J. Math. Computer Sci., 32 (2024), 137–151

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

A computational technique for computing second-type mixed integral equations with singular kernels

A. M. S. Mahdy[a,b,*], M. A. Abdou[c], D. Sh. Mohamed[a]

[a] Department of Mathematics, Faculty of Science, Zagazig University, P. O. Box 44519, Zagazig, Egypt.

[b] Department of Mathematics and Statistics, College of Science, Taif University, P. O. Box 11099, Taif, 21944, Saudi Arabia.

[c] Department of Mathematics, Faculty of Education, Alexandria University, Alexandria, 21526, Egypt.

Abstract

In the present article, we establish the numerical solution for the mixed Volterra- Fredholm integral equation (MV-FIE) in (1+1) dimensional in the Banach space $L_2[-1,1] \times C[0,T], T < 1$. The Fredholm integral term is considered in the space $L_2[-1,1]$ and it has a discontinuous kernel in position. While the Volterra integral term is considered in the class of time $C[0,T], T < 1$, and has a continuous kernel in time. The necessary conditions have been established to ensure that there is a single solution in the space $L_2[-1,1] \times C[0,T], T < 1$. By utilizing the separation of variables technique, MV-FIE is transformed to Fredholm integral equation (FIE) of the second kind with variables coefficients in time. The separation technique of variables helps the authors choose the appropriate time function to establish the conditions of convergence in solving the problem and obtaining its solution. Then, using the Boubaker polynomials method, we end up with a linear algebraic system (LAS) abbreviated. The Banach fixed point (BFP) hypothesis has been presented to determine the existence and uniqueness of the solution of the LAS. The convergence of the solution and the stability of the error are discussed. The Maple 18 software is used to perform some numerical calculations once some numerical experiments have been taken into consideration.

Keywords: Volterra-Fredholm integral equations, separation of variables, Boubaker polynomials, numerical solution, linear algebraic system.

2020 MSC: 46B45, 65R20.

1. Introduction

Many problems in mathematical physics (Hadjadj and Dussauge [19]), theory of elasticity (Abdou et al. [6, 7] and Popov [35]), contact problems in two layers of elastic materials (Bugami [9]), generalized potential theory (Alhazmi [11]), spectral relationships in laser theory (Gao et al. [18]), quantum mechanics (Lienert and Tumulka [24]), and mixed problems in the idea of elasticity (Aleksandrovsk and Covalence [10] and Georgiadis and Gourgiotis [40]) lead to one kind of integral equation. As a result, we have discovered that integral equations (IEs) have tight ties to various subfields within many scientific disciplines.

*Corresponding author

Email addresses: amr_mahdy85@yahoo.com, amr_mahdy85@zu.edu.eg, amattaya@tu.edu.sa (A. M. S. Mahdy), abdella_777@yahoo.com (M. A. Abdou), doaashokry203@yahoo.com (D. Sh. Mohamed)

doi: 10.22436/jmcs.032.02.04

Received: 2023-01-09 *Revised:* 2023-05-30 *Accepted:* 2023-07-01

Because of the various difficulties and applications, numerous forms of study have been conducted to establish multiple solutions for addressing IEs of several types and kernels.

Diego and Lima in [17], used collocation techniques to explain the numerical solution of weakly singular integral equations. While, Mirzaee and Hoseini in [31], used collocation method for solving V-FIEs with continuous kernels. Wang and Wang in [37], used Taylor polynomial technique for solving MV-FIE of the second kind with continuous kernels. In [13], Baksheesh used Galerkin approximation method to solve VIE of the first kind with convolution kernel. While, Brezinski and Zalglia in [14], used extrapolation methods to get the numerical solution of nonlinear FIE with continuous kernel. The spectral relationships of the orthogonal polynomials methods have played an important role for discussing the solution of integral equations. In [1], Abdou et al. discussed the numerical solution of quadratic IE according to Chebyshev polynomials. Matoog in [29], established an IE with generalized potential kernel from an axisymmetric contact problem and discussed its solution, using orthogonal polynomials method. In addition, Matoog in [28], discussed the solution of the nuclear integral equation in quantum physics problem, using orthogonal polynomials methods. Nemati et al. in [33] discussed the numerical solution of a class of two-dimensional nonlinear VIEs using the orthogonal polynomial approach in the Legendre form. In [4], Abdou and Alharbi, used the spectral relationships methods to examine the solution of FIE with singular kernel. In [20], Hafez and Youssri used spectral relationships in the form of Legendre-Chebyshev to examine the numerical solution of nonlinear VIE with continuous kernel. Using a computational approach for solving three dimensional MV-FIE is what Mahdy and his colleagues did in their paper [26]. Mahdy et al. employed the Chelyshkov polynomials approach in [27] to solve the nonlinear first-order integral problem in two dimensions with continuous kernels. Mohammad et al. [32] used Euler wavelets for solving nonlinear Fredholm and VIE. Micula [30] presented Trapezoidal rule to obtain the solution of two-dimensional Fredholm-Volterra integral equations. Providas [36] discussed the analytical and numerical methods for solving FIE. In [34], Noeiaghdam and Micula used Lagrange-collocation method for solving VIE with discontinuous kernel.

In several articles, Boubaker polynomials were employed, and this implementation in different physics problems in science was Boubaker polynomials augmentation method for the resolution of nonlinear high-order differential equations by Yücel and Boubaker in [39]. The Boubaker polynomial augmentation approach and variable separation were utilized by Boubaker et al. in [15] to solve the Neutron transport equation. Boubaker polynomial bases were used by Salih and Akkaya in [38] to solve linear integro-differential difference equations. The Boubaker polynomials collocation approach for resolving nonlinear VFIE of continuous kernel schemes is introduced by Davaeifar and Rashidinia in [16]. Khajehnasiri and Ezzati [22] applied Boubaker polynomials and these implementation for resolution fractional 2-dimensional nonlinear partial integro-differential equation. The advantages of Boubaker polynomial method that the CPU time used for solving examples are very small and this method more accurate than other methods.

The integral operator's form of MV-FIE will be taken into consideration in the remaining sections of this study, particularly in Section 2. The integral operator is then shown to be bounded, continuous, and a contraction operator using the fixed-point Banach theory. That is, there is one unique solution to the mixed integral problem. In Section 3, the technique of separation is implemented to transform the mixed integral equation to a FIE with variables coefficients in time, the stability of the algebraic system is considered. In Section 4, Boubaker polynomials method, as a numerical procedure is utilized to transform FIE to a linear algebraic system. The convergence of solution is discussed in Section 5. Section 6 is about illustrations. Moreover, the FIE is computed numerically, when the kernel takes some forms of singularity like logarithmic. Carleman function and Cauchy kernel are illustrated using maple 18. In addition, the error, in each case is estimated.

Consider the MV-FIE:

$$\mu u(x,t) - \lambda_1 \int_0^t \int_{-1}^1 k(|x-y|)\varphi(t,\tau)u(y,\tau)\,dy\,d\tau - \lambda_2 \int_{-1}^1 k(|x-y|)u(y,t)\,dy - \lambda_3 \int_0^t \psi(t,\tau)u(x,\tau)\,d\tau = f(x,t), \tag{1.1}$$

which $\varphi(t,\tau)$ and $\psi(t,\tau)$ have 2 kernels of continuous functions in time and belong to the class $C[0,T]$, while the discontinuous function $k(|x-y|)$ is the kernel of position and belongs to the space $L_2[-1,1]$ in set, the given function $f(x,t)$ is the free term and belongs to the space $L_2[-1,1] \times C[0,T]$, the invariable μ determines the type of the IE, $\lambda_i, i = 1,2,3$ are numerical parameters and $u(x,t)$ is the unknown function. The behaviors of the unknown function $u(x,t)$, in the space $L_2[-1,1] \times C[0,T]$ is the same behaviors of the free function.

