,d} \leqslant G^{*}_{(2)}$, and for $E = H, V$, let $0 < \left|E^{(1)}_{b\pm1,d}\right|, \left|E^{(2)}_{b,d}\right|, \left|E^{(3)}_{b,d}\right| \leqslant E$, and $\left|E^{(1)}_{x_{b,d}}\right| \leqslant E^{(1)}$, for some positive number $E^{(1)}$. For howsoever slight h, we can see

$$\begin{aligned}
\left|\vartheta_{(d-1)N_0+b,(d-1)N_0+b\pm1}\right| &< \xi_1, \quad ((b = 1(1)N_0-1, 2(1)N_0), d = 1(1)N_0),\\
\left|\vartheta_{(d-1)N_0+b,(d-1\pm1)M+b}\right| &< \xi_2, \quad ((b = 1(1)N_0), d = 1(1)N_0-1, 2(1)N_0),\\
\left|\vartheta_{(d-1)N_0+b,dN_0+b\pm1}\right| &< \xi_3, \quad ((b = 1(1)N_0-1, 2(1)N_0), d = 1(1)N_0-1),\\
\left|\vartheta_{(d-1)N_0+b,(d-2)N_0+b\pm1}\right| &< \xi_3, \quad ((b = 1(1)N_0-1, 2(1)N_0), d = 2(1)N_0).
\end{aligned}$$

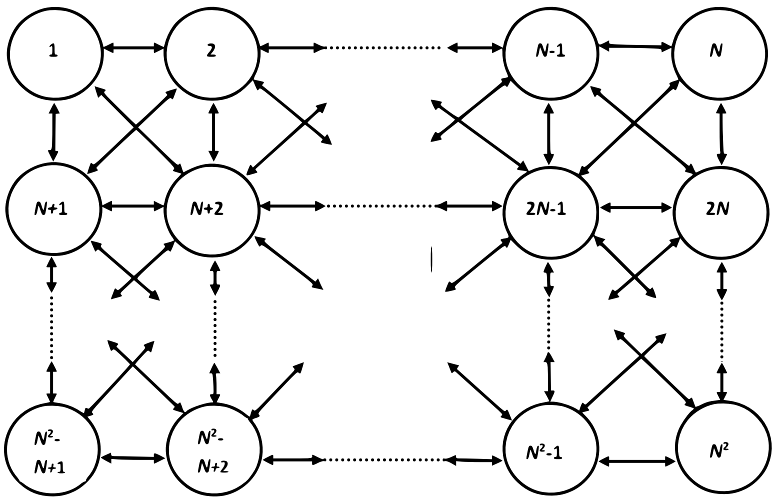

Figure 2: Directed graph of $(\mathbf{C}+\boldsymbol{\vartheta})$.

Figure 2 displays graph of $(\mathbf{C}+\boldsymbol{\vartheta})$ is directed, so, $(\mathbf{C}+\boldsymbol{\vartheta})$ is an irreducible matrix. Here, in $(\mathbf{C}+\boldsymbol{\vartheta})$ there is an arrow symbolizing the path $i \to j$ corresponding to each entry of the matrix which is non-zero. We observe that directed path is strongly connected. Thus the matrix $(\mathbf{C}+\boldsymbol{\vartheta})$ is irreducible ([19, 22]). Assuming T_q to be the summation of points in the q^{th} row of $(\mathbf{C}+\boldsymbol{\vartheta})$, then for $q = 1$ and N, we yield

$$T_q = 11\xi_3 + 1 + \frac{h}{12}(a_q + \frac{h}{2}g_q) + \frac{h^2}{24p}k_q + \frac{h^2}{12}[12G^{(2)}_{q,1} - 2G^{(3)}_{q,1} + G^{(1)}_{q,1}] + O(p^2+h^3), \tag{4.8}$$

where

$$\begin{aligned}
a_q &= \pm6H^{(2)}_{q,1} \pm H^{(1)}_{q,1} \mp H^{(3)}_{q,1},\\
g_q &= 2H^{(1)}_{q,1}H^{(2)}_{q,1} - H^{(1)}_{q,1}H^{(3)}_{q,1} - 2H^{(1)}_{x_{q,1}},\\
k_q &= 12V^{(2)}_{q,1} - 2V^{(3)}_{q,1} + V^{(1)}_{q,1},\\
S_{(N_0-1)N_0+q} &= 11\xi_3 + 1 + \frac{h}{12}(a_{(N_0-1)N_0+l} + \frac{h}{2}g_{(N_0-1)N_0+l}) + \frac{h^2}{24p}k_{(N_0-1)N_0+l}\\
&\quad + \frac{h^2}{12}[12G^{(2)}_{q,1} - 2G^{(3)}_{q,1} + G^{(1)}_{q,1}] + O(p^2+h^3),
\end{aligned} \tag{4.9}$$

where

$$a_{(N_0-1)N_0+q} = \pm 6H^{(2)}_{q,1} \pm H^{(1)}_{q,1} \mp H^{(3)}_{q,1},$$
$$g_{(N_0-1)N_0+q} = 2H^{(1)}_{q,1}H^{(2)}_{q,1} - H^{(1)}_{q,1}H^{(3)}_{q,1} - 2H^{(1)}_{x_{q,1}},$$
$$k_{(N_0-1)N_0+q} = -12V^{(2)}_{q,1} + 2V^{(3)}_{q,1} - V^{(1)}_{q,1}.$$

For $(2 \leqslant j \leqslant N_0 - 1)$:

$$T_{(j-1)N_0+q} = 1 + \frac{h}{12}\left[a_{(j-1)N_0+q} + hg_{(j-1)N_0+q}\right] + \frac{h^2}{12}[12G^{(2)}_{q,1} - 2G^{(3)}_{q,1} + G^{(1)}_{q,1}] + O(p^2 + h^3), \tag{4.10}$$

where

$$a_{(j-1)N_0+q} = \pm 6H^{(2)}_{q,j} \pm H^{(1)}_{q,j} \mp H^{(3)}_{q,1}, \qquad g_{(j-1)N_0+q} = 2H^{(1)}_{q,j}H^{(2)}_{q,j} - H^{(1)}_{q,j}H^{(3)}_{q,j} - 2H^{(1)}_{x_{q,j}}.$$

For $(i = 2(1)N_0 - 1)$:

$$S_{(q-1)N_0+i} = 1 + \frac{h}{2}\left[ha_{(q-1)N_0+i}\right] + \frac{h^2}{p}g_{(q-1)N_0+i} + h^2[\frac{3}{2}V^{(1)}_{i,q} + 3V^{(2)}_{i,q}] + O(p^2 + h^3), \tag{4.11}$$

where

$$g_{(q-1)N_0+i} = G^{(1)}_{i,q}G^{(2)}_{i,q} + \frac{1}{2}G^{(1)}_{i,q}G^{(3)}_{i,q} - G^{(1)}_{x_{i,q}}, \qquad k_{(q-1)N_0+i} = \frac{h^2}{12p}(\pm 6V^{(2)}_{q,1} \mp V^{(3)}_{q,1} \pm V^{(1)}_{q,1}),$$

and, ultimately for $((2 \leqslant i \leqslant N_0 - 1),\ 2 \leqslant j \leqslant N_0 - 1)$:

$$S_{(j-1)N_0+i} = \frac{h^2}{12}[12G^{(2)}_{i,j} - 2G^{(3)}_{i,j} + G^{(1)}_{i,j}] + \frac{h^2}{12}(2H^{(1)}_{i,j}H^{(2)}_{i,j} - 2H^{(1)}_{x_{i,j}} - H^{(1)}_{i,j}H^{(3)}_{i,j}). \tag{4.12}$$

