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Some Leindler-type inequalities on conformable fractional integrals

S. H. Saker[a], G. M. Ashry[b], M. R. Kenawy[b,*]

[a]Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt.

[b]Department of Mathematics, Faculty of Science, Fayoum University, Fayoum, Egypt.

Abstract

In this paper, we will generate some fractional Liendler type inequalities by using integration by parts, chain rule and Hölder inequality on conformable fractional calculus. As a special case, we obtain some formulas of Liendler type inequalities at $\alpha = 1$.

Keywords: Liendler type inequality, conformable fractional calculus, Hölder inequality, chain rule, integration by parts.

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1. Introduction

Functional inequalities have numerous applications in the study of qualitative theory (e.g., asymptotics, blow-up, boundedness, oscillation, periodicity, etc) of various classes of differential equations, dynamic equations on time scales, and partial differential equations; for more details see, e.g., the papers [7–9, 12, 20–24, 33].

In 1920, Hardy in [13] showed that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{j=1}^{n}u(j)\right)^{h} \leqslant \left(\frac{h}{h-1}\right)^{h}\sum_{n=1}^{\infty}u^{h}(n), \quad h>1,$$

for all $n \geqslant 1$, $u(n)$ is a positive sequence.

In 1925, Hardy in [14] used calculus of variations to prove the inequality

$$\int_{0}^{\infty}\left(\frac{1}{x}\int_{0}^{x}f(t)dt\right)^{h}dx \leqslant \left(\frac{h}{h-1}\right)^{h}\int_{0}^{\infty}f^{h}(x)dx, \tag{1.1}$$

*Corresponding author

Email addresses: shsaker@mans.edu.eg (S. H. Saker), gma07@fayoum.edu.eg (G. M. Ashry), mrz00@fayoum.edu.eg (M. R. Kenawy)

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where f is an integrable and positive function on the interval $(0,x)$, f^h is integrable and convergent over $(0,\infty)$ and $h>1$. The constant $(h/(h-1))^h$ is the best constant.

In 1928, Hardy in [15] generalized the inequality (1.1) that if $h>1$ and f is nonnegative for $x\geqslant 0$, then

$$\int_0^\infty x^{-c}\left(\int_0^x f(t)dt\right)^h dx \leqslant \left(\frac{h}{c-1}\right)^h \int_0^\infty x^{h-c}f^h(x)dx, \text{ for } c>1, \tag{1.2}$$

and

$$\int_0^\infty x^{-c}\left(\int_x^\infty f(t)dt\right)^h dx \leqslant \left(\frac{h}{1-c}\right)^h \int_0^\infty x^{h-c}f^h(x)dx, \text{ for } c<1, \tag{1.3}$$

where $(h/(c-1))^h$ and $(h/(1-c))^h$ in (1.2) and (1.3) are the best constants. In 1927, Copson in [11] showed that if

$$\int_x^\infty \frac{f(s)}{s}ds,$$

converges for $x>0$, then

$$\int_0^\infty \left(\int_x^\infty \frac{f(s)}{s}ds\right)^h dx \leqslant h^h \int_0^\infty f^h(x)dx.$$

In 1970, Leindler in [17] proved that for $h>1$ and $\lambda(n)$, $g(n)>0$, then

$$\sum_{n=1}^{\infty}\lambda(n)\left(\sum_{s=1}^{n}g(s)\right)^h \leqslant h^h\sum_{n=1}^{\infty}\lambda^{1-h}(n)\left(\sum_{s=n}^{\infty}\lambda(s)\right)^h g^h(n), \tag{1.4}$$

and

$$\sum_{n=1}^{\infty}\lambda(n)\left(\sum_{k=n}^{\infty}g(k)\right)^h \leqslant h^h\sum_{n=1}^{\infty}\lambda^{1-h}(n)\left(\sum_{k=1}^{n}\lambda(k)\right)^h g^h(n). \tag{1.5}$$

In 1990, Leindler in [18] proved the converses of inequalities (1.4) and (1.5). He proved that for $0<h\leqslant 1$, then

$$\sum_{n=1}^{\infty}\lambda(n)\left(\sum_{k=1}^{n}g(k)\right)^h \geqslant h^h\sum_{n=1}^{\infty}\lambda^{1-h}(n)\left(\sum_{k=n}^{\infty}\lambda(k)\right)^h g^h(n),$$

and

$$\sum_{n=1}^{\infty}\lambda(n)\left(\sum_{k=n}^{\infty}g(k)\right)^h \geqslant h^h\sum_{n=1}^{\infty}\lambda^{1-h}(n)\left(\sum_{k=1}^{n}\lambda(k)\right)^h g^h(n).$$

In last few years, by using the conformable calculus, the authors proved some classical inequalities for examples: Hardy's inequalities ([5, 30]), Minkowski's inequalities ([25–27]), Lyapunov's inequalities ([2, 3]), Hermite-Hadamard's inequalities ([4, 32]) and Copson's inequalities [29].

In this paper, we will prove some fractional Leindler type inequalities and their reversed inequalities using conformable calculus.

This paper is arranged as follows. Section 2 contains the preliminaries and basic lemmas of conformable calculus and Section 3 contains our main results of Leindler type inequalities and its reversed ones.

2. Preliminaries and basic lemmas

This section contains some essential definitions and lemmas on conformable calculus. The results are obtained from [16]. For more information, see [1, 6, 16].

Definition 2.1. The conformable derivative of order α of a function $w : [0, \infty) \to \mathbb{R}$ is defined by

$$D_\alpha w(s) = \lim_{\epsilon \to 0} \frac{w\left(s + \epsilon s^{1-\alpha}\right) - w(s)}{\epsilon}, \text{ for all } s > 0, \ \alpha \in (0,1).$$

Definition 2.2. The conformable integral of order α of a function $w : [0, \infty) \to \mathbb{R}$ is defined as follows

$$(I_\alpha^a w)(x) = \int_a^x w(s)\, d_\alpha s = \int_a^x s^{\alpha-1} w(s)\, ds, \ 0 < \alpha \leqslant 1.$$

Theorem 2.3. *Let $\alpha \in (0, 1]$, and w and ν be α-differentiable such that $x > 0$. Then*

1. $D_\alpha (aw + b\nu)(x) = aD_\alpha w(x) + bD_\alpha \nu(x)$;
2. $D_\alpha \left(x^\lambda\right) = \lambda x^{\lambda-\alpha}$ *for all* $\lambda \in \mathbb{R}$;
3. $D_\alpha(\theta) = 0$, *for all constant functions* $w(x) = \theta$;
4. $D_\alpha(w\nu)(x) = wD_\alpha \nu(x) + \nu D_\alpha w(x)$;
5. $D_\alpha\left(\frac{w}{\nu}\right)(x) = \frac{\nu D_\alpha w(x) - wD_\alpha \nu(x)}{\nu^2}$;
6. *if w is differentiable, then* $D_\alpha w(x) = x^{1-\alpha}\frac{dw(x)}{dx}$.