2. Existence of a unique solution of MV-FIE

Equation (1.1) allows for the derivation of numerous particular instances, such as

(a) If, in Equation (1.1), $\varphi(t,\tau) = 0$, the next integral equation is available

$$\mu u(x,t) - \lambda_2 \int_{-1}^1 k(|x-y|)u(y,t)\,dy - \lambda_3 \int_0^t \psi(t,\tau)u(x,\tau)\,d\tau = f(x,t). \tag{2.1}$$

The second kind of F-VIE is represented by the aforementioned formula, and various approaches to solving it are addressed in the area $L_2[-1,1] \times C[0,T], T < 1$, (Abdou [2, 3]).

(b) If in Equation (1.1), $\psi(t,\tau) = 0$, we have

$$\mu u(x,t) - \lambda_1 \int_0^t \int_{-1}^1 k(|x-y|)\varphi(t,\tau)u(y,\tau)\,dy\,d\tau - \lambda_2 \int_{-1}^1 k(|x-y|)u(y,t)\,dy = f(x,t). \tag{2.2}$$

The formula (2.2) represents V-FIE of the second kind in the space $L_2[-1,1] \times C[0,T], T < 1$. The numerical solution is discussed in (1+1) dimensional (Abdou [5]). The physical phenomena of Equations (2.1)-(2.2) in the contact problems in time and position is explained in Abdou [3].

(c) If, in Equation (1.1), $\mu = 0$, we have the initial iteration of V-FIE.

$$-\lambda_1 \int_0^t \int_{-1}^1 k(|x-y|)\varphi(t,\tau)u(y,\tau)\,dy\,d\tau - \lambda_2 \int_{-1}^1 k(|x-y|)u(y,t)\,dy - \lambda_3 \int_0^t \psi(t,\tau)u(x,\tau)\,d\tau = f(x,t).$$

Several spectral connections with the kernel of position having a solitary term and the kernel of time being continuous have been generated and investigated in Abdou [8]. In circumstances where the position kernel has a singular kernel, these relationships may exist. It is well known that spectral relations play an important role in explaining many physical phenomena and are often used to obtain the formula for the asymptotic solution, which is in the form of a linear combination between the eigenvalues and the eigenvectors (Abdou [8]). In this part, to establish the EUS of Equation (1.1), we suppose next cases.

(1) The kernel $k(|x-y|)$ fulfills the condition:

$$\left[\int_{-1}^1 \int_{-1}^1 k^2(|x-y|)\,dx\,dy\right]^{\frac{1}{2}} = M,$$

where M is a constant.

(2) The two kernels $\varphi(t,\tau)$ and $\psi(t,\tau)$ satisfy

$$|\,\varphi(t,\tau)\,| \leqslant M_1, \qquad |\,\psi(t,\tau)\,| \leqslant M_2, \quad \forall t,\tau \in [0,T],$$

M_1 and M_2 are constants.

(3) The free term $f(x,t) \in L_2[-1,1] \times C[0,T]$ and its norm determined:

$$\| f(x,t) \| = \max_{0 \leqslant t \leqslant T} \int_0^t \left(\int_{-1}^{1} f^2(x,t)dx \right)^{\frac{1}{2}} dt = M_3,$$

M_3 is a constant.

Theorem 2.1. *If the cases* (1)-(3) *have fulfilled, subsequently Equation* (1.1) *has a unique solution* $u(x,t)$ *in the Banach space* $\in L_2[-1,1] \times C[0,T]$, $0 \leqslant T < 1$ *under the condition*

$$|\mu| \geqslant (|\lambda_1| M M_1 + |\lambda_2| M + |\lambda_3| M_2) T.$$

Proof. To proof EUS of Equation (1.1) we utilize BFP theorem or Picard technique. The author of [21] used Picard technique. In this paper we use BFP assumption to proof the EUS of Equation (1.1).

We form the formula (1.1) in the operator create:

$$(\overline{W}u)(x,t) = \frac{1}{\mu} f(x,t) + \frac{1}{\mu}(Wu)(x,t), \quad \mu \neq 0, \tag{2.3}$$

where

$$\begin{aligned}(Wu)(x,t) = \lambda_1 \int_0^t \int_{-1}^{1} k(|x-y|)\varphi(t,\tau)u(y,\tau)dyd\tau \\ + \lambda_2 \int_{-1}^{1} k(|x-y|)u(y,t)dy + \lambda_3 \int_0^t \psi(t,\tau)u(x,\tau)d\tau.\end{aligned} \tag{2.4}$$

We must demonstrate the following in order to demonstrate the existence of a unique solution to Equation (2.3).

2.1. *The normality*

From Equation (2.4) we have

$$\begin{aligned}\| (Wu)(x,t) \| \leqslant \left| \frac{\lambda_1}{\mu} \right| \left\| \int_0^t \int_{-1}^{1} k(|x-y|)\varphi(t,\tau)u(y,\tau)dyd\tau \right\| + \left| \frac{\lambda_2}{\mu} \right| \left\| \int_{-1}^{1} k(|x-y|)u(y,t)dy \right\| \\ + \left| \frac{\lambda_3}{\mu} \right| \left\| \int_0^t \psi(t,\tau)u(x,\tau)d\tau \right\|, \quad \mu \neq 0.\end{aligned}$$

Now implementation Cauchy-Schwarz inequality and utilization the cases (1) and (2) we obtain

$$\begin{aligned}\| (Wu)(x,t) \| \leqslant \left| \frac{\lambda_1}{\mu} \right| \left\| \int_0^t |\varphi(t,\tau)| d\tau \left(\int_{-1}^{1} \int_{-1}^{1} k^2(|x-y|)dxdy \right)^{\frac{1}{2}} \max_{0 \leqslant t \leqslant T} \int_0^t \left(\int_{-1}^{1} u^2(y,\tau)dy \right)^{\frac{1}{2}} d\tau \right\| \\ + \left| \frac{\lambda_2}{\mu} \right| \left\| \left(\int_{-1}^{1} \int_{-1}^{1} k^2(|x-y|)dxdy \right)^{\frac{1}{2}} \max_{0 \leqslant t \leqslant T} \int_0^t \left(\int_{-1}^{1} u^2(y,\tau)dy \right)^{\frac{1}{2}} d\tau \right\| \\ + \left| \frac{\lambda_3}{\mu} \right| \left\| \int_0^t |\psi(t,\tau)| d\tau \max_{0 \leqslant t \leqslant T} \int_0^t \left(\int_{-1}^{1} u^2(y,\tau)dy \right)^{\frac{1}{2}} d\tau \right\|,\end{aligned}$$

Hence, the above inequality takes the form

$$\| (Wu)(x,t) \| \leqslant \left(\left| \frac{\lambda_1}{\mu} \right| M M_1 + \left| \frac{\lambda_2}{\mu} \right| M + \left| \frac{\lambda_3}{\mu} \right| M_2 \right) T \, \| u(x,t) \| .$$

Finally, we have

$$\| (Wu)(x,t) \| \leqslant \sigma \| u(x,t) \|, \quad \sigma = \left(\left| \frac{\lambda_1}{\mu} \right| M M_1 + \left| \frac{\lambda_2}{\mu} \right| M + \left| \frac{\lambda_3}{\mu} \right| M_2 \right) T < 1.$$

Consequently the integral operator W is bounded and has a normality, that means the normality of the operator $\overline{W}$ after using the condition (3).