With the help of (4.8), (4.9), (4.10), (4.11), and (4.12) for $n = 1, N_0, (N_0 - 1)N_0 + 1$ and N_0^2,

$$|a_q| \leqslant 8H, \quad |g_q| \leqslant 3H^2 + 2H^{(1)}, \quad |k_q| \leqslant 15V,$$

and for $q = i$ and $(N_0 - 1)N_0 + i$, $i = 2(1)N_0 - 1$,

$$|a_q| \leqslant H, \quad |g_q| \leqslant H, \quad |k_q| \leqslant 8V,$$

and for $n = (j - 1)N_0 + 1$ and jN, $j = 2(1)N_0 - 1$,

$$|a_q| \leqslant 8H, \quad |g_q| \leqslant 3H^2 + 2H^{(1)}, \quad |k_q| \leqslant V.$$

It conforms that for h slight enough,

$$T_q > \frac{15}{12}h^2G^{(2)}_*, \quad q = 1, N_0, (N_0 - 1)N_0 + 1 \ \text{ and } \ N_0^2, \tag{4.13a}$$
$$T_q > \frac{15}{12}h^2G^{(2)}_*, \quad q = i \ \text{ and } \ (N_0 - 1)N_0 + i, \ \ i = 2(1)N_0 - 1, \tag{4.13b}$$
$$T_q > \frac{15}{12}h^2G^{(2)}_*, \quad q = (j - 1)N_0 + 1 \ \text{ and } \ jM, \ \ j = 2(1)N_0 - 1, \tag{4.13c}$$
$$T_{(j-1)N_0+i} \geqslant \frac{15}{12}h^2G^{(2)}_*, \quad ((i = 2(1)N_0 - 1),\ j = 2(1)N_0 - 1). \tag{4.13d}$$

For sufficiently slight h, $(\mathbf{C} + \vartheta)$ is monotone. So, existence of $(\mathbf{C} + \vartheta)^{-1}$ is assured and $(\mathbf{C} + \vartheta)^{-1} = \mathbf{B} > \mathbf{0}$, where $\mathbf{B} = [B_{i,j}]_{N_0^2 \times N^2}$.

Since $\sum_{j=1}^{N_0^2} B_{l,j}T_j = 1$, $l = 1(1)N_0^2$, from(4.13a), (4.13b), (4.13c), and (4.13d), we obtain

$$B_{l,q} \leqslant \frac{1}{T_q} < \frac{12}{15h^2G_*^{(2)}}, \ q = 1, N_0, (N_0-1)N_0+1, N_0^2, \tag{4.14a}$$

$$\sum_{i=2}^{N_0-1} B_{l,q} \leqslant \frac{1}{\min\limits_{2\leqslant i\leqslant N_0-1} T_q} < \frac{12}{15h^2G_*^{(2)}}, \ q = i, (N_0-1)N_0+i, \tag{4.14b}$$

$$\sum_{j=2}^{N_0-1} B_{l,q} \leqslant \frac{1}{\min\limits_{2\leqslant j\leqslant N_0-1} T_q} < \frac{12}{15h^2G_*^{(2)}}, \ q = (j-1)N_0+1, jN_0, \tag{4.14c}$$

$$\sum_{i=2}^{N_0-1}\sum_{j=2}^{N_0-1} B_{l,q} \leqslant \frac{1}{\min\limits_{\substack{2\leqslant i\leqslant N_0-1\\ 2\leqslant j\leqslant N_0-1}} T_q} \leqslant \frac{12}{15h^2G_*^{(2)}}, \ q = (j-1)N_0+i. \tag{4.14d}$$

Further, equation (4.7) gives

$$\|\mathbf{v}\| \leqslant \|\mathbf{B}\| \, \|\mathbf{I}\|, \tag{4.15}$$

where

$$\begin{aligned} \|\mathbf{B}\| = \max_{1\leqslant l\leqslant N_0^2} \Bigg[& \left(B_{l,1} + \sum_{i=2}^{(N_0-1)} B_{l,i} + B_{l,N_0} \right) \\ & + \left(\sum_{j=2}^{N_0-1} B_{l,(j-1)N_0+1} + \sum_{i=2}^{N_0-1}\sum_{j=2}^{N_0-1} B_{l,(j-1)N_0+i} + \sum_{j=2}^{N_0-1} B_{l,jN_0} \right) \\ & + \left(B_{l,(N_0-1)N_0+1} + \sum_{i=2}^{N_0-1} B_{l,(N_0-1)N_0+i} + B_{l,N_0^2} \right) \Bigg]. \end{aligned} \tag{4.16}$$

Putting the values from (4.14a), (4.14b), (4.14c), and (4.14d) in equation (4.16), we have

$$\|\mathbf{B}\| \leqslant \frac{5}{36h^2V_*^{(2)}}. \tag{4.17}$$

Finally, using (4.15) and (4.17), for amply slight h and k, we yield

$$\|\mathbf{v}\| \leqslant O(p^2 + h^4).$$

This institutes the convergence of technique (1.1). When $p \propto h^2$, then

$$\|\mathbf{v}\| \leqslant O(h^4).$$

5. Techniques in vector form

Here we generalise the technique to solve system of nonlinear PDEs,

$$\phi_{xx}^{(i)} + \phi_{yy}^{(i)} = \rho^{(i)}(x, y, \phi^{(1)}, \phi^{(2)}, \ldots, \phi^{(n)}, \phi_x^{(1)}, \phi_x^{(2)}, \ldots, \phi_x^{(n)}, \phi_y^{(1)}, \phi_y^{(2)}, \ldots, \phi_y^{(n)}), \quad 1 \leqslant i \leqslant n, \tag{5.1}$$

where $(x, y) \in \Delta = (0,1) \times (0,1)$, and

$$\phi^{(i)}(x,y) = \phi_0^{(i)}(x,y).$$

Let $\Phi^{(i)}_{b,d}$ and $\phi^{(i)}_{b,d}$ be the exact and approximate values of $\phi^{(i)}(x_b, y_d)$, respectively. For every $i = 1(1)n$, let

$$\rho^{(i)}_{b,d} = \rho^{(i)}(x_b, y_d, \Phi^{(1)}_{b,d}, \Phi^{(2)}_{b,d}, \dots, \Phi^{(n)}_{b,d}, \Phi^{(1)}_{x_{b,d}}, \Phi^{(2)}_{x_{b,d}}, \dots, \Phi^{(n)}_{x_{b,d}}, \Phi^{(1)}_{y_{b,d}}, \Phi^{(2)}_{y_{b,d}}, \dots, \Phi^{(n)}_{y_{b,d}}).$$

We define the following approximations:

$$\begin{aligned}
\overline{\Phi}^{(i)}_{x_{b,d}} &= \frac{\Phi^{(i)}_{b+1,d} + (\zeta^2 - 1)\Phi^{(i)}_{b,d} - \zeta^2\Phi^{(i)}_{b-1,d}}{h_b\zeta(1+\zeta)},\\
\overline{\Phi}^{(i)}_{x_{b+1,d}} &= \frac{1}{\zeta(1+\zeta)h_b}((1+2\zeta)\Phi^{(i)}_{b+1,d} - (1+\zeta)^2\Phi^{(i)}_{b,d} + \zeta^2\Phi^{(i)}_{b-1,d}),\\
\overline{\Phi}^{(i)}_{x_{b-1,d}} &= \frac{1}{\zeta(1+\zeta)h_b}(-\Phi^{(i)}_{b+1,d} + (1+\zeta)^2\Phi^{(i)}_{b,d} - \zeta(2+\zeta)\Phi^{(i)}_{b-1,d}),\\
\overline{\Phi}^{(i)}_{y_{b,d}} &= \frac{1}{2p}(\Phi^{(i)}_{b,d+1} - \Phi^{(i)}_{b,d-1}),\\
\overline{\Phi}^{(i)}_{y_{b+1,d}} &= \frac{1}{2p}(\Phi^{(i)}_{b+1,d+1} - \Phi^{(i)}_{b+1,d-1}),\\
\overline{\Phi}^{(i)}_{y_{b-1,d}} &= \frac{1}{2p}(\Phi^{(i)}_{b-1,d+1} - \Phi_{b-1,d-1}),\\
\overline{\Phi}^{(i)}_{yy_{b+1,d}} &= \frac{1}{p^2}(\Phi_{b+1,d+1} - 2\Phi_{b+1,d} + \Phi_{b+1,d-1}),\\
\overline{\Phi}^{(i)}_{yy_{b-1,d}} &= \frac{1}{p^2}(\Phi_{b-1,d+1} - 2\Phi_{b+1,d} + \Phi_{b+1,d-1}).
\end{aligned}$$