Lemma 2.4. *Let $\nu(x)$ be α-differentiable with respect to x and w be α-differentiable with respect to ν. The chain rule using conformable derivative is defined as*

$$D_\alpha w(\nu(x)) = \nu^{\alpha-1}(x)\left(D_\alpha w(\nu(x))\right) D_\alpha \nu(x). \tag{2.1}$$

Lemma 2.5. *Let w and ν be α-differentiable with respect to x on $[a, b]$. The integration by parts using conformable calculus is defined as*

$$\int_a^b D_\alpha w(x)\, \nu(x)\, d_\alpha x = w(x)\,\nu(x)\,|_a^b - \int_a^b w(x)\left(D_\alpha \nu(x)\right) d_\alpha x. \tag{2.2}$$

Lemma 2.6. *Let $0 < \alpha \leqslant 1$ and $w, \nu : [a, b] \to \mathbb{R}$. The Hölder inequality using conformable integral is defined as*

$$\int_a^b |w(x)\,\nu(x)|\, d_\alpha x \leqslant \left(\int_a^b |w(x)|^\beta\, d_\alpha x\right)^{\frac{1}{\beta}} \left(\int_a^b |\nu(x)|^\gamma\, d_\alpha x\right)^{\frac{1}{\gamma}}, \tag{2.3}$$

for $\frac{1}{\beta} + \frac{1}{\gamma} = 1$ *and* $\beta > 1$.

In this paper, assume that the functions are nonnegative on $[a, \infty)$ and their fractional integrals exist and finite.

3. Main results

Theorem 3.1. *Assume that* $\alpha \in (0, 1]$, $\zeta \in [\imath, \infty)$, $\mu > 1$,

$$\Lambda(\zeta) := \int_\zeta^\infty \lambda(s)\, d_\alpha s \quad \textit{and} \quad \Phi(\zeta) := \int_\imath^\zeta f(s)\, d_\alpha s.$$

Then

$$\begin{aligned}\left(\int_\imath^\infty \lambda(\zeta)\, \Phi^{\mu-\alpha+1}(\zeta)\, d_\alpha \zeta\right)^\mu \leqslant (\mu-\alpha+1)^\mu &\left(\int_\imath^\infty \lambda^{1-\mu}(\zeta)\, \Lambda^\mu(\zeta)\, f^\mu(\zeta)\, d_\alpha \zeta\right) \\ &\times \left(\int_\imath^\infty \lambda(\zeta)\, \Phi^{\frac{\mu(\mu-\alpha)}{\mu-1}}(\zeta)\, d_\alpha \zeta\right)^{\mu-1}.\end{aligned} \tag{3.1}$$

Proof. By using the integration by parts formula (2.2) on the term

$$\int_{\imath}^{\infty} \lambda(\zeta)\, \Phi^{\mu-\alpha+1}(\zeta)\, d_{\alpha}\zeta,$$

such that $u(\zeta) = \Phi^{\mu-\alpha+1}(\zeta)$ and $D_{\alpha} v(\zeta) = \lambda(\zeta)$, we obtain

$$\int_{\imath}^{\infty} \lambda(\zeta)\, \Phi^{\mu-\alpha+1}(\zeta)\, d_{\alpha}\zeta = v(\zeta)\, \Phi^{\mu-\alpha+1}(\zeta)\,|_{\imath}^{\infty} + \int_{\imath}^{\infty} (-v(\zeta))\, D_{\alpha}\left(\Phi^{\mu-\alpha+1}(\zeta)\right) d_{\alpha}\zeta.$$

As $v(\zeta) = -\int_{\zeta}^{\infty} \lambda(s)\, d_{\alpha}s = -\Lambda(\zeta)$, then we have

$$\int_{\imath}^{\infty} \lambda(\zeta)\, \Phi^{\mu-\alpha+1}(\zeta)\, d_{\alpha}\zeta = v(\zeta)\, \Phi^{\mu-\alpha+1}(\zeta)\,|_{\imath}^{\infty} + \int_{\imath}^{\infty} \Lambda(\zeta)\, D_{\alpha}\left(\Phi^{\mu-\alpha+1}(\zeta)\right) d_{\alpha}\zeta.$$

Since $\Lambda(\infty) = 0$ and $\Phi(\imath) = 0$, then

$$\int_{\imath}^{\infty} \lambda(\zeta)\, \Phi^{\mu-\alpha+1}(\zeta)\, d_{\alpha}\zeta = \int_{\imath}^{\infty} \Lambda(\zeta)\, D_{\alpha}\left(\Phi^{\mu-\alpha+1}(\zeta)\right) d_{\alpha}\zeta. \tag{3.2}$$

Employing the chain rule formula (2.1), we find that

$$\begin{aligned} D_{\alpha}\left(\Phi^{\mu-\alpha+1}(\zeta)\right) &= \Phi^{\alpha-1}(\zeta) . \left(D_{\alpha}\Phi^{\mu-\alpha+1}(\zeta)\right)(\Phi(\zeta)) . D_{\alpha}\Phi(\zeta) \\ &= \Phi^{\alpha-1}(\zeta)\,(\mu-\alpha+1)\,\Phi^{\mu-\alpha+1-\alpha}(\zeta)\, D_{\alpha}\Phi(\zeta) = (\mu-\alpha+1)\,\Phi^{\mu-\alpha}(\zeta)\, D_{\alpha}\Phi(\zeta). \end{aligned}$$

As $D_{\alpha}\Phi(\zeta) = f(\zeta)$, then

$$D_{\alpha}\left(\Phi^{\mu-\alpha+1}(\zeta)\right) = (\mu-\alpha+1)\, f(\zeta)\, \Phi^{\mu-\alpha}(\zeta). \tag{3.3}$$