2.2. The continuity

Assume that the 2 functions $u_1(x,t), u_2(x,t) \in L_2[-1,1] \times C[0,T]$ have fulfilled (2.3), then

$$(\overline{W}u_1)(x,t) = \frac{1}{\mu}f(x,y) + \frac{\lambda_1}{\mu}\int_0^t\int_{-1}^1 k(|x-y|)\varphi(t,\tau)u_1(y,\tau)dyd\tau + \frac{\lambda_2}{\mu}\int_{-1}^1 k(|x-y|)u_1(y,t)dy \\ + \frac{\lambda_3}{\mu}\int_0^t \psi(t,\tau)u_1(x,\tau)d\tau, \tag{2.5}$$

and

$$(\overline{W}u_2)(x,t) = \frac{1}{\mu}f(x,y) + \frac{\lambda_1}{\mu}\int_0^t\int_{-1}^1 k(|x-y|)\varphi(t,\tau)u_2(y,\tau)dyd\tau + \frac{\lambda_2}{\mu}\int_{-1}^1 k(|x-y|)u_2(y,t)dy \\ + \frac{\lambda_3}{\mu}\int_0^t \psi(t,\tau)u_2(x,\tau)d\tau. \tag{2.6}$$

from (2.5) and (2.6) we obtain

$$\overline{W}(u_1(x,t)-u_2(x,t)) = \frac{\lambda_1}{\mu}\int_0^t\int_{-1}^1 k(|x-y|)\varphi(t,\tau)(u_1(y,\tau)-u_2(y,\tau))dy \\ + \frac{\lambda_2}{\mu}\int_{-1}^1 k(|x-y|)(u_1(y,t)-u_2(y,t))dy + \frac{\lambda_3}{\mu}\int_0^t \psi(t,\tau)(u_1(x,\tau)-u_2(x,\tau))_d\tau.$$

Following the application of the Cauchy-Schwarz inequality and utilizing (1) and (2), we obtain

$$\| \overline{W}(u_1(x,t)-u_2(x,t)) \| \leqslant \left| \frac{\lambda_1}{\mu} \right| MM_1T \| u_1(x,t)-u_2(x,t) \| + \left| \frac{\lambda_2}{\mu} \right| MT \| u_1(x,t)-u_2(x,t) \| \\ + \left| \frac{\lambda_3}{\mu} \right| M_2T \| u_1(x,t)-u_2(x,t) \|,$$

i.e.,

$$\| \overline{W}(u_1(x,t)-u_2(x,t)) \| \leqslant \sigma \| u_1(x,t)-u_2(x,t) \|, \tag{2.7}$$

where $\sigma = (|\frac{\lambda_1}{\mu}| \; MM_1 + |\frac{\lambda_2}{\mu}| M + |\frac{\lambda_3}{\mu}| M_2)T$.

Inequality (2.7) means the continuity of the integral operator $\overline{W}$. Therefore $\overline{W}$ is a constriction operator. Consequently via Banach fixed point theory this has a unique fixed point $u(x,t)$ that is the resolution of (1.1). Therefore, the integral operator W is a contraction operator under the condition $|\frac{\lambda_1}{\mu}| \; MM_1 + |\frac{\lambda_2}{\mu}| M + |\frac{\lambda_3}{\mu}| M_2)T < 1$. □

3. Separation of variables technique

The separation of variables technique is considered as one of the best major ways to explain mathematical physics problems. When the undetermined function is linked to the arranged and temporal variables, this is when the separation of variables technique comes into its own as an important technique. There is assistance for the researcher to select an appropriate time function, we are able to discuss the strategy behind the organize function. For this, let the next estimations:

$$u(x,t) = \Phi(x)\,\Psi(t), \quad f(x,t) = F(x)\,\Psi(t), \quad \Psi(0) \neq 0, \tag{3.1}$$

where $\Phi(x)$ and $F(x)$ are functions of the position only and $\Psi(t)$ is a function of time only.

It should be emphasized that the authors considered both the known function and the unknown function to have the same time function. The known function is called the free surface function in contact problems in elastic science or mixed problems in mechanics and engineering sciences. While the unknown function is defined as the difference in the stresses of the medium. This hypothesis, equal to the time function is very important in the theory of economics and manufacturing cost, as the time given for the known function is the same as the time for the solution of the unknown function. This was discussed in the research of Sharifa and Abdou [12]. Also in the research of Mahdy et al. [25], the separation of variables was discussed with the difference of time functions between the known function and the unknown function, and it was found that the relative error in this case increases. Substituting from (3.1) into (1.1) we obtain

$$\begin{aligned}\mu\Phi(x)\Psi(t)-\lambda_1\int_0^t\int_{-1}^1 k(|x-y|)\varphi(t,\tau)\Phi(y)\Psi(\tau)dyd\tau\\ -\lambda_2\int_{-1}^1 k(|x-y|)\Phi(y)\Psi(t)dy-\lambda_3\int_0^t\psi(t,\tau)\Phi(x)\Psi(\tau)d\tau=F(x)\Psi(t).\end{aligned}\tag{3.2}$$

Equation (3.2) may be formulated in the next establish

$$(\mu-\frac{\lambda_3}{\Psi(t)}\int_0^t\psi(t,\tau)\Psi(\tau)d\tau)\Phi(x)-(\lambda_2+\frac{\lambda_1}{\Psi(t)}\int_0^t\varphi(t,\tau)\Psi(\tau)d\tau)\int_{-1}^1 k(|x-y|)\Phi(y)dy=F(x).\tag{3.3}$$

Let

$$\alpha(t)=(\mu-\frac{\lambda_3}{\Psi(t)}\int_0^t\psi(t,\tau)\Psi(\tau)d\tau)$$

and

$$\beta(t)=(\lambda_2+\frac{\lambda_1}{\Psi(t)}\int_0^t\varphi(t,\tau)\Psi(\tau)d\tau).$$

Then Equation (3.3) takes the form

$$\alpha(t)\ \Phi(x)-\beta(t)\int_{-1}^1 k(|x-y|)\Phi(y)dy=F(x).\tag{3.4}$$

The following statements demonstrate that the condition for there existing a unique answer to Equation (3.4) has been met:

$$\| k(|x-y|)\|\leqslant\|\frac{\alpha(t)}{\beta(t)}\|.$$

The last equation explicitly states the necessary and sufficient condition to solve the problem, where the condition of the functional kernel in position was linked to another condition indicative in time. It is feasible to clearly describe the amount of time needed to continue the single solution by determining the shape of the time function on the right side of the inequality (3.4).

