J. Math. Computer Sci., 32 (2024), 332–347

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

A note on degenerate Euler polynomials arising from umbral calculus

Sae Soon Lee[a], Jin-Woo Park[b,*]

[a]Department of Education, Mathematics Education Major, Graduate School of Education, Daegu University, 38453, Republic of Korea.

[b]Department of Mathematics Education, Daegu University, 38453, Republic of Korea.

Abstract

In this paper, we derive some interesting identities that found relationships between degenerate Euler polynomials and some special polynomials by using umbral calculus and λ-analogue of the Stirling numbers of the first and the second kind, respectively. In addition, we present explicit formulas for representations of the degenerate Euler polynomials.

Keywords: Degenerate Euler polynomials, umbral calculus, λ-analogue of the Stirling numbers of the first and second kind.

2020 MSC: 33E20, 05A30, 11B83, 11S80.

1. Introduction

The *Euler polynomials* are given by

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \text{ (see [1, 9, 10, 17, 35])}. \tag{1.1}$$

In the special case $x = 0$, $E_n = E_n(0)$ are called the *Euler numbers*.

As one of the special functions, the Euler numbers and polynomials have been generalized in various ways, and applications of these polynomials and numbers have also been investigated actively by many researchers (see [1, 33]). In particular, Elmonser defined q-extension of λ-Apostol-Euler polynomial by using symmetric q-extension function as one of the generalizations of Euler polynomials and found some properties of that function arising from the q-umbral calculus in [8]. Bouzeraib-Boussayoud-Aloui found relationships between Bernoulli and Euler numbers with k-Jacobsthal numbers, k-Jacobsthal-Lucas numbers, and bivariate Fibonacci, Lucas, Pell and Pell-Lucas polynomials in [2]. In [28], Kim-Lee constructed a new type of degenerate poly type 2 Euler polynomials by using the degenerate polylogarithm function and gave some new explicit expressions and identities related to degenerate unipoly type 2 Euler polynomials and some special numbers and polynomials.

*Corresponding author

Email addresses: `0417power@hanmail.net` (Sae Soon Lee), `a0417001@daegu.ac.kr` (Jin-Woo Park)

doi: 10.22436/jmcs.032.04.04

Received: 2023-06-28 *Revised:* 2023-08-23 *Accepted:* 2023-09-01

For any nonzero real number λ, the *degenerate exponential function* is defined to be

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \ e_\lambda(t) = (1+\lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty} (1)_{n,\lambda} \frac{t^n}{n!}, \text{ (see [3, 20])}, \tag{1.2}$$

where $(x)_{0,\lambda} = 1$ and $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$, $(n \geqslant 1)$.

The study of degenerate of functions was initiated by Carlitz, and since then, various versions of degenerate of special functions have been defined and their properties have been also studied by many researchers actively. In [11], Kim defined the degenerate Cauchy polynomials and found some interesting combinatorial identities corresponding to Stirling numbers of both kinds, Bernoulli numbers, and Cauchy numbers of the first and second kinds. Kim-Kim defined degenerate poly-Bernoulli polynomials and presented explicit formulas in terms of the degenerate Bernoulli polynomials and Stirling numbers of the second kind in [18]. In [30], Kwon-Park defined the modified degenerate Changhee polynomials and derived some interesting identities and properties. Kim defined degenerate complete Bell polynomials and gave some explicit formulas related to degenerate Stirling numbers in [12]. In [13, 25], authors defined the degenerate Stirling numbers of the first and second kind and found relationships between these numbers and some special functions.

For nonzero integers n and k with $n \geqslant k$, the *Stirling numbers of the first kind* $S_1(n,k)$ and the *Stirling numbers of the second kind* $S_2(n,k)$ are defined by generating function to be

$$(x)_n = \sum_{k=0}^{n} S_1(n,k) x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \text{ (see [3, 6, 35])}, \tag{1.3}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geqslant 1)$.

As degenerate version of the Stirling numbers, the *degenerate Stirling numbers of the first kind* $S_{1,\lambda}(n,k)$ and the *degenerate Stirling numbers of the second kind* $S_{2,\lambda}(n,k)$ are introduced by Kim-Kim (see [13, 25]) as follows:

$$\frac{1}{k!}\left(\log_\lambda(1+t)\right)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!}, \text{ and } \frac{1}{k!}\left(e_\lambda(t)-1\right)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \tag{1.4}$$

where $\log_\lambda(t) = \frac{1}{\lambda}(t^\lambda - 1)$ is the compositional inverse of $e_\lambda(t)$ satisfying $\log_\lambda(e_\lambda(t)) = e_\lambda\left(\log_\lambda(t)\right) = t$.

As another degenerate version of the Stirling numbers, the λ-*analogue of the Stirling numbers of the first kind* and *the second kind*, $S_\lambda^{(1)}(n,k)$ and $S_\lambda^{(2)}(n,k)$, respectively, are defined by

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_\lambda^{(1)}(n,k) x^k \text{ and } x^n = \sum_{k=0}^{n} S_\lambda^{(2)}(n,k)(x)_{k,\lambda}, \text{ (see [21, 26])}. \tag{1.5}$$

By (1.5), we see that

$$\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k = \sum_{n=k}^{\infty} S_\lambda^{(1)}(n,k)\frac{t^n}{n!}, \text{ and } \frac{1}{k!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^k = \sum_{n=k}^{\infty} S_\lambda^{(2)}(n,k)\frac{t^n}{n!},$$

(see [21, 26]).

2. Review of umbral calculus

Let $\mathbb{C}$ be the complex numbers field,

$$\mathcal{F} = \left\{ f(t) = \sum_{n=0}^{\infty} a_k \frac{t^n}{n!} \,\middle|\, a_k \in \mathbb{C} \right\},$$

and let

$$\mathbb{P} = \mathbb{C}[x] = \left\{ \sum_{k=0}^{\infty} a_l x^k \middle| a_k \in \mathbb{C} \text{ with } a_k = 0 \text{ for all but finite number of } k \right\}.$$

Let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. For a given $f(t) \in \mathbb{P}$, linear functional $\langle f(t)| \cdot \rangle$ on $\mathbb{P}$ is defined by

$$\langle f(t)| x^n \rangle = a_n, (n \geqslant 0), \text{ (see [31, 35, 36]).} \tag{2.1}$$

From (2.1), we have

$$\left\langle t^k \middle| x^n \right\rangle = n!\delta_{n,k}, \ (n, k \geqslant 0),$$

where $\delta_{n,k}$ is Kronecker's symbol. For each $k \in \mathbb{N} \cup \{0\}$, the differential operator on $\mathbb{P}$ is defined by

$$\left(t^k\right) x^n = \begin{cases} (n)_k x^{n-k}, & \text{if } k \leqslant n, \\ 0, & \text{if } k > n, \end{cases} \tag{2.2}$$

and, by (2.2), we see that for any $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$,

$$(f(t))\, x^n = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k}, \text{ (see [31, 35, 36]).}$$

Furthermore, it can is shown that for $f(t), g(t) \in \mathcal{F}$, and $p(x) \in \mathbb{P}$,

$$\langle f(t)g(t)| p(x) \rangle = \langle g(t)| (f(t))p(x) \rangle = \langle f(t)| (g(t))p(x) \rangle .$$

The *order* of $f(t) \in \mathcal{F} - \{0\}$, denoted by $o(f(t))$, is the smallest nonnegative integer k for which the coefficient of t^k does not vanish. If $o(f(t)) = 0$, then $f(t)$ is called *invertible*, and if $o(f(t)) = 1$, then $f(t)$ is called *delta series*.

Note that the invertible series has a multiplicative inverse $\frac{1}{f(t)}$ of $f(t)$, and the delta series has a compositional inverse $\bar{f}(t)$ of $f(t)$ with $\bar{f}(f(t)) = f(\bar{f}(t)) = t$.