From (3.2) and (3.3), we have

$$\int_{\imath}^{\infty} \lambda(\zeta)\, \Phi^{\mu-\alpha+1}(\zeta)\, d_{\alpha}\zeta = (\mu-\alpha+1) \int_{\imath}^{\infty} \Lambda(\zeta)\, f(\zeta)\, \Phi^{\mu-\alpha}(\zeta)\, d_{\alpha}\zeta,$$

which can be written as

$$\int_{\imath}^{\infty} \lambda(\zeta)\, \Phi^{\mu-\alpha+1}(\zeta)\, d_{\alpha}\zeta = (\mu-\alpha+1) \int_{\imath}^{\infty} \frac{\Lambda(\zeta)\, f(\zeta)}{\lambda^{\frac{\mu-1}{\mu}}(\zeta)} \lambda^{\frac{\mu-1}{\mu}}(\zeta)\, \Phi^{\mu-\alpha}(\zeta)\, d_{\alpha}\zeta. \tag{3.4}$$

Performing Hölder's inequality (2.3), on the term

$$\int_{\imath}^{\infty} \frac{\Lambda(\zeta)\, f(\zeta)}{\lambda^{\frac{\mu-1}{\mu}}(\zeta)} \lambda^{\frac{\mu-1}{\mu}}(\zeta)\, \Phi^{\mu-\alpha}(\zeta)\, d_{\alpha}\zeta,$$

with indices $\beta = \mu$ and $\gamma = \frac{\mu}{\mu-1}$, then we have

$$\begin{aligned} &\int_{\imath}^{\infty} \frac{\Lambda(\zeta)\, f(\zeta)}{\lambda^{\frac{\mu-1}{\mu}}(\zeta)} \lambda^{\frac{\mu-1}{\mu}}(\zeta)\, \Phi^{\mu-\alpha}(\zeta)\, d_{\alpha}\zeta \\ &\leqslant \left(\int_{\imath}^{\infty} \left(\frac{\Lambda(\zeta)\, f(\zeta)}{\lambda^{\frac{\mu-1}{\mu}}(\zeta)}\right)^{\mu} d_{\alpha}\zeta\right)^{\frac{1}{\mu}} \left(\int_{\imath}^{\infty} \left(\lambda^{\frac{\mu-1}{\mu}}(\zeta)\, \Phi^{\mu-\alpha}(\zeta)\right)^{\frac{\mu}{\mu-1}} d_{\alpha}\zeta\right)^{\frac{\mu-1}{\mu}} \\ &= \left(\int_{\imath}^{\infty} \frac{\Lambda^{\mu}(\zeta)\, f^{\mu}(\zeta)}{\lambda^{(\mu-1)}(\zeta)} d_{\alpha}\zeta\right)^{\frac{1}{\mu}} \left(\int_{\imath}^{\infty} \lambda(\zeta)\, \Phi^{\frac{\mu(\mu-\alpha)}{\mu-1}}(\zeta)\, d_{\alpha}\zeta\right)^{\frac{\mu-1}{\mu}}. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we obtain that

$$\int_{\imath}^{\infty}\lambda(\zeta)\,\Phi^{\mu-\alpha+1}(\zeta)\,d_{\alpha}\zeta \leqslant (\mu-\alpha+1)\left(\int_{\imath}^{\infty}\frac{\Lambda^{\mu}(\zeta)\,f^{\mu}(\zeta)}{\lambda^{(\mu-1)}(\zeta)}d_{\alpha}\zeta\right)^{\frac{1}{\mu}}\left(\int_{\imath}^{\infty}\lambda(\zeta)\,\Phi^{\frac{\mu(\mu-\alpha)}{\mu-1}}(\zeta)\,d_{\alpha}\zeta\right)^{\frac{\mu-1}{\mu}}. \tag{3.6}$$

Taking the power μ of the inequality (3.6), we have

$$\left(\int_{\imath}^{\infty}\lambda(\zeta)\,\Phi^{\mu-\alpha+1}(\zeta)\,d_{\alpha}\zeta\right)^{\mu} \leqslant (\mu-\alpha+1)^{\mu}\left(\int_{\imath}^{\infty}\frac{\Lambda^{\mu}(\zeta)\,f^{\mu}(\zeta)}{\lambda^{(\mu-1)}(\zeta)}d_{\alpha}\zeta\right)\left(\int_{\imath}^{\infty}\lambda(\zeta)\,\Phi^{\frac{\mu(\mu-\alpha)}{\mu-1}}(\zeta)\,d_{\alpha}\zeta\right)^{\mu-1},$$

which is inequality (3.1). The proof is complete. □

Corollary 3.2. *In Theorem 3.1, if $\alpha = 1$, then we obtain*

$$\left(\int_{\imath}^{\infty}\lambda(\zeta)\,\Phi^{\mu}(\zeta)\,d\zeta\right)^{\mu} \leqslant \mu^{\mu}\left(\int_{\imath}^{\infty}\frac{\Lambda^{\mu}(\zeta)\,f^{\mu}(\zeta)}{\lambda^{(\mu-1)}(\zeta)}d\zeta\right)\left(\int_{\imath}^{\infty}\lambda(\zeta)\,\Phi^{\mu}(\zeta)\,d\zeta\right)^{\mu-1}. \tag{3.7}$$

Dividing both sides of the inequality (3.7) by

$$\left(\int_{\imath}^{\infty}\lambda(\zeta)\,\Phi^{\mu}(\zeta)\,d\zeta\right)^{\mu-1},$$

then

$$\left(\int_{\imath}^{\infty}\lambda(\zeta)\,\Phi^{\mu}(\zeta)\,d\zeta\right) \leqslant \mu^{\mu}\left(\int_{\imath}^{\infty}\frac{\Lambda^{\mu}(\zeta)\,f^{\mu}(\zeta)}{\lambda^{(\mu-1)}(\zeta)}d\zeta\right),$$

which is the Leindler type inequality of remark (1) in [28].