3.1. *The stability of the error*

To discuss the numerical solution of Equation (3.4) we assume the approximate solution, $\forall t\in[0,T], T<1$ is $\Phi_m(x)$, for the free term $F(x)$. Hence, we get

$$\alpha(t)\ R_m(x)=\beta(t)\int_{-1}^1 k(|x-y|)R_m dy,\quad (R_m(x)=\Phi(x)-\Phi_m(x)).$$

In this case, the error represents a homogeneous integral equation. In the modified error, we have

$$\alpha(t)\ R_m(x)=\beta(t)\int_{-1}^1 k(|x-y|)R_m dy=F(x)-F_m(x).$$

In the modified error, the error represents a nonhomogeneous integral equation.

4. Boubaker polynomials technique

In this part, we will study the approximate solution of Equation (3.4) via using Boubaker polynomials method, see [23].

Definition 4.1. The Boubaker polynomials of degree n are specified on $[-1,1]$ as:

$$B_n(x)=\sum_{m=0}^{\lceil \frac{n}{2}\rceil}(-1)^m\binom{n-m}{m}\frac{n-4m}{n-m}x^{n-2m}, n\geqslant 1,\quad B_0(x)=1, \tag{4.1}$$

where $\lceil \frac{n}{2}\rceil$ express the major integer in $\frac{n}{2}$.

4.1. Properties of Boubaker polynomials

(I) Recurrence relation in the form

$$B_{n+1}(x)=xB_n(x)-B_{n-1}(x), n=1,2,\dots$$

with $B_0(x)=1$ and $B_1(x)=x$.

(II) $B_n(0)=2\cos(\frac{n+2}{2}\pi),\quad n\geqslant 1$.

(III) Even and odd functions are distinct in polynomials of even and odd orders, i.e.,

$$B_n(-x)=(-1)^nB_n(x).$$

(IV) The polynomials (4.1) may be displayed in terms of Chebyshev polynomials of the primary and second types $T_n(x), U_n(x)$.

Also, we state the following two theorems without proof (see [23]).

Theorem 4.2. *For* $n\geqslant 1$, $B_n(x)=2T_n(x/2)+4U_{n-2}(x/2)$ *holds.*

Theorem 4.3. *Suppose that* $\Phi(x)\in L_2[-1,1]$ *and* $\Phi(x)$ *is approximated by* $\sum_{j=0}^{N}a_jB_j(x)$, *then we have*

$$\lim_{N\to\infty}\|\Phi(x)-\sum_{j=0}^{N}a_jB_j(x)\|_{L_2[-1,1]}=0.$$

4.2. Function approximation of numerical solution

Assume that $X=\operatorname{span}\{B_j(x), j=0,1,\dots,N\}$ and $\Phi(x)$ is an arbitrary function in $L_2[-1,1]$, then Φ has the almost convergence out of X such as $\Phi_N\in X$ such that

$$\|\ \Phi-\Phi_N\ \|\leqslant\|\ \Phi-z\ \|,\quad \forall z\in X.$$

Therefore, for $\Phi_N\in X$ there is a unique set of coefficients $a_0,a_1,\dots,a_N$ such that [8]

$$\Phi(x)\simeq\Phi_N(x)=\sum_{j=0}^{N}a_jB_j(x)=A^TB(x), \tag{4.2}$$

where A is an $(N+1)\times 1$ vector specified $A=[a_0,a_1,\dots,a_N]^T$, $B(x)$ is the vector function specified as $B(x)=[B_0(x),B_1(x),\dots,B_N(x)]^T$ and the coefficients vector A may be calculated via:

$$A^T<B(x),B(x)>=<\Phi(x),B(x)>,\qquad <\Phi(x),B(x)>=\int_0^1\Phi(x)B^T(x)dx,$$

$< .,. >$ denotes inner product on $L_2[0,1]$, by description

$$H =< B(x), B(x) >,$$

then

$$A^T = (\int_0^1 \Phi(x)B^T(x)dx)H^{-1},$$

where H is $(N+1)\times(N+1)$ matrix as

$$H =< B(x), B(x) >= \int_0^1 B(x)B^T(x)dx.$$

Now substituting from (4.2) into (3.4) we obtain

$$\alpha(t)\sum_{j=0}^{N} a_j B_j(x) - \beta(t)\int_{-1}^{1} k(| x-y |)\sum_{j=0}^{N} a_j B_j(y)dy = F(x),$$

or

$$\sum_{j=0}^{N} a_j[\alpha(t)B_j(x) - \beta(t)\int_{-1}^{1} k(| x-y |)B_j(y)dy] = F(x).$$

Collocation points are used in this sentence

$$x_l = a + \frac{(b-a)l}{N}, \quad l = 0,1,\dots,N,$$

we get the next scheme of LAE with $(N+1)$ unknowns

$$\sum_{j=0}^{N} a_j[\alpha(t)B_j(x_l) - \beta(t)\int_{\Omega} k(| x_l-y |)B_j(y)dy] = F(x_l).$$

It is possible to determine the constant values by solving the system shown above $a_j, j = 0,1,\dots,N$ and we get the convergent solution of Equation (3.4), finally by using (3.1) we obtain the solution of Equation (1.1).

5. The convergence of solution

One of the basics of searching a solution is to study the valid region and time for the existence of a single solution. Then, study the convergence of the solution in the sense of benefiting from it and realistic results can be reached. Finally, using analytical or numerical methods to find the solution. Here, the researcher must deal with the study of the resulting error, which is represented in knowing the error, the results from the approximation of the solution, as well as the error resulting from the use of the chosen method for the solution, and finally the error resulting from the programs used. Hence, researchers must pay attention to the study of convergence and error. To discuss the convergence of solution $u(x,t)$, we construct the family of solution $u(x,t) = \{u_0(x,t), u_1(x,t),\dots,u_{n-1}(x,t), u_n(x,t),\dots\}$ or in a simple form $u(x,t) = \{u_i(x,t)\}_{i=0}^{\infty}$. After this, we pick two functions $u_{n-1}(x,t), u_n(x,t)$, satisfy the general integral Equation (1.1) and to construct the sequence of integral equations as

$$\begin{aligned}\mu u_n(x,t) = {}& \lambda_1\int_0^t\int_{-1}^{1} k(|x-y|)\varphi(t,\tau)u_{n-1}(y,\tau)dyd\tau \\ & - \lambda_2\int_{-1}^{1} k(|x-y|)u_{n-1}(y,t)dy + \lambda_3\int_0^t \psi(t,\tau)u_{n-1}(x,\tau)d\tau,\end{aligned} \tag{5.1}$$

$$
\begin{aligned}
\mu u_{n-1}(x,t) = \lambda_1 \int_0^t \int_{-1}^1 k(|x-y|)\varphi(t,\tau)u_{n-2}(y,\tau)dyd\tau \\
- \lambda_2 \int_{-1}^1 k(|x-y|)u_{n-2}(y,t)dy + \lambda_3 \int_0^t \psi(t,\tau)u_{n-2}(x,\tau)d\tau.
\end{aligned} \tag{5.2}
$$

From Equations (5.1) and (5.2), we can construct a new family of corresponding function of solution

$$
\begin{aligned}
\mu v_n(x,t) = \lambda_1 \int_0^t \int_{-1}^1 k(|x-y|)\varphi(t,\tau)v_{n-1}(y,\tau)dyd\tau \\
- \lambda_2 \int_{-1}^1 k(|x-y|)v_{n-1}(y,t)dy + \lambda_3 \int_0^t \psi(t,\tau)v_{n-1}(x,\tau)d\tau.
\end{aligned} \tag{5.3}
$$