For a delta series $f(t)$ and an invertible series $g(t)$, there is a unique sequence $S_n(x)$ with $\deg S_n(x) = n$ of polynomials satisfying the orthogonality conditions

$$\left\langle g(t)\big(f(t)\big)^k \middle| S_n(x) \right\rangle = n!\delta_{n,k}, \quad (n, k \geqslant 0), \text{ (see [31, 35, 36]).} \tag{2.3}$$

In this case, $S_n(x)$ is called the *Sheffer sequence* for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$. It is a well-known fact that the sequence $S_n(x)$ is the Sheffer sequence for $(g(t), f(t))$ if and only if

$$\frac{1}{g\left(\bar{f}(t)\right)} e^{y\bar{f}(t)} = \sum_{n=0}^{\infty} S_n(y) \frac{t^n}{n!}, \text{ (see [31, 35, 36]),}$$

for all $y \in \mathbb{C}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$ such that $f(\bar{f}(t)) = \bar{f}(f(t)) = t$.

The following lemma is proved easily by (2.3).

Lemma 2.1 ([23, 27])**.** *Let $S_n(x)$ be a Sheffer sequences for $(g(t), f(t))$ and let $h(x) = \sum_{l=0}^{n} a_l S_l(x) \in \mathbb{P}$. Then*

$$a_k = \frac{1}{k!} \left\langle g(t)\, (f(t))^k \middle| h(x) \right\rangle.$$

In addition, the following theorem which is proved by Lemma 2.1, is one of the important tools for studying the properties of a special function by using umbral calculus.

Theorem 2.2 ([23, 35]). *Let S_n and r_n be Sheffer sequences for $(g(t), f(t))$ and $(h(t), l(t))$, respectively. Then we have*

$$S_n = \sum_{k=0}^{n} c_{n,k} r_k,$$

where

$$c_{n,k} = \frac{1}{k!} \left\langle \frac{h\left(\bar{f}(t)\right)}{g\left(\bar{f}(t)\right)} \left(l\left(\bar{f}(t)\right)\right)^k \middle| x^n \right\rangle.$$

The early umbral calculus consisted of symbolic techniques for sequence manipulation with no its mathematical rigor. In the 1970s, Rota made a completely solid foundations for theories based on modern ideas for linear functionals, linear operators, and adjacency functions (see [36]), and since then, many different umbral calculus has been applied in many fields such as graph theory with chromatic polynomials, topology, probability theory, combinatorial counting problem, statistics, mathematical physics, etc (see [23, 24, 27, 29, 31, 35]).

In particular, authors found some combinatorial identities between Catalan-Daehee polynomials and special polynomials by using degenerate umbral calculus (see [15]), and in [32], authors introduced a class of new integrals involving generalized Bessel functions and generalized Struve functions by using operational method and umbral formalization of Ramanujan master theorem.

In this paper, we derive some identities between degenerate Euler and Bernoulli polynomials, Bernoulli polynomials of the second kind, Daehee polynomials, Changhee polynomials, Bell polynomials, Lah-Bell polynomials, Frobenius-Euler polynomials of order r, and Mittag-Leffer polynomials by finding the coefficients of linear combinations of these polynomials.

3. Main results

In viewpoint of (1.1) and (1.2), the *degenerate Euler polynomials of order* r is defined by the generating function to be

$$\left(\frac{2}{e_\lambda(t)+1}\right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty} E_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}, \text{ (see [19]).} \tag{3.1}$$

In the special case $x = 0$, $E_{n,\lambda}^{(r)} = E_{n,\lambda}^{(r)}(0)$ are called the *degenerate Euler numbers of order* r. In particular, if r=1, then $E_{n,\lambda}(x) = E_{n,\lambda}^{(1)}(x)$ and $E_{n,\lambda} = E_{n,\lambda}^{(1)}(0)$ are called the *degenerate Euler polynomials* and *degenerate Euler numbers*, respectively. Note that, by (1.5) and (3.1), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} E_{n,\lambda}(x)\frac{t^n}{n!} = \frac{2}{e_\lambda(t)+1}e_\lambda^x(t) &= \left(\sum_{n=0}^{\infty} E_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}E_{n-m,\lambda}(x)_{m,\lambda}\right)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}E_{n-m,\lambda}S_\lambda^{(1)}(m,k)x^k\right)\frac{t^n}{n!},
\end{aligned} \tag{3.2}$$

and so, by (3.2), we see that

$$E_{n,\lambda}(x) = \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}E_{n-m,\lambda}S_\lambda^{(1)}(m,k)x^k. \tag{3.3}$$

In addition, the Sheffer sequence of $E_{n,\lambda}(x)$ is

$$E_{n,\lambda}(x) \sim \left(\frac{e^t+1}{2}, \frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right).$$

The *Bernoulli polynomials* are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\ \text{(see [2, 17])}. \tag{3.4}$$

In the special case $x=0$, $B_n=B_n(0)$ are called the *Bernoulli numbers*. By (3.4), the Sheffer sequence of the Benoulli polynomials is

$$B_n(x)\sim\left(\frac{e^t-1}{t},t\right). \tag{3.5}$$

In addition, we note that

$$\frac{e_\lambda(t)-1}{t}=\sum_{n=0}^{\infty}\frac{(1)_{n+1,\lambda}}{n+1}\frac{t^n}{n!}\ \text{and}\ \frac{e^t+1}{2}=1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{t^n}{n!}. \tag{3.6}$$

Let $E_{n,\lambda}(x)=\sum_{l=0}^{n}a_{n,l}B_n(x)$. Then by Theorem 2.2, (3.5), and (3.6), we get

$$\begin{aligned}
a_{n,l}&=\frac{1}{l!}\left\langle\frac{\frac{\lambda(e_\lambda(t)-1)}{\log(1+\lambda t)}}{\frac{e_\lambda(t)+1}{2}}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^l\middle|x^n\right\rangle\\
&=\left\langle\frac{2}{e_\lambda(t)+1}\frac{\lambda t}{\log(1+\lambda t)}\frac{e_\lambda(t)-1}{t}\middle|\left(\frac{1}{l!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^l\right)x^n\right\rangle\\
&=\sum_{m=l}^{n}\binom{n}{m}S_\lambda^{(1)}(m,l)\left\langle\frac{2}{e_\lambda(t)+1}\frac{\lambda t}{\log(1+\lambda t)}\frac{e_\lambda(t)-1}{t}\middle|x^{n-m}\right\rangle\\
&=\sum_{m=l}^{n}\sum_{a=0}^{n-m}\binom{n}{m}\binom{n-m}{a}S_\lambda^{(1)}(m,l)E_{a,\lambda}\left\langle\frac{\lambda t}{\log(1+\lambda t)}\frac{e_\lambda(t)-1}{t}\middle|x^{n-m-a}\right\rangle\\
&=\sum_{m=l}^{n}\sum_{a=0}^{n-m}\sum_{c=0}^{n-m-a}\binom{n}{m}\binom{n-m}{a}\binom{n-m-a}{c}S_\lambda^{(1)}(m,l)\lambda^cE_{a,\lambda}b_{c,\lambda}\left\langle\frac{e_\lambda(t)-1}{t}\middle|x^{n-m-a-c}\right\rangle\\
&=\sum_{m=l}^{n}\sum_{a=0}^{n-m}\sum_{c=0}^{n-m-a}\binom{n}{m}\binom{n-m}{a}\binom{n-m-a}{c}\frac{(1)_{n-m-a-c+1,\lambda}S_\lambda^{(1)}(m,l)E_{a,\lambda}\lambda^cb_{c,\lambda}}{n-m-a-c+1},
\end{aligned} \tag{3.7}$$

where $b_{n,\lambda}=\lambda^nb_n$ are the *Bernoulli numbers of the second kind* which are defined by the generating function to be

$$\frac{t}{\log(1+t)}=\sum_{n=0}^{\infty}b_n\frac{t^n}{n!},\ \text{(see [24])}.$$