Define

$$\overline{\rho}^{(i)}_{b+1,d} = \rho^{(i)}(x_{b+1}, y_d, \Phi_{b+1,d}, \overline{\Phi}_{x_{b+1,d}}, \overline{\Phi}_{y_{b+1,d}}), \quad \overline{\rho}^{(i)}_{b-1,d} = \rho^{(i)}(x_{b-1}, y_d, \Phi_{b-1,d}, \overline{\Phi}_{x_{b-1,d}}, \overline{\Phi}_{y_{b-1,d}}).$$

Next, we employ following approximations

$$\begin{aligned}
\check{\Phi}^{(i)}_{x_{b,d}} &= \overline{\Phi}^{(i)}_{x_{b,d}} - \frac{h}{12}(\overline{\rho}^{(i)}_{b+1,d} - \overline{\rho}^{(i)}_{b-1,d}) + \frac{h}{12}(\overline{\Phi}^{(i)}_{yy_{b+1,d}} - \overline{\Phi}^{(i)}_{yy_{b-1,d}}),\\
\check{\check{\Phi}}^{(i)}_{x_{b,d}} &= \overline{\Phi}^{(i)}_{x_{b,d}} - \frac{h}{4}(\overline{\rho}^{(i)}_{b+1,d} - \overline{\rho}^{(i)}_{b-1,d}) + \frac{h}{4}(\overline{\Phi}^{(i)}_{yy_{b+1,d}} - \overline{\Phi}^{(i)}_{yy_{b-1,d}}),
\end{aligned}$$

and let

$$\check{\rho}^{(i)}_{b,d} = \rho^{(i)}(x_b, y_d, \Phi^{(i)}_{b,d}, \check{\Phi}^{(i)}_{x_{b,d}}, \overline{\Phi}^{(i)}_{y_{b,d}}), \quad \check{\check{\rho}}^{(i)}_{b,d} = \rho^{(i)}(x_b, y_d, \Phi^{(i)}_{b,d}, \check{\check{\Phi}}^{(i)}_{x_{b,d}}, \overline{\Phi}^{(i)}_{y_{b,d}}).$$

Then at (x_b, y_d), the class of PDE (5.1) is discretized by the subsequentdifference scheme:

$$\begin{aligned}
&L_1(\Phi^{(i)}_{b+1,d+1} + \Phi^{(i)}_{b+1,d-1}) + L_2(\Phi^{(i)}_{b-1,d+1} + \Phi^{(i)}_{b-1,d-1}) + L_3(\Phi^{(i)}_{b,d+1} + \Phi_{b,d-1})\\
&\quad + L_4(\Phi^{(i)}_{b+1,d}) + L_5(\Phi^{(i)}_{b-1,d}) + L_6\Phi^{(i)}_{b,d}\\
&\quad = \zeta(1+\zeta)\frac{h^2}{2}\check{\rho}^{(i)}_{b,d}\exp\left(\frac{P\overline{\rho}^{(i)}_{b+1,d} + R\overline{\rho}^{(i)}_{b-1,d} - (P+R)\check{\check{\rho}}^{(i)}_{b,d}}{6\zeta(1+\zeta)\check{\rho}^{(i)}_{b,d}}\right) + I^{(i)}_{b,d},
\end{aligned} \tag{5.2}$$

where $I^{(i)}_{b,d} = O(p^2h_b^2 + p^2h_b^3 + h_b^5)$, $[b = 1(1)M, d = 1(1)N]$, where

$$\begin{aligned}
L_1 &= P\frac{h_b^2}{12p^2}, & L_2 &= R\frac{h_b^2}{12p^2}, & L_3 &= Q\frac{h_b^2}{12p^2},\\
L_4 &= 1 - 2P\frac{h_b^2}{12p^2}, & L_5 &= \zeta - 2R\frac{h_b^2}{12p^2}, & L_6 &= -(1+\zeta) - 2Q\frac{h_b^2}{12p^2}.
\end{aligned}$$

6. Method for multi-harmonic problems

Consider 2D biharmonic equation:

$$\begin{aligned}\nabla^4\phi(x,y) &\equiv \phi_{xxxx} + 2\phi_{xxyy} + \phi_{yyyy} \\ &= \rho(x,y,\phi,\phi_x,\phi_y,\nabla^2\phi,\nabla^2\phi_x,\nabla^2\phi_y), \quad (x,y)\in(0,1)\times(0,1),\end{aligned} \tag{6.1}$$

where ρ is the imposing function. Suppose $\nabla^2\phi = \theta$. Then the coupled form of equation (6.1) is as under

$$\nabla^2\phi \equiv \phi_{xx} + \phi_{yy} = \theta(x,y), \quad (x,y)\in\Delta, \tag{6.2a}$$

$$\nabla^2\theta \equiv \theta_{xx} + \theta_{yy} = \rho(x,y,\phi,\nu,\phi_x,\nu_x,\phi_y,\nu_y), \quad (x,y)\in\Delta. \tag{6.2b}$$

Note that, values of ϕ and θ are known on the boundary of Δ. We use method (5.2) to (6.2a)-(6.2b) and obtain a scheme for (6.1) as

$$\begin{aligned}&L_1(\Phi_{b+1,d+1}+\Phi_{b+1,d-1}) + L_2(\Phi_{b-1,d+1}+\Phi_{b-1,d-1}) + L_3(\Phi_{b,d+1}+\Phi_{b,d-1}) \\ &\quad + L_4(\Phi_{b+1,d}) + L_5(\Phi_{b-1,d}) + L_6\Phi_{b,d} \\ &= \zeta(1+\zeta)\frac{h^2}{2}\check{\Theta}_{b,d}\exp\left(\frac{P\overline{\Theta}_{b+1,d} + R\overline{\Theta}_{b-1,d} - (P+R)\check{\check{\Theta}}_{b,d}}{6\zeta(1+\zeta)\check{\Theta}_{b,d}}\right) + I_{b,d},\end{aligned} \tag{6.3}$$

where $I_{b,d} = O(p^2h_b^2 + p^2h_b^3 + h_b^5)$, $[b = 1(1)M, d = 1(1)N]$,

$$\begin{aligned}&L_1(\Theta_{b+1,d+1}+\Theta_{b+1,d-1}) + L_2(\Theta_{b-1,d+1}+\Theta_{b-1,d-1}) + L_3(\Theta_{b,d+1}+\Theta_{b,d-1}) \\ &\quad + L_4(\Theta b+1,d) + L_5(\Theta_{b-1,d}) + L_6\Theta_{b,d} \\ &= \zeta(1+\zeta)\frac{h^2}{2}\check{\rho}_{b,d}\exp\left(\frac{P\overline{\rho}_{b+1,d} + R\overline{\rho}_{b-1,d} - (P+R)\check{\check{\rho}}_{b,d}}{6\zeta(1+\zeta)\check{\rho}_{b,d}}\right) + I_{b,d},\end{aligned} \tag{6.4}$$

where $I_{b,d} = O(p^2h_b^2 + p^2h_b^3 + h_b^5)$, $[b = 1(1)M, d = 1(1)N]$, where $L_i, i = 1,\ldots,6$ are already defined in Section 2. Next, we consider the 2D tri-harmonic equation

$$\begin{aligned}\nabla^6\phi(x,y) &\equiv \phi_{xxxxxx} + 3(\phi_{xxxxyy} + \phi_{xxyyyy}) + \phi_{yyyyyy} \\ &= \rho(x,y,\phi,\phi_x,\phi_y,\nabla^2\phi,\nabla^2\phi_x,\nabla^2\phi_y,\nabla^4\phi,\nabla^4\phi_x,\nabla^4\phi_y), \quad (x,y)\in(0,1)\times(0,1),\end{aligned} \tag{6.5}$$

where ρ is the imposing function. Let $\nabla^2\phi = \theta$ and $\nabla^2\theta = \omega$. Then the equation (6.5) can be expressed as

$$\nabla^2\phi \equiv \phi_{xx} + \phi_{yy} = \theta(x,y), \quad (x,y)\in\Delta, \tag{6.6a}$$

$$\nabla^2\theta \equiv \theta_{xx} + \theta_{yy} = \omega(x,y), \quad (x,y)\in\Delta, \tag{6.6b}$$

$$\nabla^2\omega \equiv \omega_{xx} + \omega_{yy} = \rho(x,y,\phi,\theta,\omega,\phi_x,\theta_x,\omega_x,\phi_y,\theta_y,\omega_y), \quad (x,y)\in\Delta. \tag{6.6c}$$