In (5.3), we assumed that

$$
v_n(x,t) = u_n(x,t) - u_{n-1}(x,t). \tag{5.4}
$$

From (5.4), we deduce that

$$
u_n(x,t) = \sum_{i=0}^{n} v_i(x,t), \quad v_0(x,t) = f(x,t)/\mu. \tag{5.5}
$$

Theorem 5.1. *A continuous solution function* $u(x,t)$ *can be reached by the series* $\sum_{i=0}^{\infty} v_i(x,t)$ *in a manner that is uniformly convergent.*

Proof. The answer to the integral Equation (5.3) can be found by first the conditions (1)-(2), and applying the inequality derived from Cauchy and Schwarz,

$$
\begin{aligned}
\| v_n(x,t) \| &\leqslant \sigma \| v_{n-1}(x,t) \|, \\
\sigma &= \left(\frac{|\lambda_1| MM_1 + |\lambda_2| M + |\lambda_3| M_2}{|\mu|}\right)T, \\
\| v_n(x,t) \| &= \max_{0\leqslant t\leqslant T} \int_0^t \left(\int_{-1}^1 v_n^2(x,t)dx\right)^{\frac{1}{2}} d\tau.
\end{aligned}
$$

Using the mathematical induction, with the value of $v_0(x,t)$ and applying condition (3), we have

$$
\| v_n(x,t) \| \leqslant \frac{\sigma^n}{n!} M_3. \tag{5.6}
$$

The inequality (5.6) leads to the convergence of the sequence $\{v_n(x,t)\}$ and then the sequence $\{u_n(x,t)\}$ is also convergence uniformly. Hence, from Equation (5.5), $\forall t \in [0,T]$ and $n \to \infty$, we have

$$
u(x,t) = \lim_{n\to\infty} u_n(x,t) = \lim_{n\to\infty} \left(\sum_{i=0}^{n} v_i(x,t)\right).
$$

□

6. Numerical illustration

In here, four numerical illustrations are given to explain the above results. The results have been reported in tables and figures. All of the numerical calculations are performed by utilizing Maple 18 program.

Example 6.1. Let the next MV-FIE [21]:

$$0.9u(x,t) = f(x,t) + 0.32\int_0^t\int_{-1}^1 k(|\,x-y\,|)t^2\tau^2 u(y,\tau)\,dy\,d\tau + 0.25\int_{-1}^1 k(|\,x-y\,|)u(y,t)\,dy + 0.3\int_0^t t^2\tau u(x,\tau)\,d\tau, \tag{6.1}$$

where the function $f(x,t)$ is specified by laying $u(x,t) = (0.25 + t^2)x^2$ as an exact solution. Take the kernel Carleman

$$k(|\,x-y\,|) = |x-y|^{-\nu}, \quad 0 < \nu < 1,$$

where ν is named Poisson rate and in this case the kernel is called weakly singular kernel. First applying separation of variables method by taking $\Psi(t) = (0.25 + t^2)$, then applying Boubaker polynomials method for $N = 5$ we get the convergent solution of Equation (6.1). The numerical outcomes of Example 6.1 are shown in Table 1, as well as Figures 1 and 2, for a range of different Poisson coefficients values ν at time $T = 0.3$.

Table 1: Numerical outcomes of Example 6.1.

	$\nu = 0.12$		$\nu = 0.73$	
x	Error of our method	Error of [21]	Error of our method	Error of [21]
-1	$1.740746923 \times 10^{-10}$	3.67468×10^{-5}	$2.810800337 \times 10^{-10}$	9.14093×10^{-5}
-0.8	$8.239728786 \times 10^{-11}$	1.72775×10^{-5}	$2.028626485 \times 10^{-10}$	3.14502×10^{-5}
-0.6	$7.117410310 \times 10^{-11}$	1.73390×10^{-5}	$3.707555985 \times 10^{-10}$	3.30406×10^{-5}
-0.4	$5.647045049 \times 10^{-11}$	1.74052×10^{-5}	$1.808350656 \times 10^{-10}$	3.39897×10^{-5}
-0.2	$7.761448357 \times 10^{-11}$	1.74485×10^{-5}	$2.393089123 \times 10^{-10}$	3.45140×10^{-5}
0	$1.398491597 \times 10^{-10}$	1.74632×10^{-5}	$5.482440489 \times 10^{-10}$	3.46828×10^{-5}
0.2	$1.847372769 \times 10^{-10}$	1.74485×10^{-5}	$5.245828074 \times 10^{-10}$	3.45140×10^{-5}
0.4	$1.872793282 \times 10^{-10}$	1.74052×10^{-5}	$6.581665087 \times 10^{-11}$	3.39897×10^{-5}
0.6	$7.414581103 \times 10^{-11}$	1.73390×10^{-5}	$6.796931566 \times 10^{-10}$	3.30406×10^{-5}
0.8	$1.009664623 \times 10^{-10}$	1.72775×10^{-5}	$1.096446104 \times 10^{-9}$	3.14502×10^{-5}
1	$3.923538237 \times 10^{-11}$	3.67468×10^{-5}	$4.062254801 \times 10^{-10}$	9.14093×10^{-5}

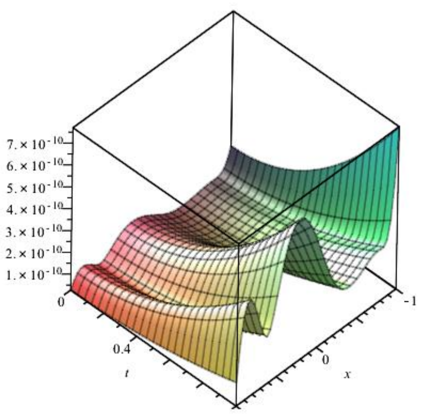

Figure 1: Error of Example 6.1, $T = 0.3$, $\upsilon = 0.12$.

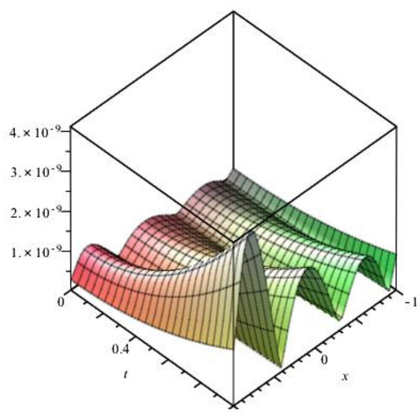

Figure 2: Error of Example 6.1, $T = 0.3$, $\upsilon = 0.0.73$.

Example 6.2. Let the next MV-FIE:

$$u(x,t) = f(x,t) + 0.01\int_0^t\int_{-1}^1 k(|\,x-y\,|)t^2\tau^2 u(y,\tau)\,dy\,d\tau + 0.01\int_0^t (t-\tau)^2 u(x,\tau)\,d\tau,$$

where the function $f(x, t)$ has presented via letting $u(x, t) = x^2t^2$ as an accurate solution. Applying the above methods by using the weakly singular kernel $k(|x - y|) = |x - y|^{-0.22}$ for about points in the area $x \in [-1, 1]$, for several values of time $T = \{0.01, 0.2, 0.8\}$ and $N = 5$ the numerical outcomes are shown in Table 2 and Figures 3, 4, and 5.