In addition, by Lemma 2.1, (3.3), and (3.6), we get

$$\begin{aligned}
a_{n,l}=\frac{1}{l!}\left\langle\frac{e(t)-1}{t}t^l\middle|E_{n,\lambda}(x)\right\rangle&=\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}E_{n-m,\lambda}S_\lambda^{(1)}(m,k)\frac{1}{l!}\left\langle\frac{e^t-1}{t}t^l\middle|x^k\right\rangle\\
&=\sum_{m=0}^{n}\sum_{k=l}^{m}\binom{n}{m}\binom{k}{l}E_{n-m,\lambda}S_\lambda^{(1)}(m,k)\left\langle\frac{e^t-1}{t}\middle|x^{k-l}\right\rangle\\
&=\sum_{m=0}^{n}\sum_{k=l}^{m}\binom{n}{m}\binom{k}{l}\frac{E_{n-m,\lambda}S_\lambda^{(1)}(m,k)(1)_{k-l+1}}{k-l+1}.
\end{aligned} \tag{3.8}$$

Conversely, assume that $B_n(x) = \sum_{l=0}^{n} b_{n,l} E_{l,\lambda}(x)$. Then, by Theorem 2.2 and (3.6), we get

$$\begin{aligned} b_{n,l} &= \frac{1}{l!}\left\langle \frac{\frac{e^t+1}{2}}{\frac{e^t-1}{t}} \left(\frac{e^{\lambda t}-1}{\lambda}\right)^l \middle| x^n \right\rangle \\ &= \left\langle \frac{t}{e^t-1}\frac{e^t+1}{2} \middle| \left(\frac{1}{l!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\right) x^n \right\rangle \\ &= \sum_{m=l} \binom{n}{m} S_\lambda^{(2)}(m,l) \left\langle \frac{t}{e^t-1}\frac{e^t+1}{2} \middle| x^{n-m} \right\rangle \\ &= \sum_{m=l}^{n}\sum_{a=0}^{n-m} \binom{n}{m}\binom{n-m}{a} S_\lambda^{(2)}(m,l) B_a \left\langle \frac{e^t+1}{2} \middle| x^{n-m-a} \right\rangle \\ &= \sum_{m=l}^{n} \binom{n}{m} S_\lambda^{(2)}(m,l) B_{n-m} + \frac{1}{2}\sum_{m=l}^{n}\sum_{a=0}^{n-m-1} \binom{n}{m}\binom{n-m}{a} S_\lambda^{(2)}(m,l) B_a. \end{aligned} \tag{3.9}$$

By (3.7), (3.8), and (3.9), we obtain the following theorem.

Theorem 3.1. *For each nonnegative integer* n, *we have*

$$\begin{aligned} E_{n,\lambda}(x) &= \sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{a=0}^{n-m}\sum_{c=0}^{n-m-a} \binom{n}{m}\binom{n-m}{a}\binom{n-m-a}{c} \frac{(1)_{n-m-a-c+1,\lambda} S_\lambda^{(1)}(m,l) E_{a,\lambda}\lambda^c b_{c,\lambda}}{n-m-a-c+1}\right) B_l(x) \\ &= \sum_{l=0}^{n}\left(\sum_{m=0}^{n}\sum_{k=l}^{m} \binom{n}{m}\binom{k}{l} \frac{E_{n-m,\lambda} S_\lambda^{(1)}(m,k)(1)_{k-l+1}}{k-l+1}\right) B_l(x), \end{aligned}$$

and

$$B_n(x) = \sum_{l=0}^{n}\left(\sum_{m=l}^{n}\binom{n}{m} S_\lambda^{(2)}(m,l) B_{n-m} + \frac{1}{2}\sum_{m=l}^{n}\sum_{a=0}^{n-m-1}\binom{n}{m}\binom{n-m}{a} S_\lambda^{(2)}(m,l) B_a\right) E_{l,\lambda}(x).$$

The *Bernoulli polynomials of the second kind* are defined by the generating function to be

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}, \quad \text{(see [24]).} \tag{3.10}$$

In the special case $x = 0$, $b_n = b_n(0)$ are the Bernoulli numbers of the second kind. By (3.10), we see that

$$b_n(x) \sim \left(\frac{t}{e^t-1}, e^t-1\right). \tag{3.11}$$

By (1.4), Theorem 2.2 and (3.11), we get

$$\begin{aligned} a_{n,l} &= \frac{1}{l!}\left\langle \frac{\frac{\frac{1}{\lambda}\log(1+\lambda t)}{e_\lambda(t)-1}}{\frac{e_\lambda(t)+1}{2}} (e_\lambda(t)-1)^l \middle| x^n \right\rangle \\ &= \left\langle \frac{2}{e_\lambda(t)+1}\frac{t}{e_\lambda(t)-1}\frac{\log(1+\lambda t)}{\lambda t} \middle| \left(\frac{1}{l!}(e_\lambda(t)-1)^l\right) x^n \right\rangle \\ &= \sum_{m=l}^{n}\binom{n}{m} S_{2,\lambda}(m,l) \left\langle \frac{2}{e_\lambda(t)+1}\frac{t}{e_\lambda(t)-1}\frac{\log(1+\lambda t)}{\lambda t} \middle| x^{n-m} \right\rangle \end{aligned} \tag{3.12}$$

$$
\begin{aligned}
&= \sum_{m=l}^{n} \sum_{a=0}^{n-m} \binom{n}{m} \binom{n-m}{a} S_{2,\lambda}(m,l) E_{a,\lambda} \left\langle \frac{t}{e_\lambda(t)-1} \frac{\log(1+\lambda t)}{\lambda t} \middle| x^{n-m-a} \right\rangle \\
&= \sum_{m=l}^{n} \sum_{a=0}^{n-m} \sum_{b=0}^{n-m-a} \binom{n}{m} \binom{n-m}{a} \binom{n-m-a}{b} S_{2,\lambda}(m,l) E_{a,\lambda} B_{b,\lambda} \left\langle \frac{\log(1+\lambda t)}{\lambda t} \middle| x^{n-m-a-b} \right\rangle \\
&= \sum_{m=l}^{n} \sum_{a=0}^{n-m} \sum_{b=0}^{n-m-a} \binom{n}{m} \binom{n-m}{a} \binom{n-m-a}{b} S_{2,\lambda}(m,l) E_{a,\lambda} B_{b,\lambda} \lambda^{n-m-a-b} D_{n-m-a-b},
\end{aligned}
$$

where $B_{n,\lambda}$ are the *degenerate Bernoulli numbers* which are defined by the generating function to be

$$\frac{t}{e_\lambda(t)-1} = \sum_{n=0}^{\infty} B_{n,\lambda} \frac{t^n}{n!}, \text{ (see [3])},$$

and D_n are the *Daehee numbers* which are defined as follows:

$$\frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}, \text{ (see [4, 27])}.$$

Conversely, assume that $b_n(x) = \sum_{l=0}^{n} b_{n,l} E_{l,\lambda}(x)$. Note that, by (1.3),

$$
\begin{aligned}
\sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} = \frac{t}{\log(1+t)} (1+t)^x &= \left(\sum_{n=0}^{\infty} b_n \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} (x)_n \frac{t^n}{n!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} b_{n-m} (x)_m \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \sum_{k=0}^{m} \binom{n}{m} S_1(m,k) b_{n-m} x^k \right) \frac{t^n}{n!},
\end{aligned}
$$

and so we see that

$$b_n(x) = \sum_{m=0}^{n} \sum_{k=0}^{m} \binom{n}{m} S_1(m,k) b_{n-m} x^k. \tag{3.13}$$

By Lemma 2.1 and (3.13), we have

$$
\begin{aligned}
b_{n,l} &= \frac{1}{l!} \left\langle \frac{e^t+1}{2} \left(\frac{e^{\lambda t}-1}{\lambda} \right)^l \middle| b_n(x) \right\rangle \\
&= \sum_{m=0}^{n} \sum_{k=0}^{m} \binom{n}{m} S_1(m,k) b_{n-m} \left\langle \frac{e^t+1}{2} \middle| \left(\frac{1}{l!} \left(\frac{e^{\lambda t}-1}{\lambda} \right)^l \right) x^k \right\rangle \\
&= \sum_{m=0}^{n} \sum_{k=l}^{m} \sum_{a=l}^{k} \binom{n}{m} \binom{k}{a} S_1(m,k) S_\lambda^{(2)}(a,l) b_{n-m} \left\langle \frac{e^t+1}{2} \middle| x^{k-a} \right\rangle \\
&= \sum_{m=0}^{n} \sum_{k=l}^{m} \binom{n}{m} S_1(m,k) S_\lambda^{(2)}(k,l) b_{n-m} + \frac{1}{2} \sum_{m=0}^{n} \sum_{k=l}^{m} \sum_{a=l}^{k-1} \binom{n}{m} \binom{k}{a} S_1(m,k) S_\lambda^{(2)}(a,l) b_{n-m}.
\end{aligned} \tag{3.14}
$$

By (3.12) and (3.14), we obtain the following theorem.