Theorem 3.3. *Assume that $\alpha \in (0,\ 1]$, $\zeta \in [\imath,\ \infty)$, $\mu > 1$,*

$$\Lambda(\zeta) := \int_{\imath}^{\zeta}\lambda(s)\,d_{\alpha}s \quad \textit{and} \quad \Phi(\zeta) := \int_{\zeta}^{\infty}f(s)\,d_{\alpha}s.$$

Then

$$\begin{aligned}\left(\int_{\imath}^{\infty}\lambda(\zeta)\,\Phi^{\mu-\alpha+1}(\zeta)\,d_{\alpha}\zeta\right)^{\mu} &\leqslant (\mu-\alpha+1)^{\mu}\left(\int_{\imath}^{\infty}\lambda^{1-\mu}(\zeta)\,\Lambda^{\mu}(\zeta)\,f^{\mu}(\zeta)\,d_{\alpha}\zeta\right)\\ &\quad\times\left(\int_{\imath}^{\infty}\lambda(\zeta)\,\Phi^{\frac{\mu(\mu-\alpha)}{\mu-1}}(\zeta)\,d_{\alpha}\zeta\right)^{\mu-1}.\end{aligned} \tag{3.8}$$

Proof. By using the integration by parts of formula (2.2) on the term

$$\int_{\imath}^{\infty}\lambda(\zeta)\,\Phi^{\mu-\alpha+1}(\zeta)\,d_{\alpha}\zeta,$$

such that $u(\zeta) = \Phi^{\mu-\alpha+1}(\zeta)$ and $D_{\alpha}v(\zeta) = \lambda(\zeta)$, we obtain

$$\int_{\imath}^{\infty}\lambda(\zeta)\,\Phi^{\mu-\alpha+1}(\zeta)\,d_{\alpha}\zeta = v(\zeta)\,\Phi^{\mu-\alpha+1}(\zeta)\,|_{\imath}^{\infty} + \int_{\imath}^{\infty}v(\zeta)\left(-D_{\alpha}\left(\Phi^{\mu-\alpha+1}(\zeta)\right)\right)d_{\alpha}\zeta.$$

As $v(\zeta) = \int_{\imath}^{\zeta}\lambda(s)\,d_{\alpha}s = \Lambda(\zeta)$, then we get

$$\int_{\imath}^{\infty}\lambda(\zeta)\,\Phi^{\mu-\alpha+1}(\zeta)\,d_{\alpha}\zeta = v(\zeta)\,\Phi^{\mu-\alpha+1}(\zeta)\,|_{\imath}^{\infty} + \int_{\imath}^{\infty}\Lambda(\zeta)\left(-D_{\alpha}\left(\Phi^{\mu-\alpha+1}(\zeta)\right)\right)d_{\alpha}\zeta.$$

Since $\Lambda(\infty) = 0$ and $\Phi(\imath) = 0$, then

$$\int_{\imath}^{\infty} \lambda(\zeta)\, \Phi^{\mu-\alpha+1}(\zeta)\, d_{\alpha}\zeta = \int_{\imath}^{\infty} \Lambda(\zeta)\left(-D_{\alpha}\left(\Phi^{\mu-\alpha+1}(\zeta)\right)\right) d_{\alpha}\zeta. \tag{3.9}$$

Employing the chain rule formula (2.1), we find that

$$\begin{aligned} -D_{\alpha}\left(\Phi^{\mu-\alpha+1}(\zeta)\right) &= -\Phi^{\alpha-1}(\zeta) . \left(D_{\alpha}\Phi^{\mu-\alpha+1}(\zeta)\right)(\Phi(\zeta)) . D_{\alpha}\Phi(\zeta) \\ &= -\Phi^{\alpha-1}(\zeta)(\mu-\alpha+1)\,\Phi^{\mu-\alpha+1-\alpha}(\zeta)\, D_{\alpha}\Phi(\zeta) \\ &= -(\mu-\alpha+1)\,\Phi^{\mu-\alpha}(\zeta)\, D_{\alpha}\Phi(\zeta) \\ &= -(\mu-\alpha+1)\,\Phi^{\mu-\alpha}(\zeta)\,(-f(\zeta)) \\ &= (\mu-\alpha+1)\, f(\zeta)\,\Phi^{\mu-\alpha}(\zeta) . \end{aligned} \tag{3.10}$$

Using (3.10) in (3.9), we obtain

$$\int_{\imath}^{\infty} \lambda(\zeta)\, \Phi^{\mu-\alpha+1}(\zeta)\, d_{\alpha}\zeta = (\mu-\alpha+1)\int_{\imath}^{\infty} \Lambda(\zeta)\, f(\zeta)\, \Phi^{\mu-\alpha}(\zeta)\, d_{\alpha}\zeta,$$

which can be written as

$$\int_{\imath}^{\infty} \lambda(\zeta)\, \Phi^{\mu-\alpha+1}(\zeta)\, d_{\alpha}\zeta = (\mu-\alpha+1)\int_{\imath}^{\infty} \frac{\Lambda(\zeta)\, f(\zeta)}{\lambda^{\frac{\mu-1}{\mu}}(\zeta)} \lambda^{\frac{\mu-1}{\mu}}(\zeta)\, \Phi^{\mu-\alpha}(\zeta)\, d_{\alpha}\zeta. \tag{3.11}$$

Performing Hölder's inequality (2.3) on the term

$$\int_{\imath}^{\infty} \frac{\Lambda(\zeta)\, f(\zeta)}{\lambda^{\frac{\mu-1}{\mu}}(\zeta)} \lambda^{\frac{\mu-1}{\mu}}(\zeta)\, \Phi^{\mu-\alpha}(\zeta)\, d_{\alpha}\zeta,$$

with indices $\beta = \mu$ and $\gamma = \frac{\mu}{\mu-1}$, then we have

$$\begin{aligned} &\int_{\imath}^{\infty} \frac{\Lambda(\zeta)\, f(\zeta)}{\lambda^{\frac{\mu-1}{\mu}}(\zeta)} \lambda^{\frac{\mu-1}{\mu}}(\zeta)\, \Phi^{\mu-\alpha}(\zeta)\, d_{\alpha}\zeta \\ &\leqslant \left(\int_{\imath}^{\infty} \left(\frac{\Lambda(\zeta)\, f(\zeta)}{\lambda^{\frac{\mu-1}{\mu}}(\zeta)}\right)^{\mu} d_{\alpha}\zeta\right)^{\frac{1}{\mu}} \left(\int_{\imath}^{\infty} \left(\lambda^{\frac{\mu-1}{\mu}}(\zeta)\, \Phi^{\mu-\alpha}(\zeta)\right)^{\frac{\mu}{\mu-1}} d_{\alpha}\zeta\right)^{\frac{\mu-1}{\mu}} \\ &= \left(\int_{\imath}^{\infty} \frac{\Lambda^{\mu}(\zeta)\, f^{\mu}(\zeta)}{\lambda^{(\mu-1)}(\zeta)} d_{\alpha}\zeta\right)^{\frac{1}{\mu}} \left(\int_{\imath}^{\infty} \lambda(\zeta)\, \Phi^{\frac{\mu(\mu-\alpha)}{\mu-1}}(\zeta)\, d_{\alpha}\zeta\right)^{\frac{\mu-1}{\mu}} . \end{aligned} \tag{3.12}$$