Table 2: Numerical outcomes of Example 6.2.

x	Error for $T = 0.01$	Error for $T = 0.2$	Error for $T = 0.8$
-1	$2.673610000 \times 10^{-14}$	$4.233982577 \times 10^{-11}$	$4.895821144 \times 10^{-11}$
-0.8	$1.401669695 \times 10^{-14}$	$2.947702201 \times 10^{-11}$	$5.137019349 \times 10^{-10}$
-0.6	$4.706521292 \times 10^{-15}$	$3.568938157 \times 10^{-11}$	$4.173224958 \times 10^{-10}$
-0.4	$1.631548240 \times 10^{-15}$	$4.567414698 \times 10^{-11}$	$1.010312735 \times 10^{-10}$
-0.2	$5.309727622 \times 10^{-15}$	$5.213004105 \times 10^{-11}$	$1.529305699 \times 10^{-10}$
0	$6.515264960 \times 10^{-15}$	$5.351560680 \times 10^{-11}$	$2.024558628 \times 10^{-10}$
0.2	$5.310377874 \times 10^{-15}$	$5.180754757 \times 10^{-11}$	$2.887428939 \times 10^{-11}$
0.4	$1.632190898 \times 10^{-15}$	$5.025906708 \times 10^{-11}$	$2.797865435 \times 10^{-10}$
0.6	$4.707327126 \times 10^{-15}$	$5.115820939 \times 10^{-11}$	$5.455095853 \times 10^{-10}$
0.8	$1.402142565 \times 10^{-14}$	$5.358619905 \times 10^{-11}$	$5.170179661 \times 10^{-10}$
1	$2.674863504 \times 10^{-14}$	$5.117578108 \times 10^{-11}$	$1.135089806 \times 10^{-10}$

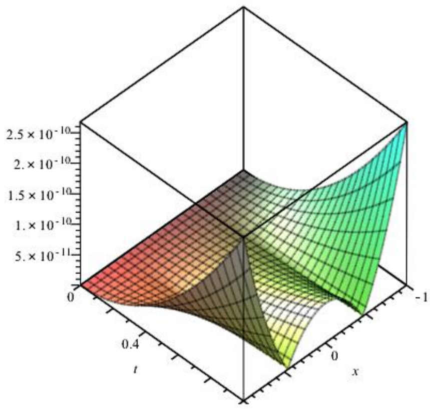

Figure 3: Error of Example 6.2, $T = 0.01$.

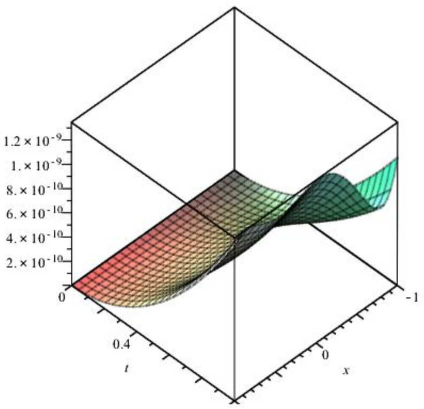

Figure 4: Error of Example 6.2, $T = 0.2$.

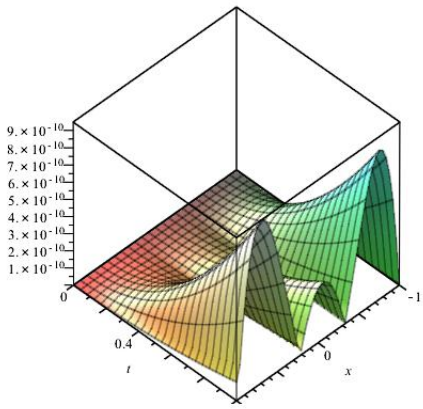

Figure 5: Error of Example 6.2, $T = 0.8$.

Example 6.3. Let the next MV-FIE with continuous kernel:

$$u(x,t) = f(x,t) + 0.01\int_0^t\int_0^1 (x-y)^3 t^2\tau^2 u(y,\tau)\,dy\,d\tau + 0.01\int_0^1 (x-y)^3 u(y,t)\,dy + 0.01\int_0^t (t-\tau)^2 u(x,\tau)\,d\tau,$$

where the function $f(x,t)$ has presented via letting $u(x,t) = x^2(0.25+t^2)$ as an accurate solution. Applying the above methods by using several values of time $T = \{0.01, 0.2, 0.8\}$ and $N = 5$ the numerical outcomes are shown in Table 3 and Figures 6, 7, and 8.

Table 3: Numerical outcomes of Example 6.3.

x	Error for $T = 0.01$	Error for $T = 0.2$	Error for $T = 0.8$
0	$1.431644929 \times 10^{-10}$	$8.0016684 \times 10^{-11}$	$1.2281466 \times 10^{-10}$
0.1	$3.390915943 \times 10^{-10}$	$2.728285825 \times 10^{-10}$	$5.552782424 \times 10^{-10}$
0.2	$1.963793915 \times 10^{-10}$	$4.522424807 \times 10^{-10}$	$5.048419568 \times 10^{-10}$
0.3	$2.063102567 \times 10^{-11}$	$4.632450729 \times 10^{-10}$	$1.486937073 \times 10^{-9}$
0.4	$4.95858108 \times 10^{-11}$	$2.918024572 \times 10^{-10}$	$1.700519170 \times 10^{-9}$
0.5	3.631203×10^{-12}	$1.3075846 \times 10^{-11}$	$1.223717355 \times 10^{-9}$
0.6	$1.23889109 \times 10^{-10}$	$2.60362714 \times 10^{-10}$	$5.4614463 \times 10^{-10}$
0.7	$2.26869682 \times 10^{-10}$	$4.30314631 \times 10^{-10}$	$2.1176461 \times 10^{-10}$
0.8	$2.4669959 \times 10^{-10}$	$4.6473909 \times 10^{-10}$	$4.6175840 \times 10^{-10}$
0.9	$1.8232045 \times 10^{-10}$	$4.4953728 \times 10^{-10}$	8.773916×10^{-10}
1	$1.4385884 \times 10^{-10}$	$6.4033671 \times 10^{-10}$	2.28807×10^{-11}

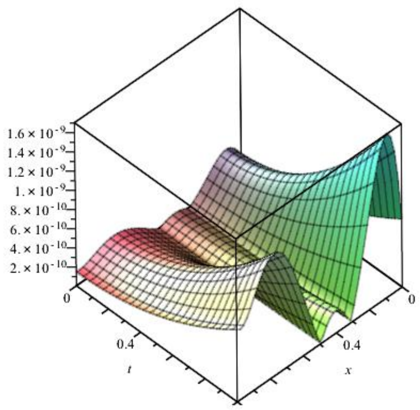

Figure 6: Error of Example 6.3, $T = 0.01$.

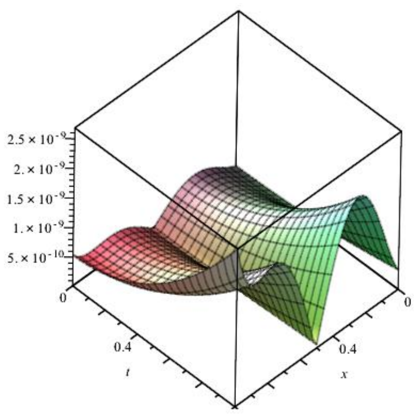

Figure 7: Error of Example 6.3, $T = 0.2$.