We use method (5.2) to (6.6a)-(6.6c), and obtain a scheme to solve equation (6.5) as

$$\begin{aligned}&L_1(\Phi_{b+1,d+1}+\Phi_{b+1,d-1}) + L_2(\Phi_{b-1,d+1}+\Phi_{b-1,d-1}) + L_3(\Phi_{b,d+1}+\Phi_{b,d-1}) \\ &\quad + L_4(\Phi_{b+1,d}) + L_5(\Phi_{b-1,d}) + L_6\Phi_{b,d} \\ &= \zeta(1+\zeta)\frac{h^2}{2}\check{\Theta}_{b,d}\exp\left(\frac{P\overline{\Theta}_{b+1,d} + R\overline{\Theta}_{b-1,d} - (P+R)\check{\check{\Theta}}_{b,d}}{6\zeta(1+\zeta)\check{\Theta}_{b,d}}\right) + I_{b,d},\end{aligned} \tag{6.7}$$

where $I_{b,d} = O(p^2h_b^2 + p^2h_b^3 + h_b^5)$, $[b = 1(1)M, d = 1(1)N]$,

$$\begin{aligned} &L_1(\Theta_{b+1,d+1} + \Theta_{b+1,d-1}) + L_2(\Theta_{b-1,d+1} + \Theta_{b-1,d-1}) + L_3(\Theta_{b,d+1} + U_{b,d-1}) \\ &\quad + L_4(\Theta b + 1, d) + L_5(\Theta_{b-1,d}) + L_6\Theta_{b,d} \\ &= \zeta(1+\zeta)\frac{h^2}{2}\check{W}_{b,d} \exp\left(\frac{P\overline{W}_{b+1,d} + R\overline{W}_{b-1,d} - (P+R)\check{\check{W}}_{b,d}}{6\zeta(1+\zeta)\check{W}_{b,d}}\right) + I_{b,d}, \end{aligned} \tag{6.8}$$

where $I_{b,d} = O(p^2h_b^2 + p^2h_b^3 + h_b^5)$, $[b = 1(1)M, d = 1(1)N]$,

$$\begin{aligned} &L_1(W_{b+1,d+1} + W_{b+1,d-1}) + L_2(W_{b-1,d+1} + W_{b-1,d-1}) + L_3(W_{b,d+1} + W_{b,d-1}) \\ &\quad + L_4(W_{b+1,d}) + L_5(W_{b-1,d}) + L_6W_{b,d} \\ &= \zeta(1+\zeta)\frac{h^2}{2}\check{\rho}_{b,d} \exp\left(\frac{P\overline{\rho}_{b+1,d} + R\overline{\rho}_{b-1,d} - (P+R)\check{\check{\rho}}_{b,d}}{6\zeta(1+\zeta)\check{\rho}_{b,d}}\right) + I_{b,d}, \end{aligned} \tag{6.9}$$

where $I_{b,d} = O(p^2h_b^2 + p^2h_b^3 + h_b^5)$, $[b = 1(1)M, d = 1(1)N]$, where $L_i, i = 1, \dots, 6$ have been described in Section 2.

7. Numerical Simulations

The interval $[0, 1]$ in y- direction is dissected into $(N_0 + 1)$ points of equal length $p > 0$, so that $y_d = dp$ for $d = 0(1)N_0 + 1$. Further, the interval $[0, 1]$ in x- direction is dissected into $(M + 1)$ points, $0 = x_0 < x_1 < ... < x_{M+1} = 1$, $M = 0, 1, 2, ...$ with non-constant grid spacing $h_b = x_b - x_{b-1}$, $b = 1(1)(M + 1)$and the mesh ratio $\zeta = (\frac{h_{b+1}}{h_b}) > 0, b = 1(1)M$.
This divides the space Δ with mesh points (x_b, y_d), $b = 0(1)(M + 1), d = 0(1)(N + 1)$.
Now,

$$1 = x_{M+1} - x_0 = (x_{M+1} - x_M) + (x_M - x_{M-1}) + ... + (x_1 - x_0)$$

$$= h_{M+1} + h_M + ... + h_1 = (1 + \zeta + \zeta^2 + ... + \zeta^M)h_1$$

Thus,

$$h_1 = 1/(1 + \zeta + \zeta^2 + ... + \zeta^M) \tag{7.1}$$

The above working , makes us known, with the first step size in x-direction, so we derive, further step sizes using $h_{b+1} = \zeta h_b$; $b = 1(1)M$. Thus we each mesh point (x_b, y_d) of the rectangular mesh is ascertained. If $\zeta = 1$ means $h_{b+1} = h_b = h$; $b = 1(1)M$, the (2.4) condenses to technique of $O(p^2 + p^2h^2 + h^4)$. For constant mesh ratio, $\sigma = \frac{p}{h^2}$, the uniform mesh $O(p^2 + p^2h^2 + h^4)$ method yields fourth order accuracy in spatial directions. The order of accuracy can be validated using:

$$\frac{\log(\frac{E_{h_1}}{E_{h_2}})}{\log(\frac{h_1}{h_2})}$$

where E_{h_1} and E_{h_2} are the maximum absolute errors (MAEs) for uniform mesh widths h_1 and h_2 respectively.

Hereby, we have computed six problems which can be directly applied to the physical models. The closed solution is stated. The RHS functions and boundary conditions are found using closed form solution. The difference equations are computed by Gauss-Seidel iterative scheme and Newton-Raphson method depending on the linearity and non-linearity respectively [19]. The simulations were aborted when the absolute error tolerance $\leqslant 10^{-12}$ was attained. MATLAB codes were employed for computation.

Numerical outcomes are compared with $O(p^2 + h_b^2)$ scheme, defined as
To discretize (1.1), we use subsequent estimations:

$$\overline{\Phi}_{x_{b+1,d}} = \frac{(1+2\zeta)\Phi_{b+1,d} - (1+\zeta)^2\Phi_{b,d} + \zeta^2\Phi_{b-1,d}}{h_b\zeta(1+\zeta)} \tag{7.2a}$$

$$\overline{\Phi}_{x_{b-1,d}} = \frac{-\Phi_{b+1,d} + (1+\zeta)^2\Phi_{b,d} - \zeta(2+\zeta)\Phi_{b-1,d}}{h_b\zeta(1+\zeta)} \tag{7.2b}$$

$$\overline{\Phi}_{xx_{b,d}} = \frac{2(\Phi_{b+1,d} - (1+\zeta)\Phi_{b,d} + \zeta\Phi_{b-1,d})}{{h_b}^2\zeta(1+\zeta)} \tag{7.3a}$$

$$\overline{\Phi}_{yy_{b,d}} = \frac{1}{p^2}(\Phi_{b,d+1} - 2\Phi_{b,d} + \Phi_{b,d-1}) \tag{7.3b}$$

$$\begin{aligned}\overline{\Phi}_{xyy_{b,d}} = \frac{1}{h_b p^2\zeta(1+\zeta)}[&(\Phi_{b+1,d+1} - (1-\zeta^2)\Phi_{b,d+1} - \zeta^2\Phi_{b-1,d+1}\\ &- 2(\Phi_{b+1,d} - (1-\zeta^2)\Phi_{b,d} - \zeta^2\Phi_{b-1,d})\\ &+ (\Phi_{b+1,d-1} - (1-\zeta^2)\Phi_{b,d-1} - \zeta^2\Phi_{b-1,d-1})]\end{aligned} \tag{7.4}$$