From (3.11) and (3.12), we obtain

$$\int_{\imath}^{\infty} \lambda(\zeta)\, \Phi^{\mu-\alpha+1}(\zeta)\, d_{\alpha}\zeta \leqslant (\mu-\alpha+1)\left(\int_{\imath}^{\infty} \frac{\Lambda^{\mu}(\zeta)\, f^{\mu}(\zeta)}{\lambda^{(\mu-1)}(\zeta)} d_{\alpha}\zeta\right)^{\frac{1}{\mu}} \left(\int_{\imath}^{\infty} \lambda(\zeta)\, \Phi^{\frac{\mu(\mu-\alpha)}{\mu-1}}(\zeta)\, d_{\alpha}\zeta\right)^{\frac{\mu-1}{\mu}} . \tag{3.13}$$

Taking the power μ of the inequality (3.13), we have

$$\left(\int_{\imath}^{\infty} \lambda(\zeta)\, \Phi^{\mu-\alpha+1}(\zeta)\, d_{\alpha}\zeta\right)^{\mu} \leqslant (\mu-\alpha+1)^{\mu}\left(\int_{\imath}^{\infty} \frac{\Lambda^{\mu}(\zeta)\, f^{\mu}(\zeta)}{\lambda^{(\mu-1)}(\zeta)} d_{\alpha}\zeta\right) \left(\int_{\imath}^{\infty} \lambda(\zeta)\, \Phi^{\frac{\mu(\mu-\alpha)}{\mu-1}}(\zeta)\, d_{\alpha}\zeta\right)^{\mu-1},$$

which is inequality (3.8). The proof is complete. □

Corollary 3.4. *In Theorem 3.3, if* $\alpha = 1$*, then we obtain*

$$\left(\int_{\imath}^{\infty} \lambda(\zeta)\,\Phi^{\mu}(\zeta)\,d\zeta\right)^{\mu} \leqslant \mu^{\mu}\left(\int_{\imath}^{\infty} \frac{\Lambda^{\mu}(\zeta)\,f^{\mu}(\zeta)}{\lambda^{(\mu-1)}(\zeta)}\,d\zeta\right)\left(\int_{\imath}^{\infty} \lambda(\zeta)\,\Phi^{\mu}(\zeta)\,d\zeta\right)^{\mu-1}. \tag{3.14}$$

Dividing both sides of the inequality (3.14), by

$$\left(\int_{\imath}^{\infty} \lambda(\zeta)\,\Phi^{\mu}(\zeta)\,d\zeta\right)^{\mu-1},$$

then

$$\left(\int_{\imath}^{\infty} \lambda(\zeta)\,\Phi^{\mu}(\zeta)\,d\zeta\right) \leqslant \mu^{\mu}\left(\int_{\imath}^{\infty} \frac{\Lambda^{\mu}(\zeta)\,f^{\mu}(\zeta)}{\lambda^{(\mu-1)}(\zeta)}\,d\zeta\right),$$

which is the Leindler type inequality of remark (3) in [28].

Theorem 3.5. *Assume that* $0 < \alpha \leqslant 1$, $0 < \mu \leqslant 1$,

$$\Omega(\zeta) := \int_{\zeta}^{\infty} \lambda(s)\,d_{\alpha}s \quad \text{and} \quad \Lambda(\zeta) := \int_{\imath}^{\zeta} f(s)\,d_{\alpha}s, \quad \text{for any } \zeta \in [\imath, \infty).$$

Then

$$\begin{aligned}\left(\int_{\imath}^{\infty} \lambda(\zeta)\,\Lambda^{\mu-\alpha+1}(\zeta)\,d_{\alpha}\zeta\right)^{\mu} \geqslant (\mu-\alpha+1)^{\mu}&\left(\int_{\imath}^{\infty} f^{\mu}(\zeta)\,\Omega^{\mu}(\zeta)\,\lambda^{1-\mu}(\zeta)\,d_{\alpha}\zeta\right)\\ &\times\left(\int_{\imath}^{\infty} \lambda(\zeta)\,\Lambda^{\frac{\mu(\mu-\alpha)}{\mu-1}}(\zeta)\,d_{\alpha}\zeta\right)^{\mu-1}.\end{aligned} \tag{3.15}$$

Proof. Employing the integration by parts formula (2.2) on the term

$$\int_{\imath}^{\infty} \lambda(\zeta)\,\Lambda^{\mu-\alpha+1}(\zeta)\,d_{\alpha}\zeta,$$

as

$$u(\zeta) = \Lambda^{\mu-\alpha+1}(\zeta) \text{ and } D_{\alpha}v(\zeta) = \lambda(\zeta),$$

we obtain that

$$\int_{\imath}^{\infty} \lambda(\zeta)\,\Lambda^{\mu-\alpha+1}(\zeta)\,d_{\alpha}\zeta = v(\zeta)\,\Lambda^{\mu-\alpha+1}(\zeta)\,|_{\imath}^{\infty} + \int_{\imath}^{\infty} (-v(\zeta))\,D_{\alpha}\left(\Lambda^{\mu-\alpha+1}(\zeta)\right)d_{\alpha}\zeta,$$

since $v(\zeta) = -\int_{\zeta}^{\infty} \lambda(s)\,d_{\alpha}s = -\Omega(\zeta)$, therefore

$$\int_{\imath}^{\infty} \lambda(\zeta)\,\Lambda^{\mu-\alpha+1}(\zeta)\,d_{\alpha}\zeta = -\Omega(\zeta)\,\Lambda^{\mu-\alpha+1}(\zeta)\,|_{\imath}^{\infty} + \int_{\imath}^{\infty} \Omega(\zeta)\,D_{\alpha}\left(\Lambda^{\mu-\alpha+1}(\zeta)\right)d_{\alpha}\zeta.$$