Theorem 3.2. *For each nonnegative integer* n, *we have*

$$E_{n,\lambda}(x) = \sum_{l=0}^{n} \left(\sum_{m=l}^{n} \sum_{a=0}^{n-m} \sum_{b=0}^{n-m-a} \binom{n}{m} \binom{n-m}{a} \binom{n-m-a}{b} S_{2,\lambda}(m,l) E_{a,\lambda} B_{b,\lambda} \lambda^{n-m-a-b} D_{n-m-a-b} \right) b_l(x),$$

and

$$b_n(x)=\sum_{l=0}^{n}\left(\sum_{m=0}^{n}\sum_{k=l}^{m}\binom{n}{m}S_1(m,k)S_\lambda^{(2)}(k,l)b_{n-m}+\frac{1}{2}\sum_{m=0}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k-1}\binom{n}{m}\binom{k}{a}S_1(m,k)S_\lambda^{(2)}(a,l)b_{n-m}\right)E_{l,\lambda}(x).$$

The *Daehee polynomials* are defined by the generating function to be

$$\frac{\log(1+t)}{t}(1+t)^x=\sum_{n=0}^{\infty}D_n(x)\frac{t^n}{n!},\ \text{(see [4, 27])}.$$

In the special case $x=0$, $D_n=D_n(0)$ are called the Daehee numbers. By the similar way to (3.13), we see that

$$D_n(x)=\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}S_1(m,k)D_{n-m}x^k, \tag{3.15}$$

and the Sheffer sequence of the Daehee polynomials is

$$D_n(x)\sim\left(\frac{e^t-1}{t},e^t-1\right). \tag{3.16}$$

Let $E_{n,\lambda}(x)=\sum_{l=0}^{n}a_{n,l}D_l(x)$. Then, by (1.4), Theorem 2.2, and (3.16), we get

$$\begin{aligned}
a_{n,l}&=\frac{1}{l!}\left\langle\frac{\frac{e_\lambda(t)-1}{\frac{1}{\lambda}\log(1+\lambda t)}}{\frac{e_\lambda(t)+1}{2}}(e_\lambda(t)-1)^l\middle|x^n\right\rangle\\
&=\frac{1}{l!}\left\langle\frac{2}{e_\lambda(t)+1}\frac{\lambda t}{\log(1+\lambda t)}\frac{(e_\lambda(t)-1)^{l+1}}{t}\middle|x^n\right\rangle\\
&=\frac{l+1}{n+1}\left\langle\frac{2}{e_\lambda(t)+1}\frac{\lambda t}{\log(1+\lambda t)}\middle|\left(\frac{1}{(l+1)!}(e_\lambda(t)-1)^{l+1}\right)x^{n+1}\right\rangle\\
&=\frac{l+1}{n+1}\sum_{m=l+1}^{n+1}\binom{n+1}{m}S_{2,\lambda}(m,l+1)\left\langle\frac{2}{e_\lambda(t)+1}\frac{\lambda t}{\log(1+\lambda t)}\middle|x^{n-m+1}\right\rangle\\
&=\frac{l+1}{n+1}\sum_{m=l+1}^{n+1}\sum_{a=0}^{n-m+1}\binom{n+1}{m}\binom{n-m+1}{a}S_{2,\lambda}(m,l+1)E_{a,\lambda}\left\langle\frac{\lambda t}{\log(1+\lambda t)}\middle|x^{n-m-a+1}\right\rangle\\
&=\frac{l+1}{n+1}\sum_{m=l+1}^{n+1}\sum_{a=0}^{n-m+1}\binom{n+1}{m}\binom{n-m+1}{a}S_{2,\lambda}(m,l+1)E_{a,\lambda}\lambda^{n-m-a+1}b_{n-m-a+1}.
\end{aligned} \tag{3.17}$$

In addition, by (3.3), we have

$$\begin{aligned}
a_{n,l}&=\frac{1}{l!}\left\langle\frac{e^t-1}{t}(e^t-1)^l\middle|E_{n,\lambda}(x)\right\rangle\\
&=\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}E_{n-m,\lambda}S_\lambda^{(1)}(m,k)\frac{1}{l!}\left\langle\frac{(e^t-1)^{l+1}}{t}\middle|x^k\right\rangle\\
&=\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\frac{(l+1)E_{n-m,\lambda}S_\lambda^{(1)}(m,k)}{k+1}\left\langle\frac{1}{(l+1)!}(e^t-1)^{l+1}\middle|x^{k+1}\right\rangle\\
&=\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\frac{(l+1)E_{n-m,\lambda}S_\lambda^{(1)}(m,k)S_1(k+1,l+1)}{k+1}.
\end{aligned} \tag{3.18}$$

Conversely, assume that $D_n(x) = \sum_{l=0}^{n} b_{n,l} E_{l,\lambda}(x)$. Then, by Lemma 2.1 and (3.15), we get

$$\begin{aligned} b_{n,l} &= \frac{1}{l!}\left\langle \frac{e^t+1}{2}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l \middle| D_n(x)\right\rangle \\ &= \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m} S_1(m,k) D_{n-m}\left\langle \frac{e^t+1}{2}\middle| \left(\frac{1}{l!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\right) x^k\right\rangle \\ &= \sum_{m=0}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k}\binom{n}{m}\binom{k}{a} S_1(m,k) D_{n-m} S_{\lambda}^{(2)}(a,l)\left\langle \frac{e^t+1}{2}\middle| x^{k-a}\right\rangle \\ &= \sum_{m=0}^{n}\sum_{k=l}^{m}\binom{n}{m} S_1(m,k) D_{n-m} S_{\lambda}^{(2)}(k,l) + \frac{1}{2}\sum_{m=0}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k-1}\binom{n}{m}\binom{k}{a} S_1(m,k) D_{n-m} S_{\lambda}^{(2)}(a,l). \end{aligned} \tag{3.19}$$

By (3.17), (3.18), and (3.19), we obtain the following theorem.

Theorem 3.3. *For each nonnegative integer n, we have*

$$\begin{aligned} E_{n,\lambda}(x) &= \sum_{l=0}^{n}\left(\frac{l+1}{n+1}\sum_{m=l+1}^{n+1}\sum_{a=0}^{n-m+1}\binom{n+1}{m}\binom{n-m+1}{a} S_{2,\lambda}(m,l+1)\lambda^{n-m-a+1} E_{a,\lambda} b_{n-m-a+1}\right) D_l(x) \\ &= \sum_{l=0}^{n}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\frac{(l+1)S_{\lambda}^{(1)}(m,k)S_1(k+1,l+1)E_{n-m,\lambda}}{k+1}\right) D_l(x), \end{aligned}$$

and

$$\begin{aligned} D_n(x) = \sum_{l=0}^{n}\Bigg(&\sum_{m=0}^{n}\sum_{k=l}^{m}\binom{n}{m} S_1(m,k) S_{\lambda}^{(2)}(k,l) D_{n-m} \\ &+\frac{1}{2}\sum_{m=0}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k-1}\binom{n}{m}\binom{k}{a} S_1(m,k) S_{\lambda}^{(2)}(a,l) D_{n-m}\Bigg) E_{l,\lambda}(x). \end{aligned}$$