Now, let

$$\overline{\rho}_{b,d} = \rho(x_b, y_d, \Phi_{b,d}, \overline{\Phi}_{x_{b,d}}, \overline{\Phi}_{y_{b,d}}), \tag{7.5a}$$

$$\overline{\rho}_{b\pm1,d} = \rho(x_{b\pm1}, y_d, \Phi_{b\pm1,d}, \overline{\Phi}_{x_{b\pm1,d}}, \overline{\Phi}_{y_{b\pm1,d}}), \tag{7.5b}$$

Using (2.1a), (2.1b), (2.1c), (2.2a), (7.2a), (7.2b), (7.3a), (7.3b), (7.4), (7.5a) and (7.5b) the $O(p^2 + h_b^2)$ method for (1.1) is obtained as:

$$\begin{aligned}&\overline{\Phi}_{xx_{b,d}} + \overline{\Phi}_{yy_{b,d}} + \frac{(\zeta-1)}{3}h_b\overline{\Phi}_{xyy_{b,d}}\\ &\qquad = \overline{\rho}_{b,d} + \frac{(\zeta-1)}{3\zeta(1+\zeta)}[\overline{\rho}_{b+1,d} - (1-\zeta^2)\overline{\rho}_{b,d} - \zeta^2\overline{\rho}_{b-1,d}]\end{aligned} \tag{7.6}$$

Example 7.1 (Convection-Diffusion equation)**.**

$$\phi_{xx} + \phi_{yy} = \beta\phi_x, \quad 0 < x,\ y < 1$$

Closed form solution is

$$e^{\beta x/2}\frac{\sin(\pi y)}{\sinh(\gamma)}[2e^{-\beta/2}\sinh(\gamma x) + \sinh(\gamma(1-x))],\ \gamma = \sqrt{\pi^2 + \frac{\beta^2}{4}}.$$

The MAEs in ϕ are recorded in Table 7.1a and Table 7.1b for $\zeta = 0.78$ and $\zeta = 1$ respectively. Graphs 7.1a, 7.1b show closed form and computed solutions for $(M = 70, N = 70)$, $\beta = 1000$ and $\zeta = 0.78$.

Table 7.1a: Example 1: The MAEs for $\zeta = 0.78$

(M,N)	Suggested method (2.4)				$O(p^2 + h_b^2)$ method (7.6)			
	$\beta = 100$	$\beta = 500$	$\beta = 1000$	$\beta = 1400$	$\beta = 100$	$\beta = 500$	$\beta = 1000$	$\beta = 1400$
(30,30)	2.4760(-04)	2.8976(-04)	5.0452(-04)	7.7537(-04)	9.1687(-01)	9.8406(-01)	9.8720(-01)	9.8698(-01)
(40,40)	1.9869(-04)	1.6837(-04)	1.7476(-04)	1.8494(-04)	9.0953(-01)	9.3798(-01)	9.5298(-01)	9.8083(-01)
(50,50)	1.8309(-04)	1.5455(-04)	1.5163(-04)	1.5199(-04)	9.8732(-01)	9.1648(-01)	9.4946(-01)	9.7196(-01)
(60,60)	1.7541(-04)	1.5161(-04)	1.4841(-04)	1.4787(-04)	9.4298(-01)	9.0864(-01)	9.6453(-01)	9.6543(-01)
(70,70)	1.7086(-04)	1.4984(-04)	1.4763(-04)	1.4708(-04)	9.3956(-01)	8.9363(-01)	8.7689(-01)	8.3497(-01)
(80,80)	1.6953(-04)	1.4763(-04)	1.4727(-04)	1.4680(-04)	9.2478(-01)	8.4198(-01)	8.86246(-01)	8.1467(-01)

Table 7.1b: Example 1: MAEs for $\zeta = 1$

h	Suggested method (2.4)				$O(p^2+h^2)$			
	$\beta = 100$	$\beta = 500$	$\beta = 1000$	$\beta = 1400$	$\beta = 100$	$\beta = 500$	$\beta = 1000$	$\beta = 1400$
1/30	3.6601(-02)	4.8397(-01)	6.9512(-01)	7.7116(-01)	Oscillations	Oscillations	Oscillations	Oscillations
1/40	1.5899(-02)	3.8128(-01)	6.1687(-01)	7.0798(-01)	Oscillations	Oscillations	Oscillations	Oscillations
1/50	7.7319(-03)	3.0079(-01)	5.4751(-01)	6.4969(-01)	Oscillations	Oscillations	Oscillations	Oscillations
1/60	4.0895(-03)	2.3800(-01)	4.8559(-01)	5.9676(-01)	Oscillations	Oscillations	Oscillations	Oscillations
1/70	2.3284(-03)	1.8855(-01)	4.3102(-01)	5.4801(-01)	Oscillations	Oscillations	Oscillations	Oscillations
1/80	1.3984(-03)	1.5069(-01)	3.8313(-01)	5.0308(-01)	Oscillations	Oscillations	Oscillations	Oscillations

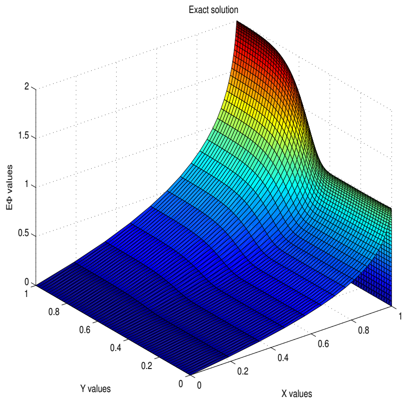

Figure 7.1a: Exact Solution of Example 1 for $(M = 70, N = 70)$ and $\beta = 1000$.

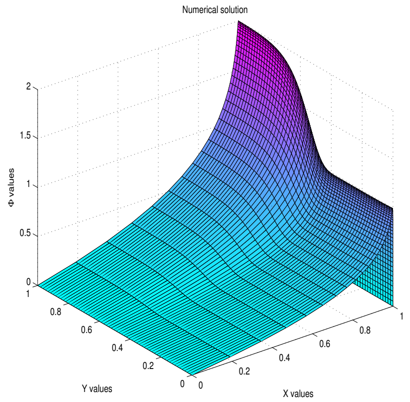

Figure 7.1b: Numerical Solution of Example 1 for $(M = 70, N = 70)$ and $\beta = 1000$.

Example 7.2 (Poisson's equation in r-z plane)**.**

$$\phi_{rr} + \frac{\alpha}{r}\phi_r + \phi_{zz} = H(r,z), \quad 0 < r,z < 1 \tag{7.7}$$

The closed form solution is $\cosh r \cosh z$. The MAEs in ϕ are recorded in Table 7.2a for $\zeta = 0.75$ and in Table 7.2b for $\zeta = 1$ and fixed mesh ratio parameter $\sigma = 20$. Graphs 7.2a, 7.2b show closed form and computed solutions for $(M = 70, N = 70)$, $\alpha = 2$ and $\zeta = 0.75$ graphically.

Table 7.2a: Example 2: MAEs for $\zeta = 0.75$

(M,N)	Suggested method (2.4)		$O(p^2+h_b^2)$ method (7.6)	
	$\alpha = 1$	$\alpha = 2$	$\alpha = 1$	$\alpha = 2$
(30,30)	3.4627(-04)	6.6427(-04)	7.5498(-02)	1.1856(-01)
(40,40)	3.4450(-04)	6.6045(-04)	7.5276(-02)	1.6956(-01)
(50,50)	3.4222(-04)	6.6033(-04)	7.5243(-02)	1.2294(-01)
(60,60)	3.3977(-04)	6.6016(-04)	7.4973(-02)	1.2294(-01)
(70,70)	3.3666(-04)	6.6011(-04)	5.4383(-02)	1.2294(-01)
(80,80)	3.3446(-04)	6.6001(-04)	5.4323(-02)	1.2294(-01)

Table 7.2b: Example 2: MAEs for $\zeta = 1$ and $\sigma = 20$.