As $v(\infty) = 0$ and $\Lambda(\imath) = 0$, then

$$\int_{\imath}^{\infty} \lambda(\zeta)\,\Lambda^{\mu-\alpha+1}(\zeta)\,d_{\alpha}\zeta = \int_{\imath}^{\infty} \Omega(\zeta)\,D_{\alpha}\left(\Lambda^{\mu-\alpha+1}(\zeta)\right)d_{\alpha}\zeta.$$

Performing the chain rule formula (2.1), we obtain

$$\begin{aligned}D_{\alpha}\left(\Lambda^{\mu-\alpha+1}(\zeta)\right) &= \Lambda^{\alpha-1}(\zeta)\,.\left(D_{\alpha}\Lambda^{\mu-\alpha+1}(\zeta)\right)(\Lambda(\zeta))\,.D_{\alpha}\Lambda(\zeta)\\ &= \Lambda^{\alpha-1}(\zeta)\,(\mu-\alpha+1)\,\Lambda^{\mu-\alpha+1-\alpha}(\zeta)\,D_{\alpha}\Lambda(\zeta) = (\mu-\alpha+1)\,\Lambda^{\mu-\alpha}(\zeta)\,D_{\alpha}\Lambda(\zeta),\end{aligned} \tag{3.16}$$

where $D_\alpha \Lambda(\zeta) = f(\zeta)$. Hence

$$D_\alpha\left(\Lambda^{\mu-\alpha+1}(\zeta)\right) = (\mu-\alpha+1)\, f(\zeta)\,\Lambda^{\mu-\alpha}(\zeta). \tag{3.17}$$

From (3.16) and (3.17), we see that

$$\begin{aligned}\int_\iota^\infty \lambda(\zeta)\,\Lambda^{\mu-\alpha+1}(\zeta)\, d_\alpha\zeta &= (\mu-\alpha+1)\int_\iota^\infty \Omega(\zeta)\, f(\zeta)\,\Lambda^{\mu-\alpha}(\zeta)\, d_\alpha\zeta \\ &= (\mu-\alpha+1)\int_\iota^\infty \left(\Omega^\mu(\zeta)\, f^\mu(\zeta)\,\Lambda^{\mu(\mu-\alpha)}(\zeta)\right)^{\frac{1}{\mu}} d_\alpha\zeta.\end{aligned} \tag{3.18}$$

Taking the power μ of the inequality (3.18), we have

$$\left(\int_\iota^\infty \lambda(\zeta)\,\Lambda^{\mu-\alpha+1}(\zeta)\, d_\alpha\zeta\right)^\mu = (\mu-\alpha+1)^\mu\left(\int_\iota^\infty \left(\Omega^\mu(\zeta)\, f^\mu(\zeta)\,\Lambda^{\mu(\mu-\alpha)}(\zeta)\right)^{\frac{1}{\mu}} d_\alpha\zeta\right)^\mu. \tag{3.19}$$

Employing Hölder's inequality formula (2.3) on the term

$$\left(\int_\iota^\infty \left(\Omega^\mu(\zeta)\, f^\mu(\zeta)\,\Lambda^{\mu(\mu-\alpha)}(\zeta)\right)^{\frac{1}{\mu}} d_\alpha\zeta\right)^\mu,$$

with indices $\beta = \frac{1}{\mu}$ and $\gamma = \frac{1}{1-\mu}$, and letting

$$F(\zeta) = \frac{\Omega^\mu(\zeta)\, f^\mu(\zeta)}{\Lambda^{\mu(\alpha-\mu)}(\zeta)},\quad G(\zeta) = \lambda^{1-\mu}(\zeta)\,\Lambda^{\mu(\alpha-\mu)}(\zeta),$$

therefore

$$\begin{aligned}\left(\int_\iota^\infty F^{\frac{1}{\mu}}(\zeta)\, d_\alpha\zeta\right)^\mu &= \left(\int_\iota^\infty \left(\frac{\Omega^\mu(\zeta)\, f^\mu(\zeta)}{\Lambda^{\mu(\alpha-\mu)}(\zeta)}\right)^{\frac{1}{\mu}} d_\alpha\zeta\right)^\mu \\ &\geqslant \frac{\int_\iota^\infty F(\zeta)\, G(\zeta)\, d_\alpha\zeta}{\left(\int_\iota^\infty G^{\frac{1}{1-\mu}}(\zeta)\, d_\alpha\zeta\right)^{1-\mu}} \\ &= \left(\int_\iota^\infty \frac{\Omega^\mu(\zeta)\, f^\mu(\zeta)\,\lambda^{1-\mu}(\zeta)\,\Lambda^{\mu(\alpha-\mu)}(\zeta)}{\Lambda^{\mu(\alpha-\mu)}(\zeta)} d_\alpha\zeta\right) \\ &\quad\times\left(\int_\iota^\infty \left(\lambda^{1-\mu}(\zeta)\,\Lambda^{\mu(\alpha-\mu)}(\zeta)\right)^{\frac{1}{1-\mu}} d_\alpha\zeta\right)^{\mu-1} \\ &= \left(\int_\iota^\infty \Omega^\mu(\zeta)\, f^\mu(\zeta)\,\lambda^{1-\mu}(\zeta)\, d_\alpha\zeta\right)\left(\int_\iota^\infty \lambda(\zeta)\,\Lambda^{\frac{\mu(\alpha-\mu)}{1-\mu}}(\zeta)\, d_\alpha\zeta\right)^{\mu-1}.\end{aligned}$$

Hence

$$\begin{aligned}&\left(\int_\iota^\infty \Omega(\zeta)\, f(\zeta)\,\Lambda^{(\mu-\alpha)}(\zeta)\, d_\alpha\zeta\right)^\mu \\ &\quad\geqslant \left(\int_\iota^\infty \Omega^\mu(\zeta)\, f^\mu(\zeta)\,\lambda^{1-\mu}(\zeta)\, d_\alpha\zeta\right)\left(\int_\iota^\infty \lambda(\zeta)\,\Lambda^{\frac{\mu(\alpha-\mu)}{1-\mu}}(\zeta)\, d_\alpha\zeta\right)^{\mu-1}.\end{aligned} \tag{3.20}$$