The *Changhee polynomials* are defined by the generating function to be

$$\frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!}, \quad \text{(see [4, 7, 30])}.$$

When $x = 0$, $Ch_n = Ch_n(0)$ are called the *Changhee numbers*. By the similar way to (3.13), we see that

$$Ch_n(x) = \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m} S_1(m,k) Ch_{n-m} x^k. \tag{3.20}$$

Let $E_{n,\lambda}(x) = \sum_{l=0}^{n} a_{n,l} Ch_l(x)$. Since

$$Ch_n(x) \sim \left(\frac{e^t+1}{2} e^t - 1\right), \tag{3.21}$$

by Theorem 2.2 and (3.21), we get

$$a_{n,l} = \frac{1}{l!}\left\langle \frac{\frac{e_\lambda(t)+1}{2}}{\frac{e_\lambda(t)+1}{2}}(e_\lambda(t)-1)^l \middle| x^n\right\rangle = \left\langle \frac{1}{l!}(e_\lambda(t)-1)^l \middle| x^n\right\rangle = S_{2,\lambda}(n,l). \tag{3.22}$$

In addition, by Lemma 2.1 and (3.3), we have

$$\begin{aligned} a_{n,l} &= \frac{1}{l!}\left\langle \frac{e^t+1}{2}\left(e^t-1\right)^l \middle| E_{n,\lambda}(x)\right\rangle \\ &= \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}E_{n-m,\lambda}S_\lambda^{(1)}(m,k)\left\langle \frac{e^t+1}{2}\middle|\left(\frac{1}{l!}\left(e^t-1\right)^l\right)x^k\right\rangle \\ &= \sum_{m=0}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k}\binom{n}{m}\binom{k}{a}E_{n-m,\lambda}S_\lambda^{(1)}(m,k)S_2(a,l)\left\langle \frac{e^t+1}{2}\middle| x^{k-a}\right\rangle \\ &= \sum_{m=0}^{n}\sum_{k=l}^{m}\binom{n}{m}E_{n-m,\lambda}S_\lambda^{(1)}(m,k)S_2(k,l)+\frac{1}{2}\sum_{m=0}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k-1}\binom{n}{m}\binom{k}{a}E_{n-m,\lambda}S_\lambda^{(1)}(m,k)S_2(a,l). \end{aligned} \tag{3.23}$$

Convesely, assume that $Ch_n(x)=\sum_{l=0}^{n} b_{n,l}E_{l,\lambda}(x)$. Then, by (3.20), we get

$$\begin{aligned} b_{n,l} &= \frac{1}{l!}\left\langle \frac{e^t+1}{2}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l \middle| Ch_n(x)\right\rangle \\ &= \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}S_1(m,k)Ch_{n-m}\left\langle \frac{e^t+1}{2}\middle|\left(\frac{1}{l!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\right)x^k\right\rangle \\ &= \sum_{m=0}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k}\binom{n}{k}\binom{k}{a}S_1(m,k)S_\lambda^{(2)}(a,l)Ch_{n-m}\left\langle \frac{e^t+1}{2}\middle| x^{k-a}\right\rangle \\ &= \sum_{m=0}^{n}\sum_{k=l}^{m}\binom{n}{m}S_1(m,k)S_\lambda^{(2)}(k,l)Ch_{n-m} \\ &\quad +\frac{1}{2}\sum_{m=0}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k-1}\binom{n}{m}\binom{k}{a}S_1(m,k)S_\lambda^{(2)}(a,l)Ch_{n-m}. \end{aligned} \tag{3.24}$$

By (3.22), (3.23), and (3.24), we obtain the following theorem.

Theorem 3.4. *For each nonnegative integer n, we have*

$$\begin{aligned} E_{n,\lambda}(x) = \sum_{l=0}^{n}\left(S_{2,\lambda}(n,l)\right)Ch_l(x) = \sum_{l=0}^{n}\Bigg(&\sum_{m=0}^{n}\sum_{k=l}^{m}\binom{n}{m}S_\lambda^{(1)}(m,k)S_2(k,l)E_{n-m,\lambda} \\ &+\frac{1}{2}\sum_{m=0}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k-1}\binom{n}{m}\binom{k}{a}S_\lambda^{(1)}(m,k)S_2(a,l)E_{n-m,\lambda}\Bigg)Ch_l(x), \end{aligned}$$

and

$$\begin{aligned} Ch_n(x) = \sum_{l=0}^{n}\Bigg(&\sum_{m=0}^{n}\sum_{k=l}^{m}\binom{n}{m}S_1(m,k)S_\lambda^{(2)}(k,l)Ch_{n-m} \\ &+\frac{1}{2}\sum_{m=0}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k-1}\binom{n}{m}\binom{k}{a}S_1(m,k)S_\lambda^{(2)}(a,l)Ch_{n-m}\Bigg)E_{l,\lambda}(x). \end{aligned}$$

The *Bell polynomials* are defined by the generating function to be

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}, \quad \text{(see [12, 20])}.$$

By the definition of the Bell polynomials, we see that

$$\mathrm{Bel}_n(x) \sim (1, \log(1+t)), \tag{3.25}$$

and

$$\mathrm{Bel}_n(x) = \sum_{m=0}^{n} S_2(n,m)x^m.$$

Note that

$$\begin{aligned}\frac{1}{l!}\left(\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)\right)^l &= \sum_{m=l}^{\infty} S_1(m,l)\frac{1}{m!}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^m \\ &= \sum_{m=l}^{\infty} S_1(m,l)\sum_{r=m}^{\infty} S_\lambda^{(1)}(r,m)\frac{t^r}{r!} = \sum_{a=l}^{\infty}\sum_{m=l}^{a} S_1(m,l)S_\lambda^{(1)}(a,m)\frac{t^a}{a!}.\end{aligned} \tag{3.26}$$

Let $E_{n,\lambda}(x) = \sum_{l=0}^{n} a_{n,l}\mathrm{Bel}_l(x)$. By Theorem 2.2, (3.25) and (3.26), we get

$$\begin{aligned}a_{n,l} &= \frac{1}{l!}\left\langle \frac{1}{\frac{e_\lambda(t)+1}{2}}\left(\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)\right)^l \middle| x^n \right\rangle \\ &= \left\langle \frac{2}{e_\lambda(t)+1} \middle| \left(\left(\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)\right)^l\right) x^n \right\rangle \\ &= \sum_{a=l}^{n}\sum_{m=l}^{a}\binom{n}{a}S_1(m,l)S_\lambda^{(1)}(a,m)\left\langle \frac{2}{e_\lambda(t)+1}\middle| x^{n-a}\right\rangle \\ &= \sum_{a=l}^{n}\sum_{m=l}^{a}\binom{n}{a}S_1(m,l)S_\lambda^{(1)}(a,m)E_{n-a,\lambda}.\end{aligned} \tag{3.27}$$

Conversely, assume that $\mathrm{Bel}_n(x) = \sum_{l=0}^{n} b_{n,l}E_{l,\lambda}(x)$. Since

$$\begin{aligned}\frac{1}{l!}\left(\frac{e^{\lambda(e^t-1)}-1}{\lambda}\right)^l &= \sum_{m=l}^{\infty} S_\lambda^{(2)}(m,l)\frac{1}{m!}\left(e^t-1\right)^m \\ &= \sum_{m=l}^{\infty} S_\lambda^{(2)}(m,l)\sum_{r=m}^{\infty} S_2(r,m)\frac{t^r}{r!} = \sum_{n=l}^{\infty}\sum_{m=l}^{n} S_\lambda^{(2)}(m,l)S_2(n,m)\frac{t^n}{n!},\end{aligned} \tag{3.28}$$

and

$$\frac{e^{e^t-1}+1}{2} = \frac{1}{2}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(e^t-1\right)^n = \frac{1}{2}+\frac{1}{2}\sum_{n=0}^{\infty}\sum_{r=n}^{\infty}S_2(r,n)\frac{t^r}{r!} = 1+\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{n}S_2(n,m)\frac{t^n}{n!}, \tag{3.29}$$

by (3.28) and (3.29), we get

$$\begin{aligned}b_{n,l} &= \frac{1}{l!}\left\langle \frac{e^{e^t-1}+1}{2}\left(\frac{e^{\lambda(e^t-1)}-1}{\lambda}\right)^l \middle| x^n\right\rangle \\ &= \left\langle \frac{e^{e^t-1}+1}{2}\middle| \left(\frac{1}{l!}\left(\frac{e^{\lambda(e^t-1)}-1}{\lambda}\right)^l\right) x^n\right\rangle \\ &= \sum_{a=l}^{n}\sum_{m=l}^{a}\binom{n}{a}S_\lambda^{(2)}(m,l)S_2(a,m)\left\langle \frac{e^{e^t-1}+1}{2}\middle| x^{n-a}\right\rangle \\ &= \sum_{m=l}^{n} S_\lambda^{(2)}(m,l)S_2(n,m) + \frac{1}{2}\sum_{a=l}^{n-1}\sum_{m=l}^{a}\sum_{b=1}^{n-a}\binom{n}{a}S_\lambda^{(2)}(m,l)S_2(a,m)S_2(n-a,b).\end{aligned} \tag{3.30}$$