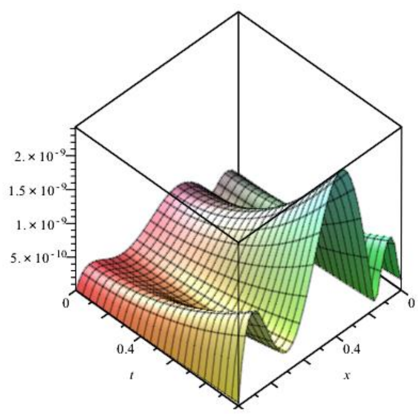

Figure 8: Error of Example 6.3, $T = 0.8$.

Example 6.4. Let the next MV-FIE with continuous kernel:

$$u(x,t) = f(x,t) + 0.01\int_0^t\int_0^1 x^3y\ t^2\tau^2 u(y,\tau)dyd\tau + 0.01\int_0^1 x^3y\ u(y,t)dy + 0.01\int_0^t (t-\tau)^2 u(x,\tau)d\tau,$$

where the function $f(x,t)$ has presented via letting $u(x,t) = ((0.25+t^2)\sin x)$ as an accurate solution. Applying the above methods by using several values of time $T = \{0.01, 0.2\}$ and $N = 5$ the numerical outcomes are shown in Table 4 and Figures 9 and 10.

Table 4: Numerical outcomes of Example 6.4.

x	Error for $T = 0.01$	Error for $T = 0.2$
0	2.501×10^{-13}	5.8×10^{-13}
0.1	1.3954×10^{-7}	1.6181×10^{-7}
0.2	6×10^{-11}	$4. \times 10^{-11}$
0.3	4.941×10^{-8}	5.725×10^{-8}
0.4	2×10^{-11}	1×10^{-10}
0.5	3.71×10^{-8}	4.33×10^{-8}
0.6	1×10^{-10}	0
0.7	5.50×10^{-8}	6.36×10^{-8}
0.8	1×10^{-10}	2×10^{-10}
0.9	1.723×10^{-7}	1.999×10^{-7}
1	0	1×10^{-10}

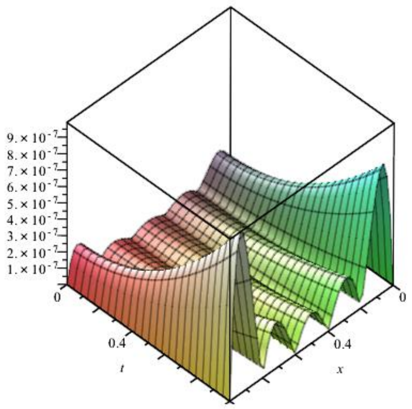

Figure 9: Error of Example 6.4, $T = 0.01$.

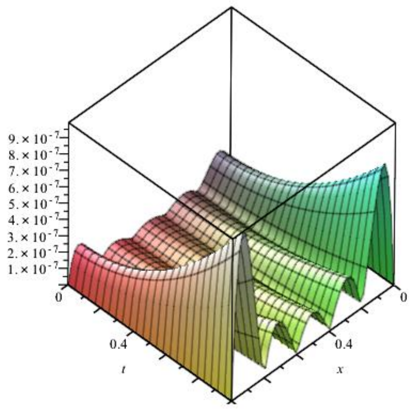

Figure 10: Error of Example 6.4, $T = 0.2$.

7. Conclusions

In this document, we calculated the numerical solution of MV-FIE in$(1+1)$ dimensional in the space $L_2[-1,1] \times C[0,T], T < 1$. By utilizing separation of variables procedure the mixed integral equation is converted to FIE of the second kind with variable coefficients in time. This procedure of separation, helps the researcher in choosing the appropriate time function to establish the condition of convergence of overcoming the problem and getting its solution. Subsequently by utilization Boubaker polynomials procedure, we get a LAS. The BFP hypothesis has introduced to calculate the existence and uniqueness of the solution of the LAS. Also convergence of the solution and stability of the error are discussed. By comparison the outcomes we note following.

1. In the first example we consider MV-FIE with weakly singular kernel, a comparison was made between the Boubaker method and Toeplitz matrix method which is used in [21] in Table 1, at the

same time $T = 0.3$ and the same values of Poisson coefficients $\nu = 0.12, 0.73$, the following was noted at the point $x = -1$, $\nu = 0.12$, the error of our method is given by $1.740746923 \times 10^{-10}$ while the error of method used in [3], at the same value of x, ν is 3.67468×10^{-5} we notice a large difference in the error, and this difference is observed for all values of x in [-1,1]. This difference was also noticed in the error when $\nu = 0.73$. This shows that Boubaker method is more accurate than Toeplitz matrix method. Also the numerical results of Example 6.1, are presented in Figures 1 and 2.

2. In the second example, for Table 2, a comparison was made of the errors at some different values of time $T = 0.01, 0.2, 0.8$, for example, at $x = -0.6$ it was noticed that the error started with $4.706521292 \times 10^{-15}$ at $T = 0.01$, it increases at $T = 0.2$ and becomes $3.568938157 \times 10^{-11}$, i.e., the difference equal to 10^{-4} also the error increase at $T = 0.8$ and becomes $4.173224958 \times 10^{-10}$ this indicates that the error is increasing with increasing time. Comparison between the error for different values of time are presented in Figures 2, 3, and 4.
3. In the third example we consider MV-FIE with continuous kernel, in Table 3, for example at $x = 0.5, T = 0.01, 0.2, 0.8$ the errors are $3.631203 \times 10^{-12}, 1.3075846 \times 10^{-11}, 1.223717355 \times 10^{-9}$, respectively, we note that also the error increases with increase of T. Similarly the same results are in Example 6.4.
4. CPU time in Example 6.1 is 0.12s, in Example 6.2 is 0.06s, in Example 6.3 is 0.07s, and in Example 6.4 is 0.09s, and the memory is 30.37M.

Future work

We will consider the nonlinear MV-FIE in the form

$$u(x,t) - \lambda_1 \int_0^t \int_{-1}^1 k(|x-y|)\varphi(t,\tau)\gamma(u(y,\tau))\,dy\,d\tau - \lambda_2 \int_{-1}^1 k(|x-y|)\gamma(u(y,t))\,dy - \lambda_3 \int_0^t \psi(t,\tau)\gamma(u(x,\tau))\,d\tau = f(x,t).$$