From (3.19) and (3.20), we obtain

$$\begin{aligned}\left(\int_\iota^\infty \lambda(\zeta)\,\Lambda^{\mu-\alpha+1}(\zeta)\, d_\alpha\zeta\right)^\mu &\geqslant (\mu-\alpha+1)^\mu\left(\int_\iota^\infty \Omega^\mu(\zeta)\, f^\mu(\zeta)\,\lambda^{1-\mu}(\zeta)\, d_\alpha\zeta\right) \\ &\quad\times\left(\int_\iota^\infty \lambda(\zeta)\,\Lambda^{\frac{\mu(\alpha-\mu)}{1-\mu}}(\zeta)\, d_\alpha\zeta\right)^{\mu-1},\end{aligned}$$

which is inequality (3.15). The proof is complete. □

Corollary 3.6. *In Theorem 3.5, if* $\alpha = 1$*, then we obtain*

$$\left(\int_{\imath}^{\infty} \lambda(\zeta)\Lambda^{\mu}(\zeta)\,d\zeta\right)^{\mu} \geqslant \mu^{\mu}\left(\int_{\imath}^{\infty} f^{\mu}(\zeta)\,\Omega^{\mu}(\zeta)\,\lambda^{1-\mu}(\zeta)\,d\zeta\right)\left(\int_{\imath}^{\infty}\lambda(\zeta)\Lambda^{\mu}(\zeta)\,d\zeta\right)^{\mu-1}. \tag{3.21}$$

Dividing both sides of the inequality (3.21) by

$$\left(\int_{\imath}^{\infty}\lambda(\zeta)\Lambda^{\mu}(\zeta)\,d\zeta\right)^{\mu-1},$$

then

$$\int_{\imath}^{\infty}\lambda(\zeta)\Lambda^{\mu}(\zeta)\,d\zeta \geqslant \mu^{\mu}\left(\int_{\imath}^{\infty} f^{\mu}(\zeta)\,\Omega^{\mu}(\zeta)\,\lambda^{1-\mu}(\zeta)\,d\zeta\right),$$

which is the Leindler type inequality of remark (5) in [28].

Theorem 3.7. *Assume that* $0 < \alpha \leqslant 1$, $0 < \mu \leqslant 1$,

$$\Omega(\zeta) := \int_{\imath}^{\zeta}\lambda(s)\,d_{\alpha}s \quad \textit{and} \quad \Lambda(\zeta) := \int_{\zeta}^{\infty} f(s)\,d_{\alpha}s, \quad \textit{for any } \zeta \in [\imath, \infty).$$

Then

$$\begin{aligned}\left(\int_{\imath}^{\infty}\lambda(\zeta)\Lambda^{\mu-\alpha+1}(\zeta)\,d_{\alpha}\zeta\right)^{\mu} &\geqslant (\mu-\alpha+1)^{\mu}\left(\int_{\imath}^{\infty} f^{\mu}(\zeta)\,\Omega^{\mu}(\zeta)\,\lambda^{1-\mu}(\zeta)\,d_{\alpha}\zeta\right)\\ &\quad\times\left(\int_{\imath}^{\infty}\lambda(\zeta)\Lambda^{\frac{\mu(\mu-\alpha)}{\mu-1}}(\zeta)\,d_{\alpha}\zeta\right)^{\mu-1}.\end{aligned} \tag{3.22}$$

Proof. Performing the integration by parts formula (2.2) on the term

$$\int_{\imath}^{\infty}\lambda(\zeta)\Lambda^{\mu-\alpha+1}(\zeta)\,d_{\alpha}\zeta,$$

such that $u(\zeta) = \Lambda^{\mu-\alpha+1}(\zeta)$ and $D_{\alpha}v(\zeta) = \lambda(\zeta)$, we get

$$\int_{\imath}^{\infty}\lambda(\zeta)\Lambda^{\mu-\alpha+1}(\zeta)\,d_{\alpha}\zeta = v(\zeta)\Lambda^{\mu-\alpha+1}(\zeta)\big|_{\imath}^{\infty} + \int_{\imath}^{\infty} v(\zeta)\left(-D_{\alpha}\left(\Lambda^{\mu-\alpha+1}(\zeta)\right)\right)d_{\alpha}\zeta.$$

As $v(\zeta) = \int_{\imath}^{\zeta}\lambda(s)\,d_{\alpha}s = \Omega(\zeta)$, therefore

$$\int_{\imath}^{\infty}\lambda(\zeta)\Lambda^{\mu-\alpha+1}(\zeta)\,d_{\alpha}\zeta = \Omega(\zeta)\Lambda^{\mu-\alpha+1}(\zeta)\big|_{\imath}^{\infty} + \int_{\imath}^{\infty}\Omega(\zeta)\left(-D_{\alpha}\left(\Lambda^{\mu-\alpha+1}(\zeta)\right)\right)d_{\alpha}\zeta.$$

Since $\Lambda(\infty) = 0$ and $\Omega(\imath) = 0$, then we obtain

$$\int_{\imath}^{\infty}\lambda(\zeta)\Lambda^{\mu-\alpha+1}(\zeta)\,d_{\alpha}\zeta = \int_{\imath}^{\infty}\Omega(\zeta)\left(-D_{\alpha}\left(\Lambda^{\mu-\alpha+1}(\zeta)\right)\right)d_{\alpha}\zeta.$$

Employing the chain rule formula (2.1), we have

$$\begin{aligned}-D_{\alpha}\left(\Lambda^{\mu-\alpha+1}(\zeta)\right) &= -\Lambda^{\alpha-1}(\zeta)\,.\left(D_{\alpha}\Lambda^{\mu-\alpha+1}(\zeta)\right)(\Lambda(\zeta))\,.D_{\alpha}\Lambda(\zeta)\\ &= -\Lambda^{\alpha-1}(\zeta)\,(\mu-\alpha+1)\,\Lambda^{\mu-\alpha+1-\alpha}(\zeta)\,D_{\alpha}\Lambda(\zeta)\\ &= -(\mu-\alpha+1)\,\Lambda^{\mu-\alpha}(\zeta)\,D_{\alpha}\Lambda(\zeta) = (\mu-\alpha+1)\,f(\zeta)\,\Lambda^{\mu-\alpha}(\zeta),\end{aligned}$$

where $D_\alpha \Lambda(\zeta) = -f(\zeta)$. Hence we get

$$\begin{aligned}\int_\iota^\infty \lambda(\zeta)\Lambda^{\mu-\alpha+1}(\zeta)\,d_\alpha\zeta &= (\mu-\alpha+1)\int_\iota^\infty \Omega(\zeta) f(\zeta)\Lambda^{\mu-\alpha}(\zeta)\,d_\alpha\zeta \\ &= (\mu-\alpha+1)\int_\iota^\infty \left(\Omega^\mu(\zeta) f^\mu(\zeta)\Lambda^{\mu(\mu-\alpha)}(\zeta)\right)^{\frac{1}{\mu}} d_\alpha\zeta. \end{aligned} \tag{3.23}$$