By (3.27) and (3.30), we obtain the following theorem.

Theorem 3.5. *For each nonnegative integer* n*, we have*

$$E_{n,\lambda}(x) = \sum_{l=0}^{n}\left(\sum_{a=l}^{n}\sum_{m=l}^{a}\binom{n}{a}S_1(m,l)S_{\lambda}^{(1)}(a,m)E_{n-a,\lambda}\right)\mathrm{Bel}_l(x),$$

and

$$\mathrm{Bel}_n(x) = \sum_{l=0}^{n}\left(\sum_{m=l}^{n}S_{\lambda}^{(2)}(m,l)S_2(n,m) + \frac{1}{2}\sum_{a=l}^{n-1}\sum_{m=l}^{a}\sum_{b=1}^{n-a}\binom{n}{a}S_{\lambda}^{(2)}(m,l)S_2(a,m)S_2(n-a,b)\right)E_{l,\lambda}(x).$$

The *unsigned Lah number* $L(n,k)$ has the explicit formula

$$L(n,k) = \binom{n-1}{k-1}\frac{n!}{k!} \quad \text{and} \quad \frac{1}{k!}\left(\frac{t}{1-t}\right)^k = \sum_{n=k}^{\infty}L(n,k)\frac{t^n}{n!}, \text{ (see [22, 37]).}$$

In [14, 20, 22], Kim-Kim defined the *Lah-Bell polynomials* by the generating function to be

$$e^{\frac{xt}{1-t}} = \sum_{n=0}^{\infty}B_n^L(x)\frac{t^n}{n!}.$$

In the special case $x = 1$, $B_n^L = B_n^L(1)$ are called the *Lah-Bell numbers*. Note that

$$e^{x\frac{t}{1-t}} = \sum_{n=0}^{\infty}x^n\frac{1}{n!}\left(\frac{t}{1-t}\right)^n = \sum_{n=0}^{\infty}\sum_{m=0}^{n}x^mL(n,m)\frac{t^n}{n!},$$

and so

$$B_n^L(x) = \sum_{m=0}^{n}L(n,m)x^m, \tag{3.31}$$

and the Sheffer sequence of the Lah-Bell polynomials is

$$B_n^L(x) \sim \left(1, \frac{t}{1+t}\right). \tag{3.32}$$

Let $E_{n,\lambda}(x) = \sum_{l=0}^{n} a_{n,l}B_l^L(x)$. Since

$$\left(\frac{t}{1+t}\right)^l = \sum_{r=0}^{\infty}(-1)^r < l >_r \frac{t^{r+l}}{r!},$$

where $< x >_0 = 1$ and $< x >_n = x(x+1)\cdots(x+n-1)$, $(n \geqslant 1)$, by Lemma 2.1, (3.3), and (3.32), we get

$$\begin{aligned} a_{n,l} = \frac{1}{l!}\left\langle \left(\frac{t}{1+t}\right)^l \middle| E_{n,\lambda}(x)\right\rangle &= \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}E_{n-m,\lambda}S_{\lambda}^{(1)}(m,k)\frac{1}{l!}\left\langle \left(\frac{t}{1+t}\right)^l \middle| x^k\right\rangle \\ &= \sum_{m=0}^{n}\sum_{k=l}^{m}\binom{n}{m}\binom{k}{l}E_{n-m,\lambda}S_{\lambda}^{(1)}(m,k)(-1)^{k-l} < l >_{k-l}. \end{aligned} \tag{3.33}$$

Conversely, assume that $B_n^L(x) = \sum_{l=0}^{n} b_{n,l}E_{l,\lambda}(x)$. Then by Lemma 2.1 and (3.31), we get

$$b_{n,l} = \frac{1}{l!}\left\langle \left(\frac{e^t+1}{2}\right)\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l \middle| B_n^L(x)\right\rangle$$

$$\begin{aligned}
&= \sum_{m=0}^{n} L(n,m) \left\langle \frac{e^t+1}{2} \middle| \left(\frac{1}{l!} \left(\frac{e^{\lambda t}-1}{\lambda} \right)^l \right) x^m \right\rangle \\
&= \sum_{m=0}^{n} \sum_{a=l}^{m} \binom{m}{a} L(n,m) S_\lambda^{(2)}(a,l) \left\langle \frac{e^t+1}{2} \middle| x^{m-a} \right\rangle \\
&= \sum_{m=0}^{n} L(n,m) S_\lambda^{(2)}(m,l) + \frac{1}{2} \sum_{m=0}^{n} \sum_{a=l}^{m-1} \binom{m}{a} L(n,m) S_\lambda^{(2)}(a,l).
\end{aligned} \tag{3.34}$$

By (3.33) and (3.34), we obtain the following theorem.

Theorem 3.6. *For each nonnegative integer n, we have*

$$E_{n,\lambda}(x) = \sum_{l=0}^{n} \left(\sum_{m=0}^{n} \sum_{k=l}^{m} \binom{n}{m} \binom{k}{l} E_{n-m,\lambda} S_\lambda^{(1)}(m,k)(-1)^{k-l} < l >_{k-l} \right) B_l^L(x),$$

and

$$B_n^L(x) = \sum_{l=0}^{n} \left(\sum_{m=0}^{n} L(n,m) S_\lambda^{(2)}(m,l) + \frac{1}{2} \sum_{m=0}^{n} \sum_{a=l}^{m-1} \binom{m}{a} L(n,m) S_\lambda^{(2)}(a,l) \right) E_{l,\lambda}(x).$$

The *Frobenius-Euler polynomials of order* r are defined by the generating function to be

$$\left(\frac{1-u}{e^t-u} \right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|u) \frac{t^n}{n!}, \text{ (see [5, 16, 29, 34]).}$$

In the special case $x = 0$, $H_n^{(r)}(u) = H_n^{(r)}(0|u)$ are called the *Frobenius-Euler numbers of order* r. By the definition of Frobenius-Euler polynomials of order r, we see the Sheffer sequence of those polynomials of order r is

$$H_n^{(r)}(x|u) \sim \left(\left(\frac{e^t-u}{1-u} \right)^r, t \right). \tag{3.35}$$