(h,p)	Suggested method (2.4)		$O(p^2+h^2)$	
	$\alpha = 1$	$\alpha = 2$	$\alpha = 1$	$\alpha = 2$
(1/20,1/20)	2.2501(-04)	2.9509(-04)	Oscillations	Oscillations
(1/40,1/80)	1.3992(-05)	1.9158(-05)	Oscillations	Oscillations
(1/80,1/320)	1.0038(-06)	1.2471(-06)	Oscillations	Oscillations

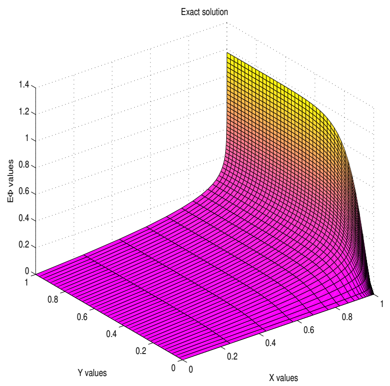

Figure 7.2a: Exact Solution of Example 2 for $(M = 70, N = 70)$ and $\alpha = 2$.

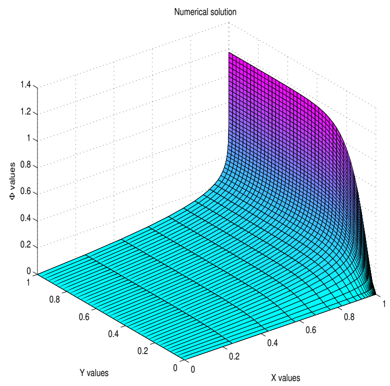

Figure 7.2b: Numerical Solution of Example 2 for $(M = 70, N = 70)$ and $\alpha = 2$.

Example 7.3 (Burgers' equation)**.**

$$\varepsilon(\phi_{xx} + \phi_{yy}) = \phi(\phi_x + \phi_y) + g(x, y), \quad 0 < x, y < 1$$

The closed form solution is $e^x \sin\left(\frac{\pi y}{2}\right)$.

The MAEs in ϕ are recorded in Table 7.3a for $\zeta = 0.85$ and in Table 7.3b for $\zeta = 1$ and fixed mesh ratio parameter $\zeta = 20$. Graphs 7.3a, 7.3b show closed form and computed solution for $(M = 60, N = 60)$, $\varepsilon = 0.01$ and $\zeta = 0.85$.

Table 7.3a: Example 3: MAEs for $\zeta = 0.85$

(M, N)	Suggested method (2.4)		$O(p^2 + h_b^2)$ method (7.6)	
	$\varepsilon = 0.1$	$\varepsilon = 0.01$	$\varepsilon = 0.1$	$\varepsilon = 0.01$
(30,30)	3.9288(-03)	5.3998(-04)	1.2943(-01)	2.3875(-01)
(40,40)	3.5232(-04)	4.8002(-04)	1.2178(-01)	2.0564(-01)
(50,50)	3.3794(-04)	4.2887(-05)	1.2067(-01)	1.9632(-01)
(60,60)	3.32216(-05)	3.9508(-05)	1.2048(-01)	1.9611(-01)
(70,70)	3.3001(-05)	3.6798(-05)	1.2023(-01)	1.9597(-01)
(80,80)	3.2892(-05)	3.4809(-05)	1.2012(-01)	1.9520(-01)

Table 7.3b: Example 3: The MAEs for $\zeta = 1$ and $\sigma = 20$

(h, p)	Suggested method (2.4)		$O(k^2 + h^2)$	
	$\varepsilon = 0.1$	$\varepsilon = 0.01$	$\varepsilon = 0.1$	$\varepsilon = 0.01$
(1/20,1/20)	4.5988(-03)	9.6298(-03)	oscillations	oscillations
(1/40,1/80)	2.8493(-04)	6.0792(-04)	oscillations	oscillations
(1/80,1/320)	1.7535(-05)	3.7824(-05)	oscillations	oscillations

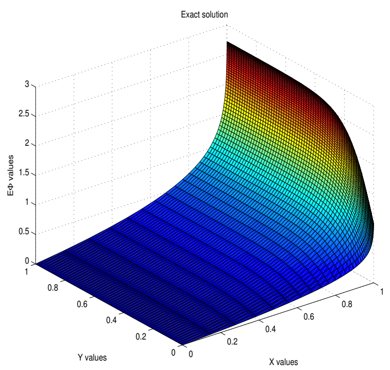

Figure 7.3a: Exact Solution of Example 3 for $(M = 60, N = 60)$, $\varepsilon = 0.01$.

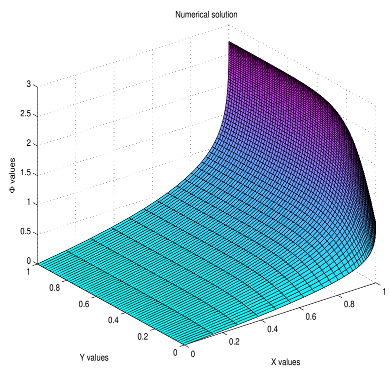

Figure 7.3b: Numerical Solution of Example 3 for $(M = 60, N = 60)$, $\varepsilon = 0.01$.

Example 7.4 (Navier-Stokes equations in cartesian coordinates)**.**

$$\frac{1}{R_e}(\phi_{xx}+\phi_{yy}) = \phi\phi_x + \theta\phi_y + f(x,y), \quad 0 < x, \ y < 1, \tag{7.8a}$$

$$\frac{1}{R_e}(\theta_{xx}+\theta_{yy}) = \phi\theta_x + \theta\theta_y + g(x,y), \quad 0 < x, \ y < 1. \tag{7.8b}$$

The closed form solutions are $\sin(\pi x)\sin(\pi y)$, $\theta(x,y) = \cos(\pi x)\cos(\pi y)$.

The MAEs in ϕ and θ are recorded in Table 7.4a for $\zeta = 1.2$ and in Table 7.4b for $\zeta = 1$ and fixed mesh ratio parameter $\sigma = 20$. Graphs 7.4a, 7.4b, 7.4c and 7.4d show closed form and computed solutions for $(M = 60, N = 60)$, $\varepsilon = 0.01$ and $\zeta = 1.2$.

Table 7.4a: Example 4: MAEs for $\zeta = 1.2$

M,N	Suggested method (5.2)			$O(p^2+h_b^2)$ method (7.6)		
	R_e=10	R_e=10^2	R_e=10^3	R_e=10	R_e=10^2	R_e=10^3
(30,30)						
ϕ	6.8419(-04)	1.4472(-04)	3.4957(-03)	5.2108(-01)	oscillations	oscillations
θ	3.9768(-04)	5.2169(-03)	4.7639(-03)	1.6934(-01)	oscillations	oscillations
(40,40)						
ϕ	6.6477(-04)	1.3820(-04)	3.4468(-03)	5.1874(-01)	oscillations	oscillations
θ	3.0955(-04)	4.5672(-03)	4.6537(-03)	1.5789(-01)	oscillations	oscillations
(50,50)						
ϕ	6.3506(-04)	1.3547(-04)	3.4253(-03)	5.1496(-01)	oscillations	oscillations
θ	2.6865(-04)	4.3531(-03)	4.4173-03)	1.5218(-03)	oscillations	oscillations
(60,60)						
ϕ	5.8635(-04)	1.3404(-03)	3.4175(-03)	5.1275(-01)	oscillations	oscillations
θ	2.4633(-04)	4.3247(-02)	4.3862(-03)	1.5031(-03)	oscillations	oscillations
(70,70)						
ϕ	4.9854(-04)	1.3318(-03)	3.3692(-03)	5.1133(-01)	oscillations	oscillations
θ	2.3292(-04)	4.3135(-02)	4.2747(-03)	1.4932(-03)	oscillations	oscillations
(80,80)						
ϕ	3.8712(-04)	1.3262(-03)	2.9845(-03)	5.1027(-01)	oscillations	oscillations
θ	2.2415(-04)	4.3132(-02)	4.2418(-03)	1.4861(-03)	oscillations	oscillations