Taking the power μ of the inequality (3.23), we obtain

$$\left(\int_\iota^\infty \lambda(\zeta)\Lambda^{\mu-\alpha+1}(\zeta)\,d_\alpha\zeta\right)^\mu = (\mu-\alpha+1)^\mu\left(\int_\iota^\infty \left(\Omega^\mu(\zeta) f^\mu(\zeta)\Lambda^{\mu(\mu-\alpha)}(\zeta)\right)^{\frac{1}{\mu}} d_\alpha\zeta\right)^\mu. \tag{3.24}$$

Performing Hölder's inequality formula (2.3) on the term

$$\left(\int_\iota^\infty \left(\Omega^\mu(\zeta) f^\mu(\zeta)\Lambda^{\mu(\mu-\alpha)}(\zeta)\right)^{\frac{1}{\mu}} d_\alpha\zeta\right)^\mu,$$

with indices $\beta = \frac{1}{\mu}$ and $\gamma = \frac{1}{1-\mu}$, and by letting

$$F(\zeta) = \frac{\Omega^\mu(\zeta) f^\mu(\zeta)}{\Lambda^{\mu(\alpha-\mu)}(\zeta)}, \quad G(\zeta) = \lambda^{1-\mu}(\zeta)\Lambda^{\mu(\alpha-\mu)}(\zeta),$$

we deduce that

$$\begin{aligned}\left(\int_\iota^\infty F^{\frac{1}{\mu}}(\zeta)\,d_\alpha\zeta\right)^\mu &= \left(\int_\iota^\infty \left(\frac{\Omega^\mu(\zeta) f^\mu(\zeta)}{\Lambda^{\mu(\alpha-\mu)}(\zeta)}\right)^{\frac{1}{\mu}} d_\alpha\zeta\right)^\mu \\ &\geqslant \frac{\int_\iota^\infty F(\zeta) G(\zeta)\,d_\alpha\zeta}{\left(\int_\iota^\infty G^{\frac{1}{1-\mu}}(\zeta)\,d_\alpha\zeta\right)^{1-\mu}} \\ &= \left(\int_\iota^\infty \frac{\Omega^\mu(\zeta) f^\mu(\zeta)\lambda^{1-\mu}(\zeta)\Lambda^{\mu(\alpha-\mu)}(\zeta)}{\Lambda^{\mu(\alpha-\mu)}(\zeta)} d_\alpha\zeta\right) \\ &\quad \times \left(\int_\iota^\infty \left(\lambda^{1-\mu}(\zeta)\Lambda^{\mu(\alpha-\mu)}(\zeta)\right)^{\frac{1}{1-\mu}} d_\alpha\zeta\right)^{\mu-1} \\ &= \left(\int_\iota^\infty \Omega^\mu(\zeta) f^\mu(\zeta)\lambda^{1-\mu}(\zeta)\,d_\alpha\zeta\right)\left(\int_\iota^\infty \lambda(\zeta)\Lambda^{\frac{\mu(\alpha-\mu)}{1-\mu}}(\zeta)\,d_\alpha\zeta\right)^{\mu-1}. \end{aligned}$$

Hence

$$\begin{aligned}\left(\int_\iota^\infty \Omega(\zeta) f(\zeta)\Lambda^{(\mu-\alpha)}(\zeta)\,d_\alpha\zeta\right)^\mu \geqslant &\left(\int_\iota^\infty \Omega^\mu(\zeta) f^\mu(\zeta)\lambda^{1-\mu}(\zeta)\,d_\alpha\zeta\right) \\ &\times \left(\int_\iota^\infty \lambda(\zeta)\Lambda^{\frac{\mu(\alpha-\mu)}{1-\mu}}(\zeta)\,d_\alpha\zeta\right)^{\mu-1}. \end{aligned} \tag{3.25}$$

From (3.24) and (3.25), we obtain

$$\begin{aligned}&\left(\int_\iota^\infty \lambda(\zeta)\Lambda^{\mu-\alpha+1}(\zeta)\,d_\alpha\zeta\right)^\mu \\ &\geqslant (\mu-\alpha+1)^\mu\left(\int_\iota^\infty \Omega^\mu(\zeta) f^\mu(\zeta)\lambda^{1-\mu}(\zeta)\,d_\alpha\zeta\right)\left(\int_\iota^\infty \lambda(\zeta)\Lambda^{\frac{\mu(\alpha-\mu)}{1-\mu}}(\zeta)\,d_\alpha\zeta\right)^{\mu-1}, \end{aligned}$$

which is inequality (3.22). The proof is complete. □

Corollary 3.8. *In Theorem 3.7, if* $\alpha = 1$, *then we obtain*

$$\left(\int_{\iota}^{\infty} \lambda(\zeta)\Lambda^{\mu}(\zeta)\,d\zeta\right)^{\mu} \geqslant \mu^{\mu}\left(\int_{\iota}^{\infty} f^{\mu}(\zeta)\,\Omega^{\mu}(\zeta)\,\lambda^{1-\mu}(\zeta)\,d\zeta\right) \times \left(\int_{\iota}^{\infty} \lambda(\zeta)\Lambda^{\mu}(\zeta)\,d\zeta\right)^{\mu-1}. \tag{3.26}$$

Dividing the inequality of (3.26) *by*

$$\left(\int_{\iota}^{\infty} \lambda(\zeta)\Lambda^{\mu}(\zeta)\,d\zeta\right)^{\mu-1},$$

then

$$\int_{\iota}^{\infty} \lambda(\zeta)\Lambda^{\mu}(\zeta)\,d\zeta \geqslant \mu^{\mu}\left(\int_{\iota}^{\infty} f^{\mu}(\zeta)\,\Omega^{\mu}(\zeta)\,\lambda^{1-\mu}(\zeta)\,d\zeta\right),$$

which is the Leindler type inequality of remark (7) in [28].

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