Since

$$(e_\lambda(t)-u)^r = \sum_{b=0}^{r} \binom{r}{b} (-u)^{r-b} e_\lambda^b(t) = \sum_{c=0}^{\infty} \sum_{b=0}^{r} \binom{r}{b} (-u)^{r-b} (b)_{c,\lambda} \frac{t^c}{c!}, \tag{3.36}$$

by Theorem 2.2, (3.35), and (3.36), we get

$$\begin{aligned}
a_{n,l} &= \frac{1}{l!} \left\langle \frac{\left(\frac{e_\lambda(t)-u}{1-u} \right)^r}{\frac{e_\lambda(t)+1}{2}} \left(\frac{\log(1+\lambda t)}{\lambda} \right)^l \middle| x^n \right\rangle \\
&= \frac{1}{(1-u)^r} \left\langle \frac{2}{e_\lambda(t)+1} (e_\lambda(t)-u)^r \middle| \left(\frac{1}{l!} \left(\frac{\log(1+\lambda t)}{\lambda} \right)^l \right) x^n \right\rangle \\
&= \frac{1}{(1-u)^r} \sum_{r=l}^{n} \binom{n}{r} S_\lambda^{(1)}(r,l) \left\langle \frac{2}{e_\lambda(t)+1} (e_\lambda(t)-u)^r \middle| x^{n-r} \right\rangle \\
&= \frac{1}{(1-u)^r} \sum_{r=l}^{n} \sum_{a=0}^{n-r} \binom{n}{r} \binom{n-r}{a} S_\lambda^{(1)}(r,l) E_{a,\lambda} \left\langle (e_\lambda(t)-u)^r | x^{n-r-a} \right\rangle \\
&= \sum_{r=l}^{n} \sum_{a=0}^{n-r} \sum_{b=0}^{r} \binom{n}{r} \binom{n-r}{a} \binom{r}{b} \frac{S_\lambda^{(1)}(r,l) E_{a,\lambda} (-u)^{r-b} (b)_{n-r-a,\lambda}}{(1-u)^r}.
\end{aligned} \tag{3.37}$$

Conversely, assume that $H_n^{(r)}(x|u) = \sum_{l=0}^{n} b_{n,l} E_{l,\lambda}(x)$. Then,

$$\begin{aligned} b_{n,l} &= \frac{1}{l!}\left\langle \frac{\frac{e^t+1}{2}}{\left(\frac{e^t-u}{1-u}\right)^r}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l \middle| x^n \right\rangle \\ &= \left\langle \frac{e^t+1}{2}\left(\frac{1-u}{e^t-u}\right)^r \middle| \left(\frac{1}{l!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\right) x^n \right\rangle \\ &= \sum_{m=l}^{n}\binom{n}{m} S_\lambda^{(2)}(m,l)\left\langle \frac{e^t+1}{2}\left(\frac{1-u}{e^t-u}\right)^r \middle| x^{n-m}\right\rangle \\ &= \sum_{m=l}^{n}\sum_{a=0}^{n-m}\binom{n}{m}\binom{n-m}{a} S_\lambda^{(2)}(m,l) H_a^{(r)}(u)\left\langle \frac{e^t+1}{2} \middle| x^{n-m-a}\right\rangle \\ &= \sum_{m=l}^{n}\binom{n}{m} S_\lambda^{(2)}(m,l) H_{n-m}^{(r)}(u) + \frac{1}{2}\sum_{m=l}^{n}\sum_{a=0}^{n-m-1}\binom{n}{m}\binom{n-m}{a} S_\lambda^{(2)}(m,l) H_a^{(r)}(u). \end{aligned} \tag{3.38}$$

By (3.37) and (3.38), we obtain the following theorem.

Theorem 3.7. *For each nonnegative integer* n*, we have*

$$E_{n,\lambda}(x) = \sum_{l=0}^{n}\left(\sum_{r=l}^{n}\sum_{a=0}^{n-r}\sum_{b=0}^{r}\binom{n}{r}\binom{n-r}{a}\binom{r}{b}\frac{S_\lambda^{(1)}(r,l)E_{a,\lambda}(-u)^{r-b}(b)_{n-r-a,\lambda}}{(1-u)^r}\right) H_l^{(r)}(x|u),$$

and

$$H_n^{(r)}(x|u) = \sum_{l=0}^{n}\left(\sum_{m=l}^{n}\binom{n}{m} S_\lambda^{(2)}(m,l) H_{n-m}^{(r)}(u) + \frac{1}{2}\sum_{m=l}^{n}\sum_{a=0}^{n-m-1}\binom{n}{m}\binom{n-m}{a} S_\lambda^{(2)}(m,l) H_a^{(r)}(u)\right) E_{l,\lambda}(x).$$

The *mittag-Leffee polynomials* are defined by the generating function to be

$$\left(\frac{1+t}{1-t}\right)^x = \sum_{n=0}^{\infty} M_n(x)\frac{t^n}{n!}, \text{ (see [14, 31])}.$$

When $x = 1$, $M_n = M_n(1)$ are called the *Mittag-Leffer numbers*. By the definition of the Mittag-Leffer polynomials, the Sheffer sequence of those polynomials is

$$M_n(x) \sim \left(1, \frac{e^t-1}{e^t+1}\right).$$

Let $E_{n,\lambda}(x) = \sum_{l=0}^{n} a_{n,l} M_l(x)$. Then by Theorem 2.2, we get

$$\begin{aligned} a_{n,l} &= \frac{1}{l!}\left\langle \frac{1}{\frac{e_\lambda(t)+1}{2}}\left(\frac{e_\lambda(t)-1}{e_\lambda(t)+1}\right)^l \middle| x^n \right\rangle \\ &= \left\langle \frac{2}{(e_\lambda(t)+1)^{l+1}} \middle| \left(\frac{1}{l!}(e_\lambda(t)-1)^l\right) x^n \right\rangle \\ &= \sum_{m=l}^{n}\binom{n}{m} S_{2,\lambda}(m,l)\frac{1}{2^l}\left\langle \left(\frac{2}{e_\lambda(t)+1}\right)^{l+1} \middle| x^{n-m}\right\rangle \\ &= \sum_{m=l}^{n}\binom{n}{m}\frac{S_{2,\lambda}(m,l)}{2^l} E_{n-m,\lambda}^{(l+1)}. \end{aligned} \tag{3.39}$$

Conversely, assume that $M_n(x) = \sum_{l=0}^{n} b_{n,l} E_{l,\lambda}(x)$. Note that

$$\left(\frac{1+t}{1-t}\right)^x = \left(1 + \frac{2t}{1-t}\right)^x = \sum_{m=0}^{\infty} (x)_m 2^m \frac{1}{m!} \left(\frac{t}{1-t}\right)^m$$
$$= \sum_{a=0}^{\infty} \sum_{m=0}^{a} 2^m L(a,m)(x)_m \frac{t^a}{a!} = \sum_{a=0}^{\infty} \sum_{m=0}^{a} \sum_{r=0}^{m} 2^m L(a,m) S_1(m,r) x^r \frac{t^a}{a!},$$

and thus we see that

$$M_n(x) = \sum_{m=0}^{n} \sum_{r=0}^{m} 2^m L(n,m) S_1(m,r) x^r. \tag{3.40}$$

By Lemma 2.1 and (3.40), we get

$$\begin{aligned} b_{n,l} &= \frac{1}{l!} \left\langle \frac{e^t+1}{2} \left(\frac{e^{\lambda t}-1}{\lambda}\right)^l \middle| M_n(x) \right\rangle \\ &= \sum_{m=0}^{n} \sum_{r=0}^{m} 2^m L(n,m) S_1(m,r) \left\langle \frac{e^t+1}{2} \middle| \left(\frac{1}{l!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\right) x^r \right\rangle \\ &= \sum_{m=0}^{n} \sum_{r=0}^{m} \sum_{a=l}^{r} \binom{r}{a} 2^m L(n,m) S_1(r,m) S_\lambda^{(2)}(a,l) \left\langle \frac{e^t+1}{2} \middle| x^{r-a} \right\rangle \\ &= \sum_{m=0}^{n} \sum_{r=0}^{m} 2^m L(n,m) S_1(r,m) S_\lambda^{(2)}(r,l) + \frac{1}{2} \sum_{m=0}^{n} \sum_{r=0}^{m} \sum_{a=l}^{r-1} \binom{r}{a} 2^m L(n,m) S_1(r,m) S_\lambda^{(2)}(a,l). \end{aligned} \tag{3.41}$$

By (3.39) and (3.41), we obtain the following theorem.