Table 7.4b: Example 4: MAEs for $\zeta = 1$ and $\sigma = 20$

(h,p)	Suggested method (5.2)			$O(p^2+h^2)$		
	R_e=10	R_e=10^2	R_e=10^3	R_e=10	R_e=10^2	R_e=10^3
(1/20,1/20)						
ϕ	1.1102(-03)	5.5715(-03)	4.7863(-03)	oscillations	oscillations	oscillations
θ	4.4622(-04)	3.8404(-03)	4.9369(-03)	oscillations	oscillations	oscillations
(1/40,1/80)						
ϕ	7.4981(-05)	3.8808(-04)	2.8792(-04)	oscillations	oscillations	oscillations
θ	2.9913(-05)	2.4199(-04)	3.1374(-04)	oscillations	oscillations	oscillations
(1/80,1/320)						
ϕ	3.5961(-06)	2.4744(-05)	1.7926(-05)	oscillations	oscillations	oscillations
θ	2.1538(-06)	1.5364(-06)	1.9549(-05)	oscillations	oscillations	oscillations

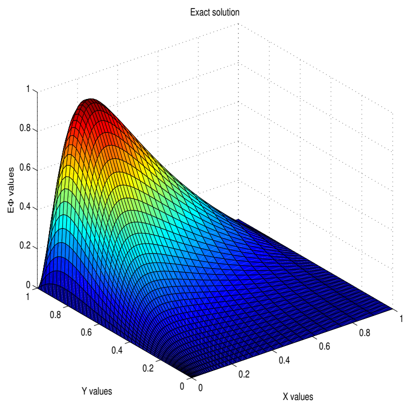

Figure 7.4a: Exact Solution of Example 4 of ϕ for $(M = 60, N = 60)$, $Re = 100$.

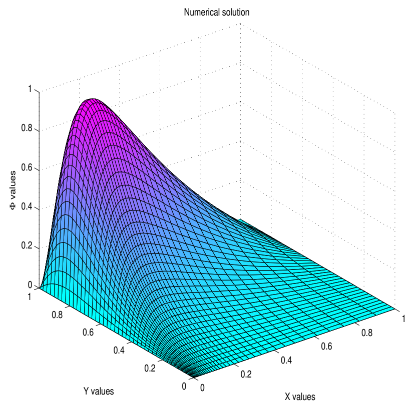

Figure 7.4b: Numerical Solution of Example 4 of ϕ for $h = (M = 60, N = 60)$, $Re = 100$.

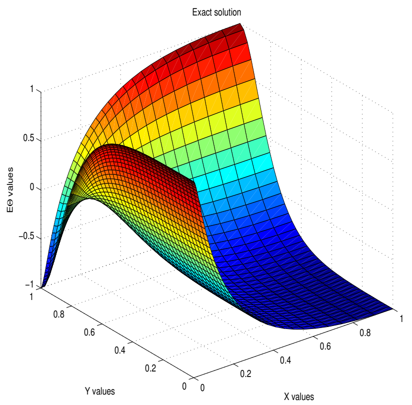

Figure 7.4c: Exact Solution of Example 4 of θ for $(M = 60, N = 60)$, $Re = 100$.

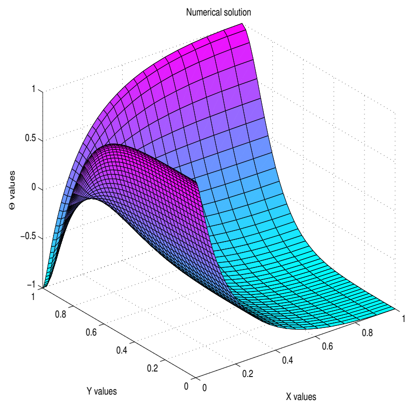

Figure 7.4d: Numerical Solution of Example 4 of θ for $(M = 60, N = 60)$, $Re = 100$.

Example 7.5 (Bi-harmonic Equation)**.**

$$\nabla^4\phi = \rho(x, y), \quad 0 < x, y < 1. \tag{7.9}$$

The closed form solution is $\sin(\pi x)\cos(\pi y)$. The MAEs in ϕ are recorded in Table 7.5a for $\zeta = 1.23$ and in Table 7.5b for $\zeta = 1$ and fixed mesh ratio parameter $\sigma = 20$. Graphs 7.5a, 7.5b show closed form and computed solutions for $(M = 60, N = 60)$ and $\zeta = 1.23$.

Table 7.5a: Example 5: MAEs for $\zeta = 0.1.23$

M,N	Proposed method (6.3)-(6.4)	$O(p^2 + h_b^2)$ method (7.6)
(30,30)	3.1830(-04)	3.1073(-03)
(40, 40)	2.5830(-04)	3.0202(-03)
(50, 50)	2.3585(-04)	2.9818(-03)
(60, 60)	2.2356(-04)	2.9605(-03)
(70, 70)	2.1609(-04)	2.9477(-03)
(80, 80)	2.1121(-04)	2.9415(-03)

Table 7.5b: Example 5: MAEs for $\zeta = 1$ and $\sigma = 20$.

(h,p)	Suggested method (6.3)-(6.4)	$O(p^2 + h^2)$
(1/20,1/20)	4.3746(-04)	8.7437(-04)
(1/40,1/80)	2.9574(-05)	1.4484(-04)
(1/80,1/320)	1.7446(-06)	2.7813(-05)

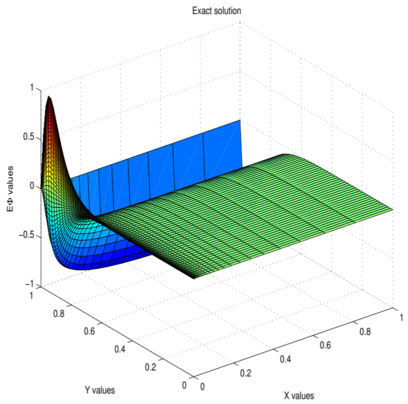

Figure 7.5a: Exact Solution of Example 5 for $(M = 60, N = 60)$.

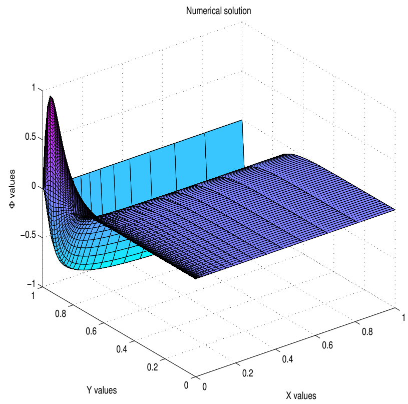

Figure 7.5b: Numerical Solution of Example 5 for $(M = 60, N = 60)$.

Example 7.6 (Tri-harmonic Equation)**.**

$$\nabla^6 \phi = \rho(x, y), \quad 0 < x, y < 1. \tag{7.10}$$

The closed form solution is $\sin(\pi x)\cos(\pi y)$. .
The MAEs in ϕ are recorded in Table 7.6a for $\zeta = 1.2$ and in Table 7.6b for $\zeta = 1$ and fixed mesh ratio parameter $\sigma = 20$. Graphs 7.6a, 7.6b show closed form and computed solutions for $(M = 60, N = 60)$ and $\zeta = 1.2$.

Table 7.6a: Example 6: MAEs for $\zeta = 1.2$

M,N	Suggested method (6.7)-(6.9)	$O(p^2 + h_b^2)$ method (7.6)
(30,30)	4.6247(-04)	3.5937(-03)
(40,40)	3.9123(-04)	3.5735(-03)
(50,50)	3.7471(-04)	3.5706(-03)
(60,60)	3.7268(-04)	3.5704(-03)
(70,70)	3.6334(-04)	3.5703(-03)
(80,80)	3.6098(-04)	3.5700(-03)

Table 7.6b: Example 6: MAEs for $\zeta = 1$ and $\sigma = 20$.