References

[1] M. A. Abdel-aty, M. A. Abdou, A. A. Soliman, *Solvability of quadratic integral equations with singular kernel*, Izv. Nats. Akad. Nauk Armenii Mat., **57** (2022), 3–18. 1
[2] M. A. Abdou, *Fredholm-Volterra integral equation and generalized potential kernel*, Appl. Math. Comput., **131** (2002), 81–94. 2
[3] M. A. Abdou, *On a symptotic methods for Fredholm-Volterra integral equation of the second kind in contact problems*, J. Comput. Appl. Math., **154** (2003), 431–446. 2, 2, 1
[4] M. A. Abdou, F. M. Alharbi, *Generalized main theorem of spectral relationships for logarithmic kernel and its applications*, J. Comput. Theor. Nanosci., **16** (2019), 2704–2711. 1
[5] M. A. Abdou, A. A. Badr, *On a method for solving an integral equation in the displacement contact problem*, Appl. Math. Comput., **127** (2002), 65–72. 2
[6] M. A. Abdou, M. Basseem, *Thermopotential function in position and time for a plate weakened by curvilinear hole*, Arch. Appl. Mech., **92** (2022), 867–883. 1
[7] M. A. Abdou, S. Raad, S. Al-Hazmi, *Fundamental contact problem and singular mixed integral equation*, Life Sci. J., **8** (2014), 323–329. 1
[8] M. A. Abdou, F. A. Salama, *Volterra-Fredholm integral equation of the first kind and spectral relationships*, Appl. Math. Comput., **153** (2004), 141–153. 2, 4.2
[9] A. M. Al-Bugami, *Numerical treating of mixed integral equation two-dimensional in surface cracks in finite layers of materials*, Adv. Math. Phys., **2022** (2022), 12 pages. 1
[10] V. M. Aleksandrovsk, E. V. Covalenko, *Problems in the Mechanics of Continuous Media with Mixed Boundary Conditions*, Nuka, Moscow, (1986). 1
[11] S. E. Alhazmi, *New model for solving mixed integral equation of the first kind with generalized potential kernel*, J. Math. Res., **9** (2017), 18–29. 1
[12] S. E. Alhazmi, M. A. Abdou, *A Physical phenomenon for the fractional nonlinear mixed integro-differential equation using a general discontinuous kernel*, Fractal Fract., **7** (2023), 1–19. 3

[13] S. J. Baksheesh, *Discontinuous Galerkin approximations for Volterra integral equations of the first kind with convolution kernel*, Indian J. Sci. Technol., **8** (2015), 33–36. 1

[14] C. Brezinski, M. Redivo-Zalglia, *Extrapolation methods for the numerical solution of nonlinear Fredholm integral equations*, J. Integral Equ. Appl., **31** (2019), 29–57. 1

[15] O. M. Dada, O. B. Awojoyogbe, M. Agida, K. Boubaker, *Variable Separation and Boubaker Polynomial Expansion Scheme for Solving the Neutron Transport Equation*, Phys. Int., **2** (2011), 25–30. 1

[16] S. Davaeifar, J. Rashidinia, *Boubaker polynomials collocation approach for solving systems of nonlinear Volterra-Fredholm integral equations*, J. Taibah Univ. Sci., **11** (2017), 1182–1199. 1

[17] T. Diego, P. Lima. *Super convergence of collocation methods for class of weakly singular integral equations*, J. Comput. Appl. Math., **218** (2008), 307–346. 1

[18] J. Gao, M. Condon, A. Iserles, *Spectral computation of highly oscillatory integral equations in laser theory*, J. Comput. Phys., **395** (2019), 351–381. 1

[19] A. Hadjadj, J.-P. Dussauge, *Shock wave boundary layer interaction*, Shock Waves, **19** (2009), 449–452. 1

[20] R. M. Hafez, Y. H. Youssri, *Spectral Legendre-Chebyshev treatment lizedof 2D linear and nonlinear mixed Volterra-Fredholm integral equation*, Math. Sci. Lett., **9** (2020), 37–47. 1

[21] A. R. Jan. *An asymptotic model for solving mixed integral equation in position and time*, J. Math., **2022** (2022), 11 pages. 2, 6.1, 1, 1

[22] A. A. Khajehnasiri, R. Ezzati, *Boubaker polynomials and their applications for solving fractional two-dimensional nonlinear partial integro-differential Volterra integral equations*, Comput. Appl. Math., **41** (2022), 1–18. 1

[23] E. Kreyszig. *Introductory functional analysis with applications*, John Wiley & Sons, New York-London-Sydney, (1978). 4, 4.1

[24] M. Lienert, R. Tumulka, *A new class of Volterra-type integral equations from relativistic quantum physics*, J. Integral Equations Appl., **31** (2019), 535–569. 1

[25] A. M. S. Mahdy, M. A. Abdou, D. Sh. Mohamed. *Computational methods for solving higher-order (1 + 1) dimensional mixed-difference integro-differential equations with variable coefficients*, Mathematics, **11** (2023), 1–25. 3

[26] A. M. S. Mahdy, A. S. Nagdy , K. M. Hashem, D. S. Mohamed, *A computational technique for solving three-dimensional mixed Volterra-Fredholm integral equations*, Fractal Fract., **7** (2023), 14 pages. 1

[27] A. M. S. Mahdy, D. Shokry, Kh. Lotfy, *Chelyshkov polynomials strategy for solving 2-dimensional nonlinear Volterra integral equations of the first kind*, Comput. Appl. Math., **41** (2022), 13 pages. 1

[28] R. T. Matoog, *Treatments of probability potential function for nuclear integral equation*, J. Phys. Math., **8** (2017), 1–7. 1

[29] R. T. Matoog, *The behavior of the generalized potential kernel of axisymmetric contact problems and the structure resolvent of the fundamental problems*, Int. J. Recent Sci. Res., **4** (2016), 11535–11541. 1

[30] S. Micula, *Numerical Solution of Two-Dimensional Fredholm-Volterra Integral Equations of the Second Kind*, Symmetry, **13** (2021), 1–12. 1

[31] F. Mirzaee, S. F. Hoseini. *Application of Fibonacci collocation method for solving Volterra-Fredholm integral equations*, Appl. Math. Comput., **273** (2016), 637–644. 1

[32] M. Mohammad, A. Trounev, M. Alshbool, *A Novel numerical method for solving fractional diffusion-wave and nonlinear Fredholm and Volterra integral equations with zero absolute error*, Axioms, **10** (2021), 1–16. 1

[33] S. Nemati, P. M. Lima, Y. Ordokhani, *Numerical solution of a class of two-dimensional nonlinear Volterra integral equations using Legendre polynomials*, J. Comput. Appl. Math., **242** (2013), 53–69. 1

[34] S. Noeiaghdam, S. Micula, *A novel method for solving second kind Volterra integral equations with discontinuous kernel*, Mathematics, **9** (2021), 1–12. 1

[35] G. Y. Popov, *Contact problems for a linearly deformable foundation*, Vishcha Schola., Kiev-Odessa, Kiev-Odessa, (1982). 1

[36] E. Providas, *A unified formulation of analytical and numerical methods for solving linear Fredholm integral equations*, Algorithms, **14** (2021), 1–20. 1

[37] K. Wang, Q. Wang, *Taylor polynomial method and error estimation for a kind of mixed Volterra-Fredholm integral equations*, Appl. Math. Comput., **229** (2014), 53–59. 1

[38] S. Yalçınbaş, T. Akkaya, *A numerical approach for solving linear integro-differential difference equations with Boubaker polynomial bases*, Ain Shams Eng. J., **3** (2012), 153–161. 1

[39] U. Yücel, K. Boubaker, *The Boubaker Polynomials expansion scheme for solving applied physics nonlinear high-order differential equations*, Stud. Nonlinear Sci., **1** (2010), 1–7. 1

[40] T. Zisis, P. A. Gourgiotis, K. P. Baxevanakis, H. G. Georgiadis, *Some basic contact problem in couple stress elasticity*, Int. J. Solids Struct., **51** (2014) 2084–2095. 1