Theorem 3.8. *For each nonnegative integer n, we have*

$$E_{n,\lambda}(x) = \sum_{l=0}^{n} \left(\sum_{m=l}^{n} \binom{n}{m} \frac{S_{2,\lambda}(m,l)}{2^l} E_{n-m,\lambda}^{(l+1)} \right) M_l(x),$$

and

$$\begin{aligned} M_n(x) = \sum_{l=0}^{n} \Bigg(&\sum_{m=0}^{n} \sum_{r=0}^{m} 2^m L(n,m) S_1(r,m) S_\lambda^{(2)}(r,l) \\ &+ \frac{1}{2} \sum_{m=0}^{n} \sum_{r=0}^{m} \sum_{a=l}^{r-1} \binom{r}{a} 2^m L(n,m) S_1(r,m) S_\lambda^{(2)}(a,l) \Bigg) E_{l,\lambda}(x), \end{aligned}$$

Acknowledgments

The authors would like to thank the referees for their valuable and detailed comments which have significantly improved the presentation of this paper. This research was supported by Daegu University Future Scholars Program, 2023.

References

[1] G. E. Andrew, R. Askey, R. Roy, *Special functions: Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, (1999). 1.1, 1

[2] M. Bouzeraib, A. Boussayoud, B. Aloui, *Convolutions with the Bernoulli and Euler numbers*, J. Integer Seq., **26** (2023), 18 pages. 1, 3.4

[3] L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math., **15** (1979), 51–88. 1.2, 1.3, 3

[4] Y.-K. Cho, T. Kim, T. Mansour, S.-H. Rim, *On a (r, s)-analogue of Changhee and Daehee numbers and polynomials*, Kyungpook Math. J., **55** (2015), 225–232. 3, 3, 3

[5] J. Choi, D. S. Kim, T. Kim, Y. H. Kim, *A note on some identities of Frobenius-Euler numbers and polynomials*, Int. J. Math. Math. Sci., **2012** (2012), 9 pages. 3

[6] L. Comtet, *Advanced Combinatorics: The art of finite and infinite expansions*, D. Reidel Publishing Co., Dordrecht, (1974). 1.3

[7] D. V. Dolgy, W. A. Khan, *A Note on type-two degenerate poly-Changhee polynomials of the second Kind*, Symmetry, **13** (2021), 1–12. 3

[8] H. Elmonser, *Symmetric q-extension of λ-Apostol-Euler polynomials via umbral calculus*, Indian J. Pure Appl. Math., **54** (2023), 583–594. 1

[9] W. A. Khan, *Construction of the type 2 degenerate poly-Euler polynomials and numbers*, Mat. Vesnik, **75** (2023), 147–156. 1.1

[10] W. A. Khan, S. K. Sharma, *A note on type 2 polyexponential Euler polynomials and numbers*, J. Math. Control Sci. Appl., **7** (2021), 36–46. 1.1

[11] T. Kim, *On the degenerate Cauchy numbers and polynomials*, Proc. Jangjeon Math. Soc., **18** (2015), 307–312. 1

[12] T. Kim, *Degenerate complete Bell polynomials and numbers*, Proc. Jangjeon Math. Soc., **20** (2017), 533–543. 1, 3

[13] T. Kim, *A note on degenerate Stirling polynomials of the second kind*, Proc. Jangjeon Math. Soc., **20** (2017), 319–331. 1, 1

[14] H. K. Kim, *Degenerate Lah-Bell polynomials arising from degenerate Sheffer sequences*, Adv. Difference Equ., **2020** (2020) 16 pages. 3, 3

[15] H. K. Kim, D. V. Dolgy, *Degenerate Catalan-Daehee numbers and polynomials of order r arising from degenerate umbral calculus*, AIMS Math., **7** (2022), 3845–3865. 2

[16] H. K. Kim, W. A. Khan, *Some identities of a new type of degenerate poly-Frobenius-Euler polynomials and numbers*, Proc. Jangjeon Math. Soc., **24** (2021), 33–45. 3

[17] D. S. Kim, T. Kim, *Some identities of Bernoulli and Euler polynomials arising form umbral calculus*, Adv. Stud. Contemp. Math. (Kyungshang), **23** (2013) 159–171. 1.1, 3.4

[18] D. S. Kim, T. Kim, *A note on degenerate poly-Bernoulli numbers and polynomials*, Adv. Difference Equ., **2015** (2015) 8 pages. 1

[19] D. S. Kim, T. Kim, *Higher-order Degenerate Euler Polynomials*, Appl. Math. Sci., **9** (2015), 57–73. 3.1

[20] T. Kim, D. S. Kim, *Degenerate polyexponential functions and degenerate Bell polynomials*, J. Math. Anal. Appl., **487** (2020), 15 pages. 1.2, 3, 3

[21] T. Kim, D. S. Kim, *Some identities on λ-analogues of r-Stirling numbers of the first kind*, Filomat, **34** (2020), 451–460. 1.5, 1

[22] D. S. Kim, T. Kim, *Lah-Bell numbers and polynomials*, Proc. Jangjeon Math. Soc., **23** (2020), 577–586. 3.31, 3

[23] D. S. Kim, T. Kim, *Degenerate Sheffer sequences and λ-Sheffer sequences*, J. Math. Anal. Appl., **493** (2021), 21 pages. 2.1, 2.2, 2

[24] T. Kim, D. S. Kim, D. V. Dolgy, J.-J. Seo, *Bernoulli polynomials of the second kind and their identities arising from umbral calculus*, J. Nonlinear Sci. Appl., **9** (2016), 860–869. 2, 3, 3.10

[25] D. S. Kim, T. Kim, G.-W. Jang, *A note on degenerate Stirling numbers of the first kind*, Proc. Jangjeon Math. Soc., **21** (2018), 393–404. 1, 1

[26] D. S. Kim, H. K. Kim, T. Kim, *Some identities on λ-analogues of r-Stirling numbers of the second kind*, Eur. J. Pure Appl. Math., **15** (2022), 1054–1066. 1.5, 1

[27] D. S. Kim, T. Kim, J. J. Seo, *Higher-order Daehee polynomials of the first kind with umbral calculus*, Adv. Stud. Contemp. Math. (Kyungshang), **24** (2014), 5–18. 2.1, 2, 3, 3

[28] H. K. Kim, D. S. Lee, *A new type of degenerate poly-type 2 Euler polynomials and degenerate unipoly-type 2 Euler polynomials*, Proc. Jangjeon Math. Soc., **24** (2021), 205–222. 1

[29] T. Kim, T. Mansour, *Umbral calculus associated with Frobenius-type Eulerian polynomials*, Russ. J. Math. Phys., **21** (2014), 484–493. 2, 3

[30] J. Kwon, J.-W. Park, *On modified degenerate Changhee polynomials and numbers*, J. Nonlinear Sci. Appl., **9** (2016), 6294–6301. 1, 3

[31] K. S. Nisar, *Umbral calculus*, LAMBERT Academic Publishing, (2012). 2.1, 2.3, 2.4, 2, 3

[32] K. S. Nisar, S. R. Mondal, P. Agawal, M. Al-Dhaifalas, *The umbral operator and the integration involving generalized Bessel-type functions*, Open Math., **13** (2015), 426–435. 2

[33] J. Quaintance, H. W. Gould, *Combinatorial identities for Stirling numbers. The unpublished notes of H. W. Gould. With a foreword by George E. Andrews*, World Scientific Publishing Co., Singapore, (2016). 1

[34] Y. Rao, W. A. Khan, S. Araci, C. S. Ryoo, *Explicit properties of Apostol-type Frobenius-Euler polynomials involving q-Trionometric functions with applications in computer modeling*, Mathematics, **2023** (2023), 1–21. 3

[35] S. Roman, *The umbral calculus*, Springer, New York, (2005). 1.1, 1.3, 2.1, 2.3, 2.4, 2.2, 2

[36] G.-C. Rota, B. D. Taylor, *The classical umbral calculus*, SIAM J. Math. Anal., **25** (1994), 694–711. 2.1, 2.3, 2.4, 2

[37] S. Tauber, *Lah numbers for Fibonacci and Lucas polynomials*, Fibonacci Quart., **6** (1968), 93–99. 3.31