(h,p)	Suggested method (6.3)-(6.4)	$O(p^2 + h^2)$
(1/20,1/20)	4.8727(-04)	9.7649(-03)
(1/40,1/80)	3.2942(-05)	1.6144(-04)
(1/80,1/320)	1.8753(-06)	2.9574(-05)

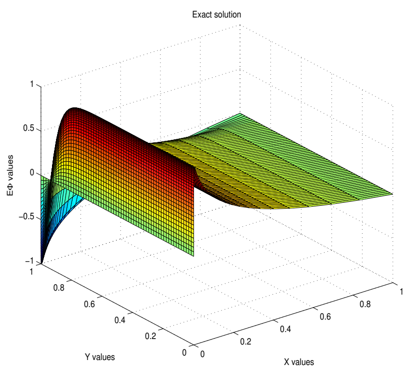

Figure 7.6a: Exact Solution of Example 6 for $(M = 60, N = 60)$.

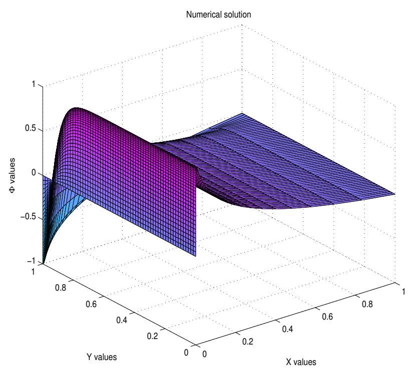

Figure 7.6b: Numerical Solution of Example 6 for $(M = 60, N = 60)$.

8. Conclusions and discussions

This paper reports a novel, implicit technique based on exponential form for computing 2D non-linear EBVPs employing nine point compact cell. This scheme leads us to diagonally dominant block tri-diagonal system of difference equations which reduces computational time to a great extent. We have performed six physically significant simulations to manifest the ability of our method both in terms of computation and accuracy. It was accomplished before that the highly accurate technique on uniform mesh behaves seamlessly for small values of $\beta(0 < \beta < 100)$, whereas the proposed technique on semi-variable mesh is stable for reasonably large values of β, that is, for $\beta = 200; 500; 1000; 1400$. For $\beta > 1500$ the proposed technique is unstable. Further, it has been observed in past that approximation techniques fails to compute for large values of R_e or small values of $\epsilon = 1/R_e$ especially in the variable mesh; whereas, the proposed method is stable for high values $R_e = 10^3$ on a semi-variable mesh. The major benefit of our research is the numerical solution on a semi-variable mesh. To establish theoretically, order of convergence, we have computed numerical solution for $\zeta = 1$ (constant mesh). Also, we have assumed $u \in C^6(\Delta)$ and we have chosen u which is at least six times continuous in the prescribed region and calculated the corresponding forcing function in terms of convection coefficient and Dirichlet boundary conditions so that the problem's analytical solution is unique irrespective of whether u is independent of convection coefficient or not.

Further, in future we plan to extend the proposed scheme to time dependent problems.

Acknowledgment

The author is thankful to reviewers for suggestions which really contributed to improving the standard of the paper.

References

[1] G. Birkhoff, R. E. Lynch, *The Numerical Solution of Elliptic Problems*, SIAM Publications, Philadelphia, (1984). 1

[2] M. Dehghan, A. Mohebbi, *High-order compact boundary value method for the solution of unsteady convection–diffusion problems*, Math. Comput. Simulation, **79** (2008), 683–699. 1

[3] M. K. Jain, R. K. Jain, M. Krishna, *Fourth order difference method for quasi-linear Poisson equation in cylindrical symmetry*, Comm. Numer. Methods Engrg., **10** (1994), 291–296. 1

[4] M. K. Jain, R. K. Jain, R. K. Mohanty, *A fourth order difference method for elliptic equations with nonlinear first derivative terms*, Numer. Methods Partial Differential Equations, **5** (1989), 87–95. 1

[5] M. K. Jain, R. K. Jain, R. K. Mohanty, *Fourth-order difference methods for the system of 2D nonlinear elliptic partial differential equations*, Numer. Methods Partial Differential Equations, **7** (1991), 227–244. 1

[6] C. T. Kelly, *Iterative Methods for Linear and Non-Linear Equations*, SIAM, Philadelphia, (1995). 1

[7] L. Li, Z. Jiang, Z. Yin, *Fourth-order compact finite difference method for solving two-dimensional convection–diffusion equation*, Adv. Difference Equ., **2018** (2018), 24 pages. 1

[8] R. K. Mohanty, *Order h^4 difference methods for a class of singular two-space dimensional elliptic boundary value problems*, J. Comput. Appl. Math., **81** (1997), 229–247. 1

[9] R. K. Mohanty, S. Karra, U. Arora, *Fourth order nine point unequal mesh discretization for the solution of 2D nonlinear elliptic partial differential equations*, Neural Parallel Sci. Comput., **14** (2006), 453–470. 1

[10] R. K. Mohanty, G. Manchanda, A. Khan, *Operator compact exponential approximation for the solution of the system of 2D second order quasilinear elliptic partial differential equations*, Adv. Difference Equ., **2019** (2019), 36 pages. 1

[11] R. K. Mohanty, G. Manchanda, A. Khan, *Compact half step approximation in exponential form for the system of 2D second-order quasi-linear elliptic partial differential equations*, J. Difference Equ. Appl., **25** (2019), 716–749. 1

[12] R. K. Mohanty, G. Manchanda, A. Khan, G. Khurana *A new high accuracy method in exponential form based on off-step discretization for non-linear two point boundary value problems*, J. Difference Equ. Appl., **26** (2020), 171–202. 1

[13] R. K. Mohanty, G. Manchanda, G. Khurana, A. Khan, *A New Third Order Exponentially Fitted Discretization for the Solution of Non-linear Two Point Boundary Value Problems on a Graded Mesh*, J. Appl. Anal. Comput., **10** (2020), 1741–1770. 1

[14] R. K. Mohanty, S. Singh,*A new fourth order discretization for singularly perturbed two dimensional non-linear elliptic boundary value problems*, Appl. Math. Comput., **175** (2006), 1400–1414. 1

[15] J. C. Strikwerda, *Finite difference schemes and partial differential equations*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, (2004). 1

[16] G. Saldanha, U. Anantha Krishnaiah, *A fourth-order finite difference scheme for two-dimensional nonlinear elliptic partial differential equations*, Numer. Methods Partial Differential Equations, **11** (1995), 33–40. 1
[17] J. W. Thomas, *Numerical partial differential equations: conservation laws and elliptic equations*, Springer-Verlag, New York, (1999). 1
[18] Z. F. Tian, S. Q. Dai, *High-order compact exponential finite difference methods for convection–diffusion type problems*, J. Comput. Phys. **220** (2007), 952–974. 1
[19] R. S. Varga, *Matrix Iterative Analysis*, Springer-Verlag, Berlin, (2000). 4, 7
[20] Y.-M. Wang, B.-Y. Guo, W.-J. Wu, *Fourth-order compact finite difference methods and monotone iterative algorithms for semilinear elliptic boundary value problems*, Comput. Math. Appl., **68** (2014), 1671–1688. 1
[21] I. Yavneh, *Analysis of a fourth order compact scheme for convection diffusion*, J. Comput. Phys., **133** (1997), 361–364. 1
[22] D. M. Young, *Iterative Solution of Large Linear Systems*, Academic Press, New York-London, (1971). 4
[23] M. U. Zapata, R. I. Balam, *High-order implicit finite difference schemes for the two-dimensional Poisson equation*, Appl. Math. Comput., **309** (2017), 222–244. 1
[24] S. Zhai, X. Feng, Y. He, *A new method to deduce high-order compact difference schemes for two-dimensional Poisson equation*, Appl. Math. Comput., **230** (2014), 9–26